A mathematical model for power output in rowing on an ergometer

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Abstract

A mathematical model relating power output of rower to stroke rate on an ergometer (the Concept II Indoor Rower TM, Model C) is studied. The model is used to analyse the ergometer performance of a particular rower. It is determined that he can be more efficient (i.e. decrease power output while maintaining fixed velocity) by decreasing stroke rate, but at the expense of increasing force during the drive. It is also shown that he can be more efficient by increasing the drag factor (using higher vent setting) without increasing force. Dependence of power output on rowing style (the shape of the force curve) is also examined. It is shown that variation of force during the drive has little effect on efficiency, but efficiency is reduced by asymmetry of the force curve that favours the legs.

Keywords: drive, ergometer, power output, recovery, stroke rate

List of symbols

P = rower's power output (W)
P_a = average power output over single stroke (W)
W_r = work done by the rower during the recovery (J)
W_d = work done by the rower during the drive $= W_f + W_m$
α = force applied to the handle by the rower (N)
t = time (s)
t_r = duration of the recovery (s)
t_d = duration of the drive (s)
ω = angular fan velocity (rad s^{-1})
ω_a = average angular fan velocity over single stroke (rad s^{-1})
ω_b = angular fan velocity at the beginning of the drive (rad s^{-1})
ω_e = angular fan velocity at the end of the drive (rad s^{-1})
ρ = velocity variation factor $= \omega_e/\omega_b$
θ = angular rotation of the fan (rad)
θ_r = angular rotation of the fan during the recovery (rad s^{-1})
θ_d = angular rotation of the fan during the drive (rad s^{-1})
d = distance travelled indicated by the monitor (m)
v = monitor velocity (m s^{-1})
v_a = average monitor velocity over single stroke (m s^{-1})
x = distance handle has moved during drive (m)
x_d = total distance the handle moves during the drive (m)

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x_l = distance handle moves during leg portion of drive (m)
x_u = distance handle moves during upper body portion of drive (m)
s = distance rower's centre of mass has moved during drive (m)
s_r = distance rower's centre of mass moves during recovery (m)
u = velocity of rower's centre of mass on the slide (m s^{-1})
r = stroke rate (strokes per minute)
I = fan moment of inertia (kg m^2)
m_r = rower's mass (kg)
m_u = rower's upper body mass (kg)
m_l = rower's leg mass $m_r - m_u$
κ_m = ratio of centre of mass velocity to handle velocity during leg portion of drive
k = gearing constant (rad m^{-1})
$\tilde{c}$ = boat drag coefficient used by monitor 2.8 kg m^{-1}
c = fan drag coefficient (kg m^2)
Φ = power output factor $= P_a/c\omega_a^3$

Introduction

The rowing stroke consists of two phases: the *drive* and the *recovery*. During the drive, the oars are pulled through the water to accelerate the boat; during the recovery, they are lifted out of the water and returned to their starting position to repeat the drive while the boat decelerates (due to water and air drag). The Concept II Indoor Rower™, Model C, is widely used by competitive rowers as a training tool to simulate on the water rowing and give feedback on the rower's performance. In place of oars, the ergometer has a handle attached by chain to the freewheel which turns a fan. The rower accelerates the fan during the drive, and it spins freely during the recovery, decelerating due to air drag. An adjustable vent allows the rower to change air drag on the fan. The monitor on the ergometer indicates distance travelled (in metres), power output (in watts), pace (time/500 m), and stroke rate (in strokes per minute). The objective of this work is to develop a model for the power output of a rower on the Concept II ergometer and use the model to determine how power output is affected by stroke rate, vent setting, and rowing style. It is also hoped that the model will lead to a better understanding of how ergometer rowing compares with a rowing boat. To this end, a similar model is being developed for rowing boats.

It should be noted that the model calculates only external power generated by the rower, and does not take into account physiological responses to the rower's effort, which may depend greatly on how power is applied. In that sense, the results of this work do not necessarily have implications for rowers who want to improve their performance on an ergometer. Still, it seems worthwhile to be able to determine external power when it differs so greatly from what is indicated on the ergometer monitor and to understand how this power is affected by stroke rate and vent setting. It seems especially so when ergometer performance is so often used to draw conclusions about boat performance, which may be affected differently by these variables.

Most research on the dynamics of rowing (ergometer or boat) has focused on experimental methods and direct measurement (MacFarlane *et al.*, 1997; Martindale & Roberston, 1984). While some effort has been made to mathematically model rowing in a boat (Brearley & De Mestre, 1996; Sanderson & Martindale, 1986), little has been done in the way of modelling ergometer dynamics or using mathematical models to compare ergometer rowing with boat rowing. Since ergometer dynamics are less complex than boat dynamics, modelling of ergometer dynamics seems like a natural starting point for the development of improved models for boat dynamics.

The power output P required to keep the fan moving at constant angular velocity ω is given by:

$$P = c\omega^3 \tag{1}$$

where c is constant called the *drag coefficient* (see Appendix 1 for derivation). There is a vent on the ergometer that controls the amount of air that passes through the fan; changing the vent setting changes the drag coefficient c. The monitor on the ergometer calculates c from fan deceleration during the recovery and uses (1) to compute the average power output P_a per stroke from the average angular velocity ω_a of the fan. In other words:

$$P_a = c\omega_a^3 \tag{2}$$

There are two aspects of rowing on an ergometer that affect the rower's power output, but that are not taken into account by (2):

i) Because (1) is nonlinear, it does not apply in an average sense if fan velocity ω varies. (A simple example to illustrate this is given later.) However, fan velocity necessarily varies during the rowing stroke; it increases during the drive and decreases during the recovery. It will be shown that variation in fan velocity actually increases the power output necessary to maintain a given average fan velocity.
ii) There is work done in accelerating and decelerating the rower's mass on the slide that does not contribute to the motion of the fan. This will also be discussed in further detail later.

The model constructed here is modification of (2) that expresses average power output as a function of average fan velocity and stroke rate, taking (i) and (ii) into account. Notice that both (i) and (ii) are affected by stroke rate; a low stroke rate increases the variation in fan velocity, and a high stroke rate causes quicker movement along the slide. This suggests there is some stroke rate that maximizes efficiency, i.e. minimises power output, for a fixed fan velocity. However, it will be shown that this stroke rate is too low to be practical, since it requires the rower to generate unreasonably large forces during the drive phase of the stroke.

The model is used to analyse the ergometer performance of a particular rower and examine ways in which he could become more efficient. It is shown that he could improve efficiency by lowering his stroke rate, but at the expense of increasing force during the drive. However, it is also shown that he could be more efficient without increasing force by using a higher vent setting (increased fan drag). Although this lowers fan velocity, the *monitor velocity* (distance travelled/elapsed time), need not decrease because of the way that the ergometer monitor corrects for fan drag in determining distance travelled. The stroke rate also remains the same with higher drag, but slightly more time is spent on the drive and less on the recovery.

