

# **Jacobian – Implications & Applications**

Singularity & Performance & Design





### **Jacobian Methods – Reference Frame - Summary**





### Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)
Explicit (Diff. the Forward Kinematic Eq.)	$^{0}{oldsymbol{J}}_{N}$	None
Iterative Velocity Eq.	$^{N}\boldsymbol{J}_{N}$	Transform Method 1: ${}^{0}v_{N} = {}^{0}_{N}R^{N}v_{N}$ ${}^{0}\omega_{N} = {}^{0}_{N}R^{N}\omega_{N}$
		Transform Method 2: ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}R & 0 \\ 0 & {}^{0}R \end{bmatrix} {}^{N}J_{N}(\theta)$
Iterative Force Eq.	$^{N}\boldsymbol{J}_{N}^{T}$	Transpose ${}^{N}J_{N} = [{}^{N}J_{N}^{T}]^{T}$
		Transform ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}R & 0 \\ 0 & {}^{0}R \end{bmatrix} {}^{N}J_{N}(\theta)$





# **Propagation to the Tip of the Tool**

**Problem Defenition** 





# **Propagation to the Tip of the Tool – Problem Definition**

- Problem
  - Practical Configuration of a robotic arm The robotic arm typically includes the following
    - F/T Sensor
    - Gripper / End Effector
    - Tool
  - Analysis The generic analysis of the robotic arm mapping position, velocities and forces / torques between the base and the wrist (last frame of the manipulators)
  - Rational -
    - Generic Analysis versus task specific elements (F/T sensor, gripper tool) -The analysis is conducted by the robot arm manufacturer however the F/T sensor, the gripper and the tool are task specific and selected by the user.
    - **Tool Change** The same arm performing different tasks may need different tools that are changed during the course of its operation
  - **Need** The need is typically to
    - Trace the position and orientation and velocities (linear and angular) of the tool tip as it follows a trajectory
    - Express force and torques applied on the environment by the tool tip and vice versa by a force sensor measuring these parameters in a different location
- **Solution** Expressing position, velocity forces and torques from the last frame (Frame 6 at the wrist) to the tip of the tool





# **Propagation to the Tip of the Tool**

Position





# Jacobian Propagation to the Tip of the Tool – Position

 In a case where the tool tips follows a trajectory, the path defines the goal position and orientation

$$T_{path} = {}^{0}_{1}T {}^{1}_{2}T {}^{2}_{3}T {}^{3}_{4}T {}^{5}_{5}T {}^{6}_{6}T {}^{7}_{T}$$
 Known

- Since the tool is attached to the end effector its position does not change as a function of time with respect to frame 6
- Multiply both sides of the equations by

$$\binom{6}{T}T^{-1} = {}^{T}_{6}T$$
$$T_{path}\binom{6}{T}T^{-1} = {}^{0}_{1}T {}^{1}_{2}T {}^{2}_{3}T {}^{3}_{4}T {}^{5}_{5}T {}^{6}_{6}T {}^{7}_{T}T\binom{6}{T}T^{-1}$$

• Solve the Inverse Kinematics

$$T_{path \ 6}^{T}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \; {}^{3}_{4}T \; {}^{5}_{5}T \; {}^{5}_{6}T$$









**Forces/Torques** 

Velocities (Linear and Angular)





Velocity





• Position







• Velocity of two rigidly connected frames (rigid body)

Vector Form

Matrix Form





• Forces / Torque

Vector Form

Matrix Form





- the end offector holds a tool

- Located at the point where the end effector attached to the manipulator is a force sensing wrist. This device can measure the forces and torgue applied to it WEFORG WRIST [W] wp. Xw





TT = ST TT - Multiply both sides by ("T)-1 -> wT - Inverting ST T ST ST - The forces & torques applied on the tool tip based on the measurment at the sensor can the calculated as









Jacobian – Singularity Problem Definition





### **Inverse Jacobian**

#### • Given

- Tool tip path (defined mathematically)
- Tool tip position/orientation
- Tool tip velocity
- Jacobian Matrix

 $\underline{\dot{x}} = J(\theta) \ \underline{\dot{\theta}}$ 

- **Problem:** Calculate the joint velocities
- Solution:
  - Compute the inverse Jacobian matrix
  - Use the following equation to compute the joint velocity

$$\underline{\dot{\theta}} = J(\theta)^{-1} \underline{\dot{x}}$$









• *Motivation:* We would like the hand of a robot (end effector) to move with a certain velocity vector in Cartesian space. Using linear transformation relating the joint velocity to the Cartesian velocity we could calculate the necessary joint rates at each instance along the path.

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1} \underline{\dot{x}}$$

- *Given:* a linear transformation relating the joint velocity to the Cartesian velocity (usually the end effector)
- **Question:** Is the Jacobian matrix invertable? (Or) Is it nonsingular?

Is the Jacobian invertable for all values of  $\theta$  ?

If not, where is it not invertable?





- Cases in which the Jacobian matrix  $J(\theta)$  is not inevitable ( $J(\theta)^{-1}$  does not exists). Non invertible matrix is called singular matrix
  - Case 1 The Jacobian matrix is not squared

In general the 6xN Jacobian matrix may be non-square in which case the inverse is not defined

- Case 2 - The determinant (  $\det(J(\theta))$  ) is equal to zero





Answer (Conceptual): Most manipulator have values of where the Ja@obian becomes singular. Such locations are called singularities of the mechanism or singularities for short







## **Singularity - The Concept**



- Lost of DOF Losing one or more DOF means that there is a some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot (end effector) no matter which joint rate are selected
- Load Balance A finite force can be applied to the end effector that produces no torque at the robot's joints
- Joint Velocity A zero end effector velocity will cause high joint velocity



# **Singularity – Physical Interpretation - Examples**

- Type of Singularities
  - Wrist
  - Elbow
  - Shoulder

















Singularities





Singularities



Jacobian – Singularity Example 1 – 3R Elbow Singularity Singularity at the Edge of the Workspace





# Jacobian Matrix by Differanciation - 3R - 1/4

• Consider the following 3 DOF Planar manipulator







• Using a matrix form we get

 $\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$ 

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

• The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees a freedom. Before tackling these problems, consider this brief review of linear algebra.





