Class Notes 9:

Power Series (1/3)

82A – Engineering Mathematics
Second Order Differential Equations – Series Solution

- Solution Anatomy

Differential Equation

\[ =0, \text{ Homogeneous} \]
\[ =g(t), \text{ Non-homogeneous} \]

Solution: \( y = y_c + y_p \)

Homogeneous \quad \text{Non-homogeneous}

Fundamental solutions

\[ \rightarrow \text{D.E. with constant coefficients} \rightarrow \text{Elementary function} \]
\[ \rightarrow \text{D.E. with variable coefficients} \rightarrow \text{Power Series} \]
Famous Differential Equation - Power Series

- Airy Eq. \( y'' - xy = 0 \)
- Chebych Eq. \( (1 - x^2) y'' - xy' + \lambda^2 y = 0 \)
- Hermit Eq. \( y'' - 2xy' + \lambda y = 0 \)
- Bessel Eq. \( t^2 y'' - ty' + (t^2 - \lambda^2) y = 0 \)
- Euler Eq. \( x^2 y'' - axy' + by = 0 \)
- Legendre Eq. \( (1 - x^2) y'' - 2xy' + \alpha(\alpha + 1) y = 0 \)
Power Series – Definition

• A power series in \((x-x_0)\) is the infinite series of the form

\[
\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + \ldots
\]

also known as a power series centered at \(x_0\)
Example - A Power Series Solution

\[ y = x + x_0 \]

\[ x_0 = 1 \]

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
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<tbody>
<tr>
<td>-2</td>
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<td>2</td>
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</tbody>
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shift to the left by \( x_0 \)

\[ y = x - x_0 \]

\[ x_0 = 1 \]

<table>
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shift to the right by \( x_0 \)
Power Series – Expansion Point at $x_0$

- A power series of Sin centered at $x_0$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

- As the degree of the Taylor polynomial rises, it approaches the correct function. This image shows $\sin x$ and its Taylor approximations, polynomials of degree 1, 3, 5, 7, 9, 11 and 13.
Power Series – Expansion Point at $x_0$

- A power series of exponent centered at $x_0$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
Power Series – Expansion Point at $x_0$ - Examples

- A power series developed around $(x=x_0)$
  \[
  \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots
  \]

- A power series developed around $(x=x_0=0)$
  \[
  \sum_{n=0}^{\infty} a_n(x)^n = a_0 + a_1(x) + a_2(x)^2 + \cdots
  \]

- Example - Power series centered around -1
  \[
  \sum_{n=0}^{\infty} (x+1)^n = \sum_{n=0}^{\infty} (x+(-1))^n
  \]

- Example - Power series centered around 0
  \[
  \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n (x-0)^n
  \]
The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full period centered at the origin.

- Error - for $-1 < x < 1$, the error is less than 0.000003
- Radius of Convergence - $-\infty < x < \infty$
Interval / Radius of Convergence & Error

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n
\]

- **Error** - The Taylor polynomials for \(\log(1 + x)\) only provide accurate approximations in the range \(-1 < x \leq 1\). Note that, for \(x > 1\), the Taylor polynomials of higher degree are **worse** approximations.
- **Radius of Convergence** - \(-1 < x < 1\)
Radius of Convergence

$x = x_0 - \rho$

Expansion point $x_0$

$x = x_0 + \rho$

Absolute Convergent

Interval of Convergence

Divergent
Interval of Convergence

- The interval of convergence is the set of \textbf{all} real numbers of $x$ for which the series converges.
Radius of Convergence

- The radius $\rho$ of the interval of convergence of a power series is called its **radius of convergence**

- If $\rho > 0$ - The power series converges if $|x - x_0| < \rho$ or $(-\rho < x - x_0 < \rho)$
  
  diverges if $|x - x_0| > \rho$ or $(x - x_0 > \rho$ or $x - x_0 < -\rho)$

- If $\rho = 0$ - The power series converges only at $x_0$

- If $\rho = \infty$ - The power series converges for all $x$
Convergence

- A power series is convergent at a specified value of $x$ if its sequence of partial sum $\{s_n(x)\}$ converges

$$\lim_{n \to \infty} s_n(x) = \lim_{n \to \infty} a_n (x - x_0)^n$$

$$\begin{aligned}
&\begin{cases}
\text{exist} \rightarrow \text{converge} \\
\text{doesn't exist} \rightarrow \text{diverge}
\end{cases}
\end{aligned}$$
Absolute Convergence Of Power Series