Appendix 5 investigates the effect on the model of changing rowing style (the distribution of force during the drive). The drive phase of the stroke is divided into three portions: the *leg portion*, the *upper body portion*, and the *arm portion*. The drive begins with the rower leaning forward, with legs bent and arms straight ahead. During the leg portion, the handle is pulled back by straightening the legs; during the upper body portion, it is pulled further by leaning back slightly, and during the arm portion, it is drawn to the chest by bending the arms. It is shown that efficiency is not necessarily improved by maintaining a constant force throughout the drive, but that a symmetric force curve is more efficient than one that favours the leg portion of the drive. This is because increasing force during the leg portion of the drive also increases acceleration of body mass, effort for which the rower is not credited by the ergometer monitor.

Distance computed by ergometer monitor

The monitor on the ergometer determines the distance d travelled from the angular rotation θ of the fan using the formula:

$$d = (c/\tilde{c})^{1/3}\theta \quad (3)$$

where $\tilde{c} = 2.8$ kg m^{-1} is a typical drag coefficient for a racing shell and c is the drag coefficient calculated for the fan. The intent of the conversion (3) is to have the distance travelled reflect what it would be if the same effort were applied in rowing a boat. This is based on the following formula for power output P required to keep a boat moving at a constant velocity:

$$P = \tilde{c}v^3 \quad (4)$$

where $\tilde{c}$ is the drag coefficient (see Appendix 1 for a derivation). Differentiating (3) with respect to time t gives:

$$v = (c/\tilde{c})^{1/3}\omega \quad (5)$$

which implies $c\omega^3 = \tilde{c}v^3$, the result of equating the right-hand sides of (1) and (4). The velocity v calculated by (5) will be referred to as *monitor velocity*.

Remark 1. *The conversion (3) is also to ensure that rowing with a higher vent setting (increased air drag on the fan) and the same effort results in essentially the same monitor velocity, in spite of a lower fan velocity. It is important to note, however, that when (2) is replaced by the model derived in this paper, this will no longer be true. In other words, the same monitor velocity may require different power output at different vent settings. As mentioned* *earlier, one goal of this work is to determine how power output is affected by vent setting.*

Effect of stroke rate on power output

This section presents examples to illustrate points (i) and (ii) of the introduction. Notice that (i) applies to (1) or (4); the following example uses (4) to illustrate (i). Suppose that two rowers row equal distances in equal times, and so their average velocities are also equal. Suppose also that Rower 1 maintains constant velocity v throughout the effort, while Rower 2 spends half of the time rowing at 80% of that velocity and the other half at 120%. By (4), the average power outputs P_1 and P_2 of Rowers 1 and 2 during the effort are

$$P_1 = \tilde{c}v^3 \text{ and}$$

$$P_2 = \frac{1}{2}[\tilde{c}(0.8v)^3 + \tilde{c}(1.2v)^3] = 1.12\tilde{c}v^3$$

Notice that Rower 2 does 12% more work during the effort than Rower 1 because of the nonlinear relationship (4) between P and v. The same argument applies to (1), which means that the average power applied to the fan by the rower is actually greater than indicated by (2) because of variation in fan velocity during the stroke. This also indicates one way in which power output is related to stroke rate. For a fixed (average) fan velocity, stroke rate is largely determined by the duration of the recovery. Specifically, decreasing stroke rate while maintaining a fixed fan velocity is accomplished mostly by increasing the duration of the recovery. However, such an increase will increase the variation in fan velocity and therefore increase the power applied to the fan. Thus, lowering stroke rate increases the power that must be applied to the fan to maintain a fixed average fan velocity.

Point (ii) is illustrated by focusing on the recovery phase of the stroke. Suppose that the rower maintains a constant velocity u on the slide during recovery. The work W_r done by the rower on an ergometer during the recovery phase is the combination of the work done accelerating and decelerating his body mass on the slide. Thus:

$$W_r = \frac{1}{2}m_r u^2 + \frac{1}{2}m_r u^2 = m_r u^2 \qquad (6)$$

where m_r is the rower's mass. Since this work is not applied to the fan, it is inefficient to recover quickly. So, in contrast to (i), this work is increased by increasing the stroke rate.

There is additional work done accelerating and decelerating body mass during the drive that will be taken into account in constructing the model.

Remark 2. *For the rower to maintain a constant velocity,* u, *during the recovery requires an instantaneous jump in velocity at the beginning and end of the slide. Although this is unrealistic, it is a reasonable approximation if the acceleration at the beginning and end of the slide is quick. It is also most efficient since slower acceleration means a higher peak velocity must be reached, requiring more work.*

Remark 3. *The work of accelerating and decelerating the rower's mass on the slide is lessened considerably in a boat because the boat and rower move relative to the centre of mass of the boat/rower system. Since the boat is much lighter, its velocity changes more than the rower's. However, this motion also causes more variation in boat velocity, which relates back to point (i). These facts will be taken into consideration in a similar project that will seek to improve on the model in (4).*

Derivation of the model

The following assumptions are made in deriving the model:

1 The determination of work done moving the rower's mass is based entirely on horizontal motion of the rower's centre of mass; vertical motion, rotational motion and motion relative to the centre of mass is neglected.
2 The rower's centre of mass moves at a constant velocity on the slide during the recovery.
3 The rower applies a constant force to the handle during the drive. (This assumption is dropped in Appendix 5.)
4 The entire body remains rigid during the arm portion of the drive, the upper body remains rigid during the leg portion of the drive and the centre of mass of the legs moves about 2/3 as fast as the upper body, and there is a continuous, linear transition between these during the upper body portion of the drive.

Assumption 4 is a modification of Pope's (1973) assumption that the velocity of the rower's centre of mass during the drive is proportional to the velocity of the handle. The assumption being made here means that the ratio between the two velocities will be constant during the leg portion of the drive, decrease linearly during the upper body portion, and be zero during the arm portion.