# **Properties of the Jacobian -Velocity Mapping and Singularities**

• **Example:** Planar 3R

$$\det(J(\theta)) = \begin{vmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{vmatrix} = L_1 L_2 s_2$$

 $\det(J(\theta)) = L_1 L_2 s_2 = 0$ 

• Note that  $\det(J(\theta))$  is not a function of  $\theta_1, \theta_3$ 





# Properties of the Jacobian -Velocity Mapping and Singularities



• The manipulator loses 1 DEF. The end effector can only move along the tangent direction of the arm. Motion along the radial direction is not possible.





# **Properties of the Jacobian -Force Mapping and Singularities**

• The relationship between joint torque and end effector force and moments is given by:

 $\underline{\tau} = J(\underline{\theta})^T \underline{F}$ 

- The rank of  $J(\theta)^T$  is equals the rank of  $J(\theta)$ .
- At a singular configuration there exists a non trivial force F such that

$$J(\underline{\theta})^T \underline{F} = 0$$

In other words, a finite force can be applied to the end effector that produces no torque at the robot's joints.
 In the singular configuration, the manipulator can "lock up."





## **Properties of the Jacobian -Force Mapping and Singularities**

• **Example:** Planar 3R



• In this case the force acting on the end effector (relative to the {0} frame) is given by

$${}^{0}F = \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix}$$





### **Properties of the Jacobian -Force Mapping and Singularities**

$$\underline{{}^{0}\tau} = {}^{0}J(\underline{\theta})^{T} \underline{{}^{0}F} = \begin{bmatrix} -L_{1}s_{1} - L_{2}s_{12} - L_{3}s_{123} & L_{1}c_{1} + L_{2}c_{12} + L_{3}c_{123} & 1 \\ -L_{2}s_{12} - L_{3}s_{123} & L_{2}c_{12} + L_{3}c_{123} & 1 \\ -L_{3}s_{123} & L_{3}c_{123} & 1 \end{bmatrix} \begin{bmatrix} Fc_{1} \\ Fs_{1} \\ 0 \end{bmatrix}$$

• For 
$$\theta_1 = \theta$$
;  $\theta_2 = \theta_3 = 0$  we get

$$\underbrace{\overset{0}{\underline{\tau}} = {}^{0}J(\underline{\theta})^{T} \underbrace{\overset{0}{\underline{F}}}_{} = \begin{bmatrix} -L_{1}s_{1} - L_{2}s_{1} - L_{3}s_{1} & L_{1}c_{1} + L_{2}c_{1} + L_{3}c_{1} & 1 \\ -L_{2}s_{1} - L_{3}s_{1} & L_{2}c_{1} + L_{3}c_{1} & 1 \end{bmatrix} \begin{bmatrix} Fc_{1} \\ Fs_{1} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -Fs_{1}c_{1}(L_{1} + L_{2} + L_{3}) + Fs_{1}c_{1}(L_{1} + L_{2} + L_{3}) \\ -Fs_{1}c_{1}(L_{2} + L_{3}) + Fs_{1}c_{1}(L_{2} + L_{3}) \\ -Fs_{1}c_{1}(L_{3}) + Fs_{1}c_{1}(L_{3}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$





Jacobian – Singularity Example 2 – 3R Shoulder Singulaity Singularity Inside the Workspace








• If we want to use the inverse Jacobian to compute the joint angular velocities we need to first find out at what points the inverse exists.

$$\underline{\dot{\theta}} = J(\theta)^{-1} \underline{\dot{x}}$$

• Considering the 3R robot

$${}^{4}J_{r}(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

• The determinate of the Jacobian is defined as follows

$$|{}^{4}J_{r}(\theta)| = -(L1 + L2c2 + L3c23)(L2s3)L3$$





$$|{}^{4}J_{r}(\theta)| = -(L1 + L2c2 + L3c23)(L2s3)L3$$

• The reduced Jacbian matrix is singular when it determinate is equal to zero

-(L1 + L2c2 + L3c23)(L2s3)L3 = 0

• The singular condition occur when either of the following are true

s3 = 0

-L1 - L2c2 - L3c23 = 0





• Case 1: s3 = 0

$$s3 = 0 \Longrightarrow \begin{cases} \theta_3 = 0^\circ \\ \theta_3 = 180^\circ \end{cases}$$

$${}^{4}J_{r}(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

- The first row of the Jacobian is zero
- The 3R robot is loosing one DOF.
- The robot can no longer move along the X-axis of frame {4}







• Case 2: -L1 - L2c2 - L3c23 = 0

L1 = -L2c2 - L3c23

• Occur only if  $L2 + L3 \ge L1$ 

	0	L2s3	0 ]
${}^{4}J_{r}(\theta) =$	0	L2c3 + L3	L3
	-L1 - L2c2 - L3c23	0	0

- The third row of the Jacobian is zero
- The origin of frame {4} intersects the Zaxis of frame {1}
- The 3R robot is loosing one DOF.
- The robot can no longer move along the Z-axis of frame {4}









## Joint Velocity Near Singular Position - 3R Example

- Robot : 3R robot
- Task: Visual inspection



Control







• Singularity (Case 2)- The origin of frame {4} intersects the Z-axis of frame {1}

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

• Solve for in terms of  $\dot{\theta}_1$  we find  $\dot{z}$ 

$$\dot{\theta}_1 = \frac{\dot{z}}{-L1 - L2c2 - L3c23}$$

$$-L1 - L2c2 - L3c23 = 0$$
$$\dot{\theta}_1 \to \infty$$





# Joint Velocity Near Singular Position - 3R Example

• Singularity -

$$\det(J(\theta)) = 0$$

- Problems:
  - Motor Constrains The robot is physically limited from moving in unusual high joint velocities by motor power constrains. Therefore, the robot will be unable to track the required joint velocity trajectory exactly resulting in some perturbation to the commanded Cartesian velocity trajectory.
  - Gears and Shafts The derivative of the angular velocity is the angular acceleration. The high acceleration of the joint resulting form approaching too close to a singularity may cause damage to the gear/shafts.
  - **DOF** At a singular configuration (specific point in space) the manipulator loses one or more DOF.
- Consequences Certain tasks can not be performed at a singular configuration





Jacobian – Singularity Example 3 – 3R Wrist Singularity Singularity Inside the Workspace





# **Mapping - Rotated Frames - Z-Y-Z Euler Angles**

Start with frame {4}.

