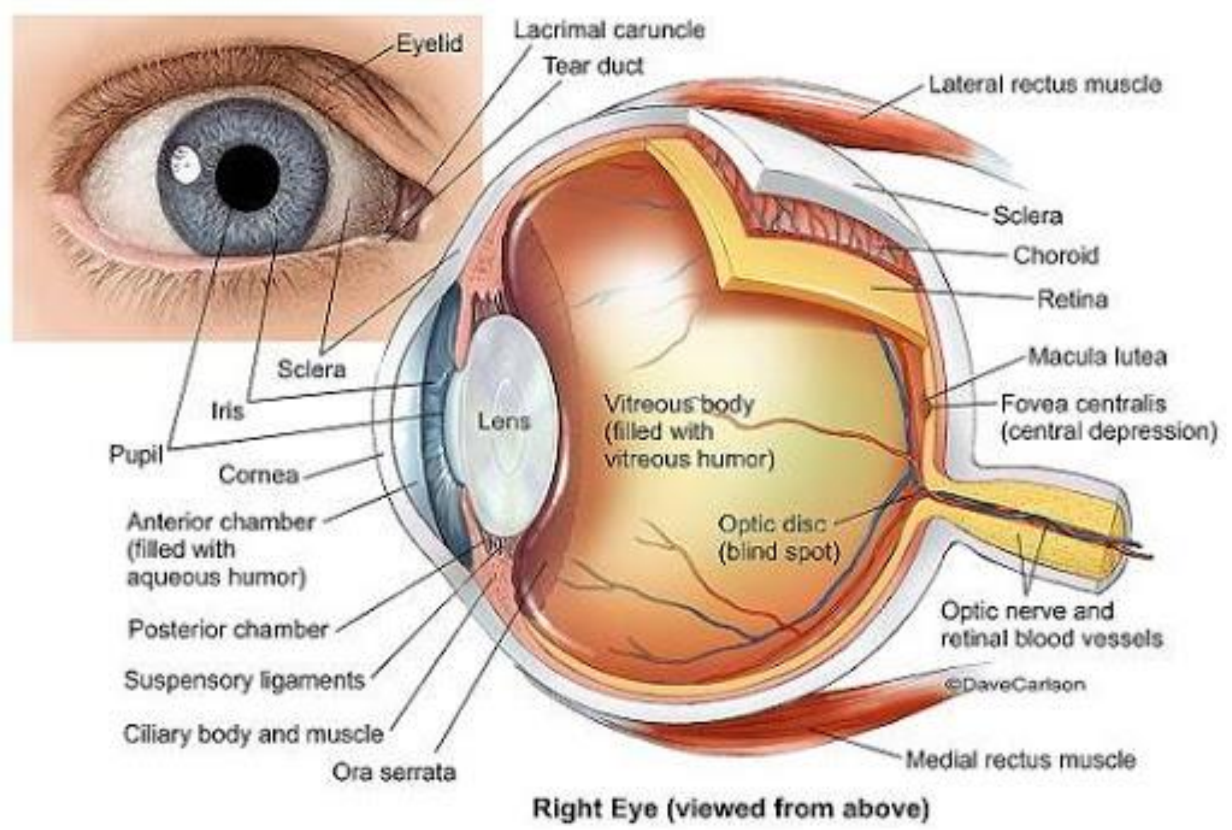
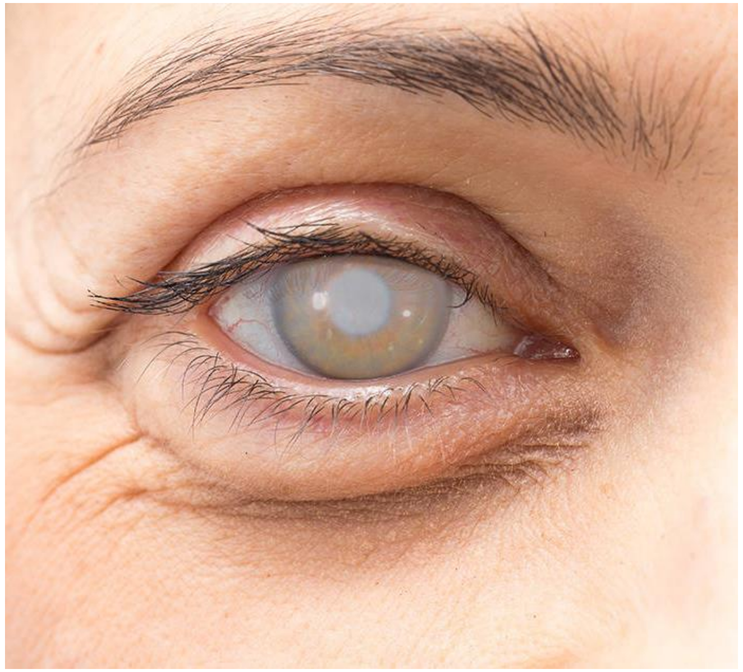
Trajectory Generation (1/2)

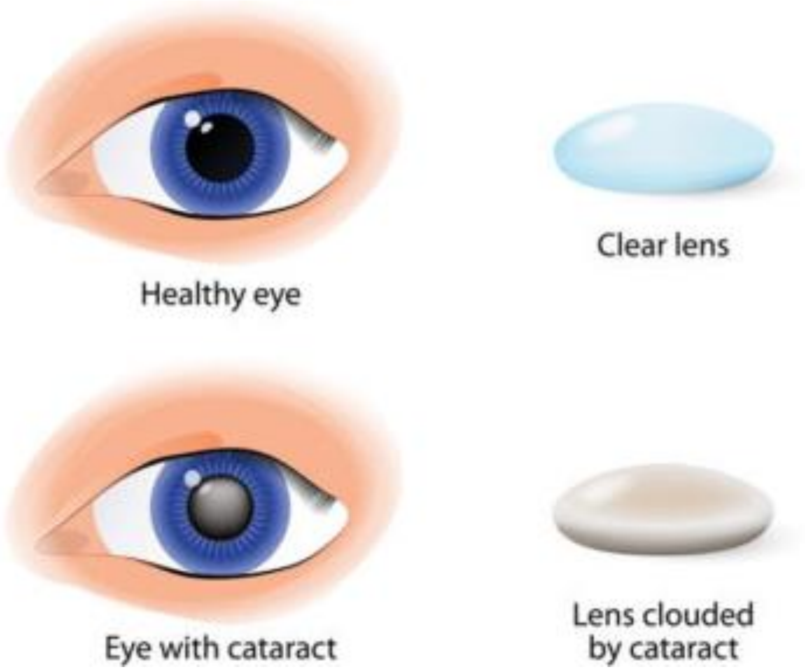
Introduction

Motion Planning



Motion Planning



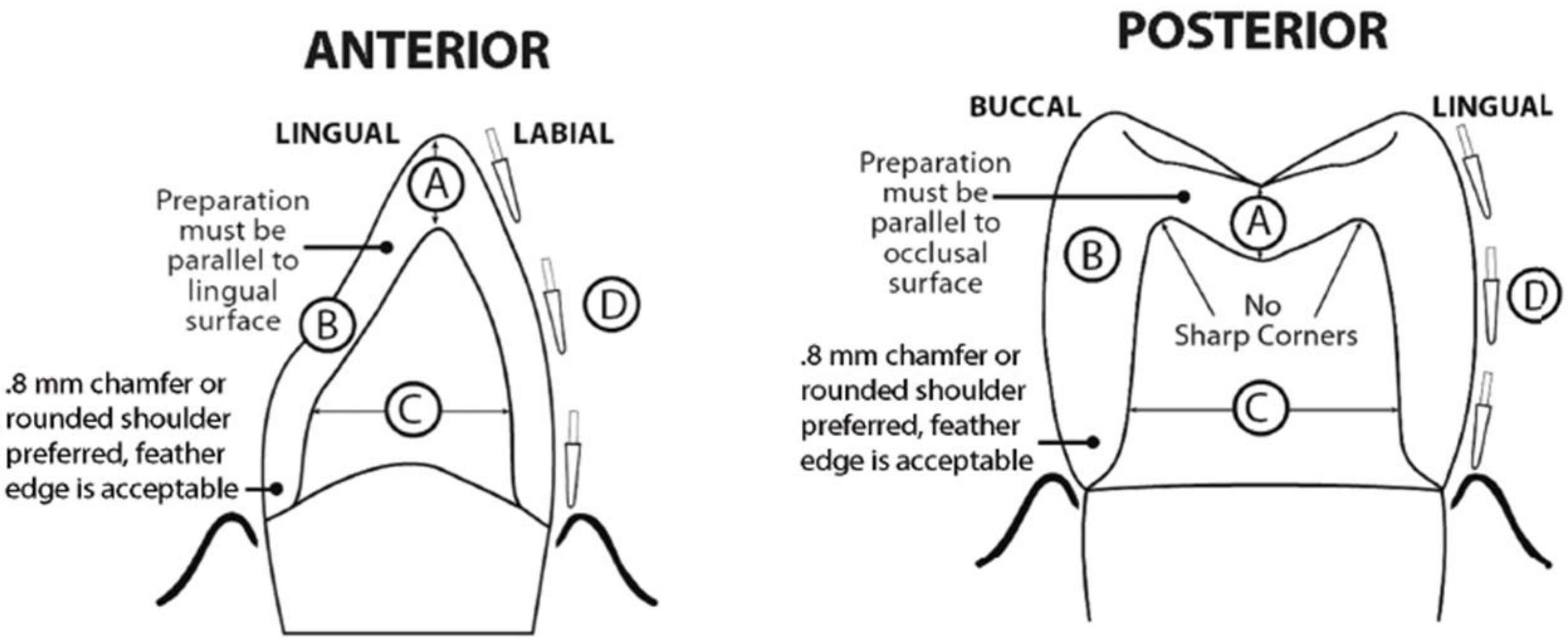


Routine Cataract Surgery

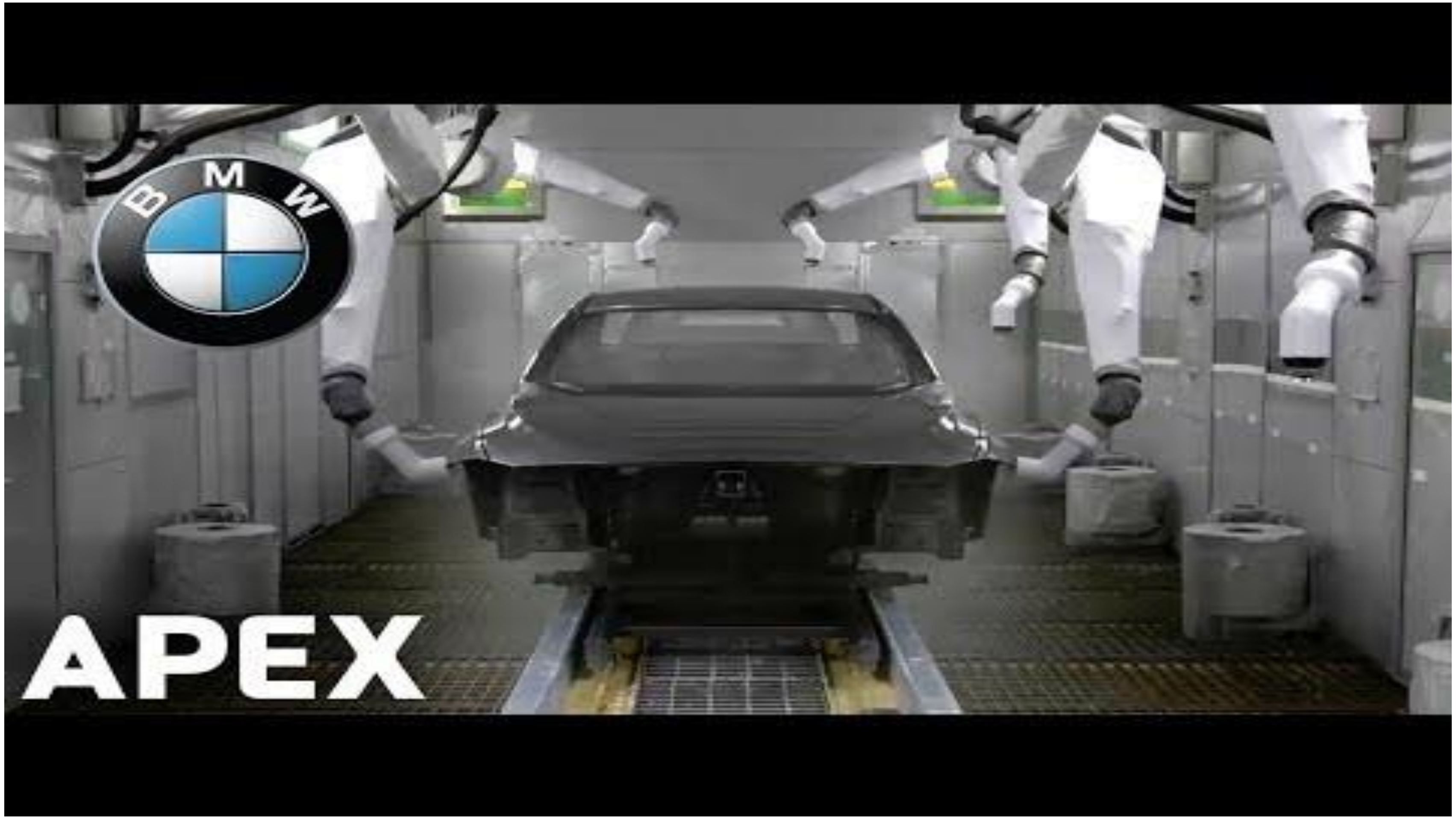
https://youtu.be/QbeI72QmFAU

Motion Planning

Motion Planning



A. 1.0 mm - 1.5 mm occlusal reduction
B. 1.0 mm middle third reduction
C. Buccal and lingual walls must be convergent
D. Preparation should be cut in three planes


BMW
APEX

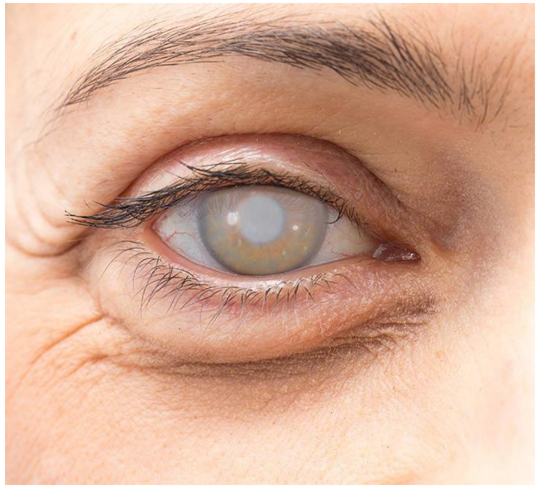
Motion Planning

Motion Planning

Problem Defenition

Motion Planning – Hierarchy

- Trajectory planning is a subset of the overall problem that is ***navigation or motion planning***. The typical hierarchy of motion planning is as follows:
    - **Task planning** – Designing a set of high-level goals, such as "go pick up the object in front of you".
    - **Path planning** – Generating a feasible path from a start point to a goal point. A path usually consists of a set of connected waypoints.
    - **Trajectory planning** – Generating a time schedule for how to follow a path given constraints such as position, velocity, and acceleration.
    - **Trajectory following** – Once the entire trajectory is planned, there is a need for a control system that can execute the trajectory in a sufficiently accurate manner.
- Q: What's the difference between path planning and trajectory planning?
- A: A trajectory is a description of how to follow a path over time



Trajectory Generation – Problem Definition

Problem

Given: *Manipulator geometry, End Effector Path (via point)*

JOINT SPACE

Compute: *The trajectory of each joint such that the end effector move in space from point A to Point B*

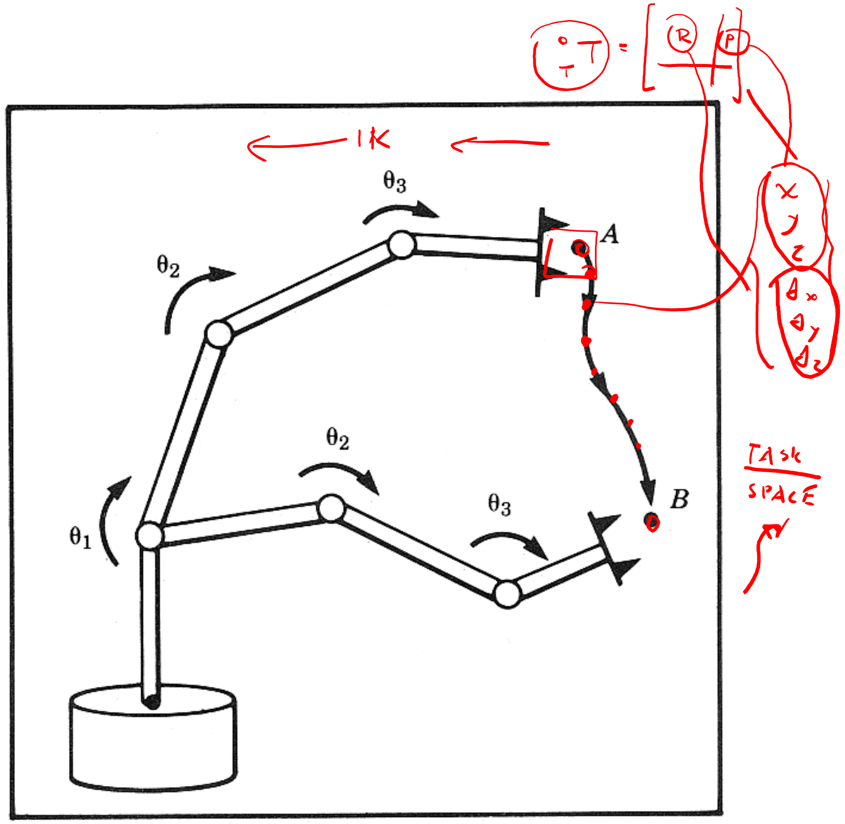
θ_1 ⋮ θ_6

Solution (Domains)

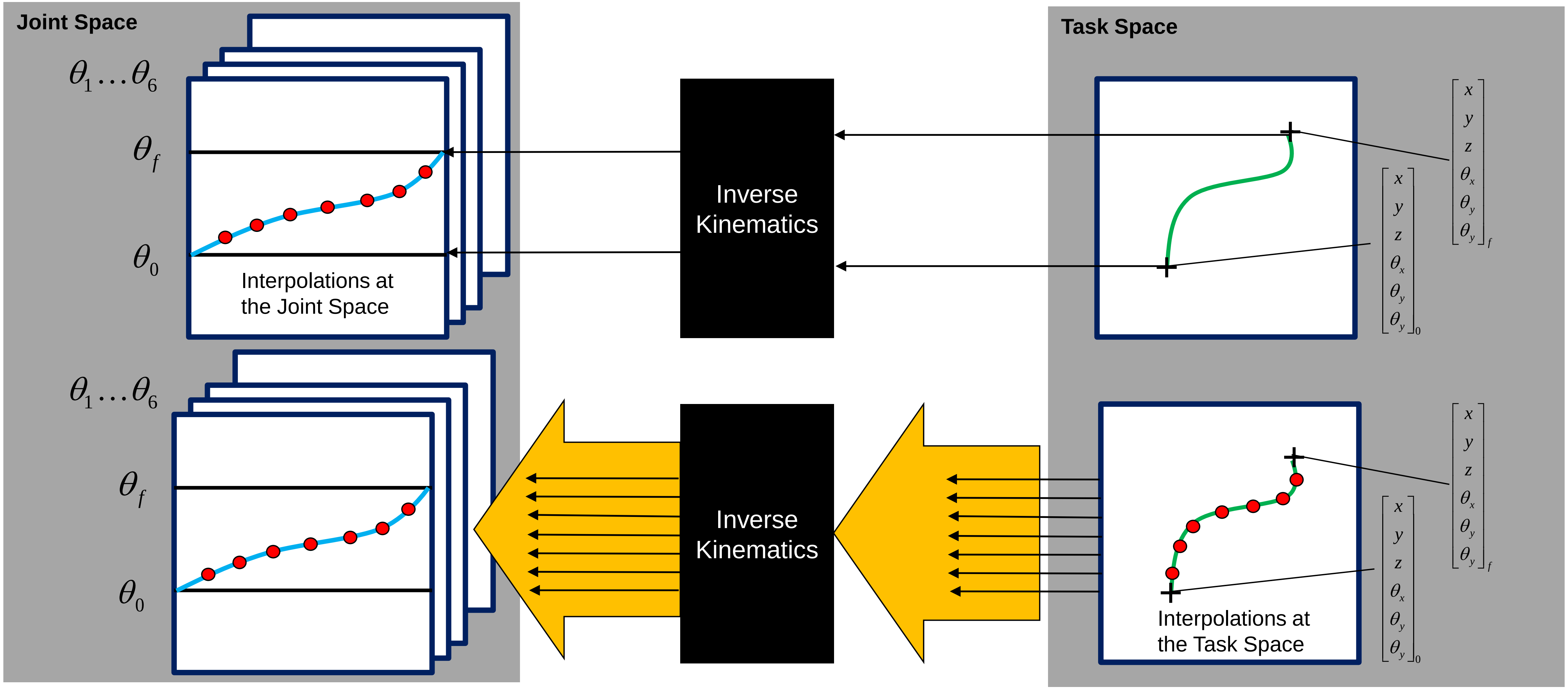
- **Joint space / Task Space**

Definitions

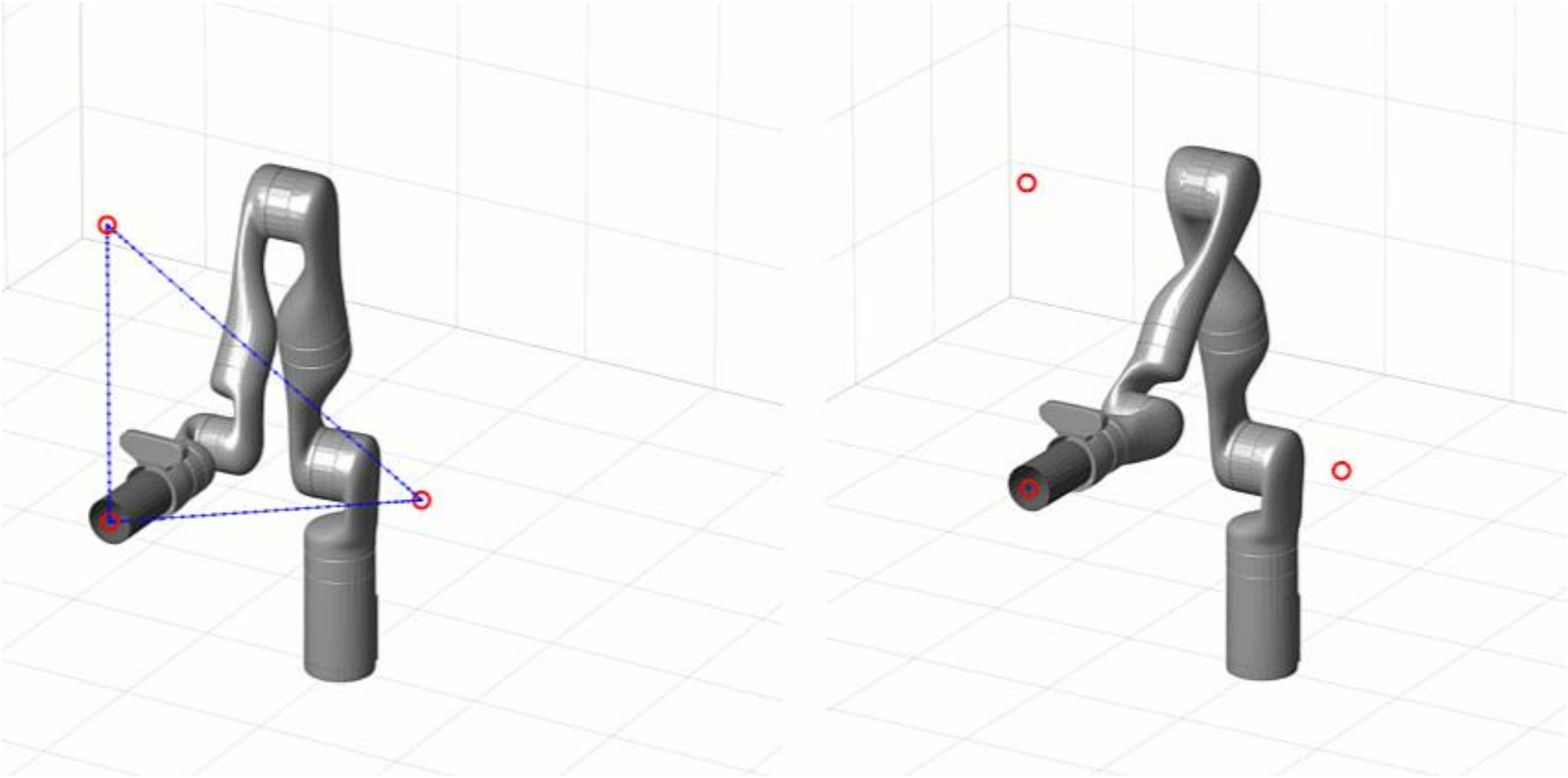
- **Trajectory (Definition)** - Time history of position, velocity, and acceleration for each DOF.
- **Trajectory Generation –** Methods of computing a trajectory that describes the desired motion of a manipulator in a multidimensional space



Task Space Versus Joint Space - Interpolations



Task Space Versus Joint Space

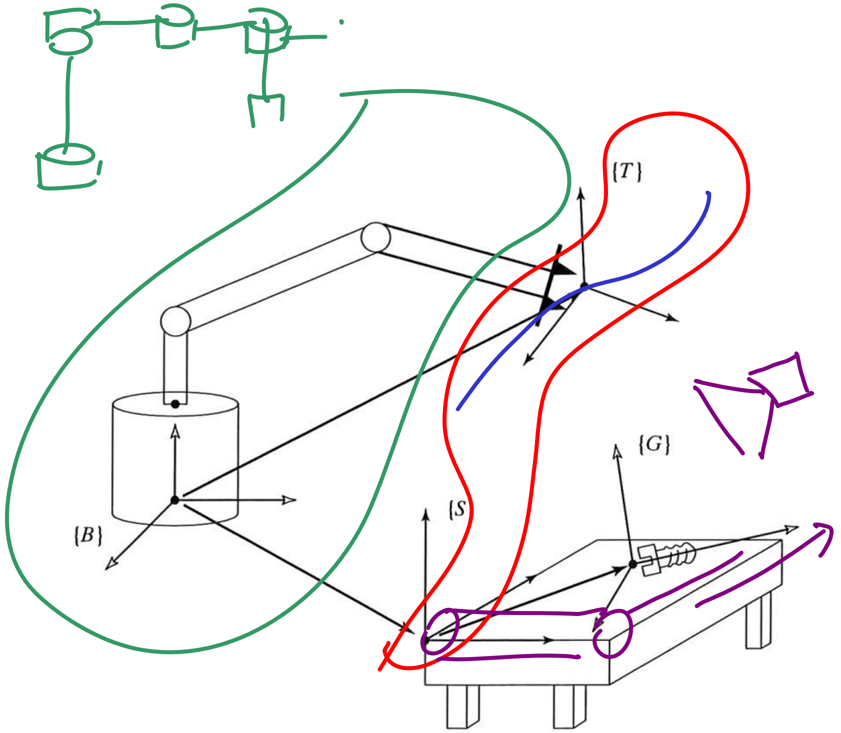


Join Space Versus Task Space – Comparison

Parameter	Joint Space	Task Space
Interpolation Space intermediate points along the trajectory	Joint Space	Task Space
Tool Trajectory Type / Length	Curved Line / Long	Straight Lines / Short
Invers Kinematics (IK) Usage	Low	High
Computation Expense (IK)	Low (IK for Start/Finish & Via Points)	High (IK for every single point / time steo on the trajectory)
Passing through Via Points	No (Correction by establishing Pseudo Points)	Yes
Via Points Defined in the Task Space	No	Yes
Path Dependency on a Specific Manipulator	Yes	No

General Consideration

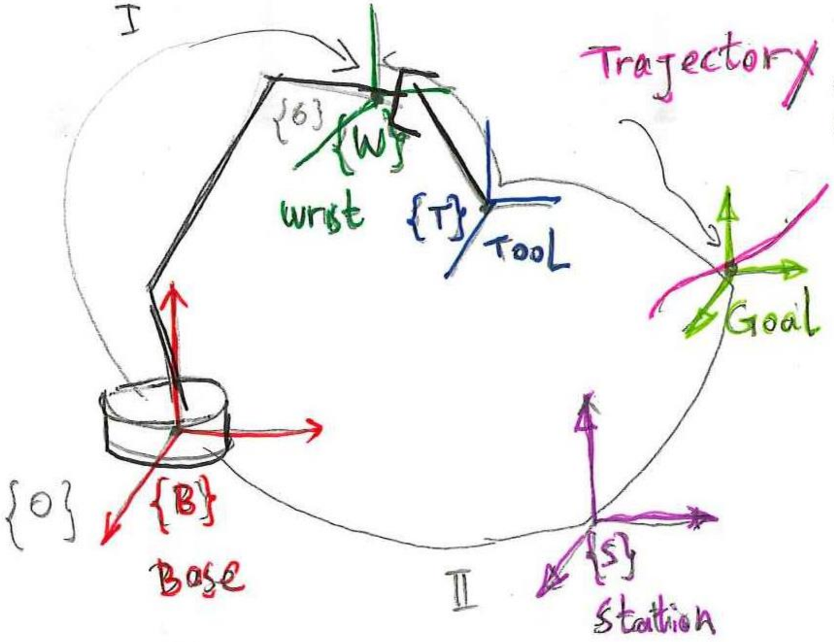
- **General approach for the motion of the manipulator**
  - Specify the path as a motion of the tool frame {*T*} relative to the station frame {*S*}. Frame {*G*} may change it position in time (e.g. conveyer belt)



- **Advantages**
  - Decouple the motion description from any particular robot, end effector, or workspace.
  - Modularity – Use the same path with:
    - Different robot
    - Different tool size

Trajectory Generation & Inverse Kinematics
General Approach

Transforming from the task space to the joint space using invers kinematics



II → Desired position and orientation

$$ {}^0_6T = \boxed{\overbrace{{}^B_W T}^{I} = \overbrace{{}^B_S T\ {}^S_G T\ {}^G_T T\ {}^T_W T}^{II}} \sim \text{Goal (KNOWN)} $$

