An \textbf{nth order ODE} has the general form

\[ P_0(t) \frac{d^n y}{dt^n} + P_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + P_{n-1}(t) \frac{dy}{dt} + P_n(t) y = G(t) \]

We assume that \( P_0, \ldots, P_n, \) and \( G \) are continuous real-valued functions on some interval \( I = (\alpha, \beta) \), and that \( P_0 \) is nowhere zero on \( I \).

Dividing by \( P_0 \), the ODE becomes

\[ L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t) \]

For an \textbf{nth order ODE}, there are typically \( n \) initial conditions:

\[ y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \]
Theorem 4.1.1

• Consider the \( n \)th order initial value problem

\[
\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t)
\]

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y^{(n-1)}_0
\]

• If the functions \( p_1, \ldots, p_n \), and \( g \) are continuous on an open interval \( I \), then there exists exactly one solution \( y = \phi(t) \) that satisfies the initial value problem. This solution exists throughout the interval \( I \).
Homogeneous Equations

• As with 2\textsuperscript{nd} order case, we begin with homogeneous ODE:

\[ L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = 0 \]

• If \( y_1, \ldots, y_n \) are solns to ODE, then so is linear combination

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) \]

• Every soln can be expressed in this form, with coefficients determined by initial conditions, iff we can solve:

\[
\begin{align*}
    c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\
    c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\
    &\vdots \\
    c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{align*}
\]
Homogeneous Equations & Wronskian

• The system of equations on the previous slide has a unique solution iff its determinant, or Wronskian, is nonzero at $t_0$:

$$W(y_1, y_2, \ldots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix}$$

• Since $t_0$ can be any point in the interval $I$, the Wronskian determinant needs to be nonzero at every point in $I$.

• As before, it turns out that the Wronskian is either zero for every point in $I$, or it is never zero on $I$. 
Theorem 4.1.2

• Consider the $n$th order initial value problem

\[
\frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t)y = 0
\]

\[
y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y^{(n-1)}
\]

• If the functions $p_1, \ldots, p_n$ are continuous on an open interval $I$, and if $y_1, \ldots, y_n$ are solutions with $W(y_1, \ldots, y_n)(t) \neq 0$ for at least one $t$ in $I$, then every solution $y$ of the ODE can be expressed as a linear combination of $y_1, \ldots, y_n$:

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
\]
Linear Dependence and Independence

• Two functions \( f \) and \( g \) are **linearly dependent** if there exist constants \( c_1 \) and \( c_2 \), not both zero, such that

\[
 c_1 f(t) + c_2 g(t) = 0
\]

for all \( t \) in \( I \). Note that this reduces to determining whether \( f \) and \( g \) are multiples of each other.

• If the only solution to this equation is \( c_1 = c_2 = 0 \), then \( f \) and \( g \) are **linearly independent**.

• For example, let \( f(x) = \sin 2x \) and \( g(x) = \sin x \cos x \), and consider the linear combination

\[
 c_1 \sin 2x + c_2 \sin x \cos x = 0
\]

This equation is satisfied if we choose \( c_1 = 1, \ c_2 = -2 \), and hence \( f \) and \( g \) are linearly dependent.
Example 1

- Are the following functions linearly independent or dependent on the interval I: $0 < t < \infty$
  
  \[ f_1(t) = 1, \quad f_2(t) = t, \quad f_3(t) = t^2 \]

- Form the linear combination and set it equal to zero
  
  \[ k_1 + k_2 t + k_3 t^2 = 0 \]

- Evaluating this at $t = 0$, $t = 1$, and $t = 1$, we get
  
  \[ k_1 = 0 \]
  \[ k_1 + k_2 + k_3 = 0 \]
  \[ k_1 - k_2 + k_3 = 0 \]

- The only solution to this system is \( k_1 = k_2 = k_3 = 0 \)

- Therefore, the given functions are linearly independent
Example 2

- Are the following functions linearly independent or dependent on any interval I:
  \[ f_1(t) = 1, \ f_2(t) = 2 + t, \ f_3(t) = 3 - t^2, \ f_4(t) = 4t + t^2 \]
- Form the linear combination and set it equal to zero
  \[ k_1 + k_2 (2 + t) + k_3 (3 - t^2) + k_4 (4t + t^2) = 0 \]
- Evaluating this at \( t = 0, t = 1, \) and \( t = -1, \) we get
  \[ k_1 + 2k_2 + k_3 = 0 \]
  \[ k_2 + 4k_4 = 0 \]
  \[ -k_3 + k_4 = 0 \]
- There are many nonzero solutions to this system of equations
- Therefore, the given functions are linearly dependent
Theorem 4.1.3

• If \( \{y_1, \ldots, y_n\} \) is a fundamental set of solutions of
\[
L(y) = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0
\]
on an interval \( I \), then \( \{y_1, \ldots, y_n\} \) are linearly independent on that interval.