- •
- Rotate frame {4} about  $\hat{Z}_4$  by an angle  $\begin{array}{c} \alpha \\ \beta \\ \end{array}$ Rotate frame {4} about  $\hat{Y}_B$  by an angle  $\begin{array}{c} \beta \\ \gamma \end{array}$

Note - Each rotation is preformed about an axis of the moving reference frame

*{4}, rather then a fixed reference.* 













#### **Mapping - Rotated Frames - Z-Y-Z Euler Angles**







$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0\\ s\alpha & c\alpha & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta\\ 0 & 1 & 0\\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0\\ s\gamma & c\gamma & 0\\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$${}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma)={}^{4}_{6}R_{\theta_{4}=0}$$





$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$
  
Goal Direct Kinematics





• Solve for  $\beta$  using element  $r_{31}, r_{32}, r_{33}$ 

$$r_{31} = -s\beta c\alpha$$

$$r_{32} = s\beta s\alpha$$

$$r_{33} = c\beta$$

$$r_{31}^{2} + r_{32}^{2} = s\beta^{2}(c\alpha^{2} + s\alpha^{2})$$

$$r_{31}^{2} + r_{32}^{2} = s\beta^{2}(c\alpha^{2} + s\alpha^{2})$$

$$r_{33} = c\beta$$

$$r_{33} = c\beta$$

$$r_{33} = c\beta$$

$$r_{31} + r_{32}^{2}$$

$$r_{31} + r_{32}^{2}$$

$$r_{31} + r_{32}^{2}$$

• Using the Atan2 function, we find

$$\beta = \text{Atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$





• Solve for  $\alpha$  using elements  $r_{23}, r_{13}$ 

$$r_{13} = c \alpha s \beta$$

$$r_{23} = s \alpha s \beta$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c \alpha c \beta c \gamma - s \alpha s \gamma & -c \alpha c \beta s \gamma - s \alpha c \gamma & c \alpha s \beta \\ s \alpha c \beta c \gamma + c \alpha s \gamma & -s \alpha c \beta s \gamma + c \alpha c \gamma & s \alpha s \beta \\ -s \beta c \gamma & s \beta s \gamma & c \beta \end{bmatrix}$$

UCLA



• Solve for  $\gamma$  using elements  $r_{32}, r_{31}$ 

 $r_{32} = s\beta s\gamma$  $r_{31} = -s\beta c\gamma$ 

$$\gamma = \operatorname{Atan2}(r_{32} / s\beta, -r_{31} / s\beta)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$





• Note: Two answers exist for angle  $\beta$  which will result in two answers each for angles  $\alpha$  and  $\gamma$ .

$$\beta = \operatorname{Atan2}\left(\pm \sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$
$$\alpha = \operatorname{Atan2}\left(r_{23} / s\beta, r_{13} / s\beta\right)$$
$$\gamma = \operatorname{Atan2}\left(r_{32} / s\beta, -r_{31} / s\beta\right)$$

• If  $\beta = 0^{\circ}, \beta = 180^{\circ} \Rightarrow s\beta = 0$  the solution degenerates





$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$$\beta = 0^{\circ}$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\gamma - s\alpha s\gamma & -c\alpha s\gamma - s\alpha c\gamma & 0 \\ s\alpha c\gamma + c\alpha s\gamma & -s\alpha s\gamma + c\alpha c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c(\alpha + \gamma) & -s(\alpha + \gamma) & 0 \\ s(\alpha + \gamma) & c(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• We are left with  $(\gamma + \alpha)$  for every case. This means we can't solve for either, just their sum.





- One possible convention is to choose  $\alpha = 0^{\circ}$
- The solution can be calculated to be



$$\beta = 0 \qquad \beta = 180$$
  

$$\alpha = 0 \qquad \alpha = 0$$
  

$$\gamma = \operatorname{Atan2}(-r_{12}, r_{11}) = \operatorname{Atan2}(s\gamma, c\gamma) \qquad \gamma = \operatorname{Atan2}(r_{12}, -r_{11}) = \operatorname{Atan2}(s\gamma, c\gamma)$$





• For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector's orientation never changes.







# **Gimbal Lock**



Normal situation The three gimbals are independent

### http://youtu.be/zc8b2Jo7mno



Gimbal lock: Two out of the three gimbals are in the same plane, one degree of freedom is lost





- In robotics, gimbal lock is commonly referred to as "wrist flip", due to the use of a "triple-roll wrist" in robotic arms, where three axes of the wrist, controlling yaw, pitch, and roll, all pass through a common point.
- An example of a wrist flip, also called a wrist singularity, is when the path through which the robot is traveling causes the first and third axes of the robot's wrist to line up. The second wrist axis then attempts to spin 180° in zero time to maintain the orientation of the end effector. The result of a singularity can be quite dramatic and can have adverse effects on the robot arm, the end effector, and the process.
- The importance of non-singularities in robotics has led the American National Standard for Industrial Robots and Robot Systems — Safety Requirements to define it as "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities".





- This situation is an old and famous one in mechanical engineering.
- For example, in the steam locomotive, "top dead center" refers to the following condition



• The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.