${}^0_6T = {}^B_W T$ – Forward kinematics

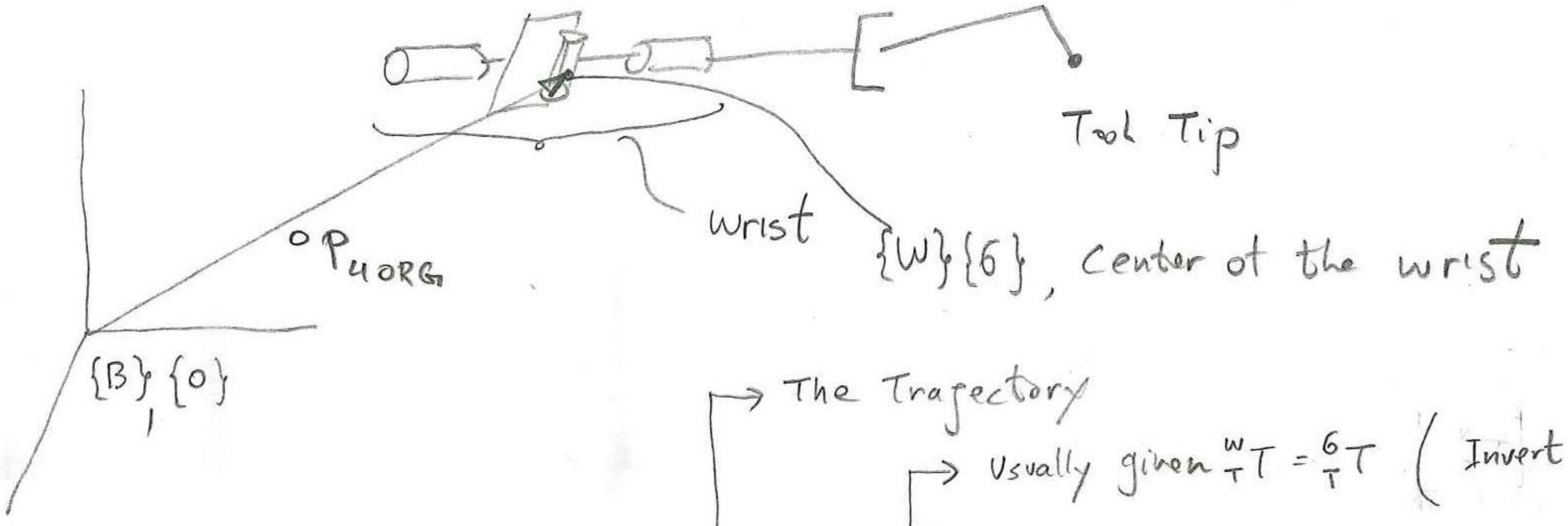
${}^B_S T$ – Location of the station with respect to the base

${}^S_G T$ – Defenition of the Trajectory

${}^G_T T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ – Full allignment (position, orientation)

${}^T_W T$ = Location of the wrist with respect to the tool → usualy given ${}^W_T T$

Trajectory Generation & Inverse Kinematics
General Approach



The Trajectory

Usually given ${}^W_T T = {}^6_T T$ (Invert ${}^T_W T = ({}^W_T T)^{-1}$)

$$ {}^B_W T = {}^0_6 T = \boxed{{}^B_S T \; {}^S_G T \; {}^G_T T \; {}^T_W T} = {}^0_1 T \; {}^1_2 T \; {}^2_3 T \; {}^3_4 T \; {}^4_5 T \; {}^5_6 T $$

$$ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow $$

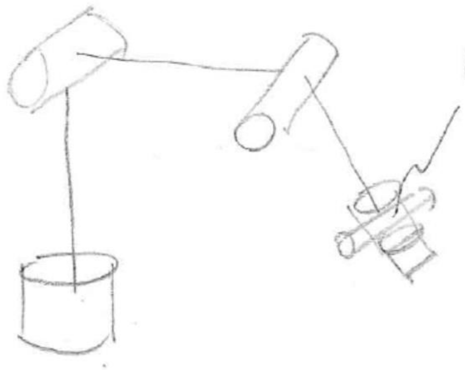
$$ \theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6 $$

KNOWN

User define $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline & & & 1 \end{array}\right]$

Trajectory Generation & Inverse Kinematics
General Approach

$$
{}^0_6T = \underset{\theta_1}{\underset{\downarrow}{{}^0_1T}}\;\underset{\theta_2}{\underset{\downarrow}{{}^1_2T}}\;\underset{\theta_3}{\underset{\downarrow}{{}^2_3T}}\;\underset{\theta_4}{\underset{\downarrow}{{}^3_4T}}\;\underset{\theta_5}{\underset{\downarrow}{{}^4_5T}}\;\underset{\theta_6}{\underset{\downarrow}{{}^5_6T}} = \underbrace{\left[\begin{array}{c|c} {}^0_6R & {}^0P_6 \\ \hline & \end{array}\right]}_{\text{GIVEN}}
$$



Given 3 intersecting axis 4,5,6 (origines of 4,56 are at the same point)

$$
{}^0P_6 = {}^6P_4
$$

Trajectory Generation & Inverse Kinematics
General Approach

$${}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T \,\Big|\, {}^3_4T \,\Big|\, {}^4_5T\,{}^5_6T$$

Problem 1

position problem

Problem 2

orientation problem

$${}^3_4T$$

$$\{\text{Frame } 3\} \xrightarrow{\alpha_3} \{\text{Frame }(R)\} \xrightarrow{a_3} \{\text{Frame }(Q)\} \xrightarrow{\theta_4} \{\text{Frame }(P)\} \xrightarrow{d_4} \{\text{Frame }(4)\}$$

$$R_{x_3}(\alpha_3) \qquad D_{x_3}(a_3) \qquad (R_{z_4}\theta_4) \qquad D_{z_4}(d_4)$$

Trajectory Generation & Inverse Kinematics
General Approach

$$ {}^0_6T = \underbrace{{}^0_1T\,{}^1_2T\,{}^2_3T\;\Big|\;R_{x_3}(\alpha_3)\,D_{x_3}(a_3)}_{\text{Problem 1: position problem}}\;\Big|\;\underbrace{R_{z_4}(\theta_4)\,D_{z_4}(d_4)\,{}^4_5T\,{}^5_6T}_{\text{Problem 2: orientation problem}} $$

Problem 1

position problem

Problem 2

Oriantation problem

$$ {}^0_6T = {}^0_1T\;{}^1_2T\;{}^2_3T\;{}^3_4T\Big|_{\theta_4=0}\;\Big|\;{}^3_4T\Big|_{\alpha_3=0}\;{}^4_5T\;{}^5_6T $$

set $\theta_4 = 0$ set $\alpha_3 = 0$

Trajectory Generation & Inverse Kinematics
General Approach

$$ {}^0_6R = {}^0_3R\; {}^3_4R\; {}^4_6R $$

$$ {}^0_6R = {}^0_3R \left(R(\alpha_3)\; \underset{\substack{\uparrow \\ D_{x_3}(a_3)}}{I} \;\Big|\; R(\theta_4)\; \underset{\substack{\uparrow \\ D_{z_4}(d_4)}}{I} \right) {}^4_6R $$

$$ {}^0_6R = \left[{}^0_3R\; R_{x_3}(\alpha_3) \right] \left[R_{z_4}(\theta_4)\; {}^4_6R \right] $$

Trajectory Generation & Inverse Kinematics
General Approach

Solving for $\theta_4, \theta_5, \theta_6$

$$\underbrace{R_{z_4}(\theta_4)\,{}^4_6R}_{} = \underbrace{\left[{}^0_3R\, R_{x_3}(\alpha_3)\right]^{-1}}_{} \underbrace{{}^0_6R}_{}$$

Solved in Problem 1

Known $\theta_1, \theta_2, \theta_3$

Desired orientation given for every point on the trajectory

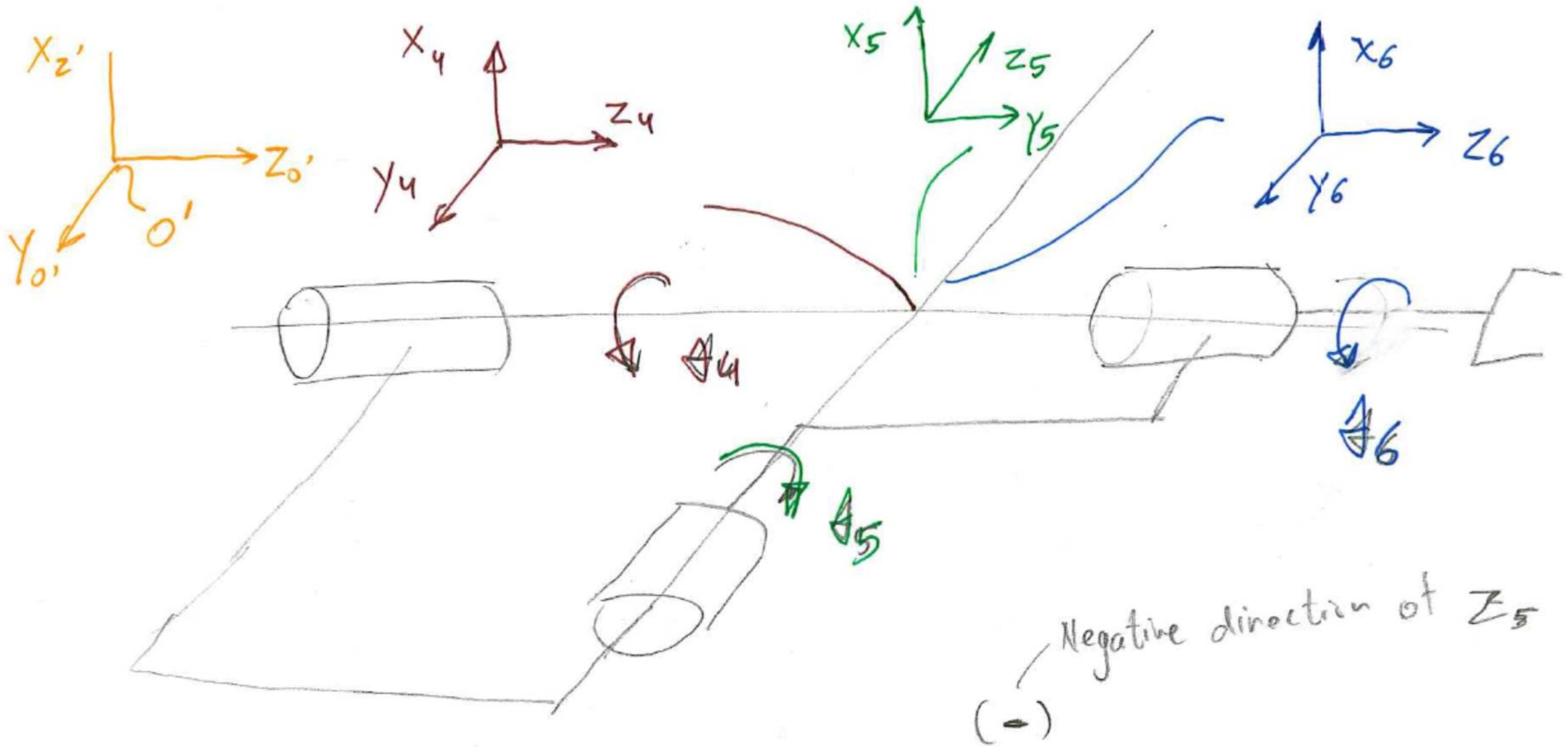
$$R_{z_4}(\theta_4)\,{}^4_5R(\theta_5)\,{}^5_6R(\theta_6) = \bar{R}_D$$

[Desired orintation of the wrist taking into account the contribution of the first 3 angles to the orintation]

$$R_{Z4}(\theta_4)\,{}^4_5R(\theta_5)\,{}^5_6R(\theta_6) = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\bar{R}_D}$$

solve for $\theta_4, \theta_5, \theta_6$ using the Z-Y-Z problem

Trajectory Generation & Inverse Kinematics
General Approach



$$R_{Euler} = R_z(\theta_4)\ R_{y4}(\theta_5)\ R_{z4}(\theta_6)$$

$$R_{ZYZ}(\alpha, \beta, \gamma) = R_Z(\alpha) R_Y(\beta) R_Z(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha \to \theta_4 \qquad \beta \to -\theta_5 \qquad \gamma \to \theta_6$$

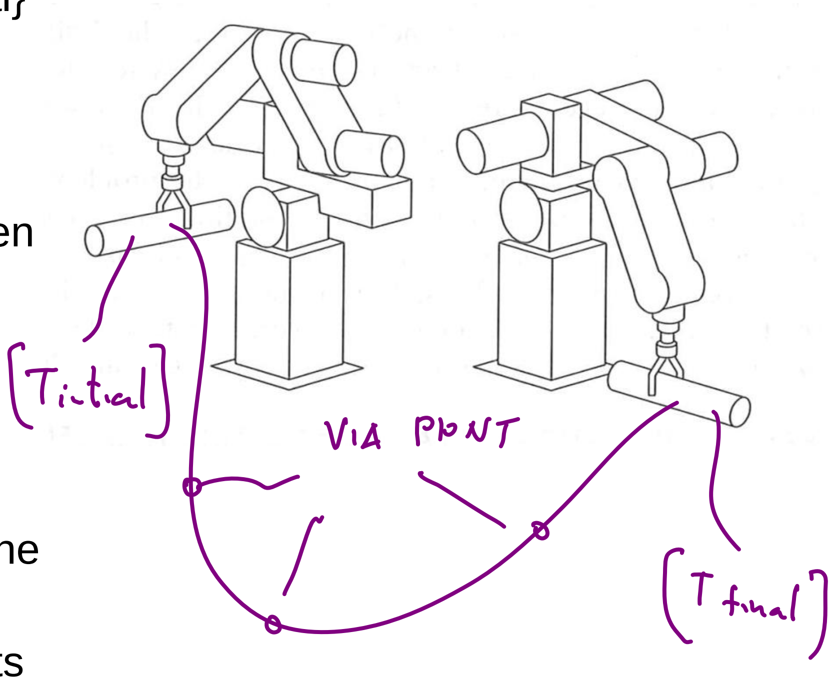
Note: ∇_0

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

General Consideration – Via Points

- **Basic Problem –** Move the tool frame {T} from its initial position / orientation {T_initial} to the final position / orientation {T_final}.

- **Specific Description**
  - **Via Point** – Intermediate points between the initial and the final end- effector locations that the end-effector mast go through and match it position and orientation along the trajectory.
  - Each via point is defined by a **frame defining the position/orintataion** of the tool with respect to the station frame
  - **Path Points** – includes all the via points along with the initial and final points
  - **Point (Frame)** – Every point on the trajectory is define by a frame (spatial description)



$$T = \left[\begin{array}{c|c} {}^{S}_{T}R & {}^{S}P_{T} \\ \hline & \end{array}\right]$$

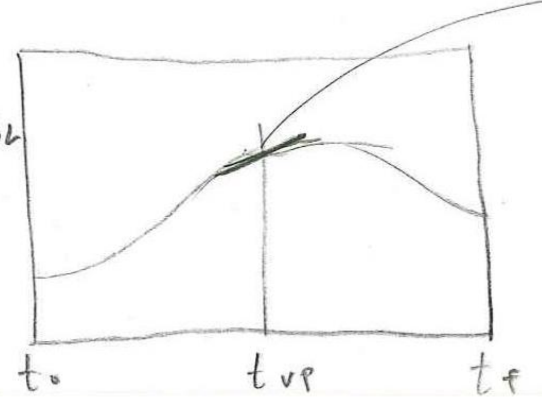
General Consideration – Smooth Path

- **"Smooth" Path or Function**
  - Continuous path / function with first and second derivatives.
  - Add constrains on the spatial and temporal qualities of the path between the via-points

- **Implications of non-smooth path**
  - Increase wear in the mechanism (rough jerky movement)
  - Vibration – exciting resonances.

- Example of Smooth Path

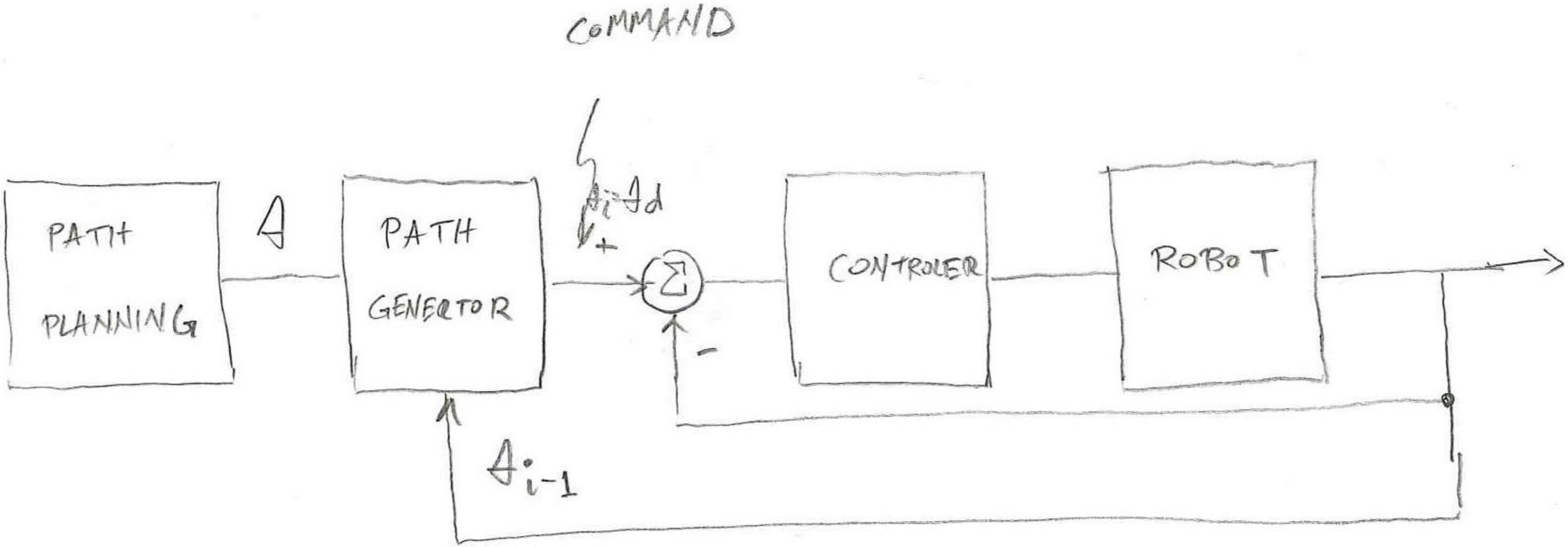
The joint angle (θ), joint velocity ($\dot{\theta}$), and joint acceleration ($\ddot{\theta}$) at the entry and exit of the via point are equal



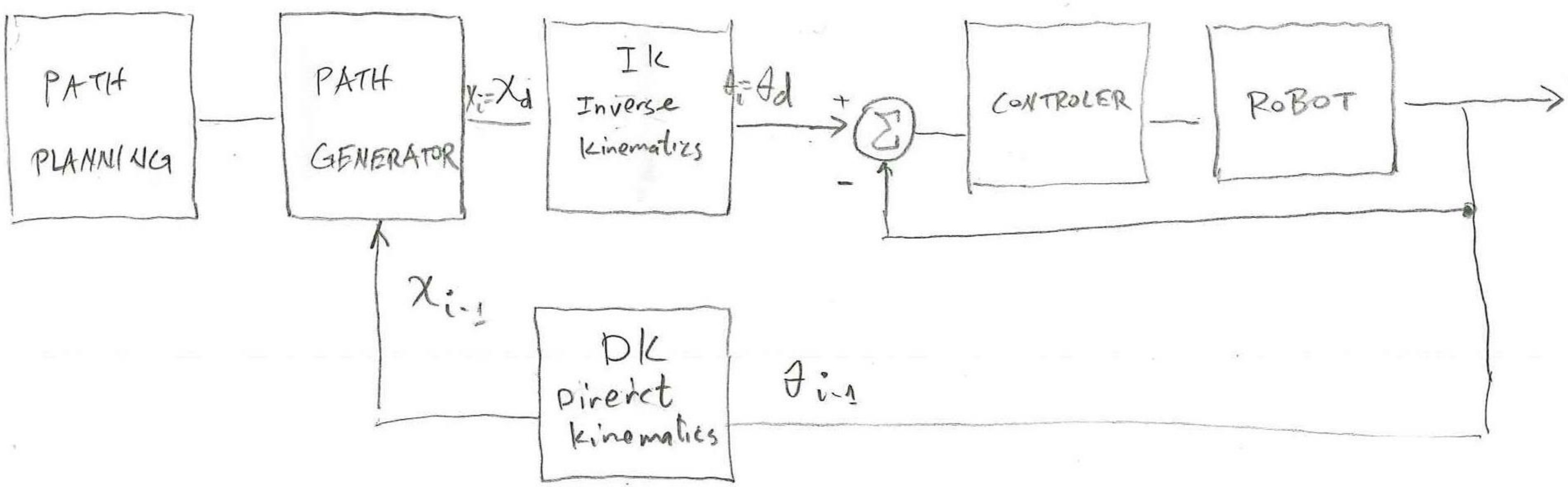
VIA POINT

$$\begin{cases} \theta_{t_{vp-}} = \theta_{t_{vp+}} \\ \dot{\theta}_{t_{vp-}} = \dot{\theta}_{t_{vp+}} \\ \ddot{\theta}_{t_{vp-}} = \dot{\theta}_{t_{vp+}} \end{cases}$$

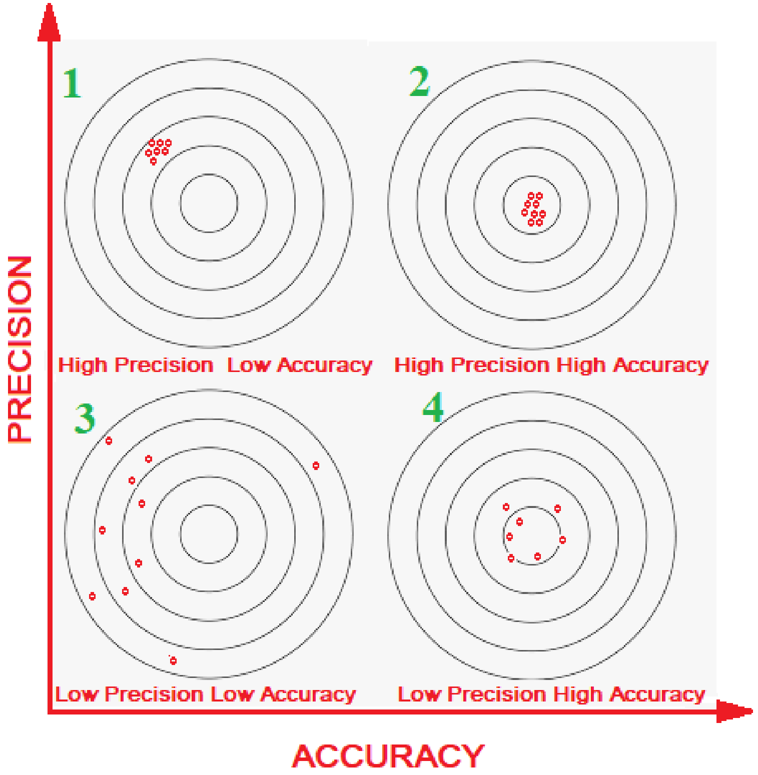
Trajectory Generation – Joint Space Space Control

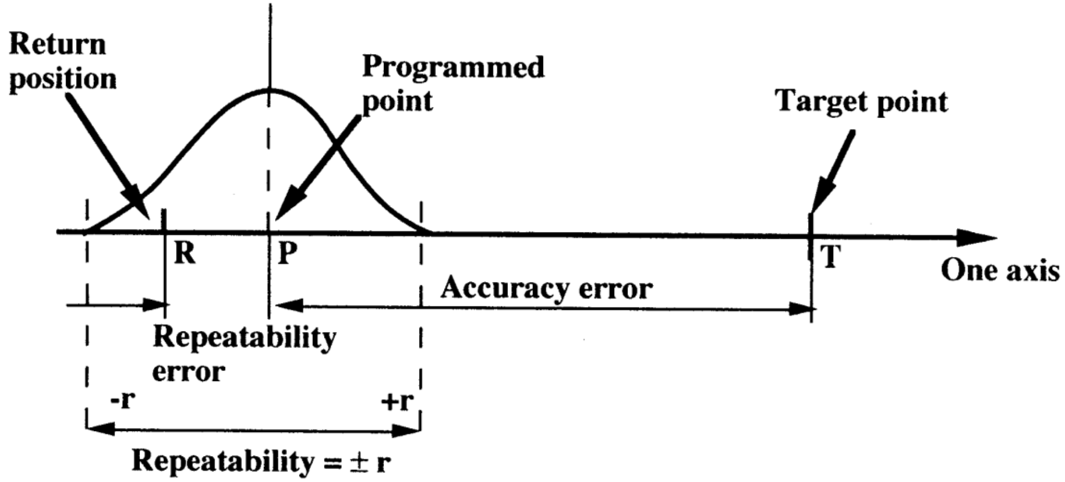


Trajectory Generation – Task Space Control

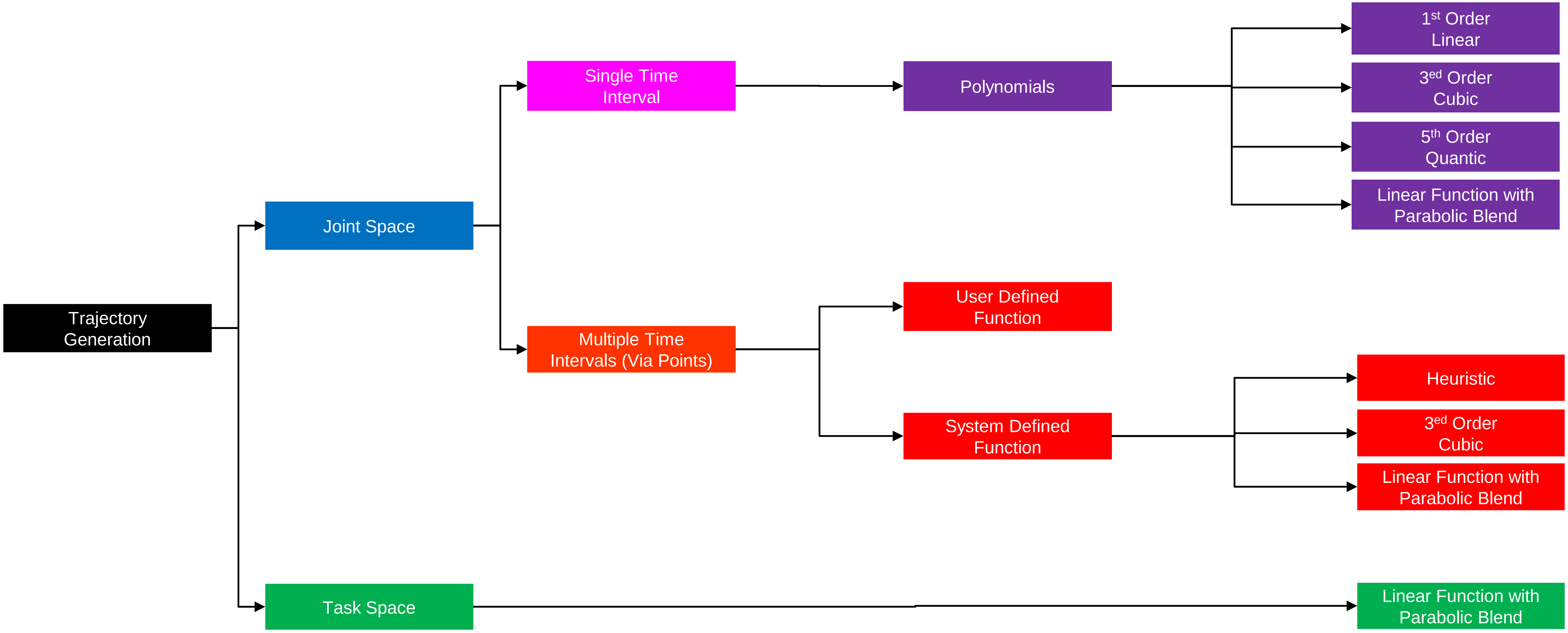


Precision / Repeatability versus Accuracy





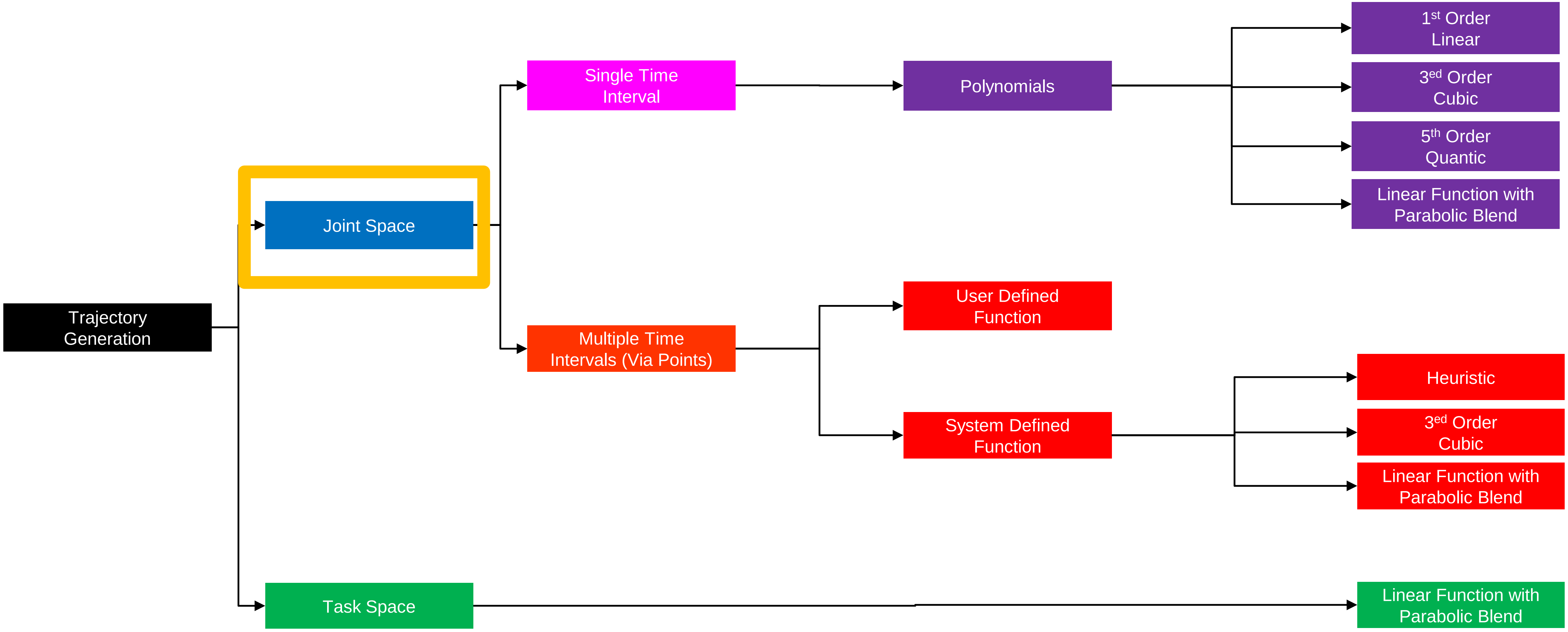
Trajectory Generation – Roadmap Diagram



Joint Space Schemes

Single Time Interval

Trajectory Generation – Roadmap Diagram



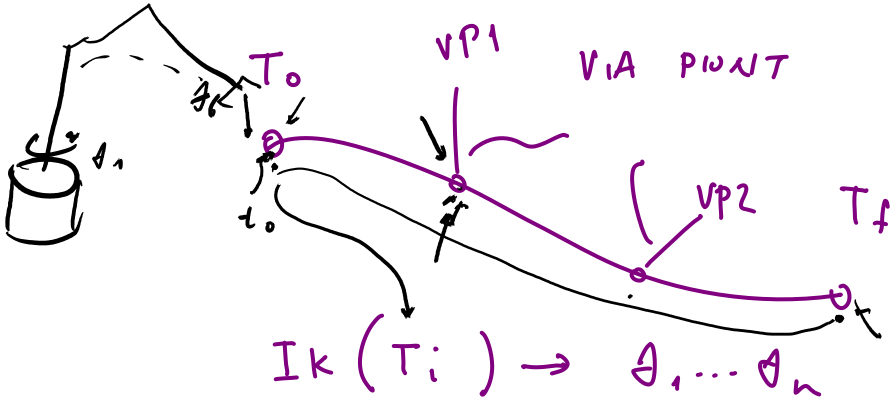
Joint Space Schemes

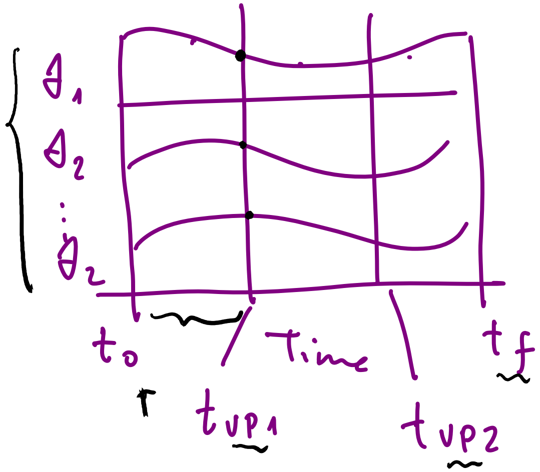
- **Joint space Schemes** – Path shapes (in space and in time) are described in terms of functions in the joint space.