- Absolute Convergence of Power Series
  - A power series \[ \sum_{n=0}^{\infty} a_n (x - x_0)^n \] is said to converge absolutely at a point \( x \) if

\[
\sum_{n=0}^{\infty} |a_n (x - x_0)^n| = \sum_{n=0}^{\infty} |a_n||x - x_0|^n
\]

converges

- Radius Of Convergence Of A Power Series (PS)
  If a power series about \( x-x_0 \) converges for all values of \( x \) in \( |x - x_0| < \rho \)

Then \( \rho \) is said to be radius of convergence of the PS
Determine The Radius Of Convergence ($\rho$) For A Given Power Series (Ratio Test)

- If $a_n \neq 0$
- If for a fixed value of $x$

$$\lim_{n \to \infty} \left| \frac{a_{n+1} (x - x_0)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_0| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = |x - x_0| \frac{1}{\rho} = L$$

$$\rho = \frac{1}{\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

- Then the power series at that value of $x$
  1) Converges if $L = |x - x_0| \frac{1}{\rho} < 1 \Leftrightarrow |x - x_0| < \rho$
  2) Diverges if $L = |x - x_0| \frac{1}{\rho} > 1 \Leftrightarrow |x - x_0| > \rho$
  3) Inconclusive if $L = |x - x_0| \frac{1}{\rho} = 1 \Leftrightarrow |x - x_0| = \rho$
Determine The Radius Of Convergence ($\rho$) For A Given Power Series (Ratio Test) – Example

- Find which values of $x$ does power series converges

$$
\sum_{n=1}^{\infty} (-1)^{n+1} n(x-2)^n
$$

$$
\lim_{n \to \infty} \left| \frac{(-1)^{n+2} (n+1)(x-2)^{n+1}}{(-1)^{n+1} n(x-2)^n} \right| = \lim_{n \to \infty} \left| \frac{(-1)(-1)^{n+1} (n+1)(x-2)(x-2)^n}{(-1)^{n+1} n(x-2)^n} \right|
$$

$$
= \left| x - 2 \right| \lim_{n \to \infty} \frac{n+1}{n} < 1 \Rightarrow \left| x - 2 \right| < 1
$$

- Thus

$$
\begin{cases}
  x - 2 < 1 \Rightarrow x < 3 \\
  -1 < x - 2 \Rightarrow 1 < x
\end{cases}
$$

Thus converges
Determine The Radius Of Convergence (ρ) For A Given Power Series (Ratio Test) – Example (Continue)

• For \( x = 1 \)
  \[
  \sum_{n=1}^{\infty} (-1)^{n+1} n(-1)^n \rightarrow \text{Diverges}
  \]

• For \( x = 3 \)
  \[
  \sum_{n=1}^{\infty} (-1)^{n+1} n(1)^n \rightarrow \text{Diverges}
  \]

• The radius of convergence is \( \rho = 1 \)
Power Series of a Given Function

- If for a given $x$ the limit

$$\lim_{m \to \infty} \sum_{n=0}^{m} a_n (x - x_0)^n$$

exist
- Then the series is said to be power series expansion of $f(x)$

$$a_0 = f(x_0)$$

The series converges for $x = x_0$
It may converge for all $x$
It may converge for some value of $x$ and not for others
Power Series of a Given Function

• A power series defines a function that is

\[ f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \]

whose domain is in the interval of convergence of the series

• If the radius of convergence is
  – R>0
  – R=\infty

• Then \( f(x) \) on the intervals is
  – Differentiable
  – Continuous
  – Integrable

• Convergence at an end point may be
  – lost by differentiation
  – gain by integration
Analytical Functions & Power Series

• **Analytical Function – Definition**
  A function $f(x)$ is said to be analytic at $x=x_0$ if $f(x)$ can be differentiated at any number of times.