• We have shown the relationship between joint space velocity and end effector velocity, given by

 $\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$ 

• It is interesting to determine the inverse of this relationship, namely

 $\underline{\dot{\theta}} = J(\underline{\theta})^{-1} \underline{\dot{x}}$ 





- Consider the square 6x6 case for  $J(\underline{\theta})$ .
- If rank < 6 (  $Det(J(\underline{\theta}))=0$  ) , then there is no solution to the inverse equation (see Brief Linear Algebra Review 1,7).

 $Rank(J(\underline{\theta})) < 6$  $\underline{\dot{\theta}} = J(\underline{\theta})^{-1} \underline{\dot{x}}$ 

• However, if the rank = 5, then there is at least one non-trivial solution to the forward equation (see Brief Linear Algebra Review - 7). That is, for

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} = 0$$





- The solution is a direction  $(\underline{\theta})$  in the injoint velocity space for which joint motion produces no end effector motion.
- We call any joint configuration  $\underline{\theta} = Q$  for which

 $Rank(J(\underline{\theta})) < 6$ 

a singular configuration.







• For certain directions of end effector motion ,  $\underline{\dot{x}}_i \quad 1 \le i \le 6$ 

$$\underline{\dot{x}} = J(\theta)\underline{\dot{\theta}} = \lambda_i(\underline{\theta})\underline{\omega}_i$$

where:

- $-\lambda_i$  are the eigenvalues of J( heta)
- $-\underline{\omega}_i$  are the eigenvectors of  $J(\theta)$
- If  $J(\theta)$  is fully ranked (see Brief Linear Algebra Review 6/), we have

$$\underline{\omega}_{i} = J(\theta)^{-1} \underline{\dot{x}} = \lambda_{i} (\underline{\theta})^{-1} \underline{\dot{x}}$$





• As the joint approach a singular configuration  $\underline{\theta} = Q$  there is at least one eigenvalue for which  $\lambda_i \to 0$ . This results in

$$\underline{\omega}_i = \frac{\underline{\dot{x}}}{\lambda_i(\underline{\theta})} \longrightarrow \frac{\underline{\dot{x}}}{0} \longrightarrow \infty$$

- In other word, as the joints approach the singular configuration, the end effector motion in a particular task direction causes the joint velocities to approach <u>i</u>nfinity. However, there are task velocities that can have solutions.
- If  $J(\underline{\theta})$  loses rank by only one, then there are n-1 eigenvectors in the task velocity space  $(\underline{\dot{x}}_j)$  for which solutions do exist. However, there can be multiple solutions.





# Jacobian – Manipulability Ellipsoid











$$V_{x tip} = \frac{d_{xtip}}{dt} = -L_{i}\dot{\theta}_{i}\varsigma_{i} - l_{2}(\dot{\theta}_{i}+\dot{\theta}_{2})\varsigma_{i2}$$

$$V_{y tip} = \frac{d_{ytip}}{dt} = L_{i}\dot{\theta}_{i}C_{i} + l_{2}(\dot{\theta}_{i}+\dot{\theta}_{2})C_{i2}$$











- As long as Ji(A) and J2(A) are not collinear,
   it is possible to generate an endeffector velocity
   Vtip in any arbitrary direction in the
   Xo.-yo plane by choosing appropriate joint
   Velocities Å, and Å.
- · Since Ji(A), and J2(D) depend on the joint values A1 and A2 there are some configurations where J1(A), J2(D) become collinear









for any 
$$\begin{cases} A_{z} = 0 & J_{1} \parallel J_{2} \\ A_{z} = 180 & J_{1} \parallel J_{2} \end{cases} \longrightarrow \text{ singularities} \\ A_{z} = 180 & J_{1} \parallel J_{2} \end{cases} \longrightarrow \text{ singularities} \\ A_{z} = 0^{\circ} & V = 0 & A_{z} = 180^{\circ} \\ A_{z} = 180^{\circ} & V = 0 & A_{z} = 180^{\circ} \\ A_{z} = 180^{\circ} & V = 0 & A_{z} = 180^{\circ} \\ A_{z} = 0^{\circ} &$$




• Substitute 
$$L_1 = 1$$
;  $L_2 = 1$   
• Consider the robot at two different  
honsingular postures  
 $A = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}$   $A = \begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix}$   
 $J \left( \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix} \right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix}$ ,  $J \left( \begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix} \right) = \begin{bmatrix} -0.71 & -0.71 \\ 0.29 & -0.71 \end{bmatrix}$ 





. The jarubian can be used to map bounds on rotational speed of the joints (2) to bounds on the enderfactor melocity (Ntip)























MANIPULABILITY ELLIPSOID & MARIPULABILITY MEASURES - GENERALIZATION

- . TASK REQUIRMENTS
  - + DESIGN MECHANISM SIZE
  - + POSTURE OF THE PUBOTIC ARM WITHINI THE WORKSPACE FOR PERFORMING A GIVENI THSK
  - AND ORINTATION OF THE ENDEFFECTOR





MANIPULABILITY ELLIPSOID - DEFENITION







## Properties of the Jacobian -Velocity Mapping and Singularities



- Note: See Mathematica Simulations
  - Two Link: <u>https://demonstrations.wolfram.com/ForwardAndInverseKinematicsForTwoLinkArm/</u>
  - Three links : <u>https://demonstrations.wolfram.com/ManipulabilityEllipsoidOfARobotArm/</u>





## Jacobian – Performance Index Design





- Kinematic Singularity The robot end effector loses its ability to translate or rotate in one ore more directions
- Kinematic Singularity Binary A kinematic singularity presents a binary proposition – a particular configuration is either kinematically singular or it is not
- **Proximity to Singularity** it is reasonable to ask if a nonsingular configuration is "close" to being singular.
- **Manipulability Ellipsoid** The manipulability ellipsoid allows one to visualize geometrically the directions in which the end-effector moves with least effort or with greatest effort