- **General process (Steps) given initial and target P/O**
  1. Select a path point or via point (desired position and orientation of the tool frame {T} with respect to the base frame {s})
  2. Convert each of the “via point” into a set of joint angles using the invers kinematics
  3. Find a smooth function for each of the *n* joints that pass trough the via points, and end the goal point.

Note 1: The **time required to complete each segment is the same for each joint** such that the all the joints will reach the via point at the same time. Thus resulting in the position and orientation of the frame {T} at the via point.

Note 2: The **joints move independently** with only one time restriction (Note 1)

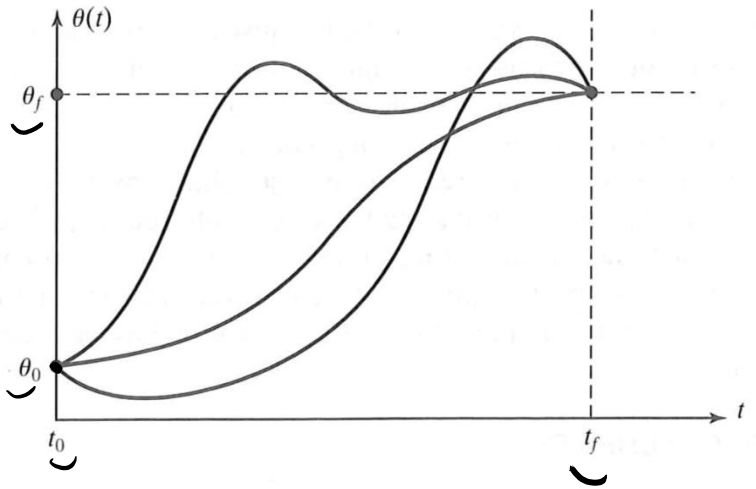


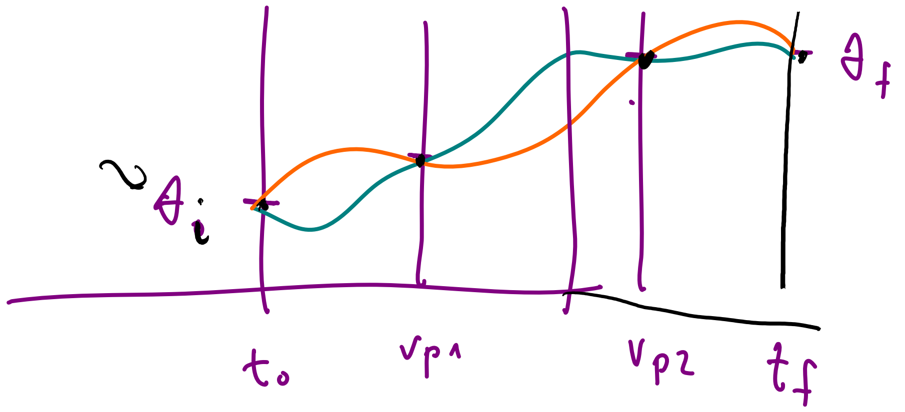


Joint Space Schemes

- Define a function for each joint such that value at t_0 is the **initial position** of the joint and whose value at t_f is the **desire goal position** of the joint

- There are many smooth functions $\theta(t)$ that may be used to interpolate the joint value.





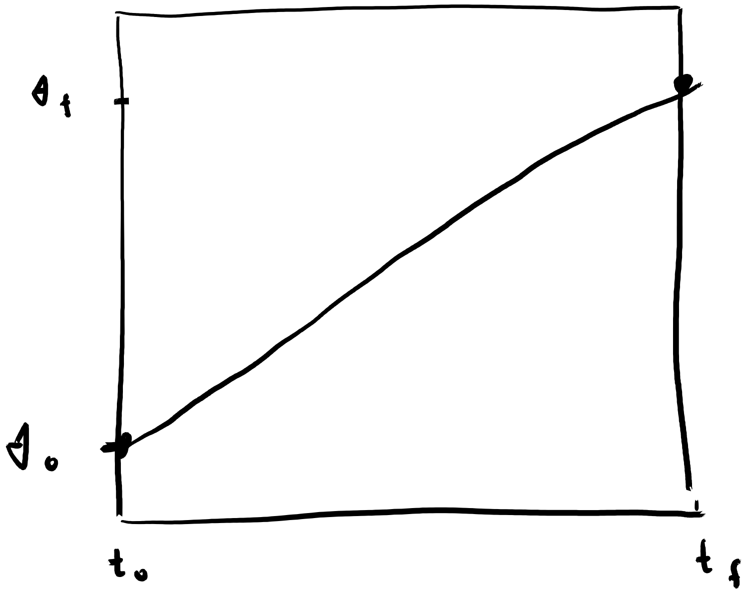
Joint Space Schemes

Single Time Interval

Polynomials

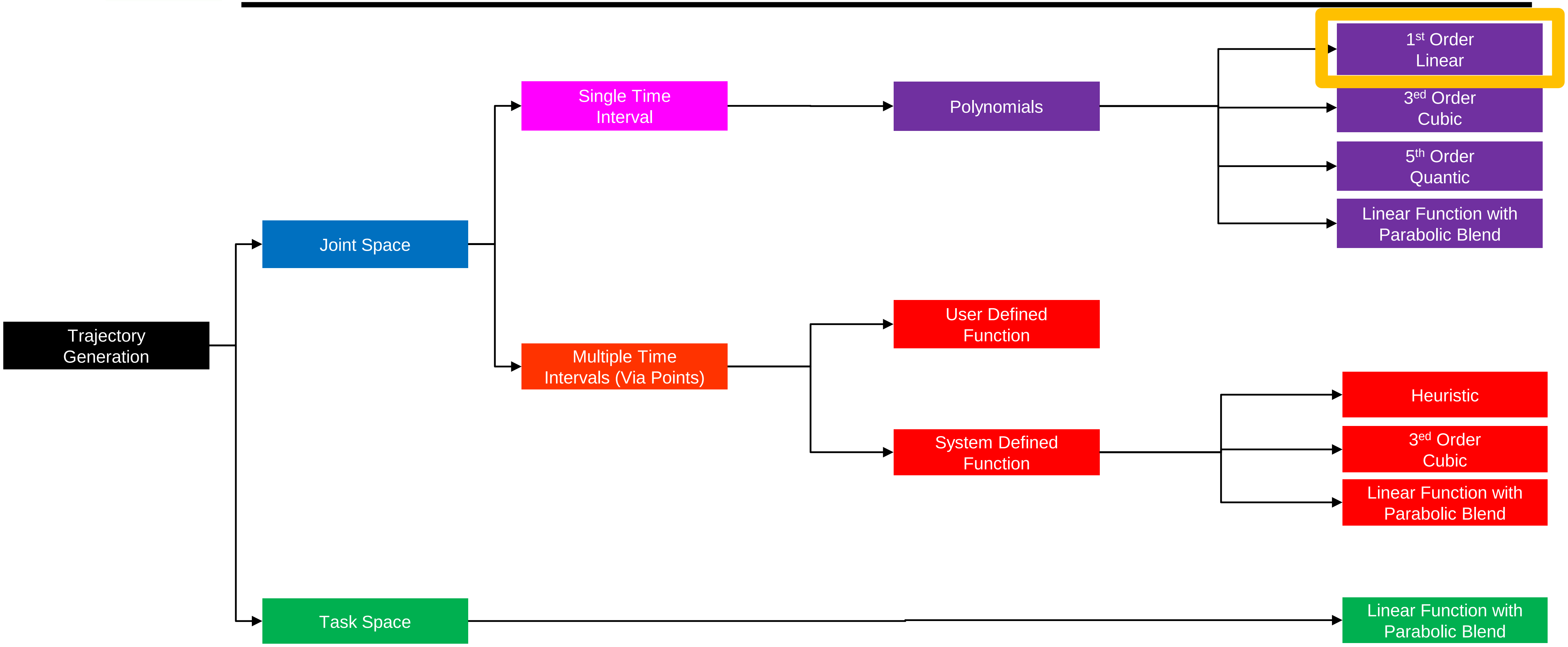
First Order Polynomial

Joint Space Schemes



$$y = a_0 + a_1 t$$

Trajectory Generation – Roadmap Diagram

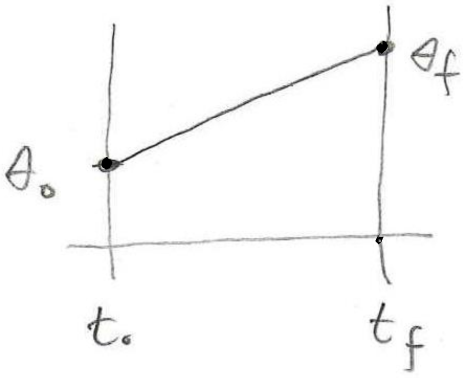


Joint Space Schemes – Linear Polynomials

- Problem – Define a function for each joint such that its value at

  – t_0 is the inital position of the joint

  t_f is the desired goal position of the joint



- Given – Constrains on $\theta(t)$

$$\begin{cases} \theta(0) = \theta_0 \\ \theta(t_f) = \theta_f \end{cases}$$

Joint Space Schemes – Linear Polynomials

- Solution – The two constraints can be satisfied by a first order polynomial

$$\theta(t) = a_0 + a_1 t$$

- Combined with the two desired constrains yields two equations in two unknown

$$\begin{cases} \theta_0(0) = a_0 \\ \theta_f(t_f) = a_0 + a_1 t_f \end{cases} \Rightarrow \begin{cases} \theta_0 = a_0 \\ a_1 = \dfrac{\theta_f - \theta_0}{t_f} \end{cases}$$

$$\rightarrow \theta = \theta_0 + \left(\frac{\theta_f - \theta_0}{t_f} \right) t$$

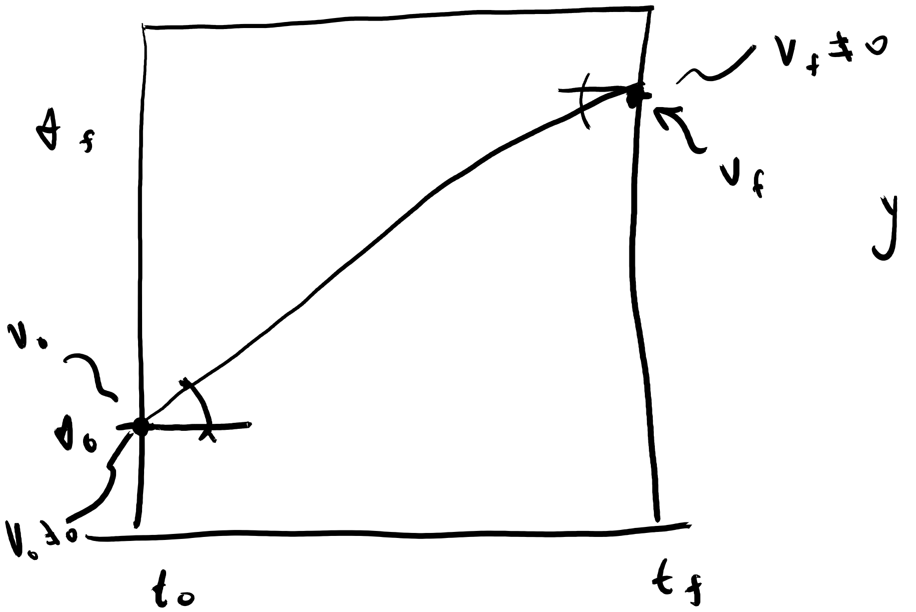
Joint Space Schemes

Single Time Interval

Polynomials

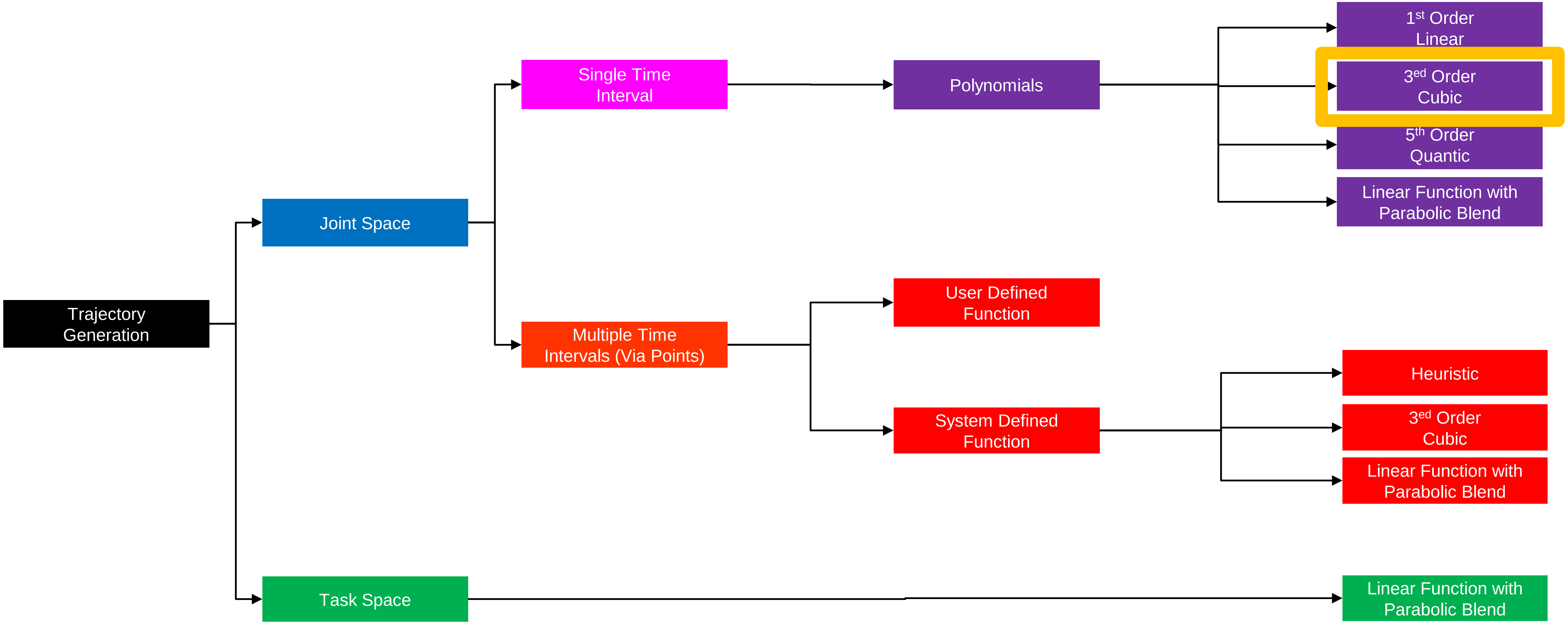
Cubic Order Polynomial

Joint Space Schemes



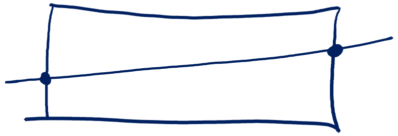
$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Trajectory Generation – Roadmap Diagram

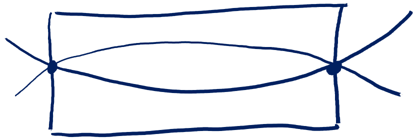


Joint Space Schemes – Order of the Polynomials

$$\theta(t) = a_0 + a_1 t$$



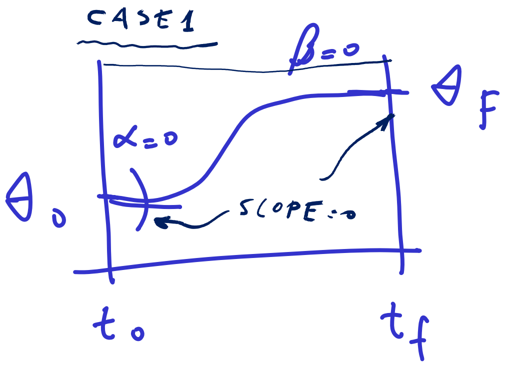
$$\theta(t) = a_0 + a_1 t + a_2 t^2$$



$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

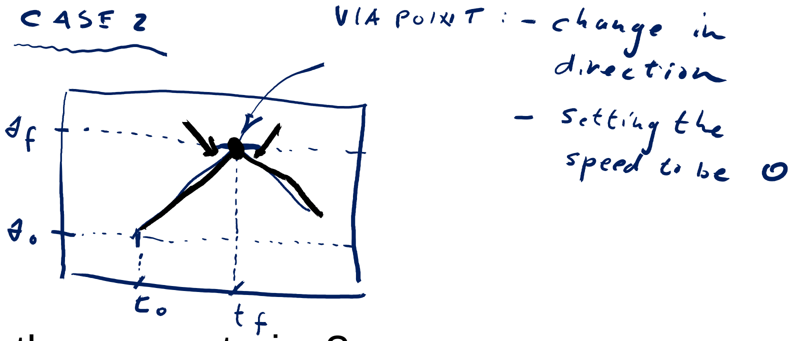
- Problem - Define a function for each joint such that it value at
  - t_0 is the **initial position** of the joint and at
  - t_f is the **desired goal position** of the joint



- Given - Constrains on $\theta(t)$

$$\theta(0) = \theta_0$$
$$\theta(t_f) = \theta_f$$
$$\dot{\theta}(0) = 0$$
$$\dot{\theta}(t_f) = 0$$

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$



- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial of at least third degree

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- The joint velocity and acceleration

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$a_0 \to a_3$

$$\begin{array}{ll} \theta(0) = \theta_0 \longrightarrow & \left\{ \begin{array}{l} \theta_0 = a_0 \\ \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ 0 = a_1 \\ 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{array} \right. \\ \theta(t_f) = \theta_f \longrightarrow & \\ \dot{\theta}(0) = 0 \longrightarrow & \\ \dot{\theta}(t_f) = 0 \longrightarrow & \end{array}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$
\begin{bmatrix} \theta_0 \\ \theta_f \\ 0 \\ 0 \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix}
\end{array}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}
$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$\begin{cases} \theta_f = \theta_0 + a_2 t_f^2 + a_3 t_f^3 \\ 0 = 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

$$\begin{cases} a_2 [t_f^2] + a_3 [t_f^3] = \theta_f - \theta_0 \\ a_2 [2t_f] + a_3 [3t_f^2] = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} \theta_f - \theta_0 & t_f^3 \\ 0 & 3t_f^2 \end{vmatrix}}{\Delta} = \frac{3t_f^2(\theta_f - \theta_0)}{t_f^4} = \frac{3(\theta_f - \theta_0)}{t_f^2}$$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & \theta_f - \theta_0 \\ 2t_f & 0 \end{vmatrix}}{\Delta} = \frac{-2t_f(\theta_f - \theta_0)}{t_f^4} = \frac{-2(\theta_f - \theta_0)}{t_f^3}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0)$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$\dot{\theta}_{max}$ – Max angular velocity at $t_f/2$

$$\dot{\theta}_{max}\left(t = t_f/2\right) = \frac{6}{t_f^{\cancel{2}}}\left(\theta_f - \theta_0\right)\left[\frac{\cancel{t_f}}{2}\right] - \frac{6}{t_f^3}\left(\theta_f - \theta_0\right)\left[\frac{t_f}{2}\right]^2$$

$$= \frac{3\left(\theta_f - \theta_0\right)}{t_f} - \frac{\overset{3}{\cancel{6}}}{\underset{2}{\cancel{4}}}\,\frac{\left(\theta_f - \theta_0\right)}{t_f}$$

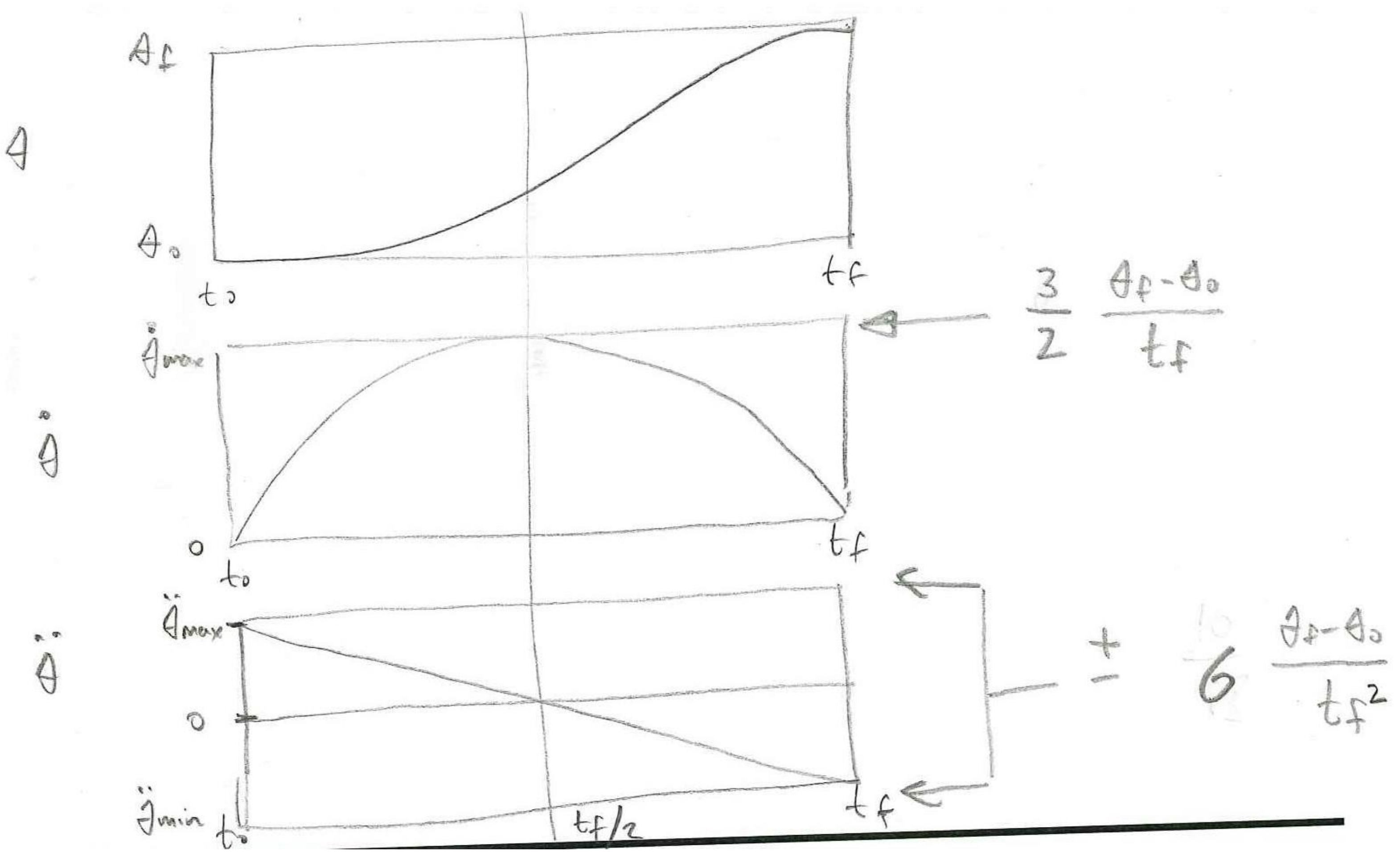
$$= \frac{3}{2}\,\frac{\theta_f - \theta_0}{t_f}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$\ddot{\theta}_{max}$ - Max angular acceleration at $t=0$ and $t=t_f$

$$\ddot{\theta}_{max} = \frac{6}{t_f^2}(\theta_f - \theta_0)$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity



Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Example – A single-link robot with a rotary joint is motionless at $\theta_0 = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta_f = 75$ degrees in 3 seconds. Find the coefficient of the cubic polynomial that accomplish this motion and brings the manipulator to rest at the goal

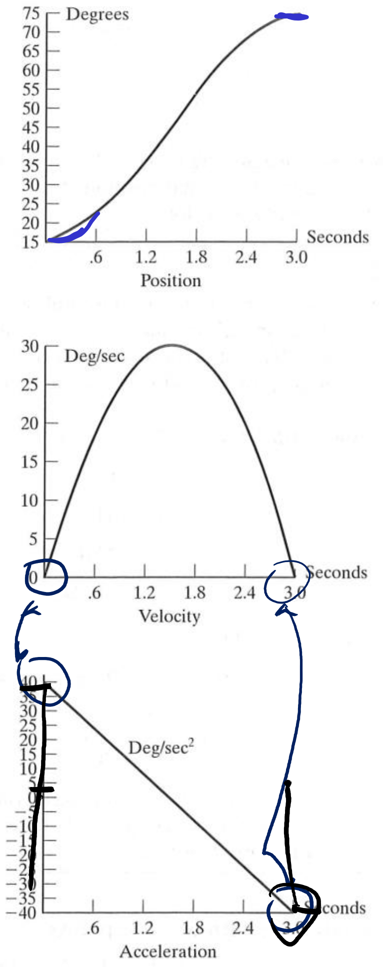
$$\theta(0) = 15$$
$$\theta(t_f) = 75$$
$$\dot{\theta}(0) = 0$$
$$\dot{\theta}(t_f) = 0$$

$$a_0 = \theta_0 = 15$$
$$a_1 = 0$$
$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) = \frac{3}{9}(75-15) = 20$$
$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) = -\frac{2}{27}(75-15) = -4.44$$

$$\theta(t) = 15 + 20t^2 - 4.44t^3$$
$$\dot{\theta}(t) = 40t - 13.33t^2$$
$$\ddot{\theta}(t) = 40 + 26.66t$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

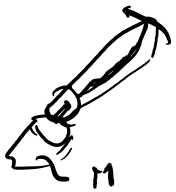
- The velocity profile of any cubic function is a parabola
- The acceleration profile of any cubic function is linear



ZERO VELOCITY

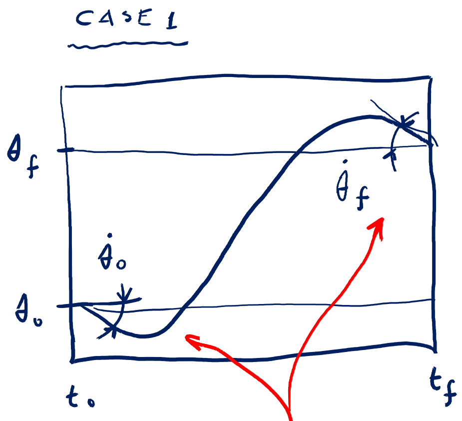
HIGHEST ACCELERATION

$M = I\ddot{\theta}$



Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

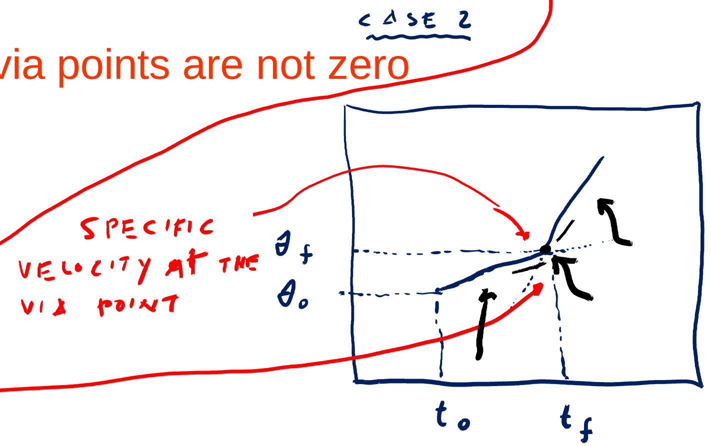
- Previous Method - The manipulator comes to rest at each via point
- General Requirement - Pass through a point without stopping
- Problem - Define a function for each joint such that it value at
  - t_0 is the **initial position** of the joint and at
  - t_f is the **desire goal position** of the joint
- Given - Constrains on $\theta(t)$ such that the velocities at the via points are not zero but rather some known velocities



$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

$$\boxed{\begin{aligned}\dot{\theta}(0) &= \dot{\theta}_0 \\ \dot{\theta}(t_f) &= \dot{\theta}_f\end{aligned}}$$



Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$$\begin{aligned}
\theta(0) &= \theta_0 & \theta_0 &= a_0 \\
\theta(t_f) &= \theta_f & \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
\dot{\theta}(0) &= \dot{\theta}_0 & \dot{\theta}_0 &= a_1 \\
\dot{\theta}(t_f) &= \dot{\theta}_f & \dot{\theta}_f &= a_1 t_f + 2a_2 t_f + 3a_3 t_f^2
\end{aligned}$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

$$\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

$$\begin{cases} \theta_f = \theta_0 + \dot{\theta}_0 t_f + a_2 t_f^2 + a_3 t_f^3 \\ \dot{\theta}_f = \dot{\theta}_0 + 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

$$\begin{cases} a_2\left[t_f^2\right] + a_3\left[t_f^3\right] = (\theta_f - \theta_0) - \dot{\theta}_0 t_f \\ a_2\left[2t_f\right] + a_3\left[3t_f^2\right] = (\dot{\theta}_f - \dot{\theta}_0) \end{cases}$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} (\theta_f - \theta_0) + \dot{\theta}_0 t_f & t_f^3 \\ (\dot{\theta}_f - \dot{\theta}_0) & 3t_f^2 \end{vmatrix}}{\Delta} = \frac{3t_f^2\left((\theta_f - \theta_0) - \dot{\theta}_0 t_f\right) - t_f^3(\dot{\theta}_f - \dot{\theta}_0)}{t_f^4}$$

$$= \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$+2t_f^2\dot{\theta}_0$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & (\theta_f - \theta_0) - \dot{\theta}_0 t_f \\ 2t_f & \dot{\theta}_f - \dot{\theta}_0 \end{vmatrix}}{\Delta} = \frac{t_f^2(\dot{\theta}_f - \dot{\theta}_0) - 2t_f\left[(\theta_f - \theta_0) - \dot{\theta}_0 t_f\right]}{t_f^4}$$

$$= -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\theta_f + \theta_0)$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)$$

- Given - velocities at each via point are
- Solution - Apply these equations for each segment of the trajectory.
- Note: The Cubic polynomials ensures the continuity of velocity but not the acceleration. Practically, the industrial manipulators are sufficiently rigid so this this continuity in acceleration

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Note:
  - The Cubic polynomials ensures the continuity of velocity but not the acceleration.
  - Practically, the industrial manipulators are sufficiently rigid so this discontinuity in acceleration is filtered by the mechanical structure
  - Therefore this trajectory is generally satisfactory for most applications

