Manipulator Dynamics 2
Forward Dynamics

Problem

Given: Joint torques and links
geometry, mass, inertia, friction

Compute: Angular acceleration of the links (solve differential equations)

Solution

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

\[ \tau = M(\Theta)\dddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta}) \]
Inverse Dynamics

Problem

*Given:* Angular acceleration, velocity and angles of the links in addition to the links geometry, mass, inertia, friction

*Compute:* Joint torques

Solution

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

\[ \tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta}) \]
To solve the Newton and Euler equations, we’ll need to develop mathematical terms for:

- The linear acceleration of the center of mass: $\dot{v}_c$
- The angular acceleration: $\dot{\omega}$
- The Inertia tensor (moment of inertia): $^cI$
- The sum of all the forces applied on the center of mass: $F$
- The sum of all the moments applied on the center of mass: $N$

\[
F = m\dot{v}_c
\]
\[
N = ^cI\dot{\omega} + \omega \times ^cI\omega
\]
Iterative Newton-Euler Equations - Solution Procedure

• **Step 1** - Calculate the link velocities and accelerations iteratively from the robot’s base to the end effector

• **Step 2** - Write the Newton and Euler equations for each link.

Outward iterations: $i: 0 \rightarrow 5$

\[
\begin{align*}
\omega_{i+1} &= R^i \omega_i + \dot{\theta}_{i+1} Z_{i+1}, \\
\dot{\omega}_{i+1} &= R^i \dot{\omega}_i + R^i \omega_i \times \dot{\theta}_{i+1} Z_{i+1} + \ddot{\theta}_{i+1} Z_{i+1}, \\
\dot{\nu}_{i+1} &= R^i (\omega_i \times P_{i+1} + \omega_i \times (\omega_i \times P_{i+1}) + \dot{\nu}_i), \\
\nu_{C_{i+1}} &= \omega_{i+1} \times P_{C_{i+1}} + \omega_{i+1} \times (\omega_{i+1} \times P_{C_{i+1}}) + \dot{\nu}_{i+1}, \\
F_{i+1} &= m_{i+1} \dot{\nu}_{C_{i+1}}, \\
N_{i+1} &= C_{i+1} I_{i+1} \omega_{i+1} + \omega_{i+1} \times C_{i+1} I_{i+1} \omega_{i+1}.
\end{align*}
\]
• **Step 3** - Use the forces and torques generated by interacting with the environment (that is, tools, work stations, parts etc.) in calculating the joint torques from the end effector to the robot’s base.

Inward iterations: \( i : 6 \rightarrow 1 \)

\[
\begin{align*}
\dot{i} f_i &= i_{i+1} R_{i+1} f_{i+1} + i F_i, \\
\dot{i} n_i &= i N_i + i_{i+1} R_{i+1} n_{i+1} + i P_{C_i} \times i F_i \\
&\quad + i P_{i+1} \times i_{i+1} R_{i+1} f_{i+1}, \\
\tau_i &= i n_i^T \dot{i} \hat{Z}_i.
\end{align*}
\]
Iterative Newton-Euler Equations - Solution Procedure

- **Error Checking** - Check the units of each term in the resulting equations

- **Gravity Effect** - The effect of gravity can be included by setting $^0\dot{\mathbf{v}}_0 = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.
Moment of Inertia / Inertia Tensor
Moment of Inertia – Intuitive Understanding

\[ F = m\dot{v}_c \]

\[ N = ^cI\dot{\omega} + \omega \times ^cI\omega \]

Easier Rotation

Difficult Rotation
Moment of Inertia – Intuitive Understanding

Easy

Hard

Very Hard
Moment of Inertia – Intuitive Understanding
Moment of Inertia – Intuitive Understanding
Moment of Inertia – Particle – WRT Axis
Moment of Inertia – Solid – WRT Axis
Moment of Inertia – Solid – WRT Frame
Moment of Inertia – Solid – WRT an Arbitrary Axis
Moment of Inertia – Solid – WRT an Arbitrary Axis
Moment of Inertia – Solid – WRT an Arbitrary Axis
Moment of Inertia – Solid – WRT an Arbitrary Axis

- For a rigid body that is free to move in a 3D space there are infinite possible rotation axes
- The intertie tensor characterizes the mass distribution of the rigid body with respect to a specific coordinate system
Inertia Tensor