• Conversely, if \( \{y_1, \ldots, y_n\} \) are linearly independent solutions to the above differential equation, then they form a fundamental set of solutions on the interval \( I \).
Fundamental Solutions & Linear Independence

- Consider the $n$th order ODE:
  \[ y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0 \]

- A set \( \{y_1, \ldots, y_n \} \) of solutions with \( W(y_1, \ldots, y_n) \neq 0 \) on \( I \) is called a **fundamental set of solutions**.

- Since all solutions can be expressed as a linear combination of the fundamental set of solutions, the **general solution** is
  \[ y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) \]

- If \( y_1, \ldots, y_n \) are fundamental solutions, then \( W(y_1, \ldots, y_n) \neq 0 \) on \( I \). It can be shown that this is equivalent to saying that \( y_1, \ldots, y_n \) are **linearly independent**:
  \[ c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0 \quad \text{iff} \quad c_1 = c_2 = \cdots = c_n = 0 \]
Nonhomogeneous Equations

• Consider the nonhomogeneous equation:

\[
L[y] = \frac{d^n y}{dt^n} + p_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \cdots + p_{n-1}(t) \frac{dy}{dt} + p_n(t) y = g(t)
\]

• If \( Y_1, Y_2 \) are solns to nonhomogeneous equation, then \( Y_1 - Y_2 \) is a solution to the homogeneous equation:

\[
L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0
\]

• Then there exist coefficients \( c_1, \ldots, c_n \) such that

\[
Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)
\]

• Thus the general solution to the nonhomogeneous ODE is

\[
y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) + Y(t)
\]

where \( Y \) is any particular solution to nonhomogeneous ODE.
Consider the $n$th order linear homogeneous differential equation with constant, real coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

As with second order linear equations with constant coefficients, $y = e^{rt}$ is a solution for values of $r$ that make characteristic polynomial $Z(r)$ zero:

$$L[e^{rt}] = e^{rt} \left[ a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n \right] = 0$$

By the fundamental theorem of algebra, a polynomial of degree $n$ has $n$ roots $r_1, r_2, \ldots, r_n$, and hence

$$Z(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_n)$$
Real and Unequal Roots

• If roots of characteristic polynomial $Z(r)$ are real and unequal, then there are $n$ distinct solutions of the differential equation:

$$e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$$

• If these functions are linearly independent, then general solution of differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

• The Wronskian can be used to determine linear independence of solutions.
Example 1: Distinct Real Roots  (1 of 3)

• Consider the initial value problem
  \[ y^{(4)} + y''' - 7y'' - y' + 6y = 0 \]
  \[ y(0) = 1, \ y'(0) = 0, \ y''(0) = -2, \ y'''(0) = -1 \]

• Assuming exponential soln leads to characteristic equation:
  \[ y(t) = e^{rt} \quad \Rightarrow \quad r^4 + r^3 - 7r^2 - r + 6 = 0 \]
  \[ \quad \Leftrightarrow \quad (r - 1)(r + 1)(r - 2)(r + 3) = 0 \]

• Thus the general solution is
  \[ y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t} \]
\[ y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_3 e^{-3t} \]

**Example 1: Solution  (2 of 3)**

- The initial conditions
  \[
  y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2, \quad y'''(0) = -1
  \]
yield
  \[
  \begin{align*}
  c_1 + c_2 + c_3 + c_4 &= 1 \\
  c_1 - c_2 + 2c_3 - 3c_4 &= 0 \\
  c_1 + c_2 + 4c_3 + 9c_4 &= -2 \\
  c_1 - c_2 + 8c_3 - 27c_4 &= -1
  \end{align*}
  \]
- Solving,
  \[
  c_1 = \frac{11}{8}, \quad c_2 = \frac{5}{12}, \quad c_3 = -\frac{2}{3}, \quad c_4 = -\frac{1}{8}
  \]
- Hence
  \[
  y(t) = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t}
  \]
Example 1: Graph of Solution  (3 of 3)

- The graph of the solution is given below. Note the effect of the largest root of the characteristic equation.

\[ y(t) = \frac{11}{8} e^t + \frac{5}{12} e^{-t} - \frac{2}{3} e^{2t} - \frac{1}{8} e^{-3t} \]
Complex Roots