The model derived there has the form:

$$P_a = c\omega_a^3\Phi(\rho) \tag{7}$$

where the function $\Phi(\rho)$, henceforth referred to as the *power output factor*, accounts for energy expended moving the rower's mass and the extra work due to variation in fan velocity during the stroke (see Eqn. (35)). The power output factor Φ is a function of the velocity variation factor ρ, i.e. the ratio between the maximum and minimum fan velocities during a stroke. Since fan velocity ω and monitor velocity v are linearly related by (5), average fan velocity ω_a and average monitor velocity v_a also satisfy (5). This means that (7) can be rewritten as:

$$P_a = \tilde{c} v_a^3\Phi(\rho) \tag{8}$$

It will also be shown (Eqn. (37)) how ρ can be expressed as function of the ratio between v_a and stroke rate r. This allows (8) to be expressed in the form:

$$P_a = \tilde{c} v_a^3\Psi(v_a/r) \tag{9}$$

Since v_a and r (unlike ω_a and ρ) are easily determined from information displayed on the ergometer monitor, (9) is a more useful formulation of the model for analysing the performance of the rower in the next section. It should be noted that Φ and Ψ also depend on the moment of inertia I and the gearing k of the fan, fan drag c, the rower's mass m_r, the mass m_l of the rower's legs, the distance x_d the handle is pulled during the drive, the distance x_l the handle is pulled during the leg portion of the drive, and the distance x_u the handle is pulled during the upper body portion of the drive.

The construction of (7) is based on the following expressions for P_a and ω_a:

$$P_a = \frac{W_d + W_r}{t_d + t_r} \tag{10}$$

$$\omega_a = \frac{\theta_d + \theta_r}{t_d + t_r} \tag{11}$$

where t_d is the duration of the drive and θ_d and W_d are the angular rotation of the fan and work done during the drive. The corresponding quantities for the recovery are t_r, θ_r and W_r. The construction begins by deriving expressions for the components of (10) and (11). Many mathematical aspects of the derivation are deferred to appendices.

Recovery phase of the stroke

The derivation begins with an analysis of the recovery phase of the stroke. It follows from (1) that the drag torque T applied to the fan by air is related to angular velocity ω by:

$$T = c\omega^2 \tag{12}$$

Combining Newton's second law (for rotational systems) with (12) leads to the following differential equation for ω:

$$I\frac{d\omega}{dt} = -c\omega^2, \qquad 0 < t < t_r \tag{13}$$

$$\omega(0) = \omega_e \tag{14}$$

$$\omega(t_r) = \omega_b \tag{15}$$

where I is the moment of inertia of the fan, ω_b and ω_e are the angular velocities of the fan at the beginning and end of the drive and t_r is the duration of the recovery. The following expressions for t_r and θ_r are derived from the solution of (13), (14) and (15) (details in Appendix 2):

$$t_r = \frac{I(\rho - 1)}{c\omega_e} \tag{16}$$

$$\theta_r = \frac{I}{c}\ln\rho \tag{17}$$

(16) is combined with (6) to obtain an expression for W_r:

$$W_r = m_r u^2 = m_r\left(\frac{s_r}{t_r}\right)^2 = \frac{c^2\omega_e^2 m_r s_r^2}{I^2(\rho - 1)^2} \tag{18}$$

where s_r is the distance moved by the rower's centre of mass during the recovery. Assumption 4 is required to determine an expression (51) for s_r, and this derivation is carried out in Appendix 4.

Drive phase of the stroke

Once the drive has begun, there is a linear relationship between θ, the angular rotation of the fan, and x, the distance the handle has moved during the drive:

$$\theta = kx \tag{19}$$

where k = 1/cog radius (in metres) is called the *gearing constant*. This implies that θ_d, the total angular rotation of the fan during the drive, is given by:

$$\theta_d = kx_d \tag{20}$$

where x_d is the total distance the handle is moved during the drive. It will also be assumed for now that the rower applies a constant force α to the handle during the drive. (Though this is not realistic, it is shown in Appendix 5 that power output remains essentially unchanged by allowing force to vary about an *average* force α.)

The drive phase of the stroke is now governed by:

$$I\frac{d\omega}{dt} = \frac{\alpha}{k} - c\omega^2, \quad 0 < t < t_d \tag{21}$$

$$\omega(0) = \omega_b \tag{22}$$

where t_d is the duration of the drive. In Appendix 3, (21) is rewritten with x as the independent variable and solved subject to (22). The following expression for α is obtained in the course of solving:

$$\alpha = \frac{ck(\beta\omega_e^2 - \omega_b^2)}{\beta - 1} = \frac{ck\omega_e^2(\beta - 1/\rho^2)}{\beta - 1} \tag{23}$$

where

$$\beta = e^{2ckx_d/I}$$

The solution of (21), (22) has the form:

$$\omega(x) = \omega_e h(\rho, x) \tag{24}$$

where h is given by (44). (24) can be written as separable differential equation (45) for θ which is solved to obtain an expression for t_d:

$$t_d = \frac{k}{\omega_e}\int_0^{x_d}\frac{dx}{h(\rho, x)} = \frac{kH(\rho)}{\omega_e} \tag{25}$$

where

$$H(\rho) = \int_0^{x_d}\frac{dx}{h(\rho,x)} \tag{26}$$

The work W_d done during the drive is the sum of W_f the work applied to the fan, and W_m, the work done accelerating and decelerating the rower's centre of mass on the slide during the drive. Notice that:

$$W_f = \alpha x_d = \frac{ckx_d\omega_e^2(\beta - 1/\rho^2)}{\beta - 1} \quad \text{by (23)} \tag{27}$$

and

$$W_m = m_r u_{\max}^2 \tag{28}$$

where $u_{\max}$ is the maximum velocity reached by the rower's centre of mass during the drive. During the drive, the velocity u of the rower's centre of mass and the velocity of the handle are related by:

$$u = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = s'(x)\omega(x)/k \quad \text{by (19)} \tag{29}$$

where s is the distance the rower's centre of mass has moved during the drive. A specific relationship between the position of the handle and the position of the rower's centre of mass is implied by Assumption 4, and this is used in Appendix 4 to derive an expression for $s'(x)$ (see (49), (50)). It is also shown there that the maximum of u occurs at the end of the leg portion of the drive. This means that:

$$u_{\max} = s'(x_l)\omega(x_l)/k \tag{30}$$

by (29). Combining this with (28) and (49), one finds:

$$W_m = \frac{m_r\kappa_m^2\omega_e^2h(\rho,x_l)^2}{k^2} \tag{31}$$

where κ_m is given by (50). Combining (27) and (31) gives an expression for W_d:

$$W_d = \frac{ckx_d\omega_e^2(\beta - 1/\rho^2)}{\beta - 1} + \frac{m_r\kappa_m^2\omega_e^2h(\rho,x_l)^2}{k^2} \tag{32}$$

An expression for average power

An expression for average power P_a over the stroke can now be obtained by combining (16), (18), (25), and (32):

$$P_a = \frac{W_d + W_r}{t_d + t_r}$$

$$= \frac{c\omega_e^3\left[\frac{ckx_d(\beta - 1/\rho^2)}{\beta - 1} + \frac{m_r\kappa_m^2 h(\rho,x_l)^2}{k^2} + \frac{c^2 m_r s_r^2}{I^2(\rho - 1)^2}\right]}{ckH(\rho) + I(\rho - 1)} \tag{33}$$