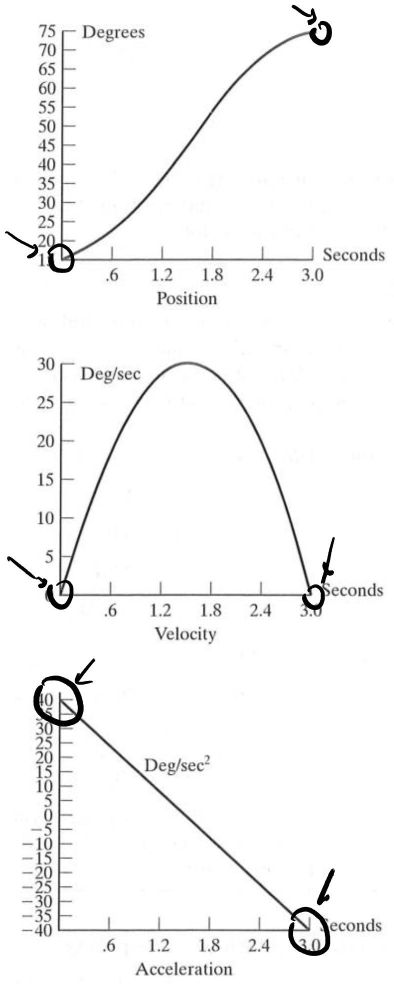
Joint Space Schemes

Single Time Interval

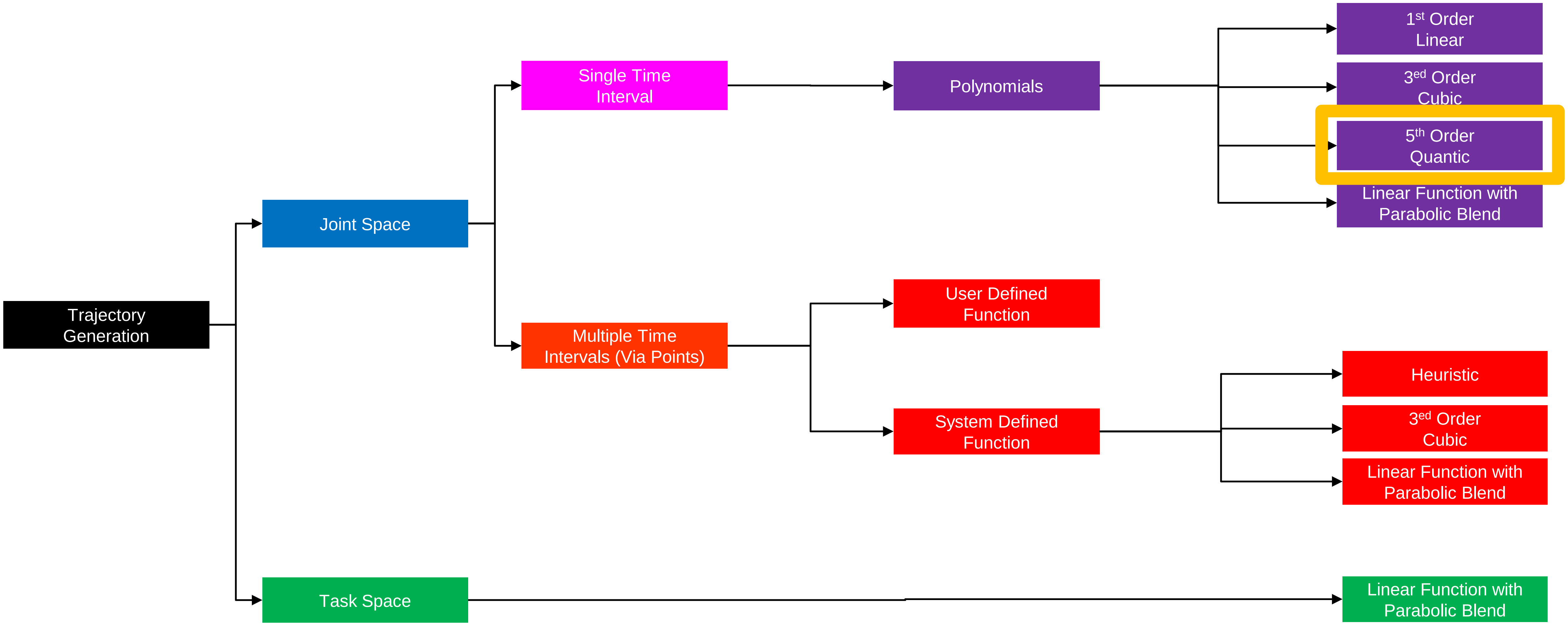
Polynomials

Quantic Order Polynomial

Joint Space Schemes



Trajectory Generation – Roadmap Diagram



Joint Space Schemes – Quantic Polynomials

- Rational for Quantic Polynomials (high order)
  - High Speed Robot
  - Robot Carrying heavy/delicate load
  - Non Rigid links
  - For high speed robots or when the robot is handling heavy or delicate loads. It is worth insuring the continuity of accelerations as well as avoid excitation of the resonance modes of the mechanism

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**

- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

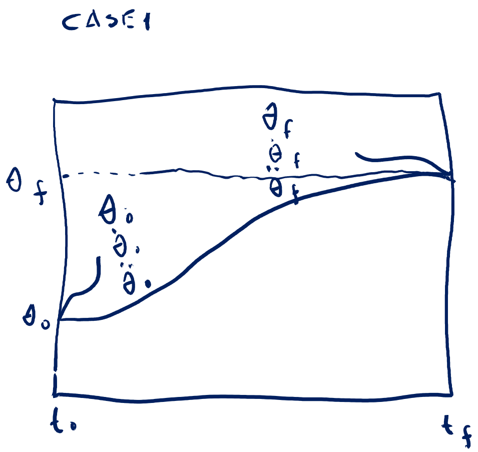
$$\dot{\theta}(0) = \dot{\theta}_0$$

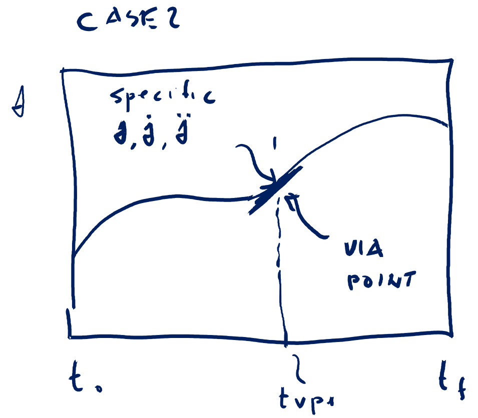
$$\dot{\theta}(t_f) = \dot{\theta}_f$$

$$\ddot{\theta}(0) = \ddot{\theta}_0$$

$$\ddot{\theta}(t_f) = \ddot{\theta}_f$$

- What should be the order of the polynomial function to meet these constrains?





Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

(1) $\theta(0) = \theta_0 \qquad \theta_0 = a_0$

(2) $\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$

(3) $\dot{\theta}(0) = \dot{\theta}_0 \qquad \dot{\theta}_0 = a_1$

(4) $\dot{\theta}(t_f) = \dot{\theta}_f \qquad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$

(5) $\ddot{\theta}(0) = \ddot{\theta}_0 \qquad \ddot{\theta}_0 = 2a_2$

(6) $\ddot{\theta}(t_f) = \ddot{\theta}_f \qquad \ddot{\theta}_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$
\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \\ \ddot{\theta}_0 \\ \ddot{\theta}_f \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3
\end{bmatrix}
\end{array}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
$$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$(2) \qquad \theta_f = \theta_0 + \dot{\theta}_0 t_f + \frac{\ddot{\theta}_0}{2} t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$(4) \qquad \dot{\theta}_f = \dot{\theta}_0 + \cancel{2}\frac{\ddot{\theta}_0}{\cancel{2}} t_f + 3a_3 t_f^3 + 4a_4 t_f^4 + 5a_5 t_f^4$$

$$(6) \qquad \ddot{\theta}_f = \cancel{2}\frac{\ddot{\theta}_0}{\cancel{2}} + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

$$\begin{bmatrix} \theta_f - \theta_0 - \dot{\theta} t_f - \frac{\ddot{\theta}_0}{2} t_f^2 \\ \dot{\theta}_f - \dot{\theta}_0 - \ddot{\theta}_0 t_f \\ \ddot{\theta}_f - \ddot{\theta}_0 \end{bmatrix} \begin{bmatrix} t_f^3 & t_f^4 & t_f^5 \\ 3t_f^3 & 4t_f^4 & 5t_f^4 \\ 6t_f & +12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

Joint Space Schemes – Cubic Polynomials - Non Zero Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2}$$

$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f - (3\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f + (3\ddot{\theta}_0 - 2\ddot{\theta}_f)t_f^2}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f - (\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^5}$$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

For a generalized case where $t_0 \neq 0$

$T = t_f - t_0$; $h = \theta_f - \theta_0$

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{1}{2} a_0$$

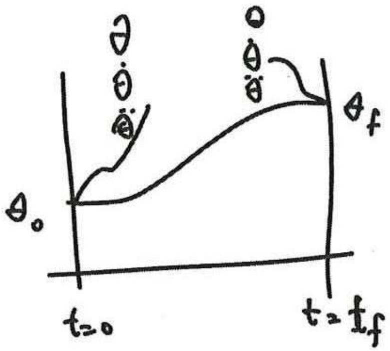
$$a_3 = \frac{1}{2T^3}\left[20h - (8\dot{\theta}_f + 12\dot{\theta}_0)T - (3a_0 - a_1)T^2\right]$$

$$a_4 = \frac{1}{2T^4}\left[-30h + (14\dot{\theta}_f + 16\dot{\theta}_0)T + (3a_0 - 2a_1)T^2\right]$$

$$a_5 = \frac{1}{2T^5}\left[12h - 6(\dot{\theta}_1 - \dot{\theta}_0)T + (a_1 - a_0)T^2\right]$$

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment



$$\theta(0)=\theta_0 \qquad \dot\theta(0)=\dot\theta_0 \qquad \boxed{\ddot\theta(0)=0}$$

$$\theta(t_f)=\theta_f \qquad \dot\theta(t_f)=\dot\theta_f \qquad \boxed{\ddot\theta(t_f)=0}$$

- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$
\begin{aligned}
\theta(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \\
\dot{\theta}(t) &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \\
\theta(t) &= 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
\end{aligned}
$$

- Combined with the six desired constraints yields six equations with six unknowns

$$
\begin{aligned}
&(1) & \theta(0) &= \theta_0 & \theta_0 &= a_0 \\
&(2) & \theta(t_f) &= \theta_f & \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \\
&(3) & \dot{\theta}(0) &= \dot{\theta}_0 & \dot{\theta}_0 &= a_1 \\
&(4) & \dot{\theta}(t_f) &= \dot{\theta}_f & \dot{\theta}_f &= a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4 \\
&(5) & \ddot{\theta}(0) &= 0 & 0 &= 2a_2 \\
&(6) & \ddot{\theta}(t_f) &= 0 & 0 &= 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
\end{aligned}
$$

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2} = 0$$

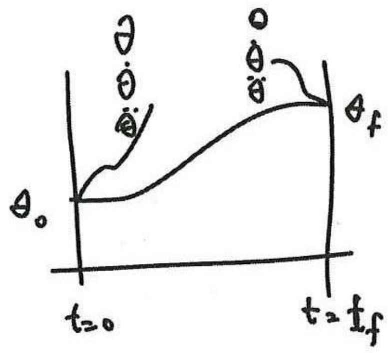
$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f}{2t_f^5}$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment



$$\theta(0) = \theta_0 \qquad \boxed{\dot{\theta}(0) = 0} \qquad \boxed{\ddot{\theta}(0) = 0}$$

$$\theta(t_f) = \theta_f \qquad \boxed{\dot{\theta}(t_f) = 0} \qquad \boxed{\ddot{\theta}(t_f) = 0}$$

- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\theta(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

$$
\begin{aligned}
&(1) \quad \theta(0) = \theta_0 && \theta_0 = a_0 \\
&(2) \quad \theta(t_f) = \theta_f && \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \\
&(3) \quad \dot{\theta}(0) = 0 && 0 = a_1 \\
&(4) \quad \dot{\theta}(t_f) = 0 && 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4 \\
&(5) \quad \ddot{\theta}(0) = 0 && 0 = 2a_2 \\
&(6) \quad \ddot{\theta}(t_f) = 0 && 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
\end{aligned}
$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = \frac{20\theta_f - 20\theta_0}{2t_f^3} = \frac{10\theta_f - 10\theta_0}{t_f^3} = 10\left[\frac{\theta_f - \theta_0}{t_f^3}\right]$$

$$a_4 = \frac{30\theta_0 - 30\theta_f}{2t_f^4} = -\frac{15\theta_f - 15\theta_0}{t_f^4} = -15\left[\frac{\theta_f - \theta_0}{t_f^4}\right]$$

$$a_5 = \frac{12\theta_f - 12\theta_0}{2t_f^5} = \frac{6\theta_f - 6\theta_0}{t_f^5} = 6\left[\frac{\theta_f - \theta_0}{t_f^5}\right]$$

$$\theta = \overset{a_0}{\overset{\downarrow}{\theta_0}} + \; + \left(\overset{a_3}{\overset{\downarrow}{10\,\frac{\theta_f-\theta_0}{t_f^3}}}\right)t^3 - \left(\overset{a_4}{\overset{\downarrow}{15\,\frac{\theta_f-\theta_0}{t_f^4}}}\right)t^4 + \left(\overset{a_5}{\overset{\downarrow}{6\,\frac{\theta_f-\theta_0}{t_f^5}}}\right)t^5$$

$$\theta = a_0 + \; + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta} = \; + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{\theta} = 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

$$\theta = \theta_0 + 10\left[\frac{\theta_f-\theta_0}{t_f^3}\right]t^3 - 15\left[\frac{\theta_f-\theta_0}{t_f^4}\right]t^4 + 6\left[\frac{\theta_f-\theta_0}{t_f^5}\right]t^5$$

$$\dot{\theta} = 30\left[\frac{\theta_f-\theta_0}{t_f^3}\right]t^2 - 60\left[\frac{\theta_f-\theta_0}{t_f^4}\right]t^3 + 30\left[\frac{\theta_f-\theta_0}{t_f^5}\right]t^4$$

$$\ddot{\theta} = 60\left[\frac{\theta_f-\theta_0}{t_f^3}\right]t - 180\left[\frac{\theta_f-\theta_0}{t^4}\right]t^2 + 120\left[\frac{\theta_f-\theta_0}{t_f^5}\right]t^3$$

$$\dddot{\theta} = 60\left[\frac{\theta_f-\theta_0}{t_f^3}\right] - 360\left[\frac{\theta_f-\theta_0}{t_f^4}\right]t + 360\left[\frac{\theta_f-\theta_0}{t_f^5}\right]t^2$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\ddot{\theta}_{max} \Longrightarrow \text{at} \quad t = t_f/2$$

$$\dot{\theta}_{max} = +3\overbrace{\left[10\,\frac{\theta_f-\theta_0}{\cancel{t_f^3}_{\,t_f}}\right]}^{a_3}\frac{\cancel{t_f^2}}{4} - 4\overbrace{\left[15\,\frac{\theta_f-\theta_0}{\cancel{t_f^4}_{\,t_f}}\right]}^{a_4}\frac{\cancel{t_f^3}}{8} + 5\overbrace{\left[6\,\frac{\theta_f-\theta_0}{\cancel{t_f^5}_{\,t_f}}\right]}^{a_5}\frac{\cancel{t_f^4}}{16}$$

$$\frac{\cancel{30}^{\,15}}{\cancel{4}_{\,2}}\left[\frac{\theta_f-\theta_0}{t_f}\right] - \frac{\cancel{60}^{\,15}}{\cancel{8}_{\,2}}\left[\frac{\theta_f-\theta_0}{t_f}\right] + \frac{\cancel{30}^{\,15}}{\cancel{16}_{\,8}}\left[\frac{\theta_f-\theta_0}{t_f}\right]$$

$$= \frac{15}{8}\,\frac{\theta_f-\theta_0}{t_f} \quad \checkmark$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\ddot{\theta}_{max} \rightarrow \quad \text{at} \quad t = t_f/4$$

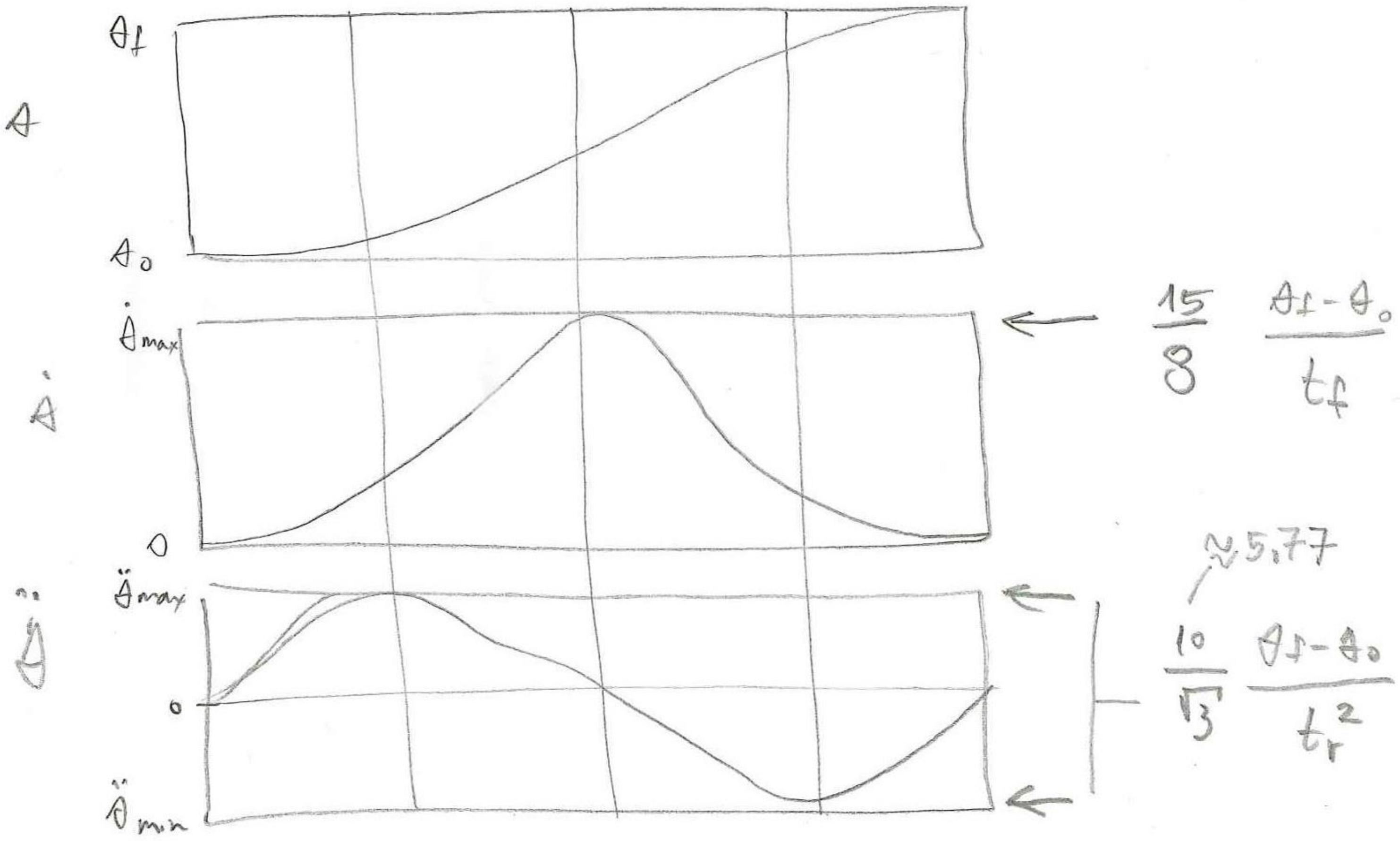
$$\ddot{\theta}_{max} = 6\left[10\,\frac{\theta_f-\theta_0}{\cancel{t_f^3}_{\,t_f^2}}\right]\frac{\cancel{t_f}}{4} - \overset{3}{\cancel{12}}\left[15\,\frac{\theta_f-\theta_0}{t_f^4}\right]\frac{t_f^2}{\underset{4}{\cancel{16}}} + \overset{5}{\cancel{20}}\left[6\,\frac{\theta_f-\theta_0}{t_f^5}\right]\frac{t_f^3}{\cancel{4}\cdot 16}$$

$$\frac{90}{8}$$

$$= 15\left[\frac{\theta_f-\theta_0}{t_f^2}\right] - \frac{45}{4}\left[\frac{\theta_f-\theta_0}{t_f^2}\right] + \frac{15}{8}\left[\frac{\theta_f-\theta_0}{t_f^2}\right] =$$

$$= \underbrace{\left[15 - \frac{75}{8}\right]}_{5.625}\frac{\theta_f-\theta_0}{t_f^2}$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration



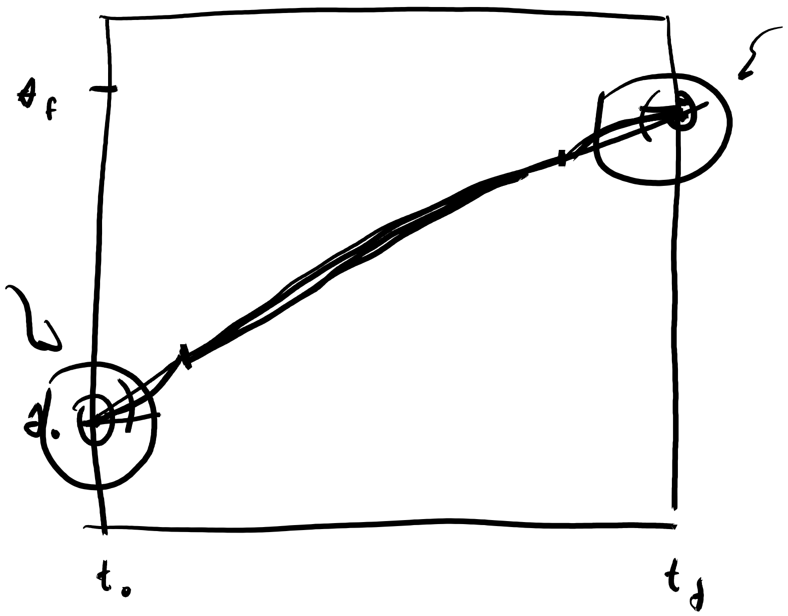
Joint Space Schemes

Single Time Interval

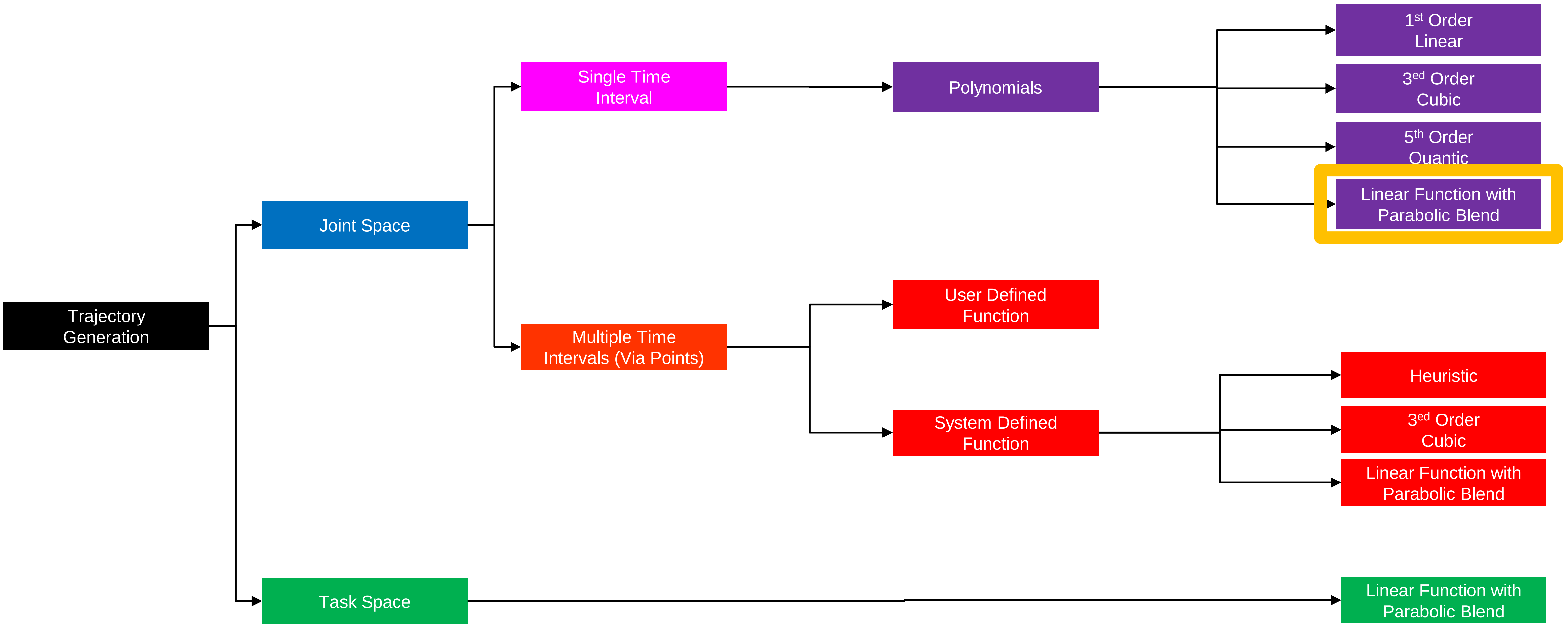
Polynomials

Linear Function with Parabolic Blend (Trapezoid Velocity Method)

Joint Space Schemes

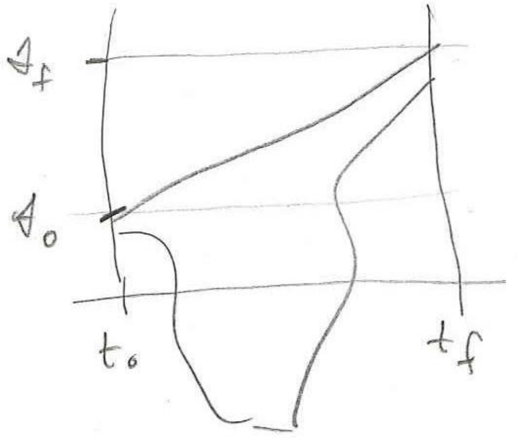


Trajectory Generation – Roadmap Diagram



Joint Space Schemes –
Linear Function With Parabolic Blend

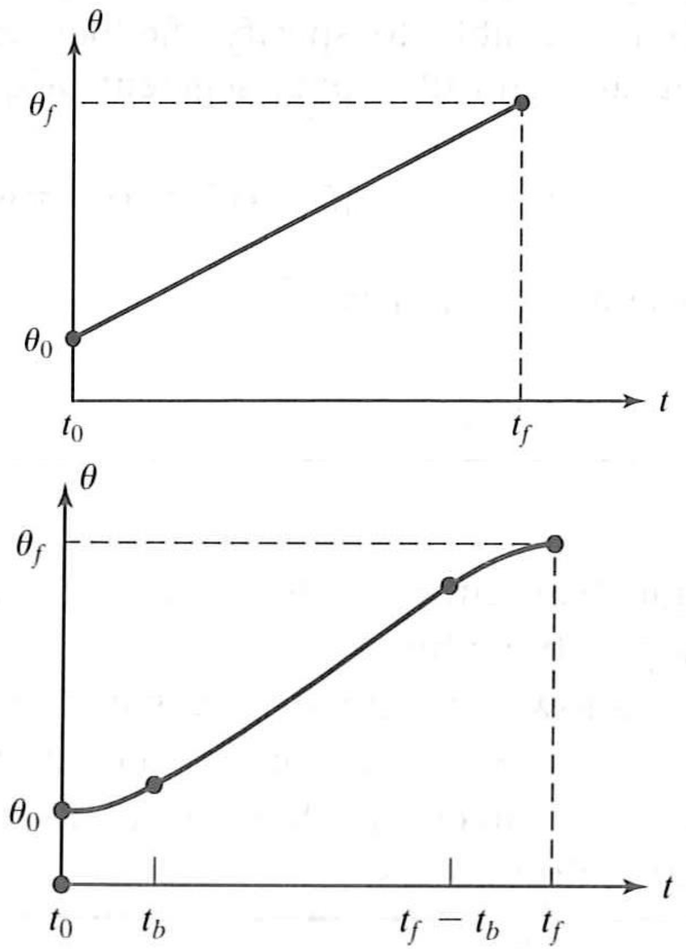
- Linear interpolation to move from the present joint position $\theta_0 (t=t_0)$ to the final position $\theta_f (t=t_f)$

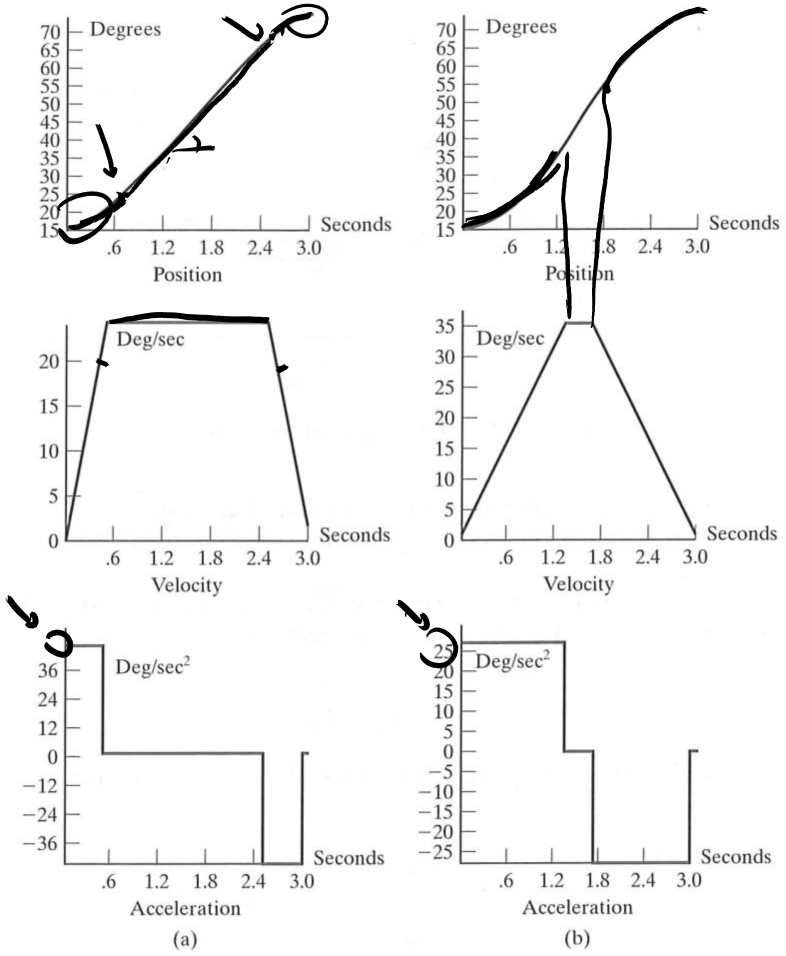


Discontinous Velocity

- Note: Although the motion of each joint is linear the EE in general does not move in a straight line in space
- Problem: Linear interpolation would cause the velocity to be discontinous at the begining / end
- Solution: Parabolic blend region.