• For an analytic function

  $$a_n = \frac{d^n}{dx^n}[f(x)]_{x=x_0}$$

  exists $a_n$ bounded for all $n$
Analytical Functions & Power Series

\[ y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]

\[ y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \]

Note that for \( n=0 \) the first term is 0. Start summing from 1

\[ y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = 2a_2 + 6a_3 x + 12a_4 x^2 + \cdots \]

Note that for \( n=0, n=1 \) the first and the second terms are 0. Start summing from 2
Power Series (PS)
Representation of an Analytic Function

• An analytic function $f(x)$ has a power series representation within the
domain of convergence an $f(x)$ can be written as

$$ f(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}[f(x)]_{x=x_0} (x-x_0)^n = \sum_{n=0}^{\infty} a_n (x-x_0)^n $$

within the domain of convergence $|x-x_0| < \rho$

$x_0$ – The expansion point of the PS
Taylor Series

- Suppose that \( \sum a_n (x - x_0)^n \) converges to \( f(x) \) for \( |x - x_0| < \rho \)
- Then the value of \( a_n \) is given by
  \[
  a_n = \frac{f^{(n)}(x_0)}{n!}
  \]
  and the series is called the **Taylor Series** for \( f \) about \( x = x_0 \)
- If
  \[
  f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
  \]
  - \( f(t) \) is continuous
  - Has derivative of all orders on the interval of convergence
  - The derivatives of \( f \) can be computed by differentiating the relevant series term by term
Taylor Series

- Taylor Series: for a point \(x_0 \neq 0\)

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \cdots
\]

- Maclaurin Series: for a point \(x_0 = 0\)

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots
\]
PS Expansions of Analytical Function (Maclaurine Series)

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
\]

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n
\]

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n
\]

These results can be used to obtain power series representations of other functions

e.g. \( e^{2x} \) (replace \( x \rightarrow x^2 \))

\[
e^{2x} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}
\]

**Interval of convergence**

\(-\infty < x < \infty\)
PS Expansions of Analytical Function (Maclaurine Series)

To obtain a Taylor series representation of $\ln x$ centered at $x_0 = 1$

Replace $x \to x - 1$

$$\ln x = \ln(1 + (x - 1)) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n$$

Interval of convergence is shifted by 1 from $(-1, 1]$ to $(0, 2]$
Arithmetic of Power Series

- **Multiplication of Power Series**

\[
e^x \sin x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right) \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right)
\]

- **Addition of power series**

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1}
\]

- **Shifting an index of summation**

\[
\sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2}
\]
Arithmetic of Power Series – Multiplication

\[ e^x \sin x = \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots \right) \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots \right) \]

\[ = (1) x + (1) x^2 + \left( -\frac{1}{6} + \frac{1}{2} \right) x^3 + \left( -\frac{1}{6} + \frac{1}{6} \right) x^4 + \left( \frac{1}{120} - \frac{1}{12} + \frac{1}{24} \right) x^5 \]

\[ = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} + \cdots \]

since \( e^x \) and \( \sin x \) both converge on \(( -\infty , \infty )\) the product converges on the same interval.
Arithmetic of Power Series – Addition

\[ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} \]

1. Both series should start with the same power

2. Both indices of summation should start with the same number
Arithmetic of Power Series – Addition

1. Both series should start with the same power

\[
\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k
\]

\[
k = n - 2 \quad \text{(} n = k + 2 \text{)}
\]

\[
k = n + 1 \quad \text{(} n = k - 1 \text{)}
\]
2. Both indices of summation should start with the same number

\[ \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k \]

\[ 2 \cdot 1 \cdot a_2 x^0 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k = 2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + a_{k-1}]x^k \]
Arithmetic of Power Series –
Shifting Index of Summation

\[ \sum_{n=2}^{\infty} a_n (x)^n = \sum_{m=0}^{\infty} a_{m+2} (x)^{m+2} \]

- The index of summation in an infinite series is a dummy parameter.
Arithmetic of Power Series – Rewriting Generic Term

\[ \sum_{n=2}^{\infty} (n+2)(n+1)a_n(x-x_0)^{n-2} = \sum_{m=0}^{\infty} (m+4)(m+3)a_{m+2}(x-x_0)^m \]