Using (16), (17), (20), and (25), ω_e can be expressed in terms of average fan velocity ω_a:

$$\omega_a = \frac{\theta_d + \theta_r}{t_d + t_r} = \frac{\omega_e(ckx_d + I\ln\rho)}{ckH(\rho) + I(\rho - 1)}$$

which can then be solved for ω_e:

$$\omega_e = \frac{\omega_a(ckH(\rho) + I(\rho - 1))}{ckx_d + I\ln\rho} \tag{34}$$

Combining (34) with (33) yields (7), where

$$\Phi(\rho) = \left[\frac{ckx_d(\beta - 1/\rho^2)}{\beta - 1} + \frac{m_r\kappa_m^2 h(\rho,x_l)^2}{k^2} + \frac{c^2 m_r s_r^2}{I^2(\rho - 1)^2}\right]\frac{(ckH(\rho) + I(\rho - 1))^2}{(ckx_d + I\ln\rho)^3} \tag{35}$$

Expressing power as function of stroke rate

Though ρ is a natural parameter to use in deriving an expression of power output P_a, the expression is not very useful since ρ is not an easily measured quantity. It is not difficult, however, to express power output in terms of stroke rate r, a quantity which is displayed on the ergometer's monitor. To accomplish this r is expressed in terms of ρ using (5), (17) and (20):

$$r = \frac{60}{t_d + t_r} = \frac{60\omega_a}{\theta_d + \theta_r} = \frac{60c^{2/3}\tilde{c}^{1/3}v_a}{ckx_d + I\ln\rho}, \tag{36}$$

which is then solved for ρ:

$$\rho = \exp\left(\frac{60c^{2/3}\tilde{c}^{1/3}v_a/r - ckx_d}{I}\right) \tag{37}$$

This can be substituted into (8) to obtain an expression for average power of the form (9).

Profile of an individual rower

In this section, the model is used to analyse the performance of a particular rower. He is a competitive heavyweight rower who weighs 93 kg, is 1.93 m tall, and has completed 2000 m in 5:57 on the ergometer using a vent setting of four. The following values are chosen based on this information: m_r = 93 kg, m_l = 43 kg, x_d = 1.4 m, x_l = 0.55 m, x_u = 0.3 m, v_a = 2000/357 ≈ 5.6 m s^{-1}, and $c = 1.4 \times 10^{-4}$ kg m^2. The values I = 0.1 kg m^2, k = 70 rad m^{-1} have been provided by Concept II for the ergometer. These parameter values are used to plot Φ versus ρ in Figure 1. Notice that the minimum value of Φ is approximately 1.29 and occurs when the velocity variation factor ρ is about 1.7. This means that the power generated by the rower is at least 29% greater than indicated on the monitor. The formulation (8) of the model is used to graph average power output P_a as function of ρ in Figure 2. Notice that the rower's power output is at least 636 W. Compare this with the power output indicated on the monitor of about 492 W. P_a is graphed as function of r in Figure 3 using (9). The profiled rower reports using a stroke rate around 31 strokes per minute during the aforementioned effort, which translates to an average power output of about 672 W.

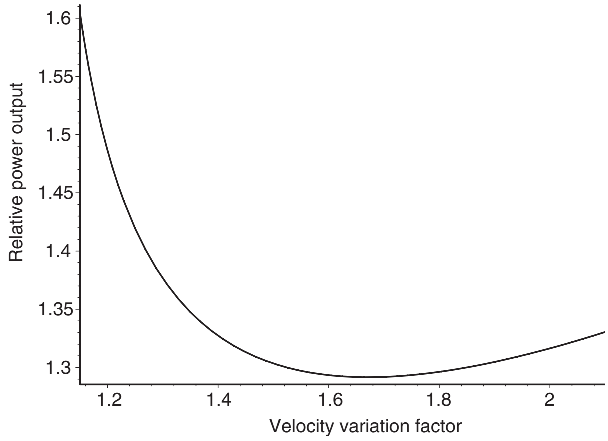

Figure 1 Relative power output Φ versus velocity variation factor ρ.

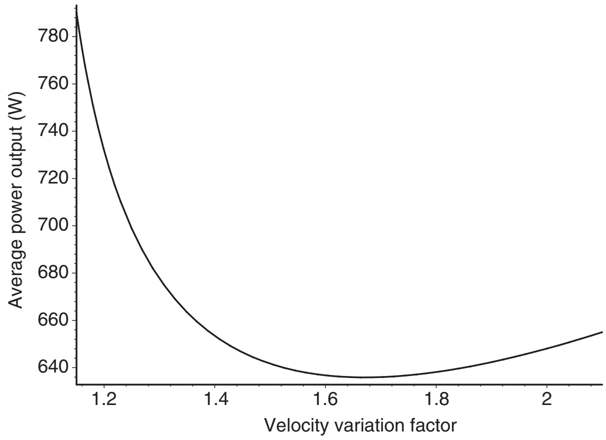

Figure 2 Average power output P_a versus velocity variation factor r.

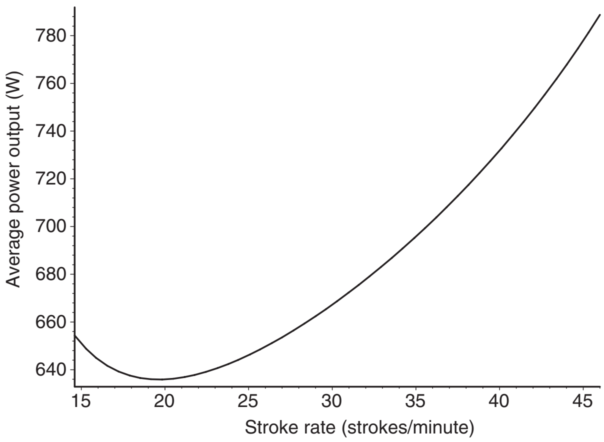

Figure 3 Average power output P_a versus stroke rate r.

Notice that the graph in Figure 3 also indicates that the rower's power output could be reduced by as much as 36 W by using a lower stroke rate. However, this would require a considerable increase in force. For comparison of forces, (23) is rewritten using (5) and (34):

$$\alpha = \frac{ck\omega_e^2(\beta - 1/\rho^2)}{\beta - 1}$$

$$= \frac{c^{1/3}\,\tilde{\tau}^{-2/3}kv_a^2(\beta - 1/\rho^2)(ckH(\rho) + I(\rho - 1))^2}{(\beta - 1)(ckx_d + I\ln\rho)^2} \quad (38)$$

This can be combined with (37) to express force α as a function of stroke rate r. Using (38) and (37), the force α for the profiled rower is found to be about 694 N at $r = 31$ strokes per minute. The force α associated with the power minimizing value $r = 19.6$ is about 1150 N, a 72% increase over the force associated with $r = 31$. This also means that work done during the drive is increased by a comparable amount. (The overall decrease in average power output is due to the increased duration of the recovery associated with the lower stroke rate.) Such a large increase in power and force applied during the drive would result in more rapid fatigue (See Alquist *et al.*, 1992; Lee *et al.*, 2000). In fact, the profiled rower reports posting a slightly slower time during which his stroke rate was around 28 strokes per minute during the first 1500 metres, but rose as high as 36 strokes per minute during the last 500. The author speculates that the lower stroke rate was too fatiguing and he was able to decrease the force applied during the drive by increasing his stroke rate (at the expense of increasing average power output).