• Manipulabity Ellipsoid - For a general n-joint serial (open chain) and a task space with coordinates the q manipulability ellipsoid corresponds to the end-effector velocities for joint rates  $\dot{q}$ 

satisfying the norm of  $\dot{\Theta}\,$  to be equal to 1

$$\dot{\Theta} = \dot{\Theta}^T \dot{\Theta} = 1$$

representing a unite sphere in the n-th dimensional joint velocity space





Assuming J is invertible, the unit joint-velocity condition can be written

$$1 = \dot{\Theta}^T \dot{\Theta}$$
  

$$1 = (J^{-1} \dot{q})^T (J^{-1} \dot{q})$$
  

$$1 = \dot{q}^T (J^{-1})^T J^{-1} \dot{q} = \dot{q}^T J^{-T} J^{-1} \dot{q}$$
  

$$1 = \dot{q}^T (J J^T)^{-1} \dot{q}$$

If J is full rank the matrix  $JJ^T$  and  $(JJ^T)^{-1}$  are

- square,
- symmetric
- positive definite





For any symmetric positive-definite  $JJ^T$ , the set of vectors  $\dot{q}$  satisfying

 $\dot{q}^T \left( J J^T \right)^{-1} \dot{q} = 1$ 

defines an ellipsoid in the m-dimensional space.

Recap

- Represent an circle / sphere  $\dot{\Theta}^T \dot{\Theta} = 1$
- Represent a ellipse / ellipsoid  $\dot{q}^T (JJ^T)^{-1} \dot{q} = 1$







 $\dot{q} = \dot{X} = J \dot{\Theta}$ 









- Performing eigenvector/eigenvalue analysis of  $JJ^T$  defining
  - eigenvectors  $V_i$
  - eigenvalues  $\lambda_i$
- The directions of the principal axes of the ellipsoid are  $V_i$  and the lengths of the principal semi-axes are  $\sqrt{\lambda_i}$









• The volume V of the ellipsoid is proportional to the product of the semi-axis lengths

$$V \propto \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(JJ^T)}$$





- Given the structure of the Jacobian matrix, it makes sense to separate it into the two sub matrixes because the units of
  - $J_{\nu}$  are linear velocities (m/s) and the unites of
  - $J_{\omega}$  are angular velocities (rad/s)

$$(\theta) = \begin{bmatrix} J_{v} \\ J_{\omega} \end{bmatrix}$$

J

• This leads to two three-dimensional manipulability ellipsoids, one for linear velocities and one for angular velocities.

$$J_{\nu}J_{\nu}^{T}$$
$$J_{\omega}J_{\omega}^{T}$$





When calculating the linear-velocity manipulability ellipsoid ( $J_{\nu}J_{\nu}^{T}$ ),

it generally makes more sense to use the Jacobian expressed in the end effector space  ${}^{N}J_{v}{}^{N}J_{v}{}^{T}$ 

instead of the Base Frame  ${}^{0}J_{\nu}{}^{0}J_{\nu}{}^{T}$ 

since we are usually interested in the linear velocity of the end effector in its own coordinate system than a fixed frame at the base





- Challenge
  - Difficulty in operating at
    - Workspace Boundaries
    - Neighborhood of singular point inside the workspace
- Goal
  - **Singularity** The further the manipulator is away from singularities the better it moves uniformly and apply forces in all directions

#### Performance Criterion

- It is useful to assign a single scalar measure defining how easily the robot can move at a given posture.





• **Isotropy** – The ratio of the longest and shortest semi-axes of the manipulability ellipsoid

$$\mu_1 \left( J J^T \right) = \frac{\sqrt{\lambda_{\max}}}{\sqrt{\lambda_{\min}}} \ge 1$$

$$1 \le \mu_1 \left( J J^T \right) \le \infty$$

- When  $\mu_1(JJ^T) \rightarrow 1$  then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable
- When  $\mu_1(JJ^T) \rightarrow \infty$  the robot approaches a singularity





### **Performance Index – Condition Number**

• **Condition Number** – Squaring the isotropy measure

$$\mu_2 \left( J J^T \right) = \left( \mu_1 \left( J J^T \right) \right)^2 = \frac{\lambda_{\max}}{\lambda_{\min}} \ge 1$$

$$1 \le \mu_2 \left( J J^T \right) \le \infty$$

- When  $\mu_1(JJ^T) \rightarrow 1$  then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable
- When  $\mu_1(JJ^T) \rightarrow \infty$  the robot approaches a singularity





• Manipulability – Proportional to the volume of the manipulability ellipsoid

$$w = \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(JJ^T)}$$

#### $0 \le w < \infty$

- A good manipulator design has large area of characterized by high value of the manipulability (w)





## **Design – Example**





# **RAVEN – A SURGICAL ROBTICS SYSTEM**

# **DESIGN – SPECIFICATIONS**

















## **Engineering Specifications - BlueDRAGON**

Device				DRAGON	UC Berkeley	UC Berkeley	UC Berkeley	DeVinchi	Zeus
Generation				R1 - E (95%)		1	2		
Referance					Measured	Traget	Obtained		
Base	Overall Geomtery	Shaft Diameter	[m]			0.01 - 0.015	0.01 - 0.015	0.01	0.005
	Position / Oriantataion	Delta Theta x	[Deg]	53.8047				+/-60	
		Delta Theta y	[Deg]	36.3807				+/-80	
		Delta Theta z	[Deg]	148.0986	90	180-270	720	+/-180	
		R	[m]	0.1027				0.2	
		Grasping Jaw s	[Deg]	24.0819				200	
		Grasping Jaw s	[m]	*	0.006	0.002-0.003	0.008 min		
		Delta X	[m]	0.1026					
		Delta Y	[m]	0.0815					
		Delta Z	[m]	0.0877					
	Velocity (Angular Linear)	Wx	[Rad/sec]	0.432					
		Wy	[Rad/sec]	0.486					
		Wz	[Rad/sec]	1.053			9.4 min		
		VR	[m/sec]	0.072					
		Wg	[Rad/sec]	0.0468					
	Force	Fx	[N]	14.7299					
		Fy	[N]	13.1981					
		Fz	[N]	184.3919					
		Fg	[N]	41.6085	15	5 min	40 min		
	Torque	Тх	[Nm]	2.3941					
		Ту	[Nm]	1.6011					
		Tz	[Nm]	0.0464	0.088	0.022			



## Kinematic Analysis – Playback Simulation using Measured Data







## **Robot Optimization - Workspace**

60°- 60°

- Dexterous Workspace (DWS)
  - High dexterity region defined by a right circular cone with a vertex angel of 60°
  - Contains 95% of the tool motions based on *in-vivo* measurements.