- For a rigid body that is free to move in a 3D space there are infinite possible rotation axes
- The intertie tensor characterizes the mass distribution of the rigid body with respect to a specific coordinate system
- The intertie Tensor relative to frame \{A\} is express as a matrix

\[
^A I = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{xy} & I_{yy} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{zz}
\end{bmatrix}
\]
Inertia Tensor

\[ \mathbf{A}\mathbf{I} = \begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{xy} & I_{yy} & -I_{yz} \\
-I_{xz} & -I_{yz} & I_{zz}
\end{bmatrix} \]

\[ I_{xx} = \iiint_V (y^2 + z^2) \rho \, dv \]
\[ I_{yy} = \iiint_V (x^2 + z^2) \rho \, dv \]
\[ I_{zz} = \iiint_V (x^2 + y^2) \rho \, dv \]

Mass moments of inertia

\[ I_{xy} = \iiint_V xy \rho \, dv \]
\[ I_{xz} = \iiint_V xz \rho \, dv \]
\[ I_{yz} = \iiint_V yz \rho \, dv \]

Mass products of inertia
Tensor of Inertia – Example

\[ ^A I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \]

- This set of six independent quantities for a given body, depend on the **position and orientation** of the frame in which they are defined.
- We are free to choose the orientation of the reference frame. It is possible to cause the product of inertia to be zero.

\[
\begin{align*}
I_{xy} &= 0 \\
I_{xz} &= 0 \\
I_{yz} &= 0
\end{align*}
\]

Mass products of inertia

\[
^A I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}
\]

- The axes of the reference frame when so aligned are called the **principle axes** and the corresponding mass moments are called the principle **moments of inertia**
Tensor of Inertia – Example
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Tensor of Inertia – Example
Tensor of Inertia – Example
Parallel Axis Theorem – 1D

- The inertia tensor is a function of the position and orientation of the reference frame
- **Parallel Axis Theorem** – How the inertia tensor changes under translation of the reference coordinate system
Parallel Axis Theorem – 3D
Parallel Axis Theorem – 3D
Inertia Tensor
Tensor of Inertia – Example
Rotation of the Inertia Tensor

- Given:
  - The inertia tensor of the a body expressed in frame A
  - Frame B is rotate with respect to frame A
- Find
  - The inertia tensor of frame B
- The angular Momentum of a rigid body rotating about an axis passing through is

\[ H_A = I_A \omega_A \]

- Let’s transform the angular momentum vector to frame B

\[ H_B = _BRH_A \]
Rotation of the Inertia Tensor
The elements for relatively simple shapes can be solved from the equations describing the shape of the links and their density. However, most robot arms are far from simple shapes and as a result, these terms are simply measured in practice.
## Inertia Tensor 2

<table>
<thead>
<tr>
<th>Object</th>
<th>Moment of Inertia Formulae</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slender rod</td>
<td>( I_y = I_z = \frac{1}{6} m L^2 )</td>
</tr>
<tr>
<td>Thin rectangular plate</td>
<td>( I_x = \frac{1}{6} m (b^2 + c^2) ) &lt;br&gt;( I_y = \frac{1}{2} m c^2 ) &lt;br&gt;( I_z = \frac{1}{2} m b^2 )</td>
</tr>
<tr>
<td>Rectangular prism</td>
<td>( I_x = \frac{1}{6} m (b^2 + c^2) ) &lt;br&gt;( I_y = \frac{1}{2} m (c^2 + a^2) ) &lt;br&gt;( I_z = \frac{1}{2} m (a^2 + b^2) )</td>
</tr>
<tr>
<td>Thin disk</td>
<td>( I_x = \frac{1}{4} m r^2 ) &lt;br&gt;( I_y = I_z = \frac{1}{4} m r^2 )</td>
</tr>
</tbody>
</table>
### Circular cylinder

<table>
<thead>
<tr>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_x = \frac{1}{2}ma^2$</td>
</tr>
<tr>
<td>$I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$</td>
</tr>
</tbody>
</table>

### Circular cone

<table>
<thead>
<tr>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_x = \frac{3}{10}ma^2$</td>
</tr>
<tr>
<td>$I_y = I_z = \frac{3}{5}m\left(\frac{1}{4}a^2 + h^2\right)$</td>
</tr>
</tbody>
</table>

### Sphere

<table>
<thead>
<tr>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_x = I_y = I_z = \frac{2}{5}ma^2$</td>
</tr>
</tbody>
</table>
Inertia Tensor – Robotic Links
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