It is worth noting that the computations for both power output and force compare favourably with those measured by MacFarlane *et al.* (1997) and Martindale & Robertson (1984). In Martindale & Robertson (1984), internal work done moving body mass was calculated for subjects rowing on air-braked stationary ergometers. For two subjects of comparable size to the profiled rower, the corresponding power output ranged from 150 to 184 for stroke rates between 22 and 26 strokes per minute. For the profiled rower, power output applied to the fan is

$$\frac{\alpha x_d}{t_d + t_r} = \frac{\alpha r x_d}{60} \approx 502 \text{ W}$$

which means that power applied in moving his mass is about 170 W. (Notice moving the body on the slide accounts for the vast majority of the effort not accounted for by the ergometer monitor.) The model predicts that, if he were to lower his stroke rate to 26 strokes per minute, the power applied in moving his mass would decrease to about 159 W.

MacFarlane *et al.* (1997) measured handle force for subjects rowing on Concept II for 90 s at about the same pace and stroke rate as the profiled rower. They applied an average of about 975 J to the handle during the drive phase of each stroke, which corresponds to an average force of 696 N (assuming the handle travels 1.4 m during the drive). Compare this with the calculated value of 694 N for the profiled rower.

Effect of changing the drag coefficient

It turns out that the profiled rower can improve efficiency without increasing force by increasing the drag coefficient c. Figure 4 shows a graph of P_a with $c = 2 \times 10^{-4}$ kg m^2 (this typically corresponds to a vent setting of 8 on the Concept II) along with the graph from Figure 3 for comparison. Notice that, for higher stroke rates, power output is lowered significantly with the higher drag. Corresponding graphs of α in Figure 5 show that force is only marginally increased. Suppose the profiled rower increases the drag coefficient to $c = 2 \times 10^{-4}$ kg m^2, maintains the same monitor velocity, and increases his rate slightly to 31.2 strokes per minute. Then force α is

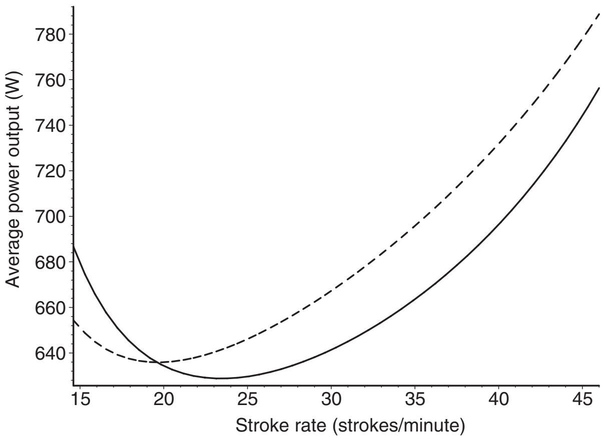

Figure 4 Average power output P_a versus stroke rate r with fan drag $c = 1.4 \times 10^{-4}$ kg m^2 (dots) and $c = 2 \times 10^{-4}$ kg m^2 (solid).

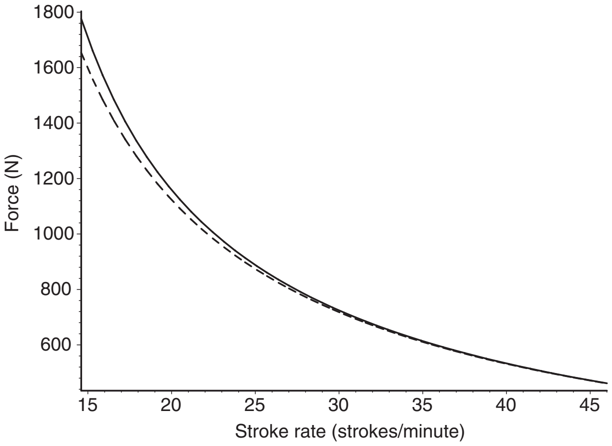

Figure 5 Force α versus stroke rate r with fan drag $c = 1.4 \times 10^{-4}$ kg m^2 (dots) and $c = 2 \times 10^{-4}$ kg m^2 (solid).

unchanged, but P_a falls from 672 W to 646 W, a 4% decrease in power output.

Another consequence of this change is an increase in the duration of the drive. Using (25), one finds that

$$t_d = \frac{kH(\rho)}{\omega_e} = \frac{kH(\rho)(ckx_d + I\ln\rho)}{\omega_a(ckH(\rho) + I(\rho - 1))}$$

$$= \frac{c^{1/3}kH(\rho)(ckx_d + I\ln\rho)}{\tilde{c}^{1/3}v_a(ckH(\rho) + I(\rho - 1))} \tag{39}$$

Combining (39) with (37), t_d is calculated for both drag factors and found to increase from 0.636 to 0.713; a 12% increase. Higher drag allows the rower to apply roughly the same torque to the fan more slowly, decreasing the work done accelerating and decelerating body mass during the drive (which is more significant than during the recovery since the rower's mass reaches a higher velocity during the drive). Lower body mass acceleration during the drive also means that the rower is applying a slightly smaller force with his feet during the leg portion of the drive. It is also noted that, with this change in drag coefficient, the rower's velocity variation factor ρ increases from 1.32 to 1.38, which means that there is actually more power being applied to the fan. However, at such a high stroke rate, velocity variation contributes far less to the power output factor Φ than acceleration of body mass. It should be noted, however, that the physiological implications of increasing vent setting are not clear. In the last section it was shown that decreasing power output by decreasing stroke rate and increasing force during the drive was likely to have a fatiguing effect on the rower. Increasing vent setting and applying the same force more slowly during the drive could have the same effect.

Conclusion

The model derived here shows that there is a significant amount of power generated by a rower on a Concept II ergometer that is not accounted for by the ergometer's monitor. The majority of that power is the result of accelerating the rower's mass on the slide, and can thus be reduced by lowering stroke rate. However, since there is a corresponding increase in force required to maintain the same average fan velocity, there is a limit to how useful this may be in improving performance on the ergometer. It is also shown that the rower can slow acceleration during the drive (and thus reduce power output) without decreasing monitor velocity or increasing force by increasing the vent setting. This causes force to be applied more slowly during the drive, which may also not be ideal for physiological reasons. Of course, the goal in using an ergometer is usually to improve performance on the water, not on the ergometer. The usefulness of the model may then be as a first step to determining how to relate ergometer performance to performance on the water.