## **Spherical Mechanism - Robot Optimization**







## Optimization of Raven IV – Problem & Parameters (7) Definitions








# **RAVEN – A SURGICAL ROBTICS SYSTEM**

# **DESIGN – KINEMATIC ANALYSIS & OPTIMIZATION**





## Direct Kinematics – Coordinate Systems Assignment







# Direct Kinematics – Coordinate Systems Assignment







## Direct Kinematics: DH Parameters - Left and Right Robot

Robot	i-1	i	$\alpha_i$	$a_i$	$d_i$	$ heta_i$
Left	0	1	$\pi - \alpha$	0	0	$\theta_1(t)$
Robot	1	2	$-\beta$	0	0	$-\theta_2(t)$
(1,3)	2	3	0	0	0	$\pi/2 -  heta_3(t)$
	3	4	$-\pi/2$	0	$d_4(t)$	0
	4	5	$\pi/2$	$a_5$	0	$\pi/2 -  heta_5$
	5	6	$-\pi/2$	0	0	$\pi/2 + \theta_6$
Right	0	1	$\pi - \alpha$	0	0	$\pi - \theta_1(t)$
Robot	1	2	$-\beta$	0	0	$\theta_2(t)$
(2,4)	2	3	0	0	0	$\pi/2 + \pi + \theta_3(t)$
	3	4	$-\pi/2$	0	$d_4(t)$	0
	4	5	$-\pi/2$	$a_5$	0	$\pi/2 + \theta_5$
	5	6	$-\pi/2$	0	0	$\pi/2 -  heta_6$





## Direct Kinematics: Transform Matrix for Left Robot







# Direct Kinematics: Transform Matrix for Right Robot







### **Direct Kinematics: Solution**

$${}^{0}_{1}T = \begin{bmatrix} -c_{1} & s_{1}c\alpha & s_{1}s\alpha & 0\\ s_{1} & c_{1}c\alpha & c_{1}s\alpha & 0\\ 0 & s\alpha & -c\alpha & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}_{2}T = \begin{bmatrix} c_{2} & -s_{2}c\beta & -s_{2}s\beta & 0\\ s_{2} & c_{2}c\beta & c_{2}s\beta & 0\\ 0 & -s\beta & c\beta & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{2}_{3}T = \begin{bmatrix} s_{3} & c_{3} & 0 & 0\\ -c_{3} & s_{3} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{4}_{5}T = \begin{bmatrix} -s_{5} & 0 & -c_{5} & -a_{5}s_{5}\\ c_{5} & 0 & -s_{5} & a_{5}c_{5}\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{5}_{6}T = \begin{bmatrix} s_{6} & 0 & -c_{6} & 0\\ c_{6} & 0 & s_{6} & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}$$







- 6 DOFs for positioning and orienting  $\rightarrow$  Inverse Kinematics
- 1 DOF for the opening and closing of the grasper  $\rightarrow$  Redundancy
- Joint Limit Range

$\theta_i$	range	sin	cos
$\theta_1$	$[0^{\circ}, 90^{\circ}]$	+	+
$\theta_2$	$[20^{\circ}, 140^{\circ}]$	+	+/-
$\theta_3$	$[-86^{\circ}, 86^{\circ}]$	+/-	+
$d_4$	[-250, -0] mm	N/A	N/A
$\theta_5$	$[-86^{\circ}, 86^{\circ}]$	+/-	+
$\theta_6$	$[-86^{\circ}, 86^{\circ}]$	+/-	+





### Inverse Kinematics: Homogeneous Transformation Matrix and Its Inverse

Homogenous Transform Matrix → Inverse

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$${}^{0}_{6}T = {}^{0}_{1}T {}^{1}_{2}T {}^{2}_{3}T {}^{3}_{4}T {}^{5}_{5}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_{x} \\ r_{21} & r_{22} & r_{23} & P_{y} \\ r_{31} & r_{32} & r_{33} & P_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow {}^{6}_{0}T = [{}^{0}_{1}T {}^{1}_{2}T {}^{2}_{3}T {}^{3}_{4}T {}^{5}_{5}T]^{-1} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_{xinv} \\ r_{21} & r_{22} & r_{23} & P_{yinv} \\ r_{31} & r_{32} & r_{33} & P_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For the left robot,  $P_{xinv} = (-d_4c_5 + a_5)c_6$   $P_{yinv} = s_5d_4$   $P_{zinv} = (-d_4c_5 + a_5)s_6$ For the right robot • Define  $P_{2inv}^2 = P_{2inv}^2 + P_{2inv}^2 = (a_5 - d_4c_5)^2 c_6^2 + (a_5 - d_4c_5)s_6^2 + s_5^2 d_4^2$   $\Rightarrow$   $P_{2inv}^2 = (a_5 - d_4c_5)^2 + s_5^2 d_4^2 = a_5^2 - 2a_5d_4c_5 + d_4^2 c_5^2 + d_4^2 s_5^2$   $\Rightarrow$  $P_{2inv}^2 = a_5^2 - 2a_5d_4c_5 + d_4^2$
- For the right robot,

 $P_{xinv} = (d_4c_5 - a_5)c_6$  $P_{yinv} = s_5d_4$  $P_{zinv} = -(d_4c_5 - a_5)s_6$ 

• Which gives

$$c_5^2 = \left(\frac{a_5^2 + d_4^2 - P_{inv}^2}{2a_5 d_4}\right)^2$$



### **Inverse Kinematics**

• For the left robot,





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• Four Possible Solutions of  $d_4$ 

$$\begin{cases} d_4 = \sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = -\sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = \sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = -\sqrt{a_5^2 + P_{inv}^2 - 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \end{cases}$$

$$d_4 = -\sqrt{a_5^2 + P_{inv}^2 - 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}}$$