• If the characteristic polynomial $Z(r)$ has complex roots, then they must occur in conjugate pairs, $\lambda \pm i\mu$.
• Note that not all the roots need be complex.
• Solutions corresponding to complex roots have the form

$$e^{(\lambda+i\mu)t} = e^{\lambda t} \cos \mu t + ie^{\lambda t} \sin \mu t$$

$$e^{(\lambda-i\mu)t} = e^{\lambda t} \cos \mu t - ie^{\lambda t} \sin \mu t$$

• As in Chapter 3.4, we use the real-valued solutions

$$e^{\lambda t} \cos \mu t, \ e^{\lambda t} \sin \mu t$$
Example 2: Complex Roots  (1 of 2)

• Consider the initial value problem

\[ y^{(4)} - y = 0, \quad y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -2 \]

• Then

\[ y(t) = e^{rt} \quad \Rightarrow \quad r^4 - 1 = 0 \quad \Leftrightarrow \quad (r^2 - 1)(r^2 + 1) = 0 \]

• The roots are 1, -1, i, -i. Thus the general solution is

\[ y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t) \]

• Using the initial conditions, we obtain

\[ y(t) = 0e^t + 3e^{-t} + \frac{1}{2} \cos(t) - \sin(t) \]

• The graph of solution is given on right.
Example 2: Small Change in an Initial Condition (2 of 2)

• Note that if one initial condition is slightly modified, then the solution can change significantly. For example, replace
  
  \[ y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -2 \]

  with
  
  \[ y(0) = \frac{7}{2}, \quad y'(0) = -4, \quad y''(0) = \frac{5}{2}, \quad y'''(0) = -\frac{15}{8} \]

  then
  
  \[ y(t) = \frac{1}{32} e^t + \frac{95}{32} e^{-t} + \frac{1}{2} \cos(t) - \frac{17}{16} \sin(t) \]

• The graph of this soln and original soln are given below.
Repeated Roots

• Suppose a root \( r_k \) of characteristic polynomial \( Z(r) \) is a repeated root with multiplicity \( s \). Then linearly independent solutions corresponding to this repeated root have the form

\[
e^{r_k t}, \ te^{r_k t}, \ t^2 e^{r_k t}, \ldots, \ t^{s-1} e^{r_k t}
\]

• If a complex root \( \lambda + i\mu \) is repeated \( s \) times, then so is its conjugate \( \lambda - i\mu \). There are \( 2s \) corresponding linearly independent solns, derived from real and imaginary parts of

\[
e^{(\lambda+i\mu)t}, \ te^{(\lambda+i\mu)t}, \ t^2 e^{(\lambda+i\mu)t}, \ldots, \ t^{s-1} e^{(\lambda+i\mu)t}
\]
or

\[
e^{\lambda t} \cos \mu t, \ e^{\lambda t} \sin \mu t, \ te^{\lambda t} \cos \mu t, \ te^{\lambda t} \sin \mu t, \ldots,
\]

\[
t^{s-1} e^{r_k t} \cos \mu t, \ t^{s-1} e^{r_k t} e^{\lambda t} \sin \mu t,
\]
Example 4: Repeated Roots

- Consider the equation
  \[ y^{(4)} + 2y'' + y = 0 \]
- Then
  \[ y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 2r + 1 = 0 \iff (r^2 + 1)(r^2 + 1) = 0 \]
- The roots are \( i, i, -i, -i \). Thus the general solution is
  \[ y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t) \]

Sample Solution: \( y = (1 + t) \cos t + (1 + t) \sin t \)
Example 4: Complex Roots of -1  (1 of 2)

• For the general solution of \( y^{(4)} + y = 0 \), the characteristic equation is \( r^4 + 1 = 0 \).

• To solve this equation, we need to use Euler’s equation to find the four 4\(^\text{th}\) roots of -1:

\[-1 = \cos \pi + i \sin \pi = e^{i\pi} \text{ or} \]

\[-1 = \cos(\pi + 2m\pi) + i \sin(\pi + 2m\pi) = e^{i(\pi + 2m\pi)} \text{ for any integer } m \]

\((-1)^{1/4} = e^{i(\pi+2m\pi)/4} = \cos(\pi/4 + m\pi/2) + i \sin(\pi/4 + m\pi/2)\)

• Letting \( m = 0, 1, 2, \text{ and } 3 \), we get the roots:

\[
\frac{1+i}{\sqrt{2}}, \quad \frac{-1+i}{\sqrt{2}}, \quad \frac{-1-i}{\sqrt{2}}, \quad \frac{1-i}{\sqrt{2}}, \text{ respectively.}
\]
Given the four complex roots, extending the ideas from Chapter 4, we can form four linearly independent real solutions.