Appendix 1: Derivation of basic power formulae

The rate p (in kg s^{-1}) at which air passes through the ergometer fan and the velocity v (in m s^{-1}) with which it is expelled from the fan are both proportional to the fan velocity ω:

$$p = c_1\omega, \qquad v = c_2\omega$$

where c_1 (kg) and c_2 (m) are constants. The power P required to keep the fan moving is equal the rate at which the air passing through the fan acquires kinetic energy:

$$P = \frac{1}{2}pv^2 = \frac{1}{2}c_1c_2{}^2\omega^3$$

which is (1) with $c = \frac{1}{2}c_1c_2{}^2$.

According to (7), the drag force D on the boat due to fluid resistance is proportional to the square of its velocity v:

$$D = \tilde{c}v^2$$

where $\tilde{c}$ is the drag coefficient. Power output P is the product of force and velocity:

$$P = Dv = \tilde{c}v^3$$

which is (4).

Appendix 2: Recovery phase calculations

Solving (13) by separation of variables and using (14), one obtains:

$$\omega(t) = \frac{I\omega_e}{I + c\omega_e t} \tag{40}$$

(15) and (40) imply that

$$\omega_b = \frac{I\omega_e}{I + c\omega_e t_r}$$

which is solved for t_r:

$$t_r = \frac{I(\omega_e - \omega_b)}{c\omega_e\omega_b} = \frac{I(\rho - 1)}{c\omega_e}$$

Since

$$\frac{d\theta}{dt} = \omega \tag{41}$$

an expression for θ_r is obtained by integrating (40):

$$\begin{aligned}\theta_r &= \int_0^{t_r} \omega\, dt \\ &= \int_0^{t_r} \frac{I\omega_e\, dt}{I + c\omega_e t} \\ &= \frac{I}{c}\ln\left(1 + \frac{c\omega_e t_r}{I}\right) \\ &= \frac{I}{c}\ln\rho \text{ by (16)}\end{aligned}$$

Appendix 3: Drive phase calculations

Using (41), (19), and the chain rule, (21) is rewritten in terms of ω and x:

$$I\frac{d\omega}{dt} = I\frac{d\omega}{dx}\frac{dx}{d\theta}\frac{d\theta}{dt} = \frac{I\omega}{k}\frac{d\omega}{dx} = \frac{\alpha}{k} - c\omega^2$$

which implies:

$$\omega\frac{d\omega}{dx} + ck\omega^2/I = \alpha/I, \qquad 0 < x < x_d \tag{42}$$

This equation is made exact by multiplying by an integrating factor:

$$2e^{2ckx/I}\omega\frac{d\omega}{dx} + 2ck\omega^2 2e^{2ckx/I}/I$$

$$= \frac{d}{dx}\left[\omega^2 2e^{2ckx/I}\right] = 2\alpha e^{2ckx/I}/I$$

Integrating and using (22) gives

$$\omega^2 e^{2ckx/I} - \omega_b^2 = \frac{\alpha(e^{2ckx/I} - 1)}{ck} \tag{43}$$

A formula for force can be obtained by letting $x = x_d$ in (43):

$$\omega^2 e^{2ckx_d/I} - \omega_b^2 = \frac{\alpha(e^{2ckx_d/I} - 1)}{ck}$$

which is solved for α to obtain (23). Also, (43) can be used to obtain an expression for ω:

$$\omega(x) = e^{-ckx/I}\sqrt{\left(\omega_b^2 + \frac{\alpha(e^{2ckx/I} - 1)}{ck}\right)}$$

$$= e^{-ckx/I}\sqrt{\left(\omega_b^2 + \frac{\omega_e^2(e^{2ckx/I} - 1)(\beta - 1/\rho^2)}{\beta - 1}\right)} \text{ by (23)}$$

$$= \omega_e h(\rho, x)$$

where

$$h(\rho, x) = e^{-ckx/I}\sqrt{\left(\frac{1}{\rho^2} + \frac{(e^{2ckx/I} - 1)(\beta - 1/\rho^2)}{\beta - 1}\right)} \quad (44)$$

This, combined with (41) and (19), implies

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = \omega/k = \omega_e h(\rho, x)/k \quad (45)$$

Separating variables and integrating gives

$$k\int_0^{x_d} \frac{dx}{h(\rho, x)} = \omega_e \int_0^{t_d} dt = \omega_e t_d$$

which is solved for t_d to obtain (25).

Appendix 4: Motion of the rower's centre of gravity

The purpose of this section is to derive relationship between the motions of the rower's centre of mass and the handle. First, notice that:

$$s'(x) = \frac{m_u s'_u(x) + m_l s'_l(x)}{m_r} \quad (46)$$

where m_l and m_u are the masses of the legs and upper body respectively, and $s_l(x)$ and $s_u(x)$ are the positions of their centres of mass. Let x_l be the distance the handle moves during of the leg portion of the drive and x_u be the distance the handle moves during the upper body portion of the drive (excluding the arms). Thus the leg portion of the drive will be represented by $0 \le x < x_l$, the upper body portion by $x_l \le x < x_l + x_u$, and the arm portion by $x_l + x_u \le x \le x_d$.

Based on Assumption 4, the following specifications are made:

$$s'_u(x) = \begin{cases} 1 & \text{if } 0 \le x < x_l \\ 1 - \dfrac{x - x_l}{x_u} & \text{if } x_l \le x < x_l + x_u \\ 0 & \text{if } x_l + x_u \le x \le x_d \end{cases} \quad (47)$$

$$s'_l(x) = \begin{cases} \dfrac{2}{3} & \text{if } 0 \le x < x_l \\ \dfrac{2}{3}\left(1 - \dfrac{x - x_l}{x_u}\right) & \text{if } x_l \le x < x_l + x_u \\ 0 & \text{if } x_l + x_u \le x \le x_d \end{cases} \quad (48)$$

Combining (46), (47) and (48) gives

$$s'_u(x) = \begin{cases} \kappa_m & \text{if } 0 \le x < x_l \\ \kappa_m\left(1 - \dfrac{x - x_l}{x_u}\right) & \text{if } x_l \le x < x_l + x_u \\ 0 & \text{if } x_l + x_u \le x \le x_d \end{cases} \quad (49)$$

where

$$\kappa_m = 1 - \frac{m_l}{3m_r} \quad (50)$$

is the ratio of the velocity of the rower's centre of mass to the velocity of the handle during the leg portion of the drive. (Using the values of m_l and m_r given for the profiled rower, $\kappa_m \approx 0.85$.) Also, (49) yields and expression for s_r:

$$s_r = \int_0^{x_d} s'(x)\, dx = \frac{1}{2}\kappa_m(2x_l + x_u) \quad (51)$$

(Using the values of x_l and x_u given for the profiled rower, $s_r \approx 0.59$ m.)