• Resolve 
$$\theta_6$$
  
For the left robot,  $s_6 = \frac{P_{zinv}}{(-c_5d_4 + a_5)}$   
For the right robot,  $s_6 = \frac{-P_{xinv}}{(-c_5d_4 + a_5)}$   
 $\theta_6 = A \tan 2(s_6, c_6)$   
• Resolve  $\theta_5$   
•  $s_6 = \frac{P_{yinv}}{d_4}$   
 $c_6 = \sqrt{1 - s_6^2}$   
 $\theta_6 = A \tan 2(s_6, c_6)$ 





### **Inverse Kinematics**

• With resolved 
$$d_4$$
,  $\theta_5$  and  $\theta_6$   
 ${}^{0}_{3}T = {}^{0}_{1}T {}^{1}_{2}T {}^{2}_{3}T = {}^{6}_{0}T [{}^{3}_{4}T {}^{4}_{5}T {}^{5}_{6}T]^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{x} \\ a_{21} & a_{22} & a_{23} & a_{y} \\ a_{31} & a_{32} & a_{33} & a_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$ 

• Where

$$a_{32} = s_2 s_\alpha c_3 + (c_2 s_\alpha c_\beta + c_\alpha s_\beta) s_3$$
$$a_{33} = c_2 s_\alpha c_\beta - c_\alpha s_\beta$$

• Resolve 
$$\theta_2$$

$$c_2 = \frac{c_{\alpha}s_{\beta} + a_{_{33}}}{s_{\alpha}c_{\beta}}$$
$$s_2 = \sqrt{1 - c_2^2}$$
$$\theta_2 = A \tan 2(s_2, c_2)$$

• **Define**  $a = s_{\alpha}$ 

 $b = c_2 s_\alpha c_\beta + c_\alpha s_\beta$ 

• We have

 $a_{32} = ac_3 + bs_3$ 

• According to [1]

$$\theta_3 = 2A \tan(\frac{b + \sqrt{a^2 + b^2 - a_{32}}}{a + a_{32}})$$





Check  $a_{13}$  to select between the two solution of  $heta_3$ 

For the left robot,  $a_{13} = -s_2 s_\alpha s_3 + c_2 s_\alpha c_3 c_\beta + s_\alpha c_3 s_\beta$ 

For the right robot,  $a_{13} = s_2 s_\alpha s_3 - c_2 s_\alpha c_3 c_\beta - s_\alpha c_3 s_\beta$ 

- With resolved  $\theta_5$  ,  $\theta_5$  ,  $d_4$  ,  $\theta_5$  and  $\theta_6$ 

$${}^{0}_{1}T = {}^{0}_{0}T[{}^{3}_{4}T {}^{4}_{5}T {}^{5}_{6}T]^{-1}[{}^{1}_{2}T {}^{2}_{3}T]^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{x} \\ b_{21} & b_{22} & b_{23} & b_{y} \\ b_{31} & b_{32} & b_{33} & b_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• Where

**For the left robot,**  $s_1 = b_{11}, c_1 = b_{21}$ 

**For the right robot,**  $s_1 = b_{11}, c_1 = -b_{21}$ 

$$\Rightarrow$$

 $\theta_1 = A \tan 2(s_1, c_1)$ 





- The mechanism isotropy is determined by the eigen-values of Jacobian matrix, which can be derived by velocity propagation
- General equations for velocity propagation:  $\overset{\bullet}{X} = J \Theta$

For the angular velocity,
$$i+1 \\ \omega_{i+1} = i+1 \\ i \\ R^i \\ \omega_i + \theta_{i+2} \\ Z_{i+1}$$
For the linear velocity, $i+1 \\ v_{i+1} = i+1 \\ i \\ R^i \\ \omega_i \\ P_{i+1} + i \\ v_i) + d_{i+2} \\ Z_{i+1}$ For the revolute joint, $\theta_{i+2} = 0$ For the prismatic joint, $d_{i+2} = 0$ 





Initial Condition

Link 1 is rotating at  $\dot{\theta}_1$  about  $z_0$ :  ${}^0\omega_0 = [0,0,\dot{\theta}_1]^T {}^0v_0 = [0,0,0]^T$ Link 2 is rotating at  $\dot{\theta}_2$  about  $z_1$ Link 3 is frozen with  $\dot{\theta}_3 = 0$ Translation in homogeneous transformation matrix:  ${}^0P_1 = {}^1P_2 = {}^2P_3 = [0,0,0]^T$ Link 4 is translating at  $\dot{d}_4$  along  $z_3$ 

• Rotation Matrices  ${}^{i+1}_{i}R = {}^{i}_{i+1}R^{T}$ , which leads to

For the left robot	${}_{0}^{1}R = \begin{bmatrix} c_{1} & s_{1} & 0\\ s_{1}c\alpha & -c_{1}c\alpha & s\alpha\\ s_{1}s\alpha & -c_{1}s\alpha & -c\alpha \end{bmatrix}$	${}_{1}^{2}R = \begin{bmatrix} c_{2} & -s_{2} & 0\\ s_{2}c\beta & c_{1}c\beta & -s\beta\\ s_{2}s\beta & c_{1}s\beta & c\beta \end{bmatrix}$	${}_{2}^{3}R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
For the right robot	${}_{0}^{1}R = \begin{bmatrix} -c_{1} & s_{1} & 0\\ s_{1}c\alpha & c_{1}c\alpha & s\alpha\\ s_{1}s\alpha & c_{1}s\alpha & -c\alpha \end{bmatrix}$	${}_{1}^{2}R = \begin{bmatrix} c_{2} & s_{2} & 0 \\ -s_{2}c\beta & c_{1}c\beta & -s\beta \\ -s_{2}s\beta & c_{1}s\beta & c\beta \end{bmatrix}$	${}_{2}^{3}R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$