For the complex conjugate pair \( \frac{1\pm i}{\sqrt{2}} \), we get the solutions

\[
y_1 = e^{t/\sqrt{2}} \cos(t/\sqrt{2}), \quad y_2 = e^{t/\sqrt{2}} \sin(t/\sqrt{2})
\]

For the complex conjugate pair \( \frac{-1\pm i}{\sqrt{2}} \), we get the solutions

\[
y_3 = e^{-t/\sqrt{2}} \cos(t/\sqrt{2}), \quad y_4 = e^{-t/\sqrt{2}} \sin(t/\sqrt{2})
\]

So the general solution can be written as

\[
c_1y_1 + c_2y_2 + c_3y_3 + c_4y_4
\]
The method of undetermined coefficients can be used to find a particular solution $Y$ of an $n$th order linear, constant coefficient, nonhomogeneous ODE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

provided $g$ is of an appropriate form.

As with 2nd order equations, the method of undetermined coefficients is typically used when $g$ is a sum or product of polynomial, exponential, and sine or cosine functions.

Section 4.4 discusses the more general variation of parameters method.
Example 1

• Consider the differential equation
  \[ y''' - 3y'' + 3y' - y = 4e^t \]
• For the homogeneous case,
  \[ y(t) = e^{rt} \implies r^3 - 3r^2 + 3r - 1 = 0 \iff (r - 1)^3 = 0 \]
• Thus the general solution of homogeneous equation is
  \[ y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t \]
• For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with
  \[ Y(t) = At^3e^{2t} \]
• As in Chapter 3, it can be shown that
  \[ Y(t) = \frac{2}{3}t^3e^{2t} \implies y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^{2t} \]
Example 2

• Consider the equation \( y^{(4)} + 2y'' + y = 3\sin t - 5\cos t \)

• For the homogeneous case,

\[
y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 2r^2 + 1 = 0 \iff (r^2 + 1)(r^2 + 1) = 0
\]

• Thus the general solution of the homogeneous equation is

\[
y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos(t) + c_4 t \sin(t)
\]

• For the nonhomogeneous case, because of the form of the solution for the homogeneous equation, we need

\[
Y(t) = t^2 (A \sin t + B \cos t)
\]

• As in Chapter 3, it can be shown that \( Y(t) = -\frac{3}{8} \sin t + \frac{5}{8} \cos t \)

• Thus, the general solution for the nonhomogeneous equation is

\[
y(t) = y_c(t) + Y(t)
\]
Example 3

- Consider the equation
\[ y''' - 4y' = t + 3\cos t + e^{-2t} \]

- For the homogeneous case,
\[ y(t) = e^{rt} \quad \Rightarrow \quad r^3 - 4r = 0 \quad \Leftrightarrow \quad r(\sqrt{r^2 - 4}) \quad \Leftrightarrow \quad r(\sqrt{r^2 - 4}) = 0 \]

- Thus the general solution of homogeneous equation is
\[ y_c(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t} \]

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases:
\[ Y_1(t) = (A + Bt)t, \quad Y_2(t) = C\cos t + D\sin t, \quad Y_3(t) = Ete^{2t}, \]

- As in Chapter 3, can be shown that \( Y(t) = -\frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t} \)

- The general solution is \( y(t) = y_c(t) + Y(t) \)
The variation of parameters method can be used to find a particular solution of the nonhomogeneous \( n \)th order linear differential equation

\[
L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t),
\]

provided \( g \) is continuous.

As with 2\(^{nd}\) order equations, begin by assuming \( y_1, y_2, \ldots, y_n \) are fundamental solutions to homogeneous equation.

Next, assume the particular solution \( Y \) has the form

\[
Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) + \cdots + u_n(t)y_n(t)
\]

where \( u_1, u_2, \ldots u_n \) are functions to be solved for.

In order to find these \( n \) functions, we need \( n \) equations.
Variation of Parameters Derivation (2 of 5)

• First, consider the derivatives of $Y$:
  \[ Y' = (u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n) + (u_1 y'_1 + u_2 y'_2 + \cdots + u_n y'_n) \]

• If we require
  \[ u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n = 0 \]
  then
  \[ Y'' = (u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n) + (u'_1 y''_1 + u'_2 y''_2 + \cdots + u'_n y''_n) \]

• Thus we next require
  \[ u'_1 y'_1 + u'_2 y'_2 + \cdots + u'_n y'_n = 0 \]

• Continuing in this way, we require
  \[ u'_1 y'_{(k-1)} + u'_2 y'_{(k-1)} + \cdots + u'_n y'_{(k-1)} = 0, \quad k = 1, \ldots, n-1 \]
  and hence
  \[ Y^{(k)} = u_1 y^{(k)}_1 + u_2 y^{(k)}_2 + \cdots + u_n y^{(k)}_n, \quad k = 0, 1, \ldots, n-1 \]
Variation of Parameters Derivation (3 of 5)