It remains to verify (30). Since it follows from (29) and (49) that u increases during the leg portion of the drive, it will suffice to show that $du/dx < 0$ during the upper body portion of the drive. It follows from from (29) that:

$$\frac{du}{dx} = \frac{s''(x)\,\omega(x) + s'(x)\,\omega'(x)}{k} \quad (52)$$

During the upper body portion of the drive, (49) implies that:

$$s''(x) = -\frac{\kappa_m}{x_u} \text{ and } s'(x) \le \kappa_m \quad (53)$$

and (42) implies that

$$\omega'(x) = \frac{\alpha}{I\omega} - \frac{ck\omega}{I} < \frac{\alpha}{I\omega} \quad (54)$$

Combining (53) and (54) with (52) yields

$$\frac{du}{dx} < -\frac{\kappa_m \omega}{k x_u} + \frac{\kappa_m \alpha}{I k \omega}$$

which means $du/dx < 0$ if

$$\alpha < \frac{I\omega^2}{x_u}. \tag{55}$$

Using the data provided for the profiled rower, ω is at least 132 rad s^{-1}, which makes the right-hand side of (55) over 5800 N, clearly greater than the force α generated by the rower.

Appendix 5: Nonconstant force

In this section, the model is modified under the assumption that the rower applies force as a continuously varying function of handle position during the drive. Specifically, the force applied to the handle by the rower x metres into the drive will be represented by $F(x)$, $0 \le x \le x_d$. Now α will represent the average force applied:

$$\alpha = \frac{1}{x_d}\int_0^{x_d} F(x)\, dx \tag{56}$$

F can be written in the form:

$$F(x) = \alpha f(x/x_d)$$

where $f(z) = F(x_d z)/\alpha$ for $0 \le z \le 1$. Notice then that (56) implies that:

$$\int_0^1 f(z)\, dz = 1 \tag{57}$$

The function f represents the shape (independent of magnitude or length) of the force curve F and can be thought of as a parameter representing rowing style. f will be referred to as *relative force*, and z as *relative handle position*.

The drive phase of the stroke is now governed by:

$$I\frac{dw}{dt} = F(x)/k - c\omega^2,\ 0 < t < t_d,\ 0 < x < x_d \tag{58}$$

$$\omega(0) = \omega_b \tag{59}$$

First, (58) is rewritten in terms of ω and x in the same manner as in Appendix 2:

$$\omega\frac{dw}{dx} + ck\omega^2/I = F(x)/I = \alpha f(x/x_d)/I$$

Multiplying by an integrating factor, one obtains:

$$2e^{2ckx/I}\omega\frac{dw}{dx} + 2ck\omega^2 e^{2ckx/I}/I$$

$$= \frac{d}{dx}\left[\omega^2 e^{2ckx/I}\right]$$

$$= 2\alpha e^{2ckx/I} f(x/x_d)/I$$

Integrating and using (59) gives:

$$\omega^2 e^{2ckx/I} - \omega_b^2 = 2\alpha x_d G(x)/I \tag{60}$$

where

$$G(x) = \frac{1}{x_d}\int_0^x e^{2ck\xi/I} f(\xi/x_d)\, d\xi$$

$$= \int_0^{x/x_d} e^{2ckx_d z/I} f(z)\, dz$$

A formula for α is obtained by letting $x = x_d$ in (60):

$$\omega_e^2 e^{2ckx_d/I} - \omega_b^2 = 2\alpha x_d G(x_d)/I$$

where

$$G_0 = G(x_d) = \int_0^1 e^{2ckx_d z/I} f(z)\, dz$$

This implies:

$$\alpha = \frac{I(\beta\omega_e^2 - \omega_b^2)}{2x_d G_0} = \frac{I\omega_e^2(\beta - 1/\rho^2)}{2x_d G_0} \tag{61}$$

(60) also implies that:

$$\omega(x) = e^{-ckx/I}\sqrt{\left(\omega_b^2 + 2\alpha x_d G(x)/I\right)}$$

$$= e^{-ckx/I}\sqrt{\left(\omega_b^2 + \omega_e^2(\beta - 1/\rho^2)G(x)/G_0\right)}$$

$$= \omega_e \tilde{h}(\rho, x)$$

where

$$\tilde{h}(\rho, x) = e^{-ckx/I}\sqrt{\left(1/\rho^2 + (\beta - 1/\rho^2)G(x)/G_0\right)} \tag{62}$$

(61) implies that

$$W_f = \alpha x_d = \frac{I\omega_e^2(\beta - 1/\rho^2)}{2G_0} \tag{63}$$

Using (63) in place of (27) gives:

$$P_a = cv_a^3\left[\frac{I(\beta - 1/\rho^2)}{2G_0} + \frac{m_r\kappa_m^2\tilde{h}(\rho,x_l)^2}{k^2} + \frac{c^2 m_r s_r^2}{I^2(\rho - 1)^2}\right]\frac{\left(ck\tilde{H}(\rho) + I(\rho - 1)\right)^2}{(ckx_d + I\ln\rho)^3} \qquad (64)$$

where

$$\tilde{H}(\rho) = \int_0^{x_d}\frac{dx}{\tilde{h}(\rho, x)}$$

(37) holds as before and (38) is replaced by

$$\alpha = \frac{I\tilde{c}^{2/3}v_a^2(\beta - 1/\rho^2)\left(ck\tilde{H}(\rho) + I(\rho - 1)\right)^2}{2c^{2/3}x_d G_0(ckx_d + I\ln\rho)^2} \qquad (65)$$

which can be derived from (61) in the same manner as before.

(64) and (65) can now be used to recalculate the power output and force generated by the rower profiled earlier with a nonconstant force curve. Figure 7 shows graphs of P_a (as functions of r) with constant force ($f(z) = f_1(z) = 1$), and with

$$f(z) = f_2(z) = \left(1 - |2z - 1|^3\right)/\gamma$$

where $\gamma = \int_0^1\left(1 - |2z - 1|^3\right)dz$

(graphed in Figure 6) and other parameter values the same as in Figure 3. (f_2 is normalized by γ so that (57) is satisfied.) Notice that there is no significant change in power output at any stroke rate. Figure 8 shows the corresponding graphs of α. It can be shown by careful analysis (omitted) of the elements of the derivation in this section that the power output is largely unchanged because of the symmetry of f_1 and f_2 about $z = 1/2$.