Joint Space Schemes –
Linear Function With Parabolic Blend



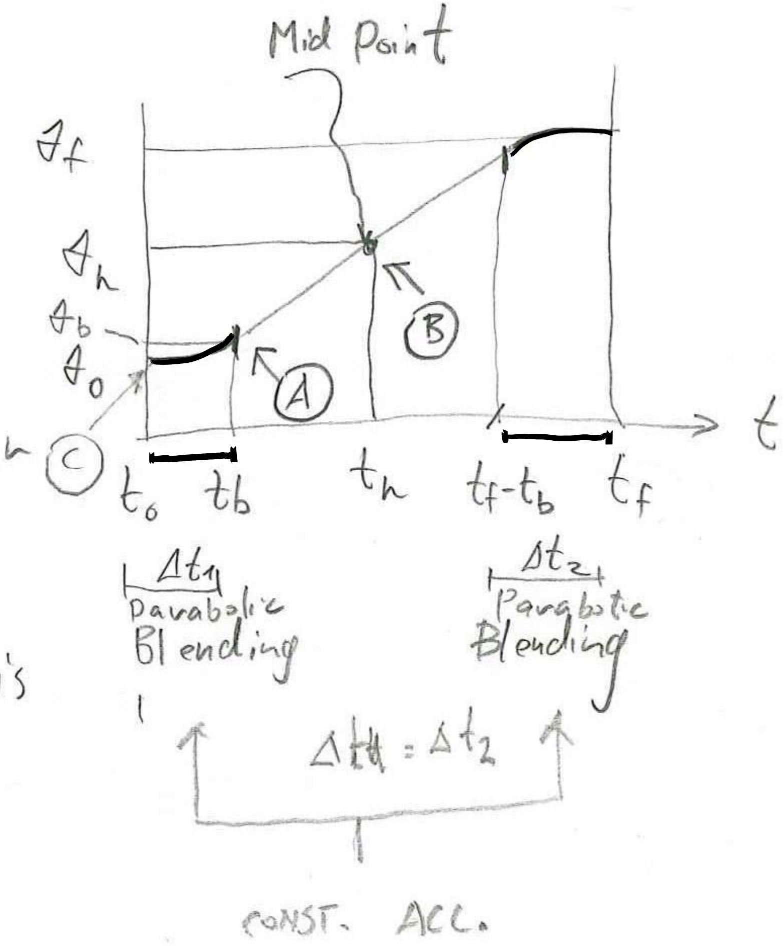


Joint Space Schemes – Linear Function With Parabolic Blend

- During the blend – Constant Acceleration to change the velocity smoothly
- Assumptions (1) the parabolic blend segments (Δt_1, Δt_2) have the same duration

  $\Delta t_1 = \Delta t_2$

  (2) The same constant acceleration is used during both blends



Joint Space Schemes – Linear Function With Parabolic Blend

• Conditions: Point A

I – Constant acceleration during the blend (Point C to A)

– Initial velocity is zero (Point C)

$$(1) \quad \theta_b = \theta_0 + \overset{0}{V_0} t + \frac{1}{2}\ddot{\theta} t_b^2$$ → Constant Acceleration

I

Initial velocity (= 0)

II

II The slope at point A must be equal on both sides

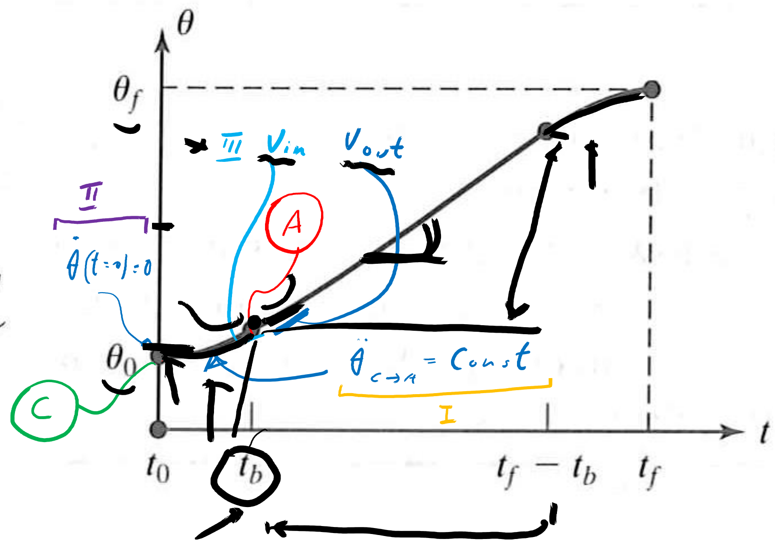
Acc ← → time

$$(2) \quad \underbrace{\ddot{\theta}\, t_b}_{\text{III}} = \underbrace{\frac{\theta_h - \theta_b}{t_h - t_b}}_{\text{III}}$$

Velocity from the left (V_{in}) Velocity from the right (V_{out})

$V_{in} = V_{out}$

$V(t=0)=0$



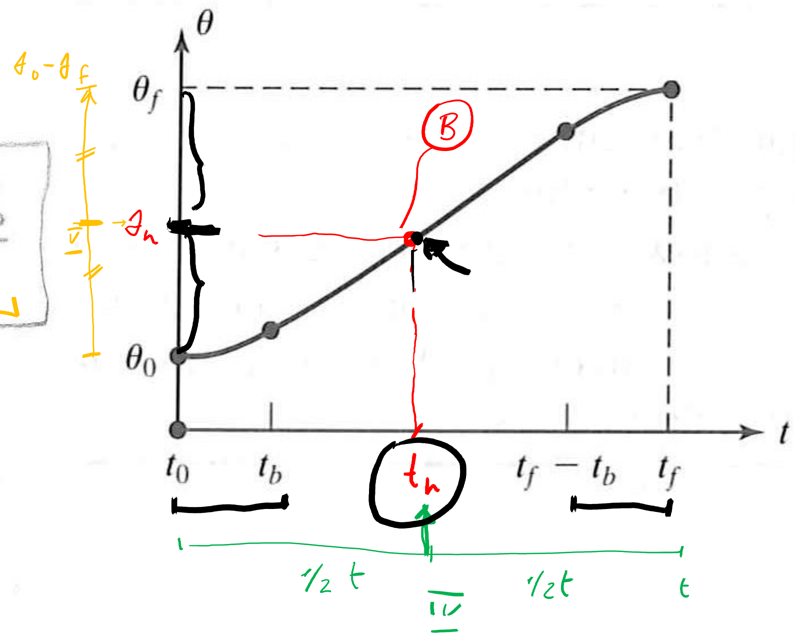
Joint Space Schemes – Linear Function With Parabolic Blend

Point (B)

- Point (B) is at the midle of the segment

(3) $\boxed{t_h = \frac{t}{2}}$ IV

(4) $\boxed{\theta_h = \frac{\theta_f - \theta_0}{2} + \theta_0 = \frac{\theta_f - \theta_0 + 2\theta_0}{2} = \underbrace{\frac{\theta_f + \theta_0}{2}}_{V}}$



plug Eq (4) into Eq(2) and Eq(3) into Eq(2)

$$\ddot{\theta} t_b = \frac{\theta_h - \theta_b}{t_h - t_b} = \frac{\left(\frac{\theta_f + \theta_0}{2}\right) - \theta_b}{\frac{t}{2} - t_b}$$

Joint Space Schemes –
Linear Function With Parabolic Blend

$$\ddot{\theta}\, t_b \left(\frac{t}{2} - t_b\right) = \frac{\theta_f + \theta_0}{2} - \theta_b$$

$$\ddot{\theta}\, t_b \left(\frac{t - 2t_b}{\cancel{2}}\right) = \frac{\theta_f + \theta_0 - 2\theta_b}{\cancel{2}}$$

$$\ddot{\theta}\, t_b t - 2\ddot{\theta}\, t_b^2 = \theta_f + \theta_0 - 2\theta_b$$

$$\ddot{\theta}(t_b t) - 2\ddot{\theta}\, t_b^2 - \theta_f - \theta_0 + 2\theta_b = 0$$

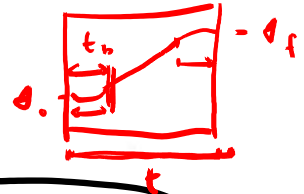
↑ plug Eq (1)

$$\ddot{\theta}(t_b t) - 2\ddot{\theta}\, t_b^2 - \theta_f - \theta_0 + 2\theta_0 + \ddot{\theta}\, t_b^2 = 0$$

$$\ddot{\theta}(t_b t) - \ddot{\theta}\, t_b^2 + \theta_f + \theta_0 = 0$$

Joint Space Schemes – Linear Function With Parabolic Blend

(5) $$\underbrace{(\ddot{\theta})}_{a} t_b^2 + \underbrace{(\ddot{\theta} t)}_{b} t_b + \underbrace{(\theta_f - \theta_0)}_{c} = 0$$



option 1
- Given : θ_f, θ_0, t, t_b (desired duration of motion)
- Calculate : $\ddot{\theta}$ (Eq. 5)

m, I → $\ddot{\theta}$

option 2
- Given : $\ddot{\theta}$ (chosen), t, θ_f, θ_0
- Calculate : t_b

$$t_b = \frac{+\ddot{\theta} t \pm \sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}} = \frac{t}{2} \pm \frac{\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

> 0

Joint Space Schemes – Linear Function With Parabolic Blend

- Constraint on the acceleration used in the blend

$$\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)} > 0$$

$$\ddot{\theta}^{\cancel{2}} t^2 > 4\cancel{\ddot{\theta}}(\theta_f - \theta_0)$$

$$\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t^2}$$

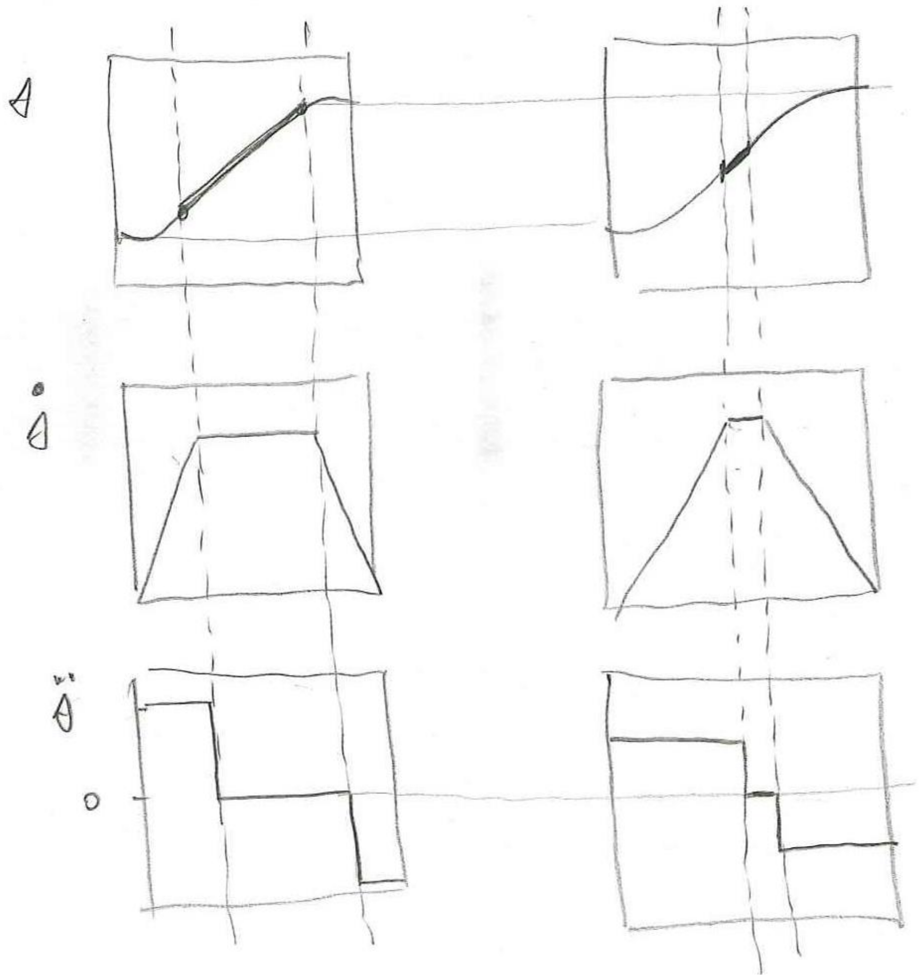
If equal $\quad t_b = \frac{t}{2} \pm \frac{\sqrt{0}}{2\ddot{\theta}} \Rightarrow t_b = \frac{t}{2}$

Joint Space Schemes – Linear Function With Parabolic Blend

- The length of the linear portion and the parabolic portion may vary
- High Acceleration ($\ddot{\theta}$) $\longrightarrow$ Short Blend

  Low Acceleration ($\ddot{\theta}$) $\longrightarrow$ Long Blend

Joint Space Schemes –
Linear Function With Parabolic Blend

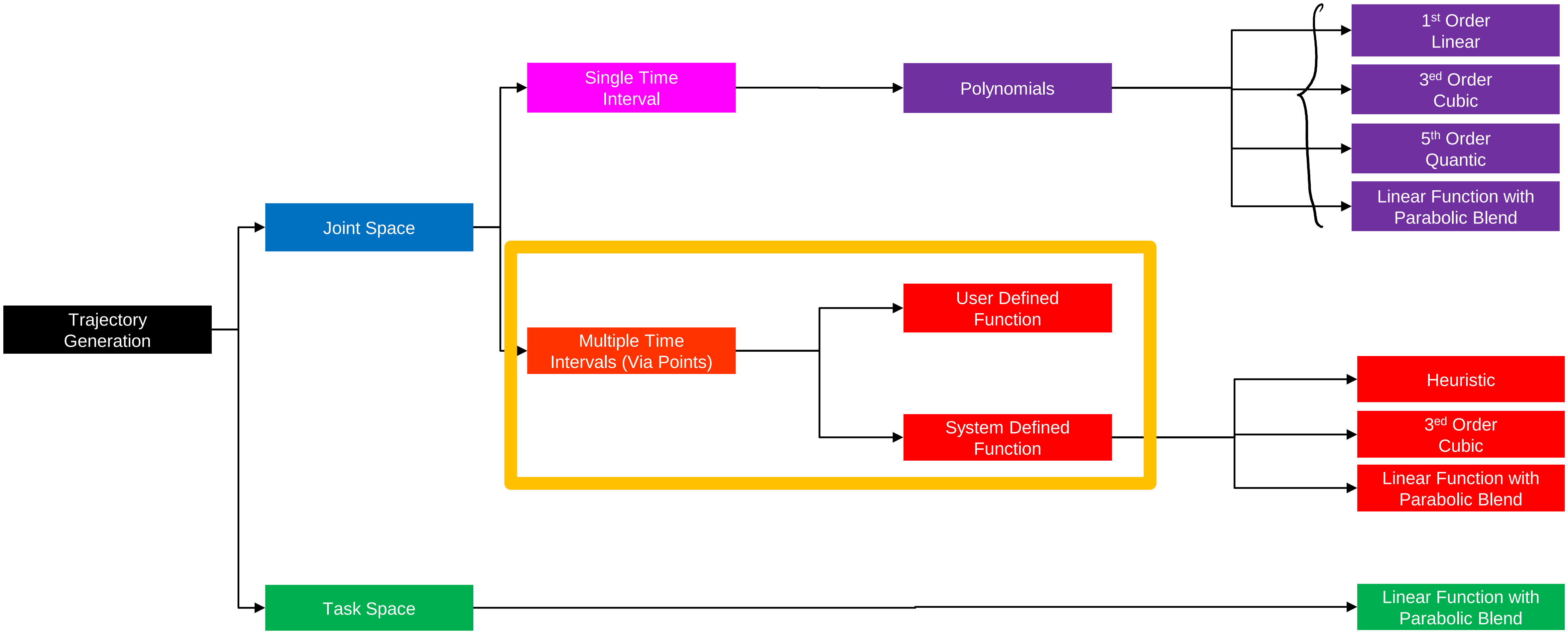


Joint Space Schemes

Multiple Time Interval

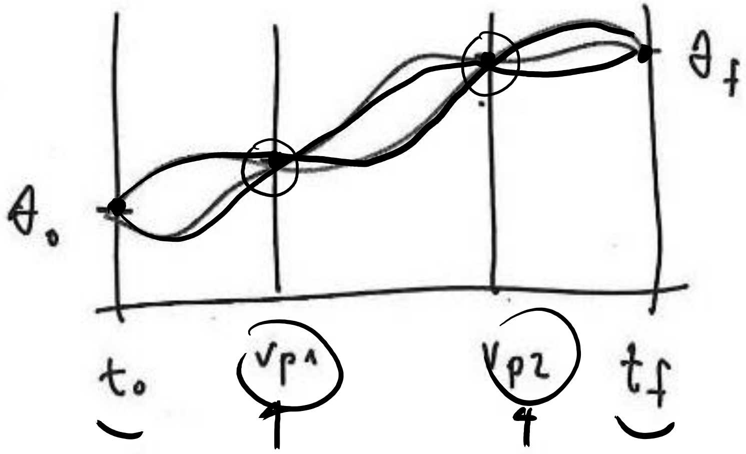
Via Point

Trajectory Generation – Roadmap Diagram



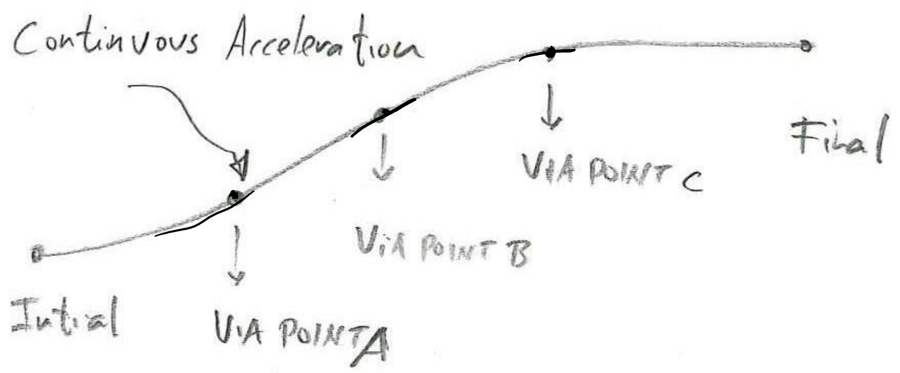
Joint Space Schemes – Multiple Time – Via Points

- Define multiple functions for each joint such that
  - The value at t_0 is the **initial position** of the joint and whose value at t_f is the **desire goal position** of the joint $\theta(t)$ at the end of the time interval
  - In between the beginning/ending points the user define via points that each joint must pass trough.
  - Note that all the joints reach the via point at the same time to guarantee a specific position and orientation of the end effector
- There are many smooth functions that may be used to interpolate the joint value.

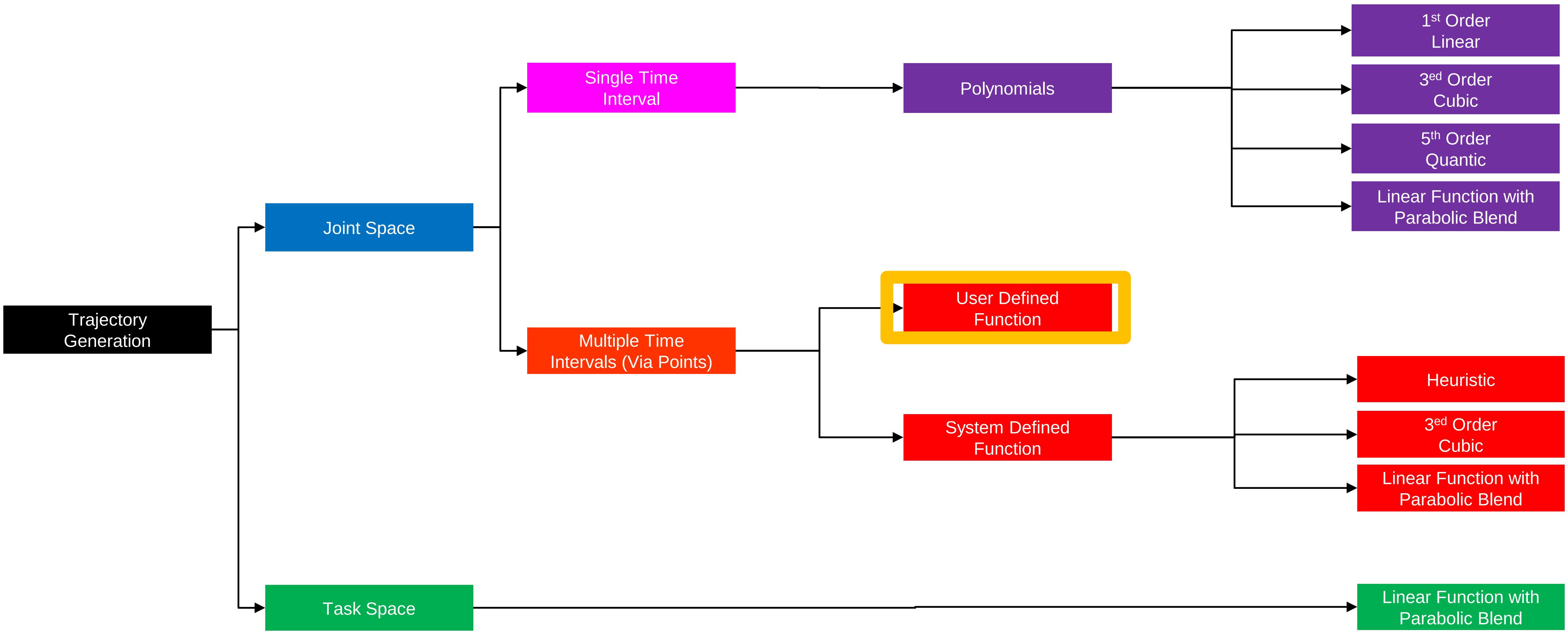


Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

- Specify the desired velocities at each segment:
  1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.
  2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) automatically using a suitable heuristic method.
  3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.



Trajectory Generation – Roadmap Diagram



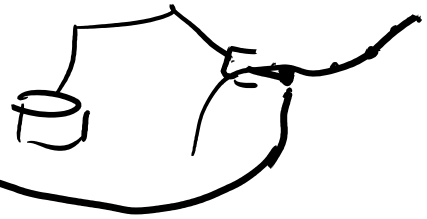
Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.

– **Mapping** (Cartesian Space to Joint Space) - Cartesian velocities at the via point are "mapped" to desired joint rates by using the inverse Jacobian

$$\dot{\theta}_f = J^{-1}\dot{X}_f \qquad \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix}$$

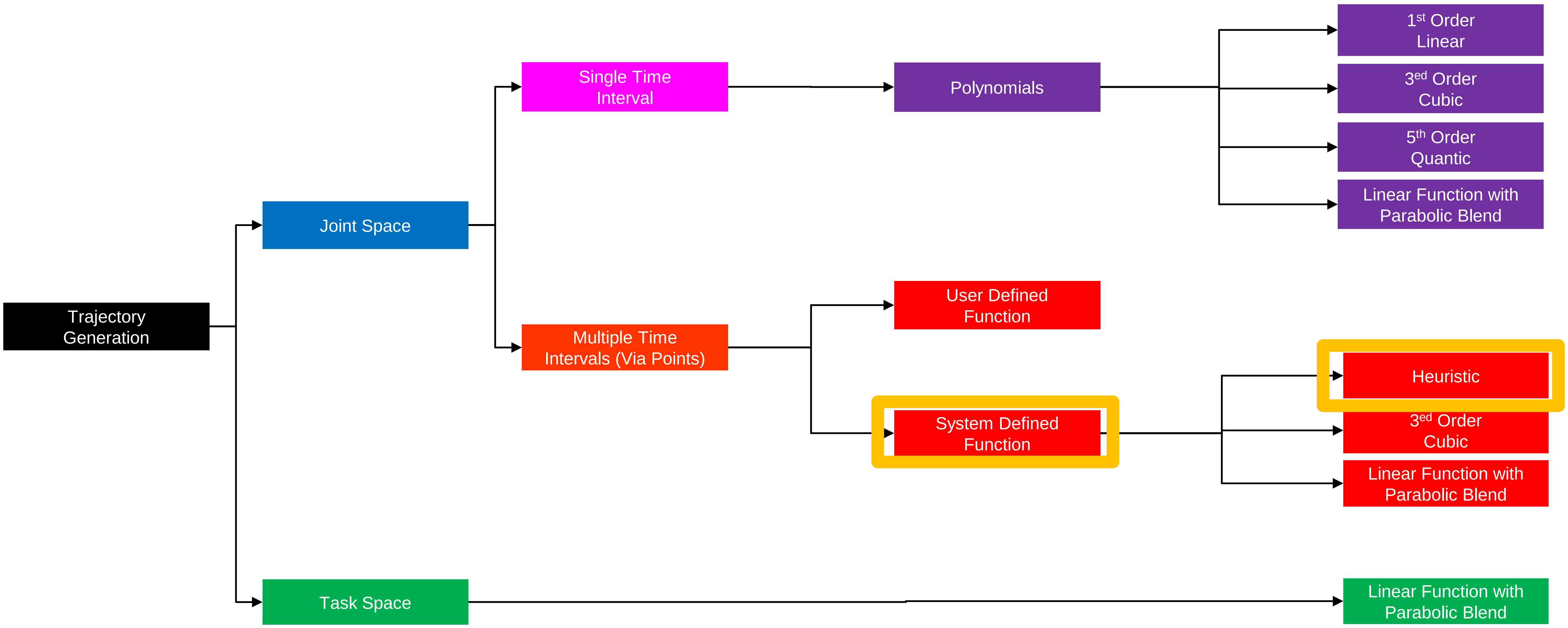
$\dot{\theta} \to \infty$



– **Singularity** - If the manipulator is at a singular point at a particular via point then the user is not free to choose an arbitrarily velocity at this point.

– **Difficult**

Trajectory Generation – Roadmap Diagram

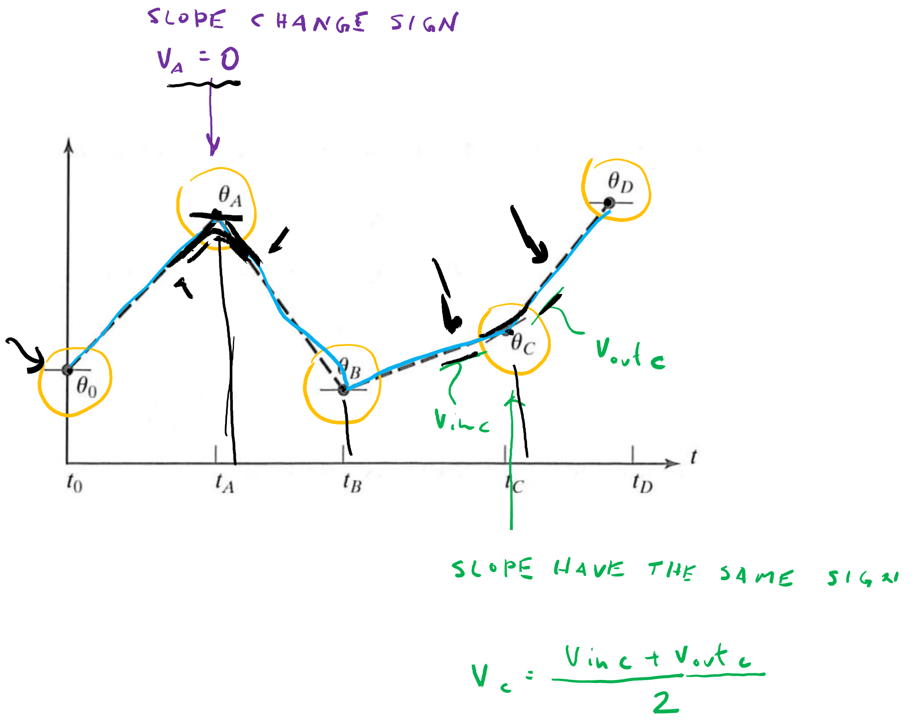


Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) using a suitable heuristic method given a trajectory .

Heuristic method

- – Consider a path defined by via points
- – Connect the via points with straight lines
- – If the slope change sign
  - – Set the velocity at the via point to be zero
- – If the slope have the same sign
  - – Calculate the average between the to velocities at the via point.



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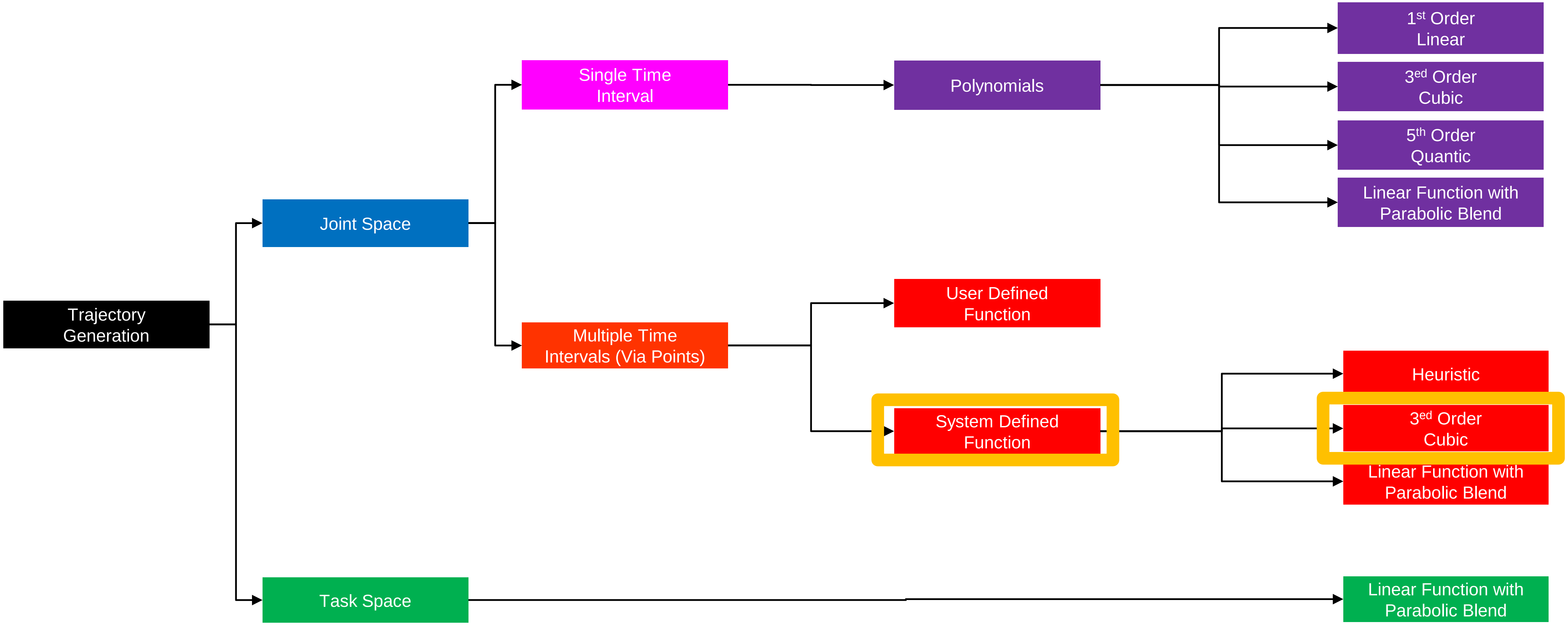
Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.

