Chapter 2  First Order DE

\[ \frac{dy}{dt} = f(t, y) \quad \text{or} \quad \frac{dy}{dt} + pt y = g(t) \]

If we can't separate variable and integrate we can use an integrating factor \( \mu(t) \)

Procedure
- Multiply through by \( \mu(t) \)
- Product rule for derivatives

\[ \frac{d}{dt} [\mu(t) y] = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt} y \]

First term set equal to second term

Example

\[ \frac{dy}{dt} + ay = g(t) \]

\[ \mu(t) \frac{dy}{dt} + a \mu(t) y = g(t) \frac{dy}{dt} \]

First Term

\[ a \mu(t) y = \frac{d\mu(t)}{dt} y \]

Separate variables can be a factor
If \( a \) is a function of \( t \), \( a = p(t) \)

\[
\ln |M(t)| = \int p(t) \, dt + C
\]

\[
M(t) = e^{\int p(t) \, dt}
\]

If you use these equations for integrating factor make sure the initial form of the DE is correct!!

Finally we can solve

\[
\frac{d}{dt} [M(t)y] = M(t) g(t)
\]

by direct integration.

Example: pg 40 problem 15

put eq. in correct form...

\( ty' + dy = t^2 - t + 1 \quad \Rightarrow \quad y' + \frac{2}{t} y = t - 1 + \frac{1}{t} \)
\[
\frac{d}{dt} \left[ t^3 y \right] = t^3 - t^2 + t
\]
\[
y = \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + c
\]
\[
y = \frac{t^2}{4} - \frac{t}{3} + \frac{1}{2} + \frac{c}{t^2}
\]

Applying \( t = 1 \) \( y(1) = \frac{1}{2} \) gives \( c = \frac{1}{12} \).
Separable Equations

\[ \frac{dy}{dx} = f(x, y) \]

What if non-linear

\[ M(x, y) + N(x, y) \frac{dy}{dx} < 0 \]

If \( M \) is \( fn \ x \)

\( N \) is \( fn \ y \)

\[ \Rightarrow M(x) + N(y) \frac{dy}{dx} = 0 \]

Seperable

\[ M(x) \; dx + N(y) \; dy = 0 \]

Example prob 3 pg 48

\[ y^1 + y^2 \sin x = 0 \]

Non-linear!

\[ \frac{1}{y^2} \; dy = -\sin x \; dx \]
Modeling w/ FODE

Example problem 18 pg 63

Box w/ internal temp \( u(t) \).

\[
\frac{du}{dt} = -k [u - T(t)]
\]

\[
\text{ambient Temp.}
\]

\[
\text{const. } \left[ \frac{1}{\rho c p} \right]
\]

\[ T(t) = T_0 + T_1 \cos wt \]

\[
\frac{du}{dt} + ku = kT(t) \quad \text{Linear}
\]

Integrating factor: \( e^{kt} \)

\[
\frac{d}{dt}[e^{kt} u] = k (T_0 + T_1 \cos wt) e^{kt}
\]

\[
e^{kt} u = T_0 e^{kt} + kT_1 \int \cos wt e^{kt} dt + c
\]

Integration by parts.

\[
u(t) = T_0 + kT_1 \left( \frac{k \cos wt + w \sin wt}{k^2 + w^2} \right) + c e^{-kt}
\]
Exact Equations

\[ M(x, y) + N(x, y)y' = 0 \]

General Nonlinear ODE
Homogeneous

This equation is exact if and only if

\[ My(x, y) = Nx(x, y) \]

Then a function \( \psi \) exist where

\[
\begin{align*}
\psi_x(x, y) &= M(x, y) \\
\psi_y(x, y) &= N(x, y)
\end{align*}
\]

\[ \psi(k, y) = C \]

Example problem 1 pg 101

\[ (2x+3) + (2y-3)y' = 0 \]

\[ M = 2x+3, \quad N = 2y-3 \]

\[ My = Nx = 0 \Rightarrow \text{exact} \]

\[ \psi_x = M = 2x+3 \]

\[ \psi_y = N = 2y-3 \]

\[ \psi = 2x+3 + k(y) \]
If equation is not exact an integrating factor can be used.

\[ M(x,y)M(x,y) + M(x,y)N(x,y) y' = 0 \]

\[ (uM)_y = (uN)_x \]

\[ M N_y - N N_x + (M y - N x) m = 0 \]

Just as hard to solve

Assume \( m \) is function of one variable

\[ M = M(x) \rightarrow \frac{dm}{dx} = \frac{M y - N x}{N} \]

\[ M = M(y) \rightarrow \frac{dm}{dM} = \frac{N x - M y}{M} \]

Example: prob. 25 pg 102.

\[ (3 x^2 y + 2 xy + y^2) + (x^2 + y^3) y' = 0 \]

\[ \frac{M y - N x}{M} = 3 \text{ for only of } x \rightarrow m = e^{3x} \]
\[ Y_x = \mu M = e^{3x} \left( 3x^2 y + 2xy + y^3 \right) \]
\[ Y = (x^2 y + y^3) e^{3x} + h(y) \]
\[ Y_y = (x^2 + y^3) e^{3x} + h'(y) = \mu N = e^{3y} \left( x^2 - y^3 \right) \]
\[ h'(y) = 0 \]
\[ Y = (3x^2 y + y^3) e^{3x} = c \]