Consider the force curve $f_3(z) = 12z(1 - z)^2$ (graphed in Figure 9), which represents a rowing style in which the effort during the drive is shifted more towards the legs and away from the upper body. Figure 10 shows graphs of P_a with $f(z) = 1$ and $f(z) = f_3(z)$; Figure 11 shows the corresponding graphs of α. If it is stipulated that average force α must be 694 N, the same as for f_1 at 31 strokes per minute, then the corresponding stroke rate is $r = 31.2$ strokes per minute. At this stroke rate, power output is about 690 W, a 2.7% increase over that with constant force curve f_1 at 31 strokes per minute. This can be explained by noting

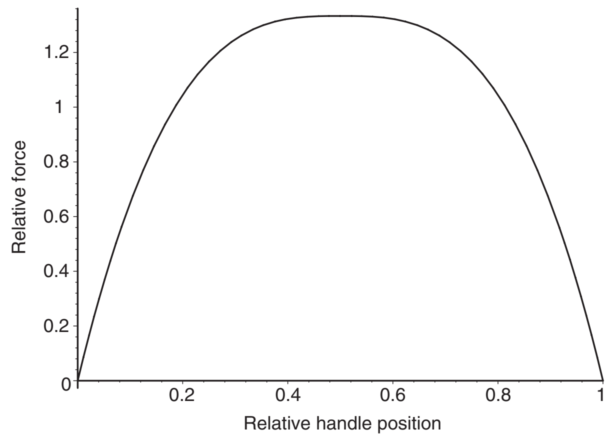

Figure 6 Nonconstant relative force f_2 versus relative handle position z.

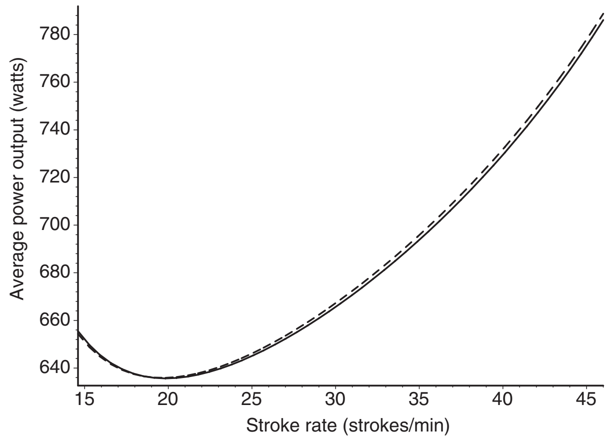

Figure 7 Average power output P_a versus stroke rate r with constant relative force $f = 1$ (dots) and nonconstant relative force $f = f_2$ (solid).

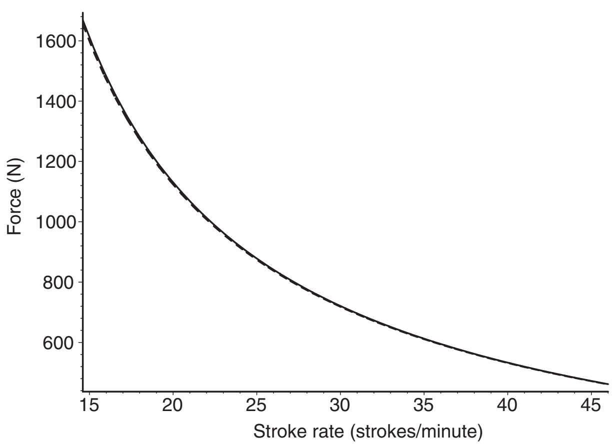

Figure 8 Average force α versus stroke rate r with constant relative force $f = 1$ (dots) and nonconstant relative force $f = f_2$ (solid).

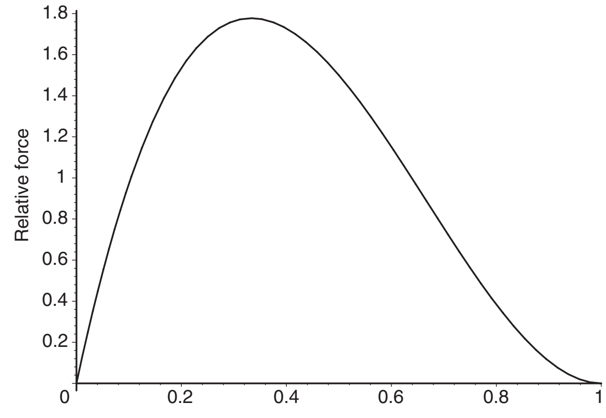

Figure 9 Asymmetric relative force f_3 versus relative handle position z.

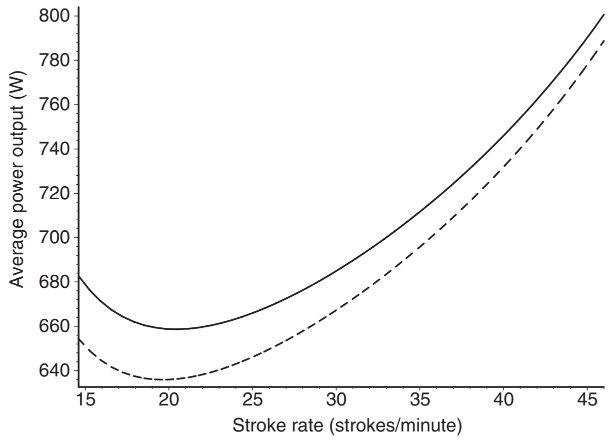

Figure 10 Average power output P_a versus stroke rate r with constant relative force $f = 1$ (dots) and asymmetric relative force $f = f_3$ (solid).

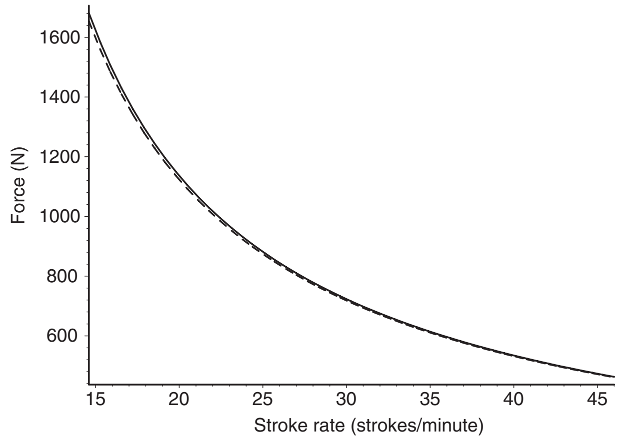

Figure 11 Average force α versus stroke rate r with constant relative force $f = 1$ (dots) and asymmetric relative force $f = f_3$ (solid).

that this shift in effort necessarily increases the work done accelerating the rower's mass during the leg portion of the drive without benefiting fan velocity.

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