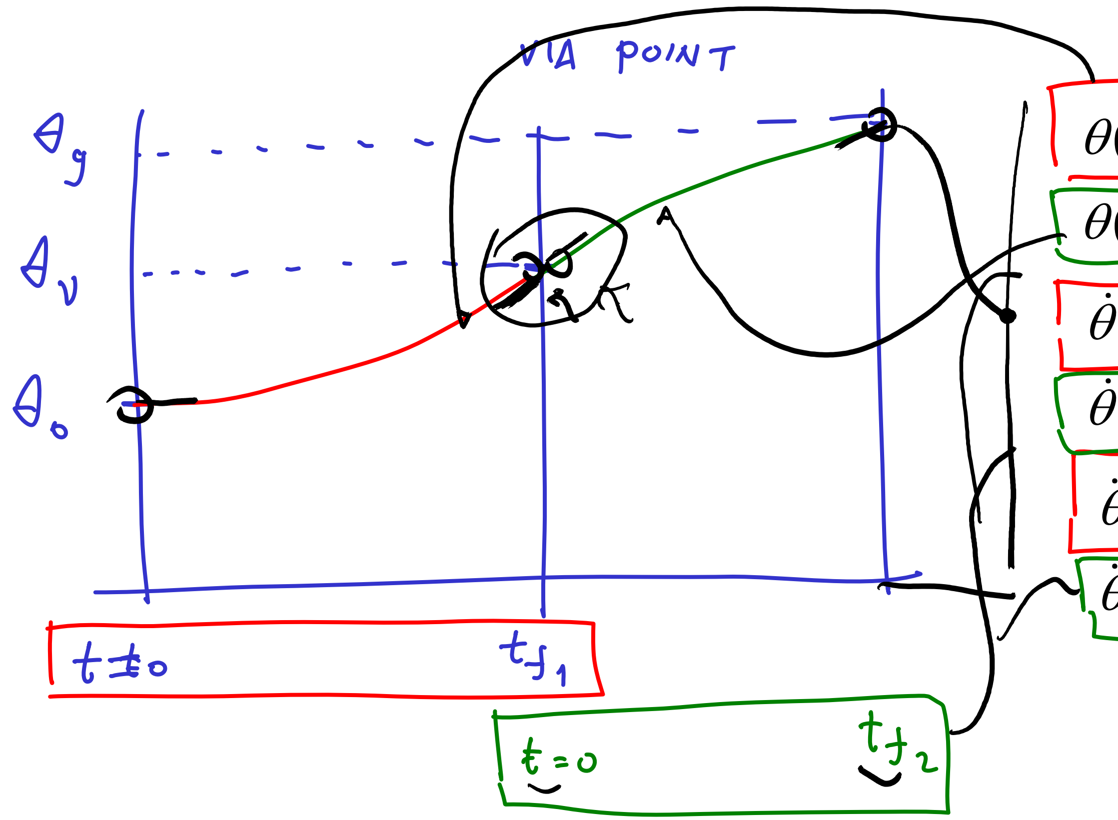
– **Spline** - Enforcing the velocity and the acceleration to be continuous at the via point

Trajectory Generation – Roadmap Diagram



Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Solve for the coefficients of two cubic functions that are connected in a two segment spline with a continuous acceleration at the intermediate via point.



$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$

$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t + 3a_{13}t^2$$

$$\dot{\theta}(t) = a_{21} + 2a_{22}t + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$

$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- The joint angle velocity and acceleration for each segment (8 unknowns)

$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$

$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t + 3a_{13}t^2$$

$$\dot{\theta}(t) = a_{21} + 2a_{22}t + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$

t_{f1}

$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

$t=0$

Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Position at the beginning and end of each segment

- Segment 1

$$\theta_0(t=0)=a_{10}$$ eq 1

$$\theta_{via}(t=t_{f_1})=a_{10}+a_{11}t_{f_1}+a_{12}t_{f_1}^{\ 2}+a_{13}t_{f_1}^{\ 3}$$ eq 2

- Segment 2

$$\theta_{via}(t=0)=a_{20}$$ eq 3

$$\theta_g(t=t_{f_2})=a_{20}+a_{21}t_{f_2}+a_{22}t_{f_2}^{\ 2}+a_{23}t_{f_2}^{\ 3}$$ eq 4

Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Velocity at the beginning of the interval

$$\dot{\theta}(t=0)=a_{11} \quad \text{eq 5}$$

- Velocity at the end of the interval

$$\dot{\theta}(t=t_{f_2})=a_{21}+2a_{22}t_{f_2}+3a_{23}t_{f_2}^{\,2} \quad \text{eq 6}$$

- Velocity at the mid point between the intervals

$$\dot{\theta}\left[Function1(t=t_{f_1})\right]=\dot{\theta}\left[Function2(t=0)\right]$$

$$a_{11}+2a_{12}t_{f_1}+3a_{13}t_{f_1}^{\,2}=a_{21} \quad \text{eq 7}$$

Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Acceleration at the mid point between the intervals

$$\ddot{\theta}\left[Function1(t=t_{f_1})\right]=\ddot{\theta}\left[Function2(t=0)\right]$$

$$2a_{12}+6a_{13}t_{f_1}=2a_{22} \qquad \text{eq. 8}$$

- Solve 8 equations with 8 unknown

Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Solution for the 8 equations

$$a_{10} = \theta_0$$

$$a_{11} = 0$$

$$a_{12} = \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2}$$

$$a_{13} = \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3}$$

$$a_{10} = \theta_v$$

$$a_{21} = \frac{3\theta_g - 3\theta_0}{4t_f}$$

$$a_{22} = \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2}$$

$$a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}$$

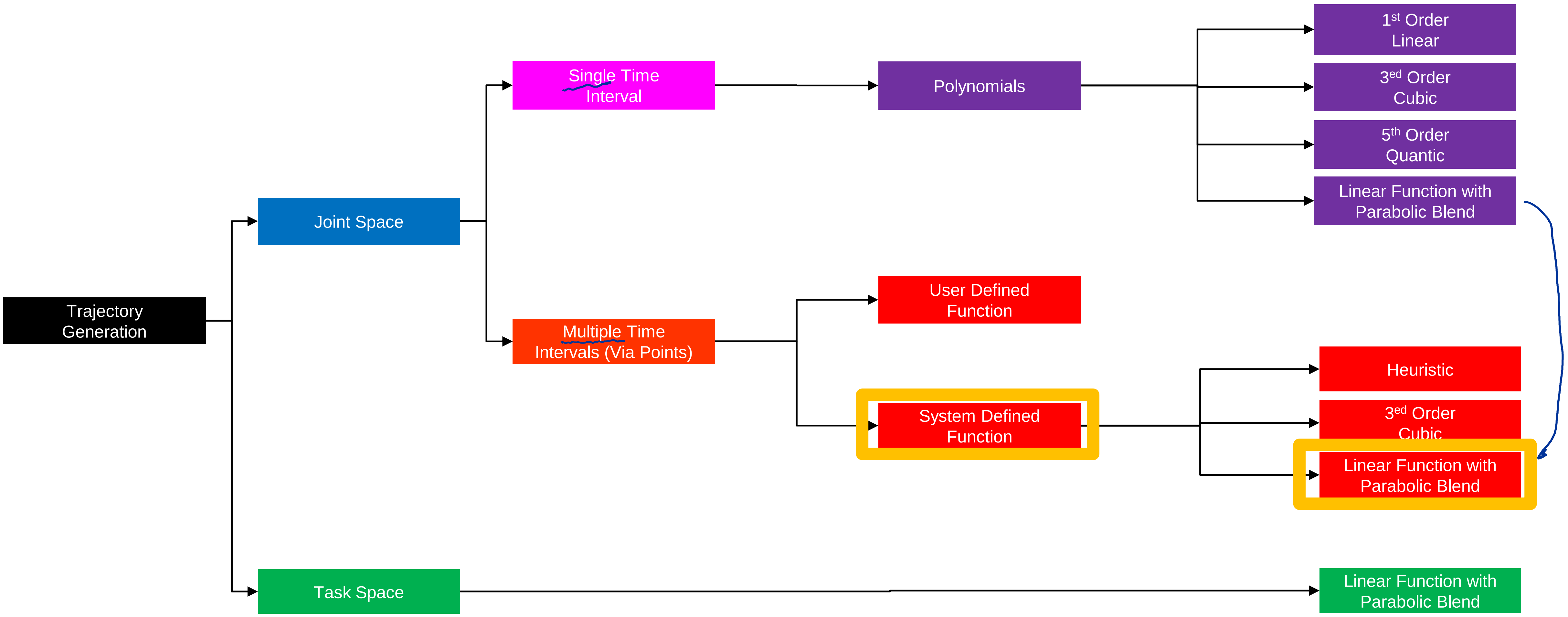
Joint Space Schemes

Multiple Time Intervals

Via Point

System Defined Function – Linear Function With Parabolic Blend

Trajectory Generation – Roadmap Diagram

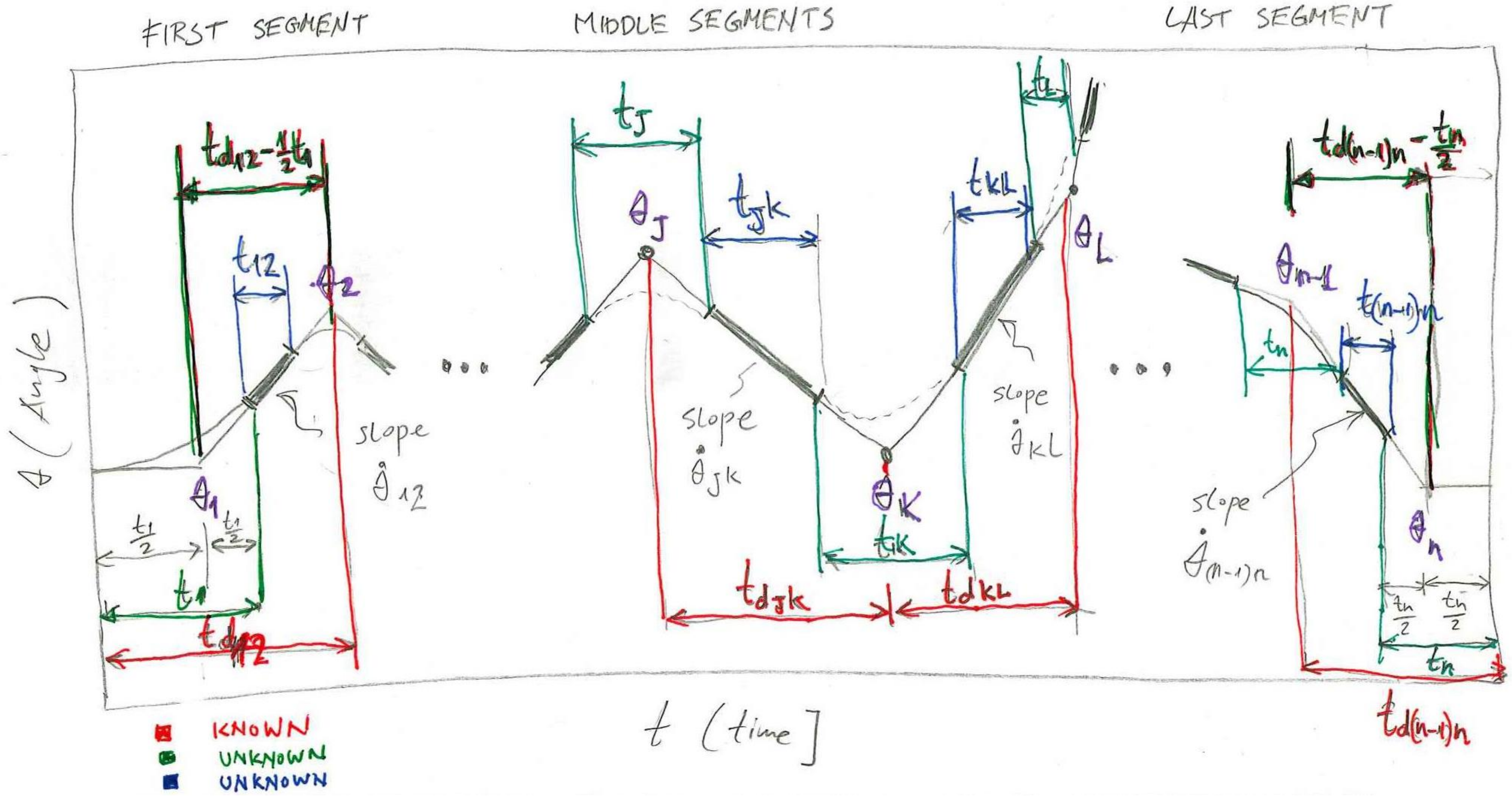


Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- The Need
  - Linear path with parabolic blends is used in cases where there are **arbitrary number of via points specified**

- Method Anatomy
  - **Linear Functions** – Connecting the via points
  - **Parabolic Blend** – Connecting the linear functions around the via points

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend



Joint Space Schemes

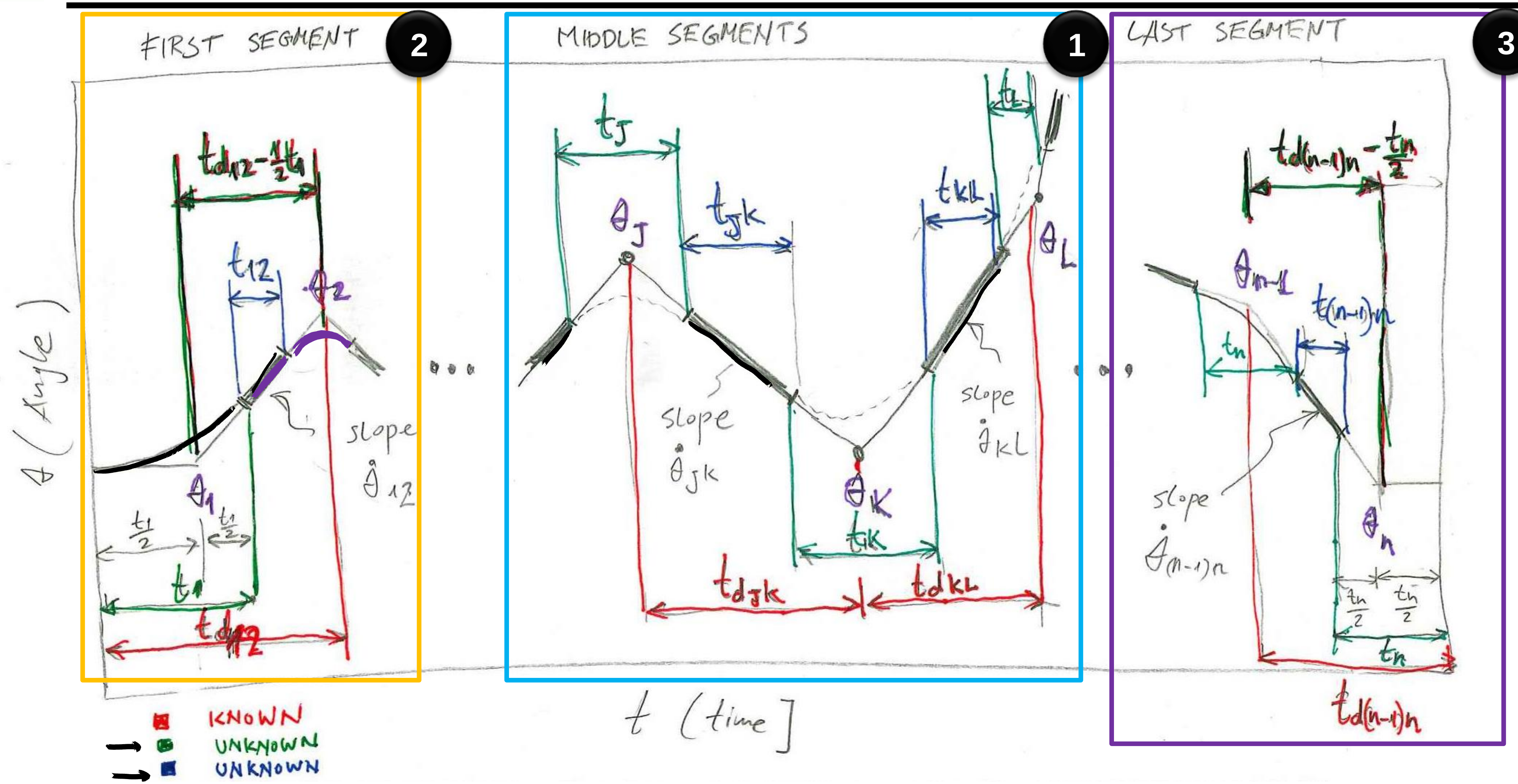
Multiple Time Intervals

Via Point

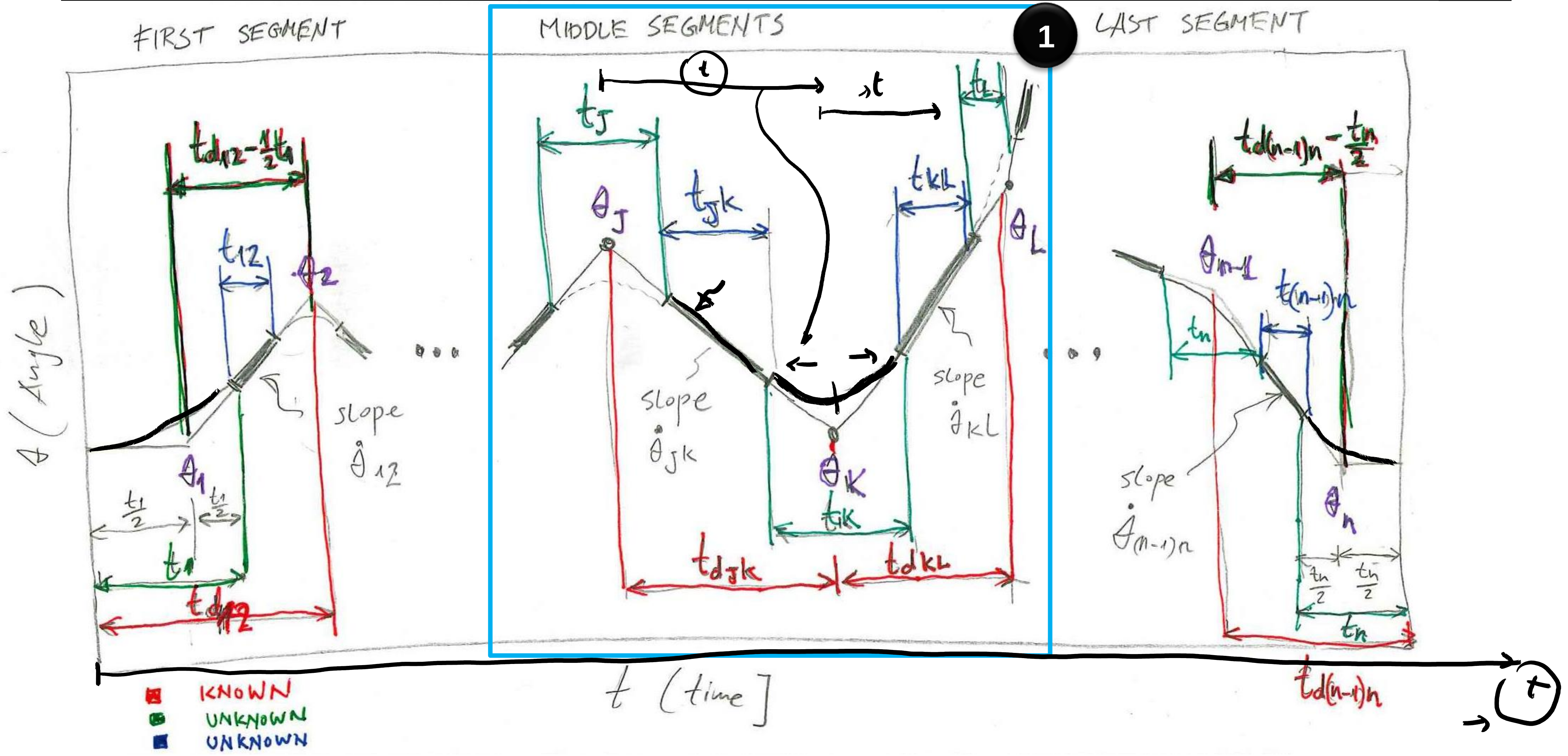
System Defined Function – Linear Function With Parabolic Blend

Time Interval Analysis

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

1

Mid Points (j, k, L)

Given: → $\theta_J, \theta_k, \theta_L$

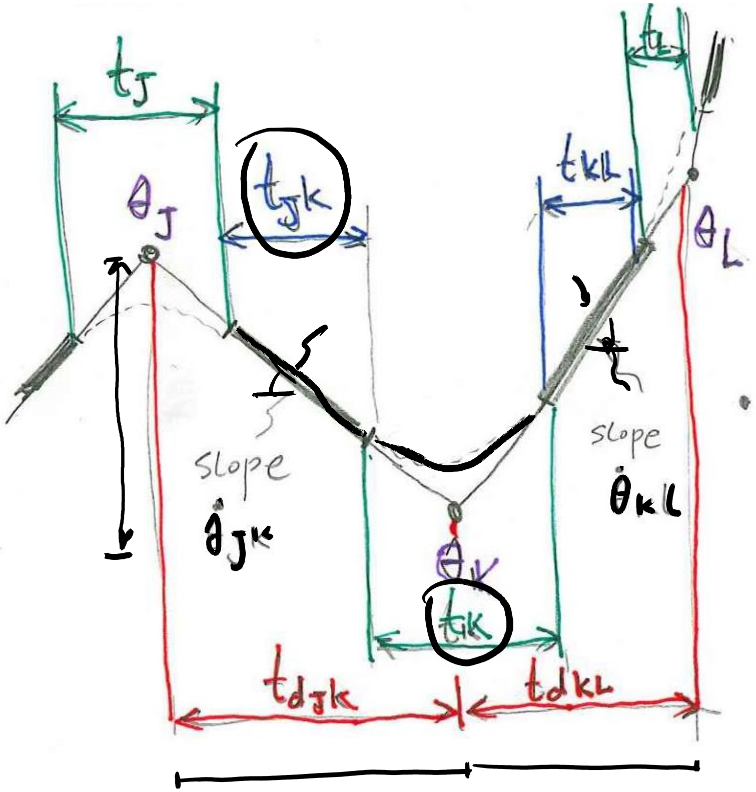
td_{jk}, td_{kL}

→ $|\ddot{\theta}_k|$ Desired acceleration at each path point

- Velocity

→ $\dot{\theta}_{jk} = \dfrac{\theta_k - \theta_J}{td_{jk}}$

$\dot{\theta}_{kL} = \dfrac{\theta_L - \theta_k}{td_{kL}}$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

– Acceleration

$$\ddot{\theta}_K = \left[\mathrm{SGN}\left(\dot{\theta}_{KL} - \dot{\theta}_{JK}\right) \right] \ddot{\theta}_K$$

$\oplus$ Acceleration

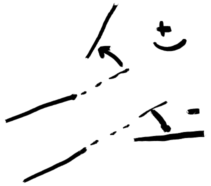
$\ominus$ Deceleration

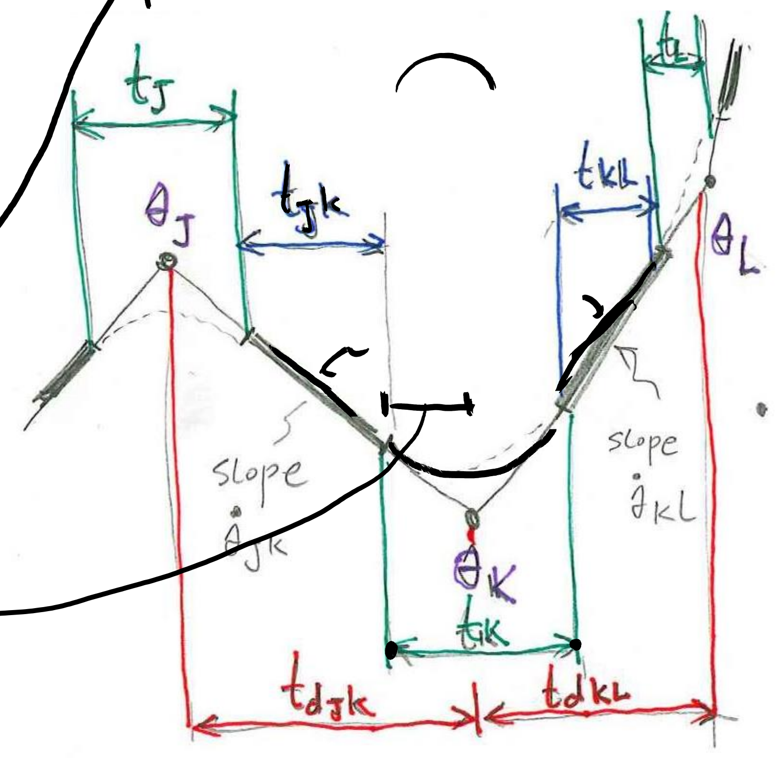
$$t_K = \frac{\dot{\theta}_{KL} - \dot{\theta}_{JK}}{\ddot{\theta}_K}$$

$v = at$

$t = \frac{v}{a}$

$$t_{JK} = t_{dJK} - \frac{1}{2} t_J - \frac{1}{2} t_K$$

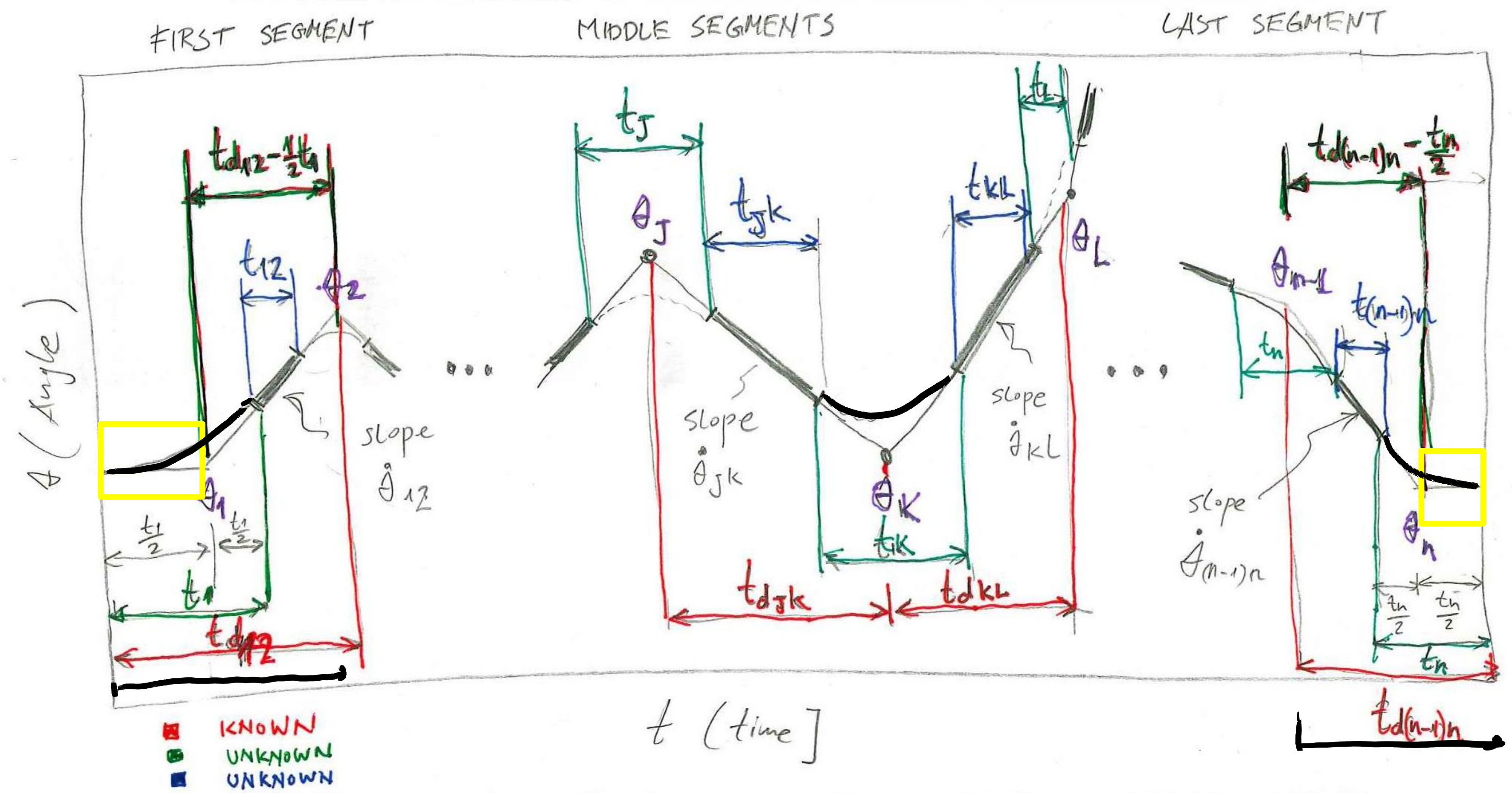




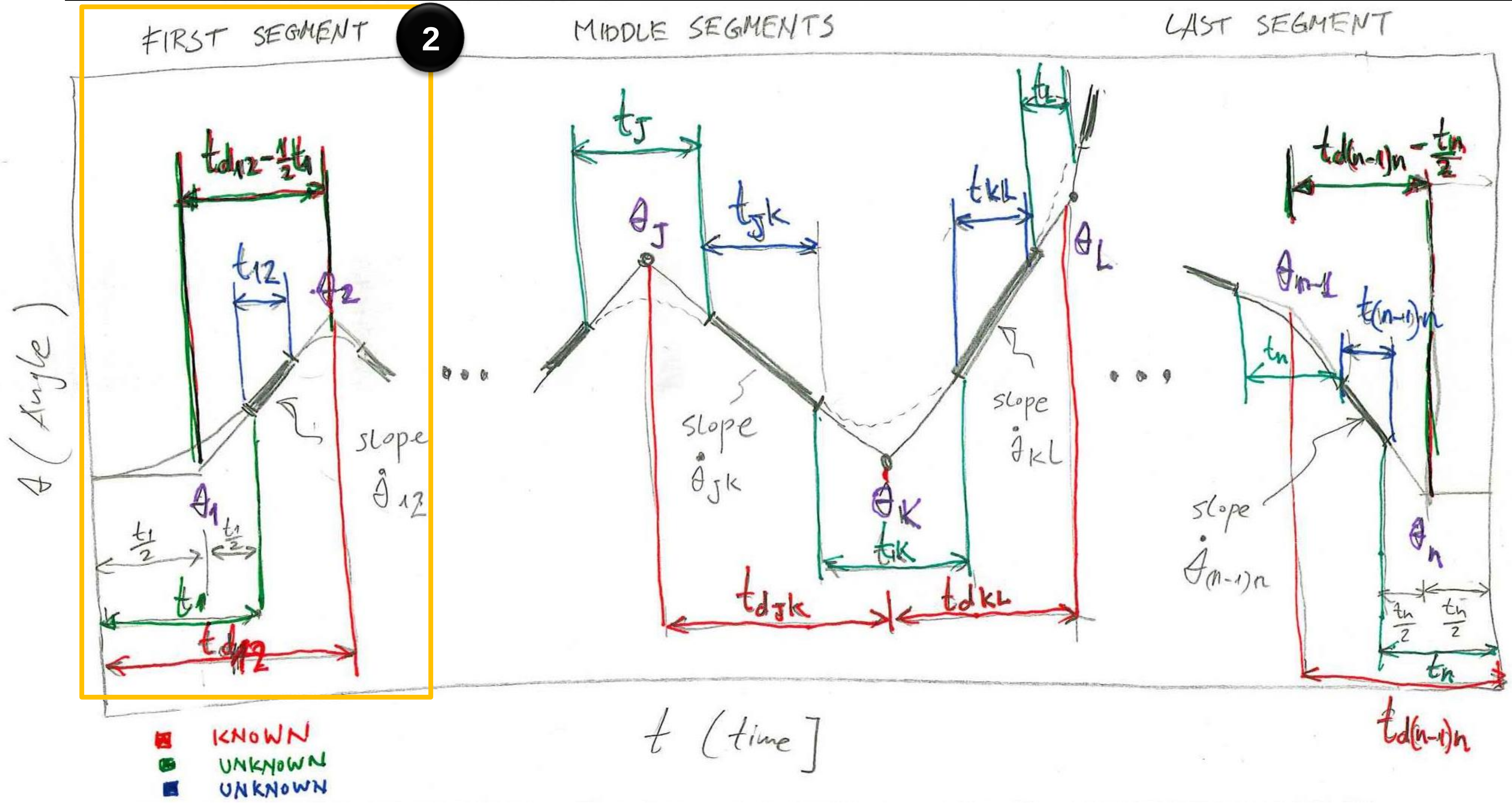
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

- **Note: First and Last Segments** – The first and the last segments must be handled slightly differently because the entire bland regeion at one end of the segment must be counted in the total segment's time duration.