- From the previous slide,
  \[ Y^{(k)} = u_1 y_1^{(k)} + \cdots + u_n y_n^{(k)}, \quad k = 0,1,\ldots,n-1 \]

- Finally,
  \[ Y^{(n)} = \left( u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \right) + \left( u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} \right) \]

- Next, substitute these derivatives into our equation
  \[ y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t) \]

- Recalling that \( y_1, y_2 \ldots, y_n \) are solutions to homogeneous equation, and after rearranging terms, we obtain
  \[ u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g \]
Variation of Parameters Derivation (4 of 5)

• The $n$ equations needed in order to find the $n$ functions $u_1, u_2, \ldots, u_n$ are

\[
\begin{align*}
  u'_1 y_1 + \cdots + u'_n y_1 &= 0 \\
  u'_1 y'_1 + \cdots + u'_n y'_1 &= 0 \\
  &\vdots \\
  u'_1 y_1^{(n-1)} + \cdots + u'_n y_n^{(n-1)} &= g
\end{align*}
\]

• Using Cramer’s Rule, for each $k = 1, \ldots, n$,

\[
u'_k(t) = \frac{g(t)W_k(t)}{W(t)}, \quad \text{where } W(t) = W(y_1, \ldots, y_n)(t)
\]

and $W_k$ is determinant obtained by replacing $k$th column of $W$ with $(0, 0, \ldots, 1)$. 
Variation of Parameters Derivation (5 of 5)

• From the previous slide,

\[ u'_k(t) = \frac{g(t)W_k(t)}{W(t)}, \quad k = 1, \ldots, n \]

• Integrate to obtain \( u_1, u_2, \ldots, u_n \):

\[ u_k(t) = \int_{t_0}^{t} \frac{g(s)W_k(s)}{W(s)} ds, \quad k = 1, \ldots, n \]

• Thus, a particular solution \( Y \) is given by

\[ Y(t) = \sum_{k=1}^{n} \left[ \int_{t_0}^{t} \frac{g(s)W_k(s)}{W(s)} ds \right] y_k(t) \]

where \( t_0 \) is arbitrary.
Example 1 (1 of 3)

• Consider the equation below, along with the given solutions of corresponding homogeneous solutions $y_1, y_2, y_3$:

$$y''' - y'' - y' + y = g(t), \quad y_1(t) = e^t, \quad y_2(t) = te^t, \quad y_3(t) = e^{-t}$$

• Then a particular solution of this ODE is given by

$$Y(t) = \sum_{k=1}^{3} \left[ \int_{t_0}^{t} \frac{e^{2s}W_k(s)}{W(s)} \, ds \right] y_k(t)$$

• It can be shown that

$$W(t) = \begin{vmatrix} e^t & te^t & e^{-t} \\ e^t & (t+1)e^t & -e^{-t} \\ e^t & (t+2)e^t & e^{-t} \end{vmatrix} = 4e^t$$
Example 1 (2 of 3)

• Also,

\[
W_1(t) = \begin{vmatrix}
0 & te^t & e^{-t} \\
0 & (t + 1)e^t & -e^{-t} \\
1 & (t + 2)e^t & e^{-t}
\end{vmatrix} = -2t - 1
\]

\[
W_2(t) = \begin{vmatrix}
e^t & 0 & e^{-t} \\
e^t & 0 & -e^{-t} \\
e^t & 1 & e^{-t}
\end{vmatrix} = 2
\]

\[
W_3(t) = \begin{vmatrix}
e^t & te^t & 0 \\
e^t & (t + 1)e^t & 0 \\
e^t & (t + 2)e^t & 1
\end{vmatrix} = e^t
\]
Thus a particular solution in integral form is

\[ Y(t) = \sum_{k=1}^{3} \left[ \int_{t_0}^{t} \frac{g(s)W_k(s)}{W(s)} \, ds \right] y_k(t) \]

\[ = e^{t} \int_{t_0}^{t} \frac{g(s)(-2s-1)}{4e^{s}} \, ds + te^{t} \int_{t_0}^{t} \frac{g(s)2}{4e^{s}} \, ds + e^{-t} \int_{t_0}^{t} \frac{g(s)e^{2s}}{4e^{s}} \, ds \]

\[ = \frac{1}{4} \int_{t_0}^{t} \left[ e^{t-s}(-1+2(t-s))+e^{-(t-s)} \right]g(s)\,ds \]