Generic term

\[ m = n - 2 \]
\[ n = m + 2 \]

\[ n = 2 \text{ corresponds to } m = 0 \]
Arithmetic of Power Series – Rewriting Generic Term

put $x^2$ into the sum

$$x^2 \sum_{n=0}^{\infty} (r + n)a_n x^{r+n-1} = \sum_{n=0}^{\infty} (r + n)a_n x^{n+r+1} = \sum_{m=1}^{\infty} (r + m - 1)x^{r+m}$$

$$x^2 x^{r+m-1} = x^{r+m-1+2} = x^{r+m+1}$$

$$m = n + 1$$
$$n = m - 1$$
Series Equality

- If two power series are equal

\[ \sum_{n=1}^{\infty} a_n (x - x_0)^n = \sum_{n=1}^{\infty} b_n (x - x_0)^n \]

for each \( x \) in some open interval with center \( x(0) \)

Then \( a_n = b_n \) for \( n = 0, 1, 2, \ldots \)
Determining Coefficients

Assume \[ \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n \]

- What this implies about the coefficients
- Rewriting both series with the same power of x

\[ \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{m=0}^{\infty} (m+1)a_{m+1} x^m = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n \]

\[ n - 1 = m \quad \text{replace } m \rightarrow n \]
\[ n = m + 1 \]

\[ (n+1)a_{n+1} = a_n \rightarrow a_{n+1} = \frac{a_n}{n+1} \quad \text{for } n = 0, 1, 2, 3, \ldots \]
Determining Coefficients

\[ a_1 = \frac{a_0}{2} \]

\[ a_2 = \frac{a_1}{3} = \frac{a_0}{2 \times 3} = \frac{a_0}{6} \]

\[ a_3 = \frac{a_2}{4} = \frac{a_0}{4 \times 6} = \frac{a_0}{24} \]

\[ a_n = \frac{a_0}{n!} \]

\[ a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x \]
A Power Series Solution – Example

\[ y' + y = 0 \]

\[ y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]

**Step 1**: calculate derivative of the assumed solution

\[ y' = \sum_{n=1}^{\infty} a_n n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 \]

**Step 2**: substitute \( y \) & \( y' \) into the diff eq.

\[ y' + y = \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0 \]
A Power Series Solution – Example

Step 3: shift indices of summation

\[
y' + y = \sum_{n=1}^{\infty} a_n nx^{n-1} + \sum_{n=0}^{\infty} a_n x^n
\]

\[
= \sum_{k=0}^{\infty} a_{k+1} (k+1) x^k + \sum_{k=0}^{\infty} a_k x^k
\]

\[
= \sum_{k=0}^{\infty} [a_{k+1} (k+1) + a_k] x^k
\]

Step 4: Because \( y' + y = 0 \) for all \( x \) in some interval

\[
\sum_{k=0}^{\infty} [a_{k+1} (k+1) + a_k] x^k = 0
\]

\[
a_{k+1} (k+1) + a_k = 0
\]
A Power Series Solution – Example

\[ a_{k+1} = -\frac{a_k}{(k+1)} \]

\[ a_1 = -\frac{1}{1} a_0 = -a_0 \]

\[ a_2 = -\frac{1}{2} a_1 = -\frac{1}{2} (-a_0) = \frac{1}{2} a_0 \]

\[ a_3 = -\frac{1}{3} a_2 = -\frac{1}{3*2} a_0 \]

\[ a_4 = -\frac{1}{4} a_3 = -\frac{1}{4*3*2} a_0 \]

**Step 5:** Define the solution

\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = a_0 - a_0 x + \frac{1}{2} a_0 x^2 - \frac{1}{3*2} a_0 x^3 + \frac{1}{4*3*2} a_0 x^4 \]

\[ = a_0 \left[ 1 - x + \frac{1}{2} x^2 - \frac{1}{3*2} x^3 + \frac{1}{4*3*2} x^4 \right] = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k \]

\[ y = a_0 e^{-x} \]
A Power Series Solution – Example

\[
\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!}
\]
Famous Series Solutions

Air y Eq. \( y'' - xy = 0 \)

Chebychev Eq. \( (1 - x^2) y'' - xy' + \lambda^2 y = 0 \)

Hermite Eq. \( y'' - 2xy' + \lambda y = 0 \)

Bessel's Eq. \( t^2 x'' + tx' + (t^2 - \lambda^2) x = 0 \)

Euler's Eq. \( x^2 y'' + axy' + \beta y = 0 \)

Legendre's Eq. \( (1 - x^2) y'' - 2xy' + \alpha(\alpha + 1)y = 0 \)