- Note the difference between
  - Mid Points t_{djk} t_{djk}
  - First Point t_{d12}
  - Last Point $t_{d(n-1)n}$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

2

FIRST SEGMENT

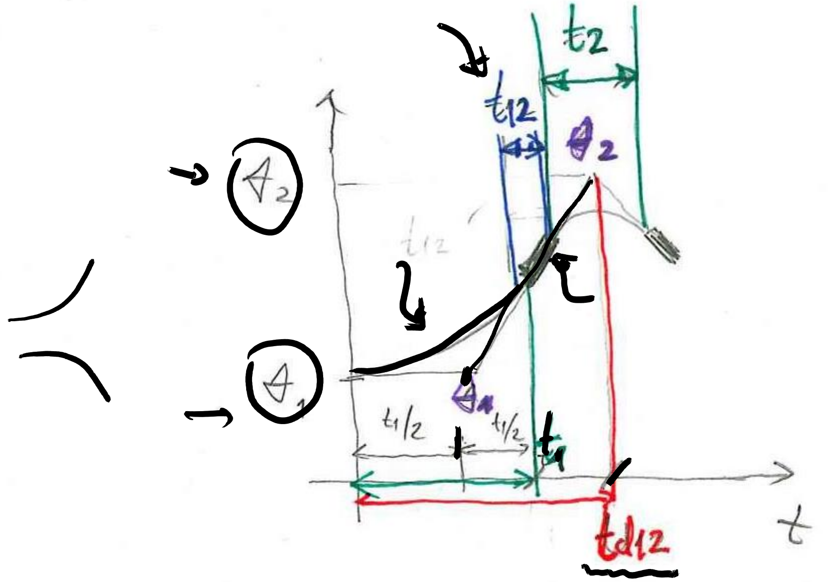
Velocity $\rightarrow$ $\dfrac{\theta_2-\theta_1}{t_{d12}-\frac{1}{2}t_1} = \ddot{\theta}_1 t_1$ $\quad$ $\ddot{\theta}_1 t_1 \rightarrow V = a \cdot t$

Acceleration $\quad \ddot{\theta}_1 = \underbrace{SGN}_{\text{SIGN}}(\theta_2-\theta_1)\,|\ddot{\theta}_1|$

$$\theta_2-\theta_1 = \ddot{\theta}\, t_1 \left(t_{d12}-\frac{1}{2}t_1\right)$$

$$\left(+\ddot{\theta}_1 \frac{1}{2}\right) t_1^2 - \left(\ddot{\theta}_1 t_{d12}\right) t + (\theta_2-\theta_1) = 0$$

$$t_1 = \frac{\ddot{\theta}_1 t_{d12} \pm \sqrt{\ddot{\theta}_1^2 t_{d12}^2 - \overset{2}{\cancel{4}}\,\ddot{\theta}_1 \cancel{\tfrac{1}{2}}\,(\theta_2-\theta_1)}}{\ddot{\theta}_1}$$



$t_1 \rightarrow$ Blending time

$t_{12} \rightarrow$ Straight line time

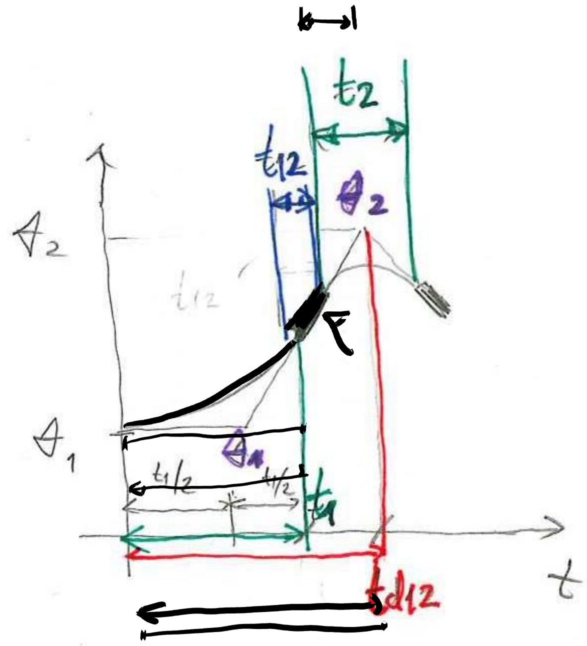
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

$$t_1 = t_{d12} \pm \sqrt{t_{d12}^2 + \frac{2(\theta_1 - \theta_2)}{\ddot{\theta}_1}}$$

Taking only the (−) sign since $t_1 < t_{d12}$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1}$$

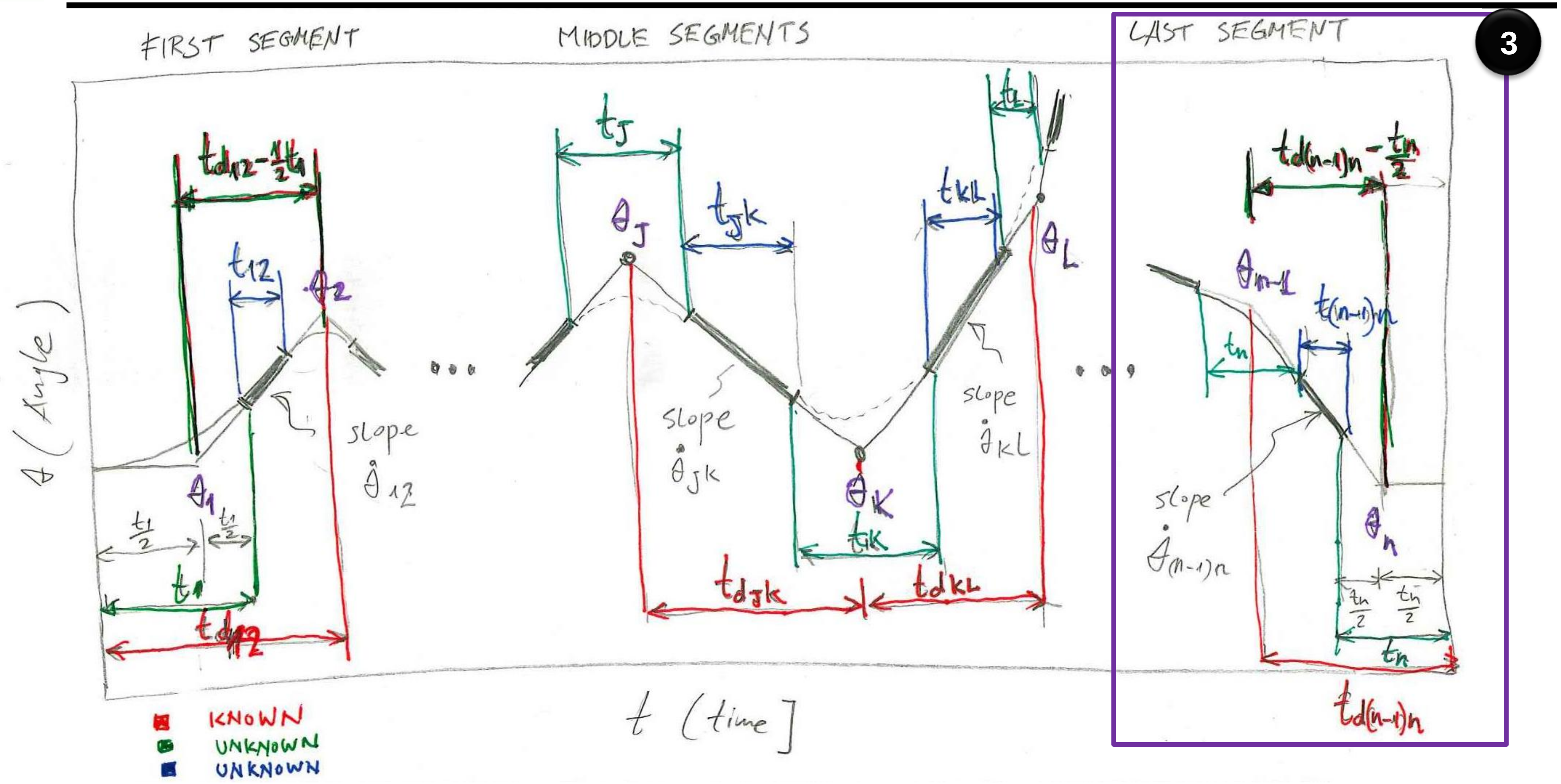
$$t_{12} = t_{d12} - t_1 - \frac{1}{2} t_2$$



$t_1 \rightarrow$ Blending time

$t_{12} \rightarrow$ Straight line time

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

3 LAST SEGMENT

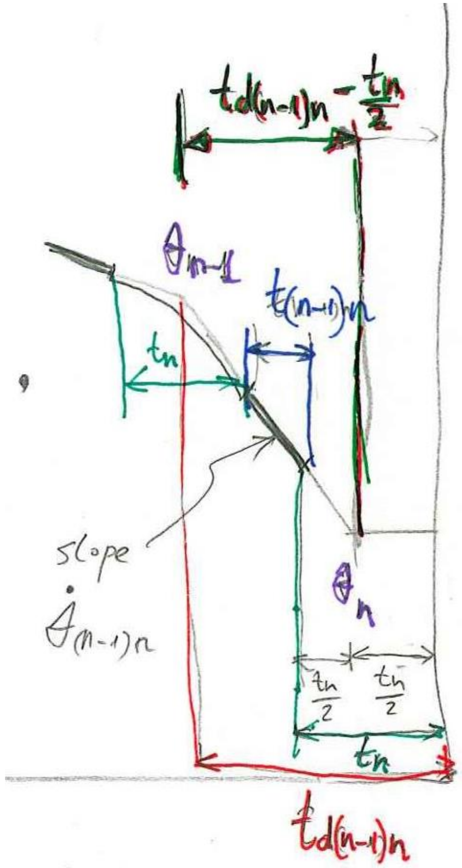
Velocity $\rightarrow$ $$\frac{\theta_{n-1} - \theta_n}{t_{d(n-1)n} - \frac{1}{2} t_n} = \ddot{\theta}_n t_n$$

Acceleration $\rightarrow$ $$\ddot{\theta}_n = SGN(\theta_{n-1} - \theta_n)\,|\ddot{\theta}_n|$$

$$\theta_{n-1} - \theta_n = \ddot{\theta}_n t_n \left(t_{d(n-1)n} - \frac{1}{2} t_n\right)$$

$$+\left(\ddot{\theta}_n \frac{1}{2}\right) t_n^2 - \left(\ddot{\theta}_n t_{d(n-1)n}\right) t_n + \left(\theta_{n-1} - \theta_n\right) = 0$$

$$t_n = \frac{\ddot{\theta}_n t_{d(n-1)n} \pm \sqrt{\ddot{\theta}_n^2 t_{d(n-1)n}^2 - 4\ddot{\theta}_n \frac{1}{2}(\theta_{n-1} - \theta_n)}}{\ddot{\theta}_n}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

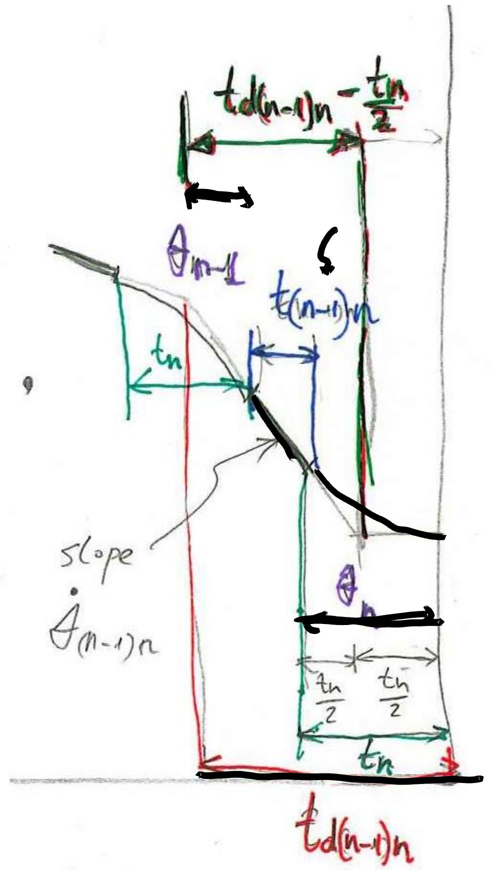
$$t_n = t_{d(n-1)n} \pm \sqrt{t^2_{d(n-1)n} - \frac{2(\theta_{n-1} - \theta_n)}{\ddot{\theta}_n}}$$

Taking only the (–) sign since

$$t_n < t_{d(n-1)n}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2} t_n}$$

$$t_{(n-1)n} = t_{d(n-1)} - t_n - \frac{1}{2} t_{n-1}$$



Joint Space Schemes

Multiple Time Intervals

Via Point

System Defined Function – Linear Function With Parabolic Blend

Linear & Parabolic Spline Analysis

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- Middle Segment

LINEAR SPLINE $t \in t_{jk}$

$$\theta = \theta_j + \dot{\theta}_{jk} t$$

$$\dot{\theta} = \dot{\theta}_{jk}$$

$$\ddot{\theta} = 0$$

PARABOLIC SPLINE $t \in t_k$

$$t_{inb} = t - \left(\tfrac{1}{2} t_j + t_{jk}\right)$$

$$\theta = \theta_j + \dot{\theta}_{jk}(t - t_{inb}) + \tfrac{1}{2} \ddot{\theta}_k t_{inb}^2$$

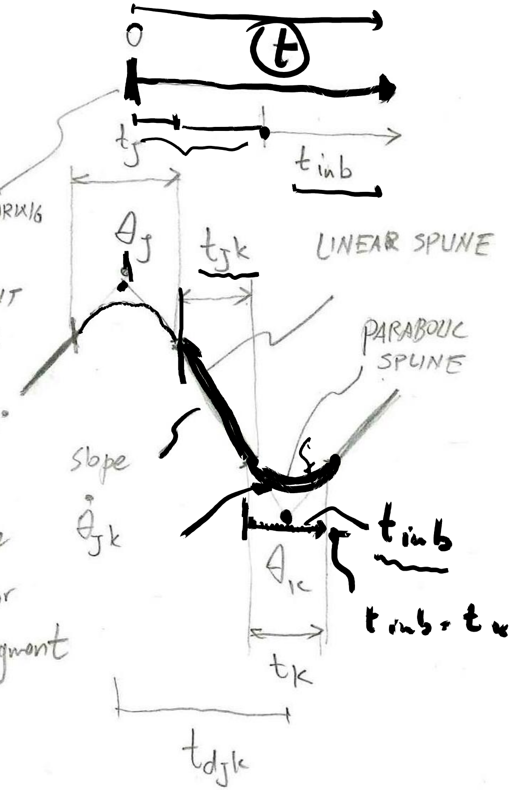
$$\dot{\theta} = \dot{\theta}_{jk} + \ddot{\theta}_k t_{inb}$$

$$\ddot{\theta} = \ddot{\theta}_k$$

E PATH

+ START MEASURING TIME FROM THE VIA POINT θ_j FOR THE SEGMENT JK

+ Reset the time to 0 at θ_k for the following segment



Note: $\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}$; $\ddot{\theta}_k$ - Max allowed acceleration (Used to define t_k)

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- First Segment

LINEAR SPLINE $t \in t_{d12}$

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

$$\dot{\theta} = \dot{\theta}_{12}$$

$$\ddot{\theta} = 0$$

PARABOLIC SPLINE $t \in t_1$

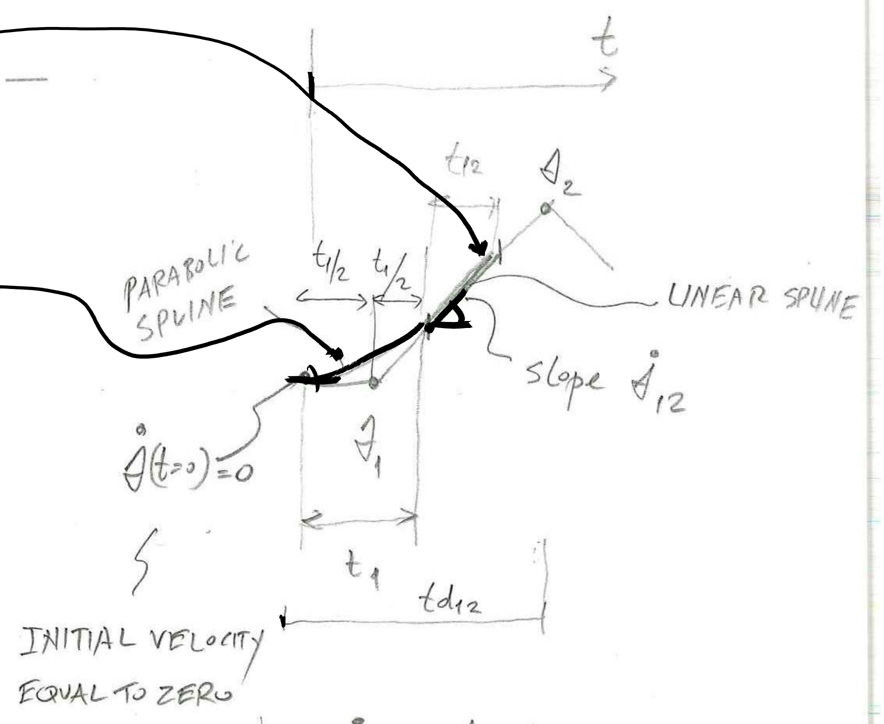
$$\theta = a_0 + a_1 t + a_2 t^2$$

CONDITIONS

$$\theta(t=0) = \theta_0$$

$$\dot{\theta}(t=0) = 0$$

$$\dot{\theta}(t=t_1) = \dot{\theta}_{12}$$



$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

$$\begin{cases} \theta = a_0 + a_1 t + a_2 t^2 \\ \dot{\theta} = a_1 + 2a_2 t \\ \ddot{\theta} = 2a_2 \end{cases}$$

$$\theta(t=0) = \theta_0 = a_0 + a_1 0 + a_2 0 \Rightarrow a_0 = \theta_0$$

$$\dot{\theta}(t=0) = 0 = a_1 + 2a_2 0 \Rightarrow a_1 = 0$$

$$\dot{\theta}(t=t_1) = \dot{\theta}_{12} = +2a_2 t_1 \Rightarrow a_2 = \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1}$$

$$\rightarrow \theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

$$\dot{\theta} = \frac{\dot{\theta}_{12}}{t_1} t$$

$$\ddot{\theta} = \frac{\dot{\theta}_{12}}{t_1}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- Last Segment

LINEAR SPLINE $t \in t_{(n-1)n}$

$$\left\{\begin{aligned} \theta &= \theta_{n-1} + \dot{\theta}_{(n-1)n}\, t \\ \dot{\theta} &= \dot{\theta}_{(n-1)n} \\ \ddot{\theta} &= 0 \end{aligned}\right.$$

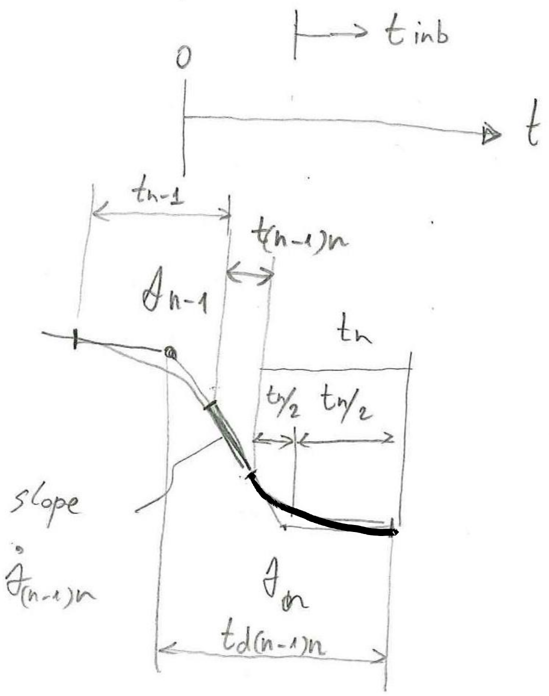
PARABOLIC SPLINE

$$t_{inb} = t - \left(\tfrac{1}{2} t_{(n-1)} + t_{(n-1)n}\right)$$

$$\theta = a_0 + a_1 t_{inb} + a_2 t_{inb}^2$$

CONDITIONS

$$\left\{\begin{aligned} \theta(t_{inb}=0) &= \theta_{n-1} + \dot{\theta}_{(n-1)n}\, t_{(n-1)n} = \theta_{inb} \\ \dot{\theta}(t_{inb}=0) &= \dot{\theta}_{(n-1)n} \\ \ddot{\theta}(t = t_{inb}) &= 0 \end{aligned}\right.$$



$$\dot{\theta}_{(n-1)n} = \frac{\theta_{(n-1)} - \theta_n}{t_{d(n-1)n} - \frac{1}{2} t_n}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

$$
\begin{cases}
\theta = a_0 + a_1 t + a_2 t^2 \\
\dot{\theta} = a_1 + 2a_2 t \\
\ddot{\theta} = 2a_2
\end{cases}
$$

$$\theta(t_{inb}=0) = \theta_{inb} = a_0 + a_1 0 + a_2 0 \Rightarrow a_0 = \theta_{inb}$$

$$\dot{\theta}(t_{inb}=0) = \dot{\theta}_{(n-1)n} = a_1 + 2a_2 0 \Rightarrow a_1 = \dot{\theta}_{(n-1)n}$$

$$\dot{\theta}(t=t_{inb}) = 0 = \dot{\theta}_{(n-1)n} + 2a_2 t_{inb} \Rightarrow a_2 = -\frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}}$$

$$\rightarrow \theta = \theta_{inb} + \dot{\theta}_{(n-1)n} t_{inb} - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Calculation Approaches

- **Calculation Approach No. 1**
  - User Defines
    - Via Points
    - Desired time duration of segments
  - System Defines
    - Use default value of acceleration for each joint

- **Calculation Approach No. 2**
  - System calculate time durations based on default velocities

- **Note for both Approaches –** At all the blends, sufficiently large acceleration must be used so that the system has sufficient time to get into the linear portion of the segment before the next blend region starts

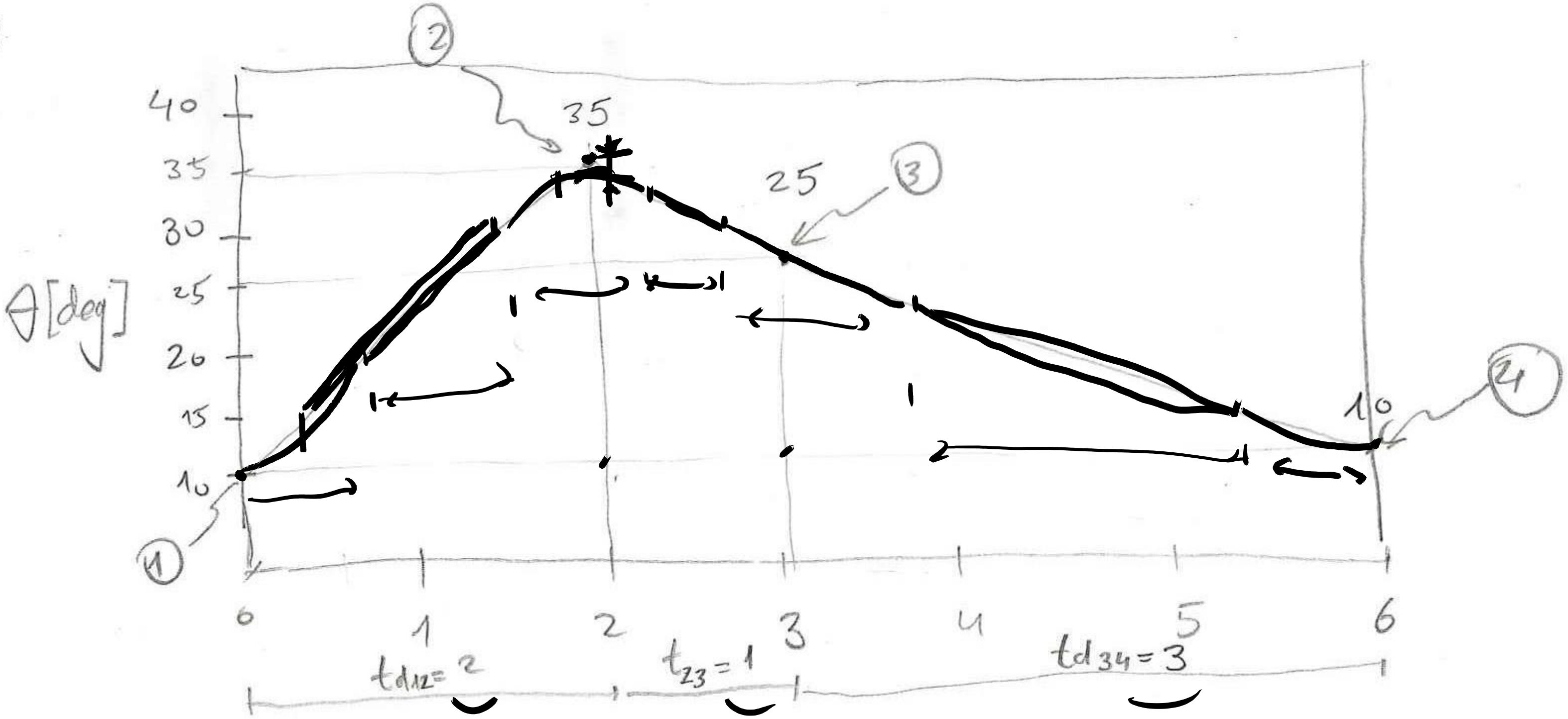
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Example (Book) 7.2/7.3

The trajectory of a particular joint is specified as follows:

$\ddot{\theta} = 50 \; deg/s^2$

GIVEN:



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Acceleration – Magnitude of the default acceleration to use at all blend points is

$$|\ddot{\theta}|_{max} = 50 \ \mathrm{Deg/sec^2}$$

CALCULATE :
- Segment velocities $\dot{\theta}$
- Blend times t_i (t_1, t_2, t_3, t_4)
- Linear segments times

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

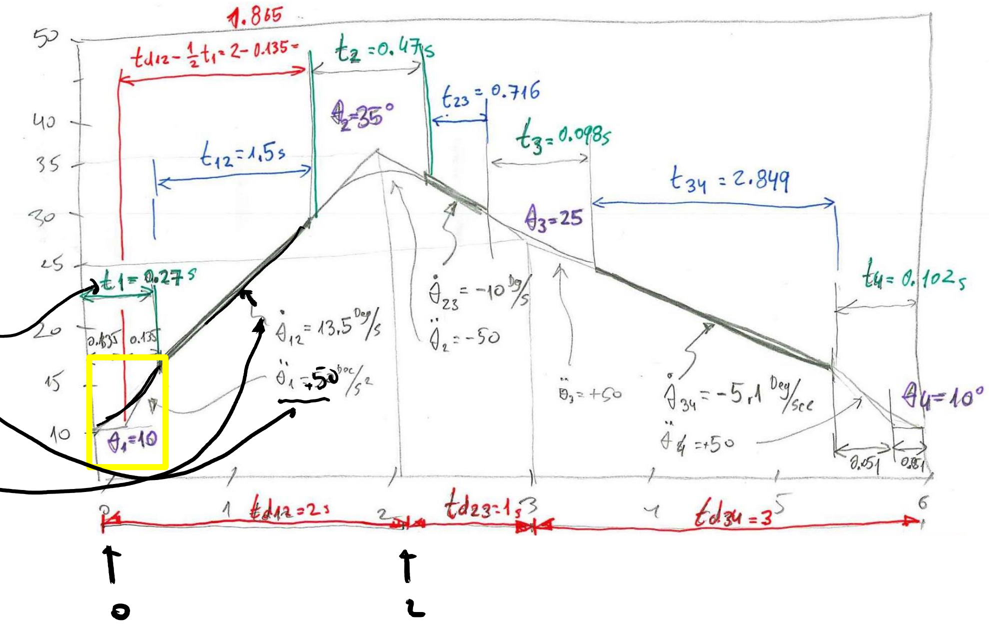
- **Solution Step 1 – Time Interval Analysis**

First Segment (t: 0→2)

$$\ddot{\theta}_1 = 50 = SGN(\theta_2 - \theta_1)|\ddot{\theta}_1| = SGN(35 - 10)|50| = +50 \frac{Rad}{s^2}$$

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}} = 2 - \sqrt{4 - \frac{2(35-10)}{50}} = \boxed{0.27 \ sec}$$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1} = \frac{35 - 10}{2 - \frac{1}{2}(0.27)} = 13.50 \ Deg/sec$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Second Segment (t: 2→3)

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{td_{jk}} = \dot{\theta}_{23} = \frac{\theta_3 - \theta_2}{td_{23}} = \frac{25-35}{1} = -10.0 \ \text{deg/sec}$$

$j=2,\ k=3$

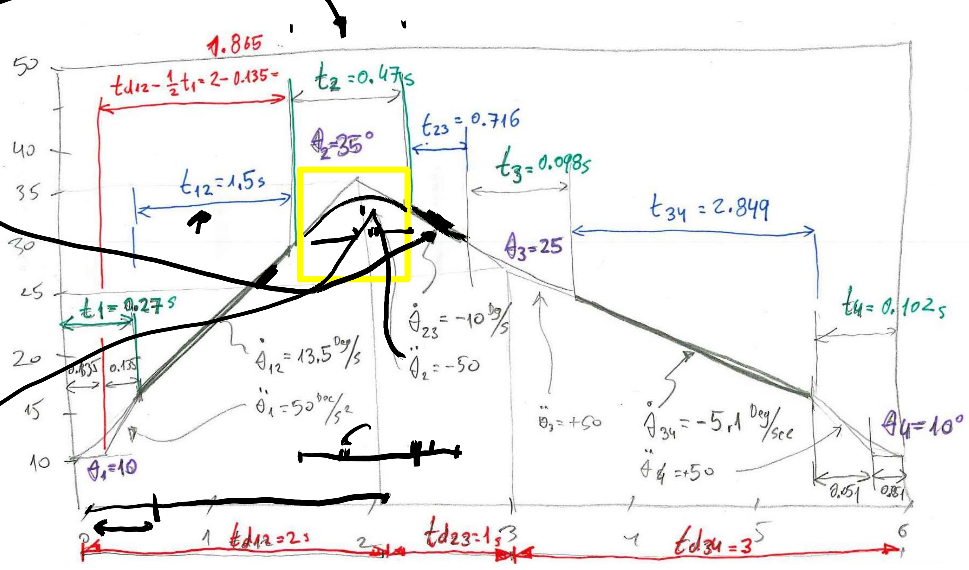
$$\ddot{\theta}_k = SIG(\dot{\theta}_{kl} - \dot{\theta}_{jk})\,|\ddot{\theta}_k| = \ddot{\theta}_2 = SIG(\dot{\theta}_{23} - \dot{\theta}_{12})\,|\ddot{\theta}_2| = SIG(-10-13)\,|\ddot{\theta}_2| = -50 \ \text{deg/sec}$$

$j=1,\ k=2,\ l=3$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k} = t_2 = \frac{\dot{\theta}_{23} - \dot{\theta}_{12}}{\ddot{\theta}_2} = \frac{(-10)-(13.5)}{-50} = \boxed{0.47 \ sec}$$

$j=1,\ k=2,\ l=3$

$$t_{12} = td_{12} - t_1 - \frac{1}{2}t_2 = 2 - 0.27 - \frac{1}{2}(0.47) = 1.5$$



UCLA

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

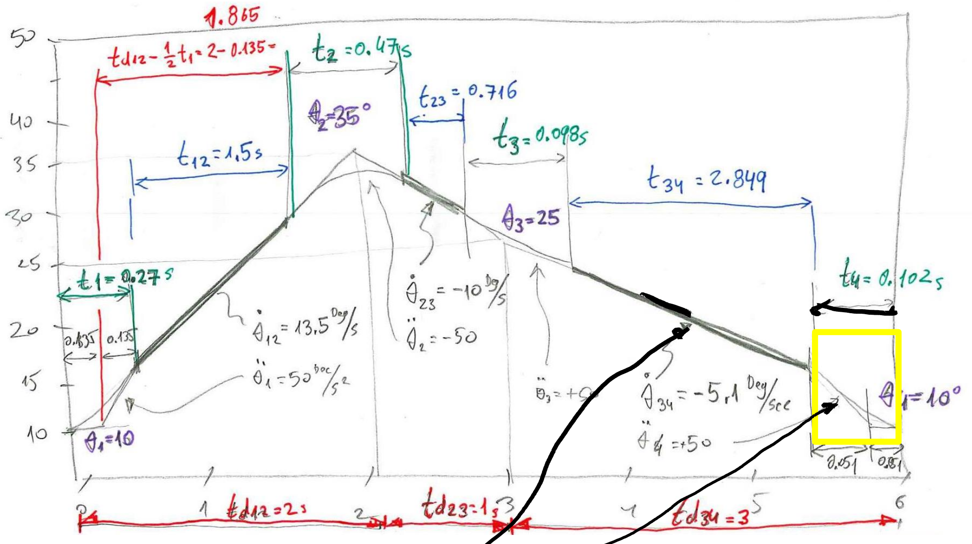
- **Third Segment (t: 3→6)**

$$\ddot{\theta}_n = SGN(\theta_{n-1}-\theta_n)|\ddot{\theta}_n| \underset{n=4}{=} SIG(\theta_3-\theta_4)|\ddot{\theta}_4| = (25-10)|50| = +50\ Deg/s^2$$

$$\ddot{\theta}_4 = 50\ Deg/s^2$$

$$t_n = t_{d(n-1)n} - \sqrt{t^2_{d(n-1)n} - \frac{2(\theta_{n-1}-\theta_n)}{\ddot{\theta}_n}} \underset{n=4}{=} t_4 = t_{d34} - \sqrt{t^2_{d34} - \frac{2(\theta_3-\theta_4)}{\ddot{\theta}_4}} = 3 - \sqrt{9 - \frac{2(25-10)}{50}} = \boxed{0.102\ sec}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n-\theta_{n-1}}{t_{d(n-1)n}-\frac{1}{2}t_n} \underset{n=4}{=} \dot{\theta}_{34} = \frac{\theta_4-\theta_3}{t_{d34}-\frac{1}{2}t_4} = \frac{10-25}{3-\frac{1}{2}0.102} = -5.10\ Deg/sec$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- **Third Segment (t: 3→6) Cont.**

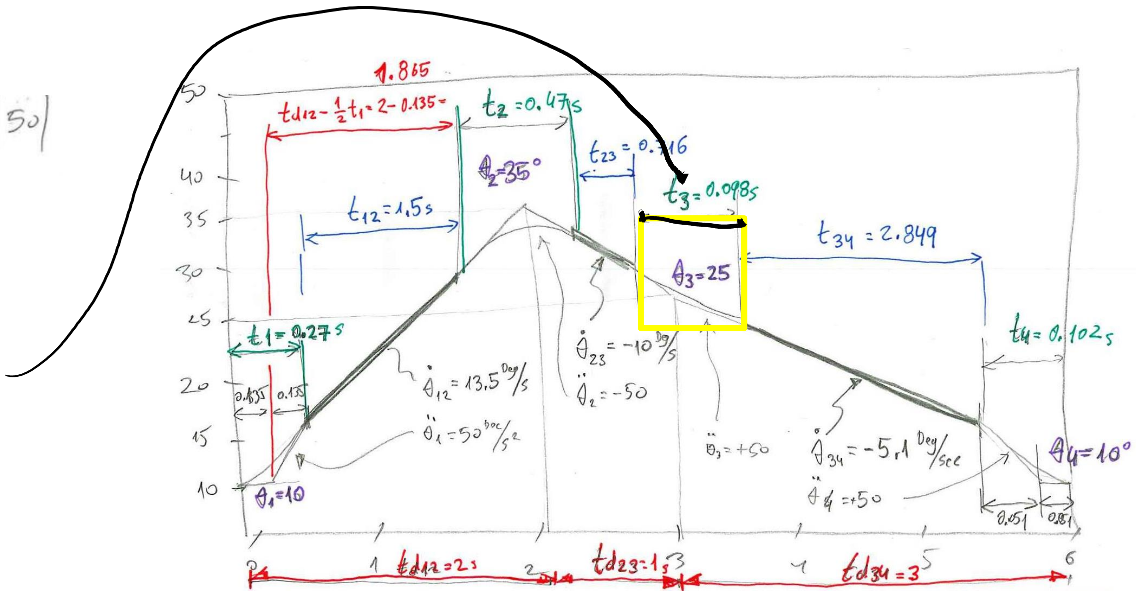
$$\ddot{\theta}_k = SGN\left(\dot{\theta}_{kL} - \dot{\theta}_{jk}\right)\left(\ddot{\theta}_k\right) \Rightarrow \ddot{\theta}_3 = SGN\left(\dot{\theta}_{34} - \dot{\theta}_{23}\right)\left|\ddot{\theta}_3\right| = SGN\left((-5.10)-(-10)\right)\left|50\right|$$

J = 2
K = 3
L = 4

$$\ddot{\theta}_3 = +50\ \text{Deg}/\text{sec}^2$$

$$t_k = \frac{\dot{\theta}_{kL} - \dot{\theta}_{jk}}{\ddot{\theta}_k} \Rightarrow t_3 = \frac{\dot{\theta}_{34} - \dot{\theta}_{23}}{\ddot{\theta}_3} = \frac{(-5.10)-(-10)}{50} = \boxed{0.098\ \text{sec}}$$

j = 2
k = 3
L = 4



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- **Third Segment (t: 3→6) Cont.**

$$t_{jk} = t_{djk} - \frac{1}{2} t_j - \frac{1}{2} t_k = t_{23} = t_{d23} - \frac{1}{2} t_2 - \frac{1}{2} t_3 = 1 - \frac{1}{2}(0.47) - \frac{1}{2}(0.098)$$

$j=2$

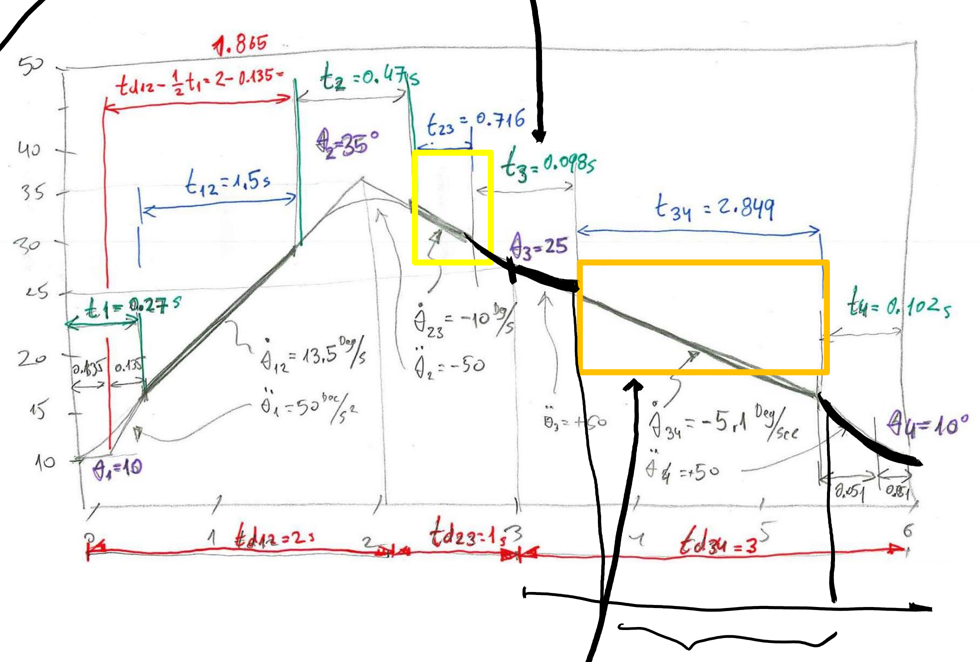
$k=3$

$$= \boxed{0.716 \text{ sec}}$$

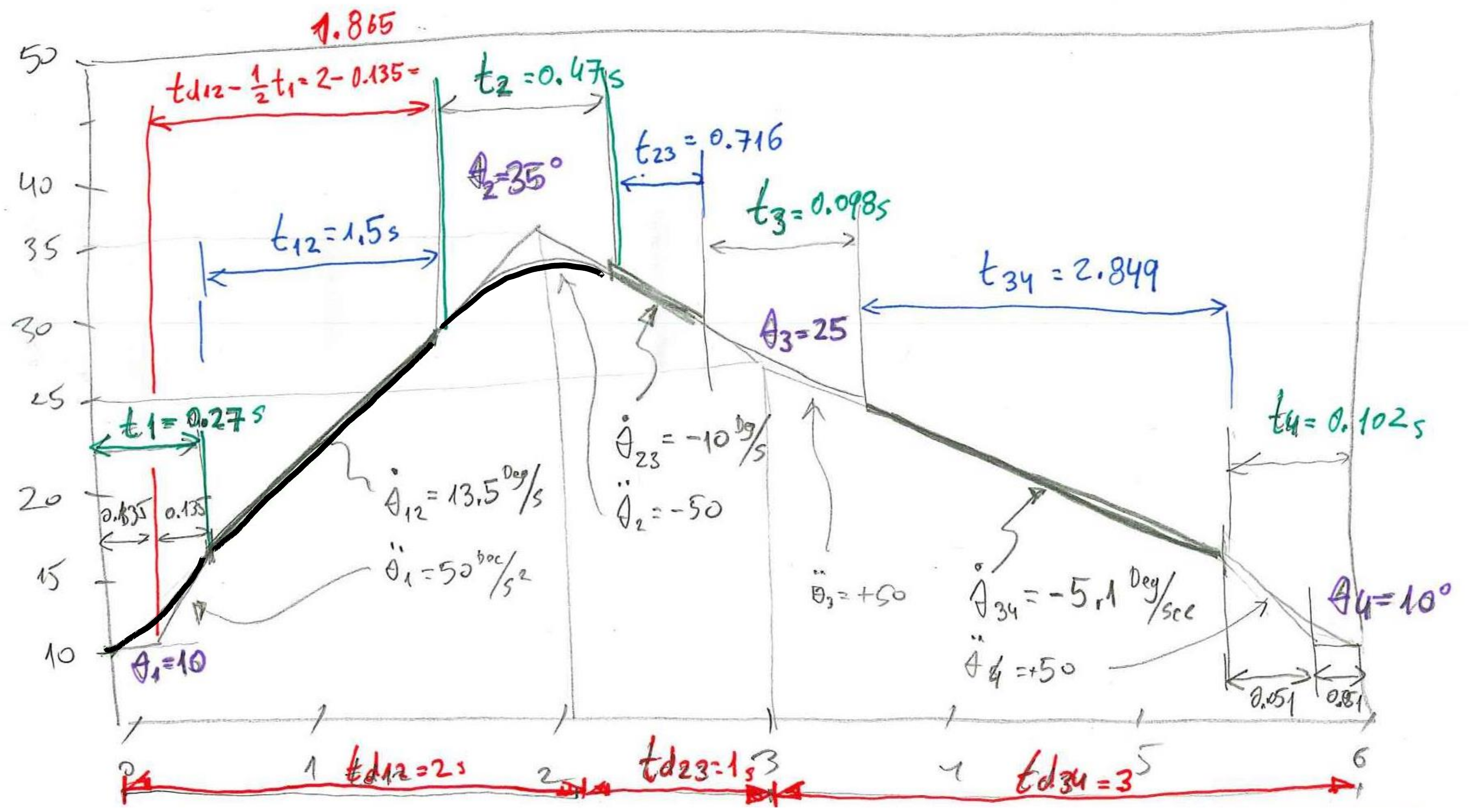
$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2} t_{n-1} = t_{34} = t_{d34} - t_4 - \frac{1}{2} t_3 =$$

$n=4$

$$= 1 - 0.102 - \frac{1}{2}\, 0.098 = \boxed{2.849 \text{ sec}}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example



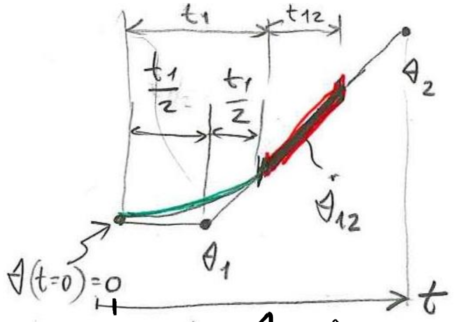
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear / parabolic Splines – Summary Reminder

FIRST SEGMENT

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

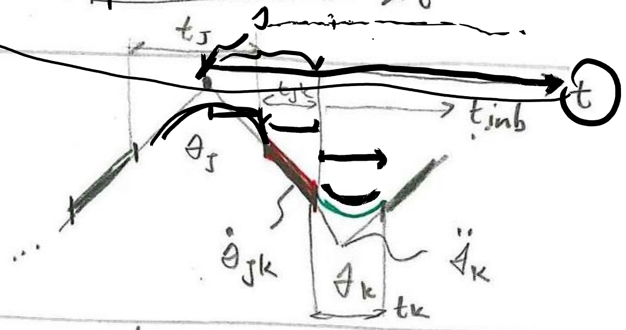


MID SEGMENT

$$\theta = \theta_J + \dot{\theta}_{JK} t$$

$$\theta = \theta_J + \dot{\theta}_{JK}(t - t_{inb}) + \frac{1}{2}\ddot{\theta}_K^2 t_{inb}$$

$$t_{inb} = t - \left(\frac{1}{2} t_J + t_{JK}\right)$$

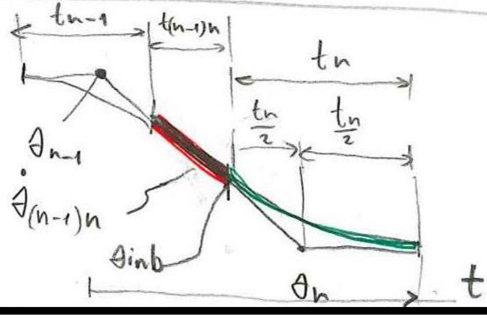


LAST SEGMENT

$$\theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t$$

$$\theta = \theta_{inb} + \dot{\theta}_{(n-1)n} t - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

$$t_{inb} = t - \left(\frac{1}{2} t_{(n-1)} + t_{(n-1)n}\right)$$



Linear Polynomial

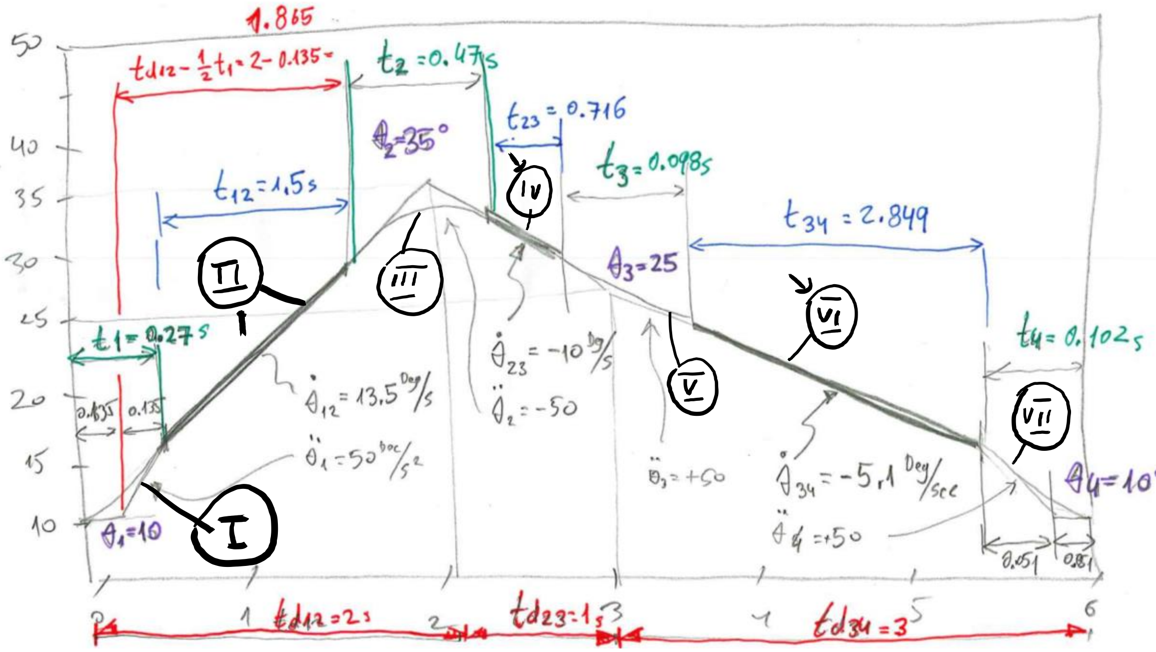
LINEAR SEGMENTS

$$\text{II} \quad \theta = 10 + 13.5t$$

$$\text{IV} \quad \theta = 35 - 10t$$

$$\text{VI} \quad \theta = 25 - 5.1t$$

$$\cancel{\theta_{inb} = 25 - 5.1\left(\frac{0.098}{2} + 2.849\right)}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Parabolic Blend

I $\quad \theta = 10 + \frac{1}{2}\,\frac{13.5}{0.27}\,t^2 \qquad$ (FIRST)

III $\quad \theta = 10 + 13.5\,(t - t_{inb}) + \frac{1}{2}(50)^2\, t_{inb}$

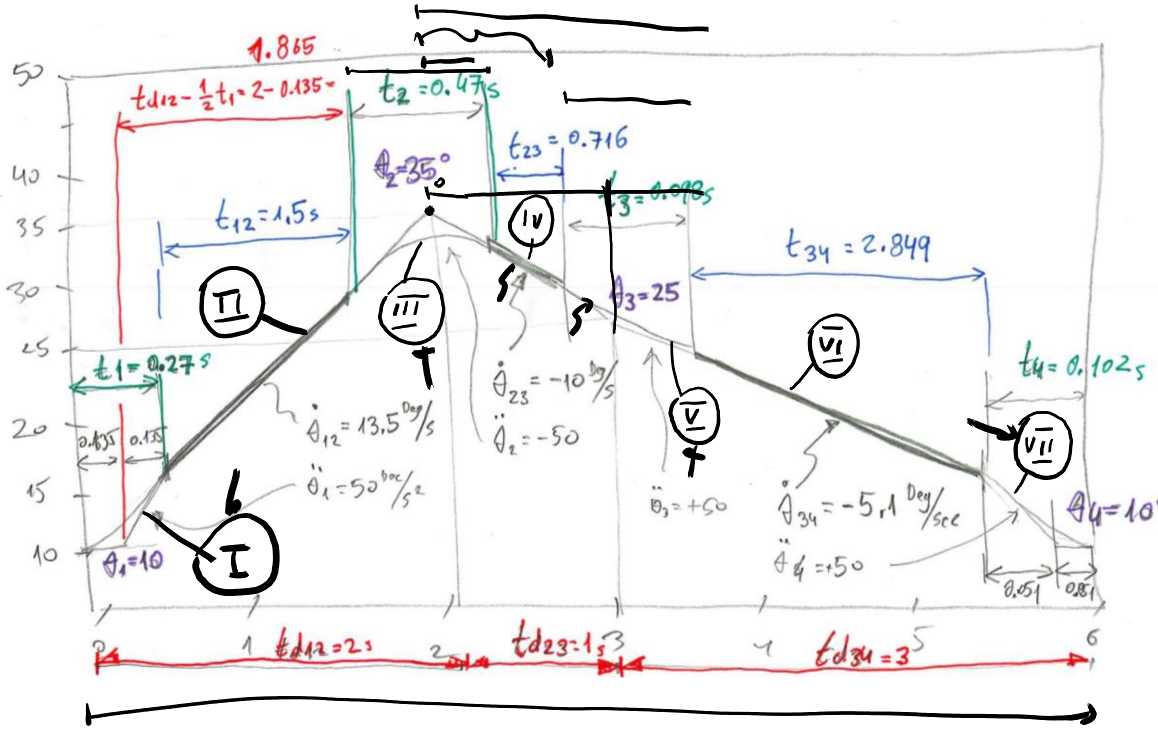
$t_{inb} = t - \left(\frac{1}{2}\,0.27 + 1.5\right)$

V $\quad \theta = 35 - 10\,(t - t_{inb}) + \frac{1}{2}(50)^2\, t_{inb}$

$t_{inb} = t - \left(\frac{1}{2}\,0.47 + 0.716\right)$

VII $\quad \theta = \theta_{inb} + 35\,t - \frac{1}{2}\,\frac{(-5.1)}{t_{inb}}\,t^2 \qquad$ (LAST)

$t_{inb} = t - \left(\frac{1}{2}\,0.998 + 2.849\right)$



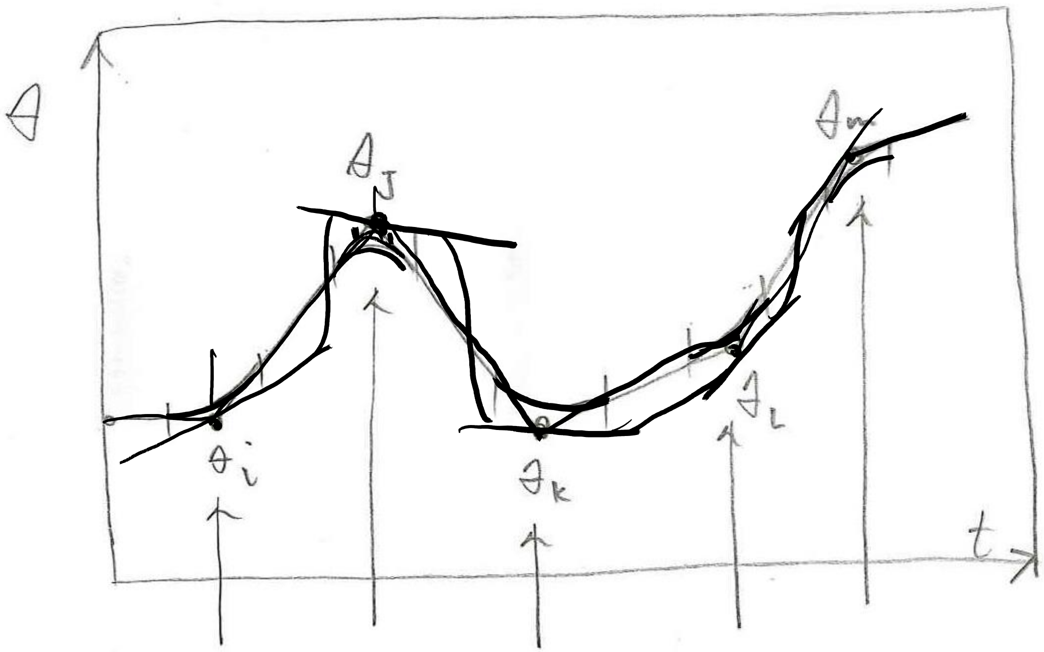
Joint Space Schemes

Multiple Time Intervals

Pseudo Via Point

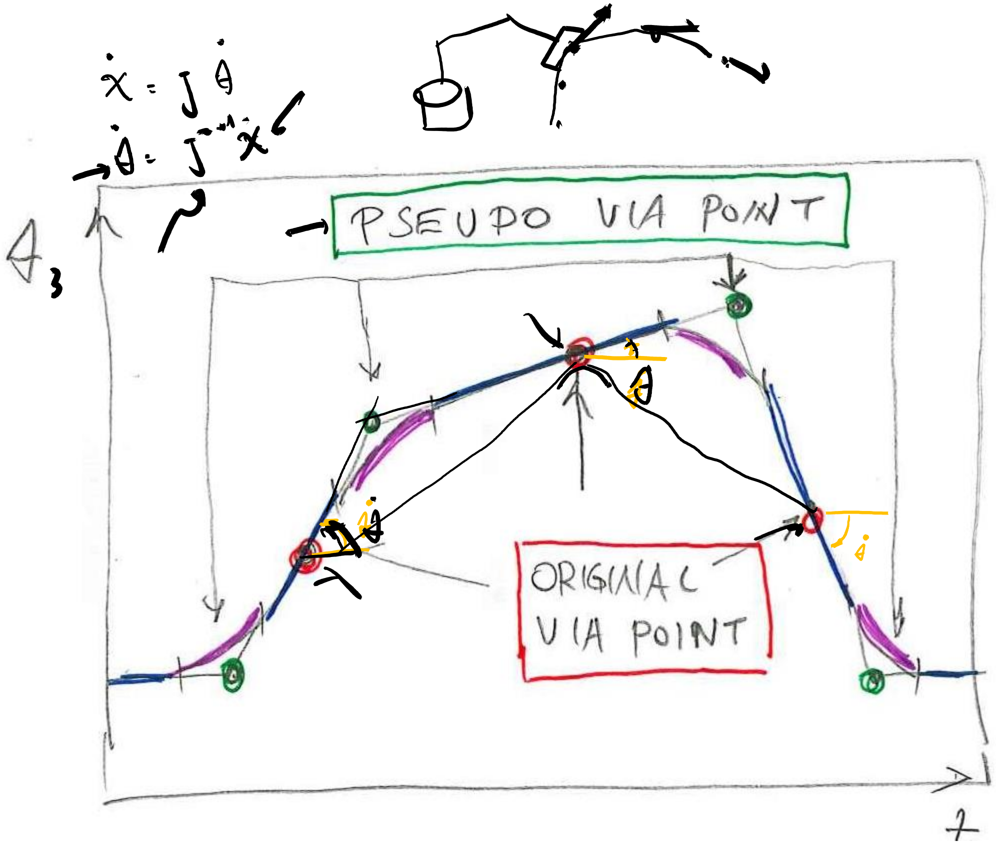
Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

- **Problem**:
  - In the linear parabolic blend spline, note that the via points are not actually reached unless the manipulator come to a stop
  - Often when the acceleration is sufficiently high the path will come quite close to the desire via point

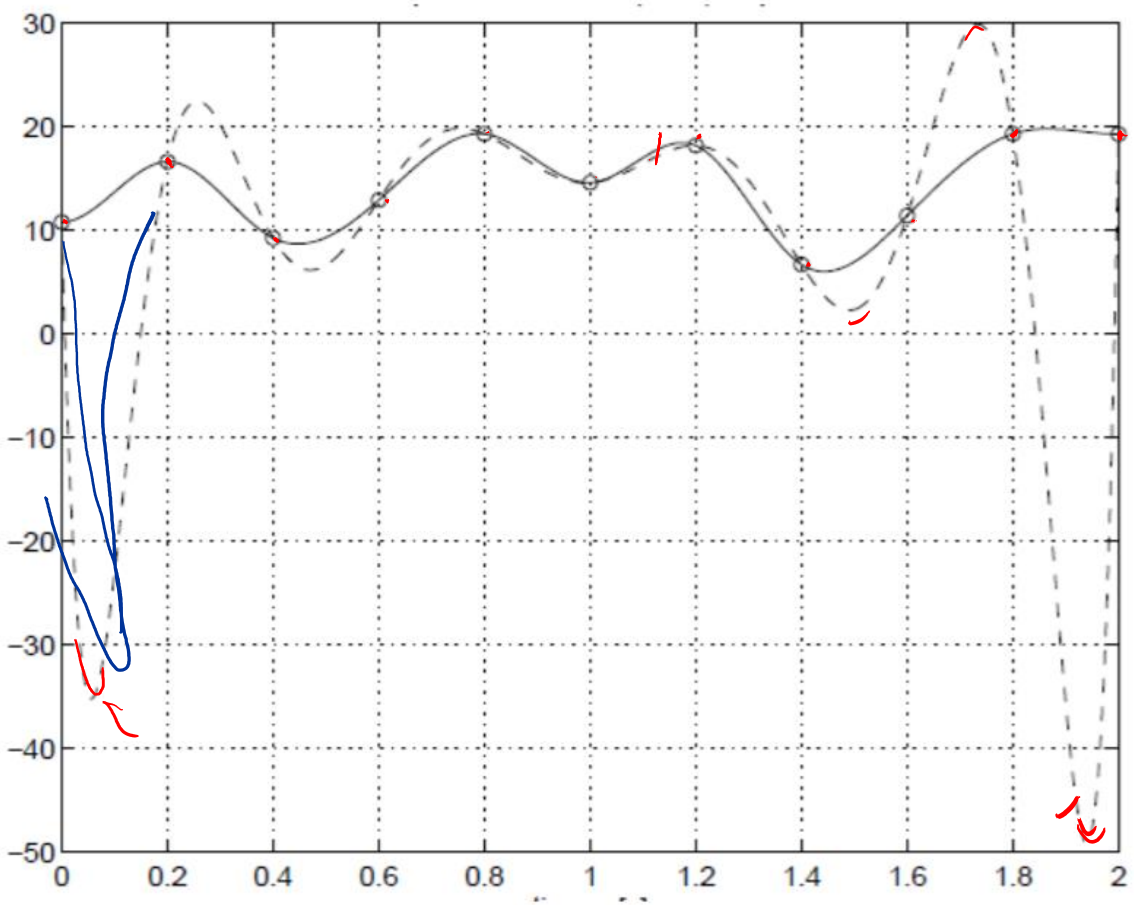


Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

- **Solution**
  - **Define Pseudo via Points** - The system automatically replace the via point through which we wish the manipulator to pass through with two pseudo via points one on each side of the original.
  - The original via point will now lie in the linear region of the path connecting the two pseudo via points
  - **Define Velocity at the original Via Points** - In addition to requesting that the manipulator pass exactly through the via point, the user can also request that it pass through with a certain velocity. If the user does not specify this velocity the system chooses it by means of suitable heuristic
  - **Define Through Point** - Through Point rather than via point is used to specify a path through which we force the manipulator to pass exactly through



Joint Space Schemes – High Order Polynomials



In general, given:

2 points $\Longrightarrow$ unique line

3 points $\Longrightarrow$ unique quadric

...

n points $\Longrightarrow$ unique polynomial with degree $n-1$

Joint Space Schemes – High Order Polynomials