• Angular velocity propagation

$${}^{1}\omega_{1} = {}^{1}_{0}R^{0}\omega_{0} + \dot{\theta}_{2}\dot{Z}_{1}$$
$${}^{2}\omega_{2} = {}^{2}_{1}R^{1}\omega_{1}$$
$${}^{3}\omega_{3} = {}^{3}_{2}R^{2}\omega_{2}$$

• Linear velocity propagation

$${}^{1}v_{1} = {}^{1}_{0}R({}^{0}\omega_{0} \times P_{1} + {}^{0}v_{0})$$
$${}^{2}v_{2} = {}^{2}_{1}R({}^{1}\omega_{1} \times P_{2} + {}^{1}v_{1})$$
$${}^{3}v_{3} = {}^{3}_{2}R({}^{2}\omega_{2} \times P_{3} + {}^{2}v_{2}) + \dot{d}_{4}\dot{Z}_{3}$$





• Hence, the velocity of the end-point of Link 3 is with reference to Frame 3 is



**Linear Velocity** 

For both the left robot and right robot

$$^{3}_{3}v = \begin{bmatrix} 0\\0\\ \cdot\\d_{4} \end{bmatrix}$$





# **Jacobian & Isotropy**

- Hence, the Jacobian Matrix is For the left robot  $\begin{bmatrix}
  3 & \omega_x \\
  3 & \omega_y \\
  3 & d_z
  \end{bmatrix} = {}^{3}J \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2 \\
  \dot{\theta}_4
  \end{bmatrix} = \begin{bmatrix}
  c_2 c \beta s \alpha + s \beta c \alpha & -s \beta & 0 \\
  s_2 s \alpha & 0 & 0 \\
  c_2 s \beta s \alpha - c \beta c \alpha & c \beta & 1
  \end{bmatrix} \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2 \\
  \dot{\theta}_4
  \end{bmatrix}$ For the right robot  $\begin{bmatrix}
  3 & \omega_x \\
  3 & \omega_y \\
  3 & d_z
  \end{bmatrix} = {}^{3}J \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2 \\
  \dot{\theta}_4
  \end{bmatrix} = \begin{bmatrix}
  -(c_2 c \beta s \alpha + s \beta c \alpha) & -s \beta & 0 \\
  s_2 s \alpha & 0 & 0 \\
  c_2 s \beta s \alpha - c \beta c \alpha & c \beta & 1
  \end{bmatrix} \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2 \\
  \dot{\theta}_4
  \end{bmatrix}$
- The mechanism isotropy only depends on the 2X2 sub-matrix to the left corner

$${}^{3}J_{s} = \begin{bmatrix} \pm (c_{2}c\beta s\alpha + s\beta c\alpha) & -s\beta \end{bmatrix}$$
$$s_{2}s\alpha & 0 \end{bmatrix}$$





- **Mechanism isotropy -** the end-effector's ability of moving in all direction given a specific manipulator configuration.
- Definition

$$Iso = \frac{\lambda_{\min}}{\lambda_{\max}}$$

Range

 $0 \le Iso \le 1$ 





• The eigen-values of the Jacobian matrix can be found by solving

$$\det({}^{3}J_{s}{}^{3}J_{s}^{T}-\lambda I_{2\times 2})=0$$

- Which gives  $\det({}^{3}J_{s}{}^{3}J_{s}^{T} - \lambda I_{2\times 2}) = \lambda^{2} - [(c_{2}c\beta s\alpha + s\beta c\alpha)^{2} + (s\beta)^{2}]\lambda - (c_{2}c\beta s\alpha + s\beta c\alpha)^{2}(s_{2}s\alpha)^{2}$
- Define  $B = (c_2 c \beta s \alpha + s \beta c \alpha)^2 + (s \beta)^2$  $C = -(c_2 c \beta s \alpha + s \beta c \alpha)^2 (s_2 s \alpha)^2$

$$Iso = \frac{\lambda_{\min}}{\lambda_{\max}} = \frac{B - \sqrt{B^2 - 4C}}{B + \sqrt{B^2 - 4C}} = 1 - \frac{2\sqrt{B^2 - 4C}}{B + \sqrt{B^2 - 4C}}$$









## Optimization of Raven IV – Problem & Parameters (7) Definitions





#### Cost Function

- Geometry Largest circular common workspace (Area Circumference Ratio)
- Manipulations Best Isotropy
  - Across the common workspace
  - Worst case value (min/max problem)
- Mechanics Stiff mechanism (Smallest Mechanism)

$$C = \max_{(\alpha,\beta,\phi_x,\phi_y,\phi_z,b_x,b_y)} \left\{ \begin{array}{c} \varsigma \cdot \sum Iso \cdot Iso_{\min} \\ \alpha^3 + \beta^3 \end{array} \right\}$$

- Method
  - Brute force search across all the free parameters





### **Common Workspace – Reference Plane**







• Definition

 $\varsigma = \frac{Area}{Circumference}$ 

• According to the **Isoperimetric Inequality**, a circle has the largest possible area among all the figures with the length of boundary

$$\varsigma_c = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$$





# **Effect of Limiting Minimum Isotropy Performance**



Fig. 13. Four Raven Arms: Distribution of  $\varsigma$  for different  $\alpha$  and  $\beta$ ,  $Iso_{min} = 0.2$ .





• Workspace propagation – Minimum Mechanism Isotropy = 0.2





# Optimization of Raven IV surgical System Overall simulation result

• Parameter ranges, resolutions and optimal values

	Range	Optimal Value	Resolution	
$\alpha$	$[5^{\circ}, 90^{\circ}]$	85°	20°	
$\beta$	$[5^{\circ}, 90^{\circ}]$	65°	20°	
$\phi_x$	$[-20^{\circ}, 20^{\circ}]$	20°	10°	
$\phi_y$	$[-20^{\circ}, 20^{\circ}]$	10°	10°	
$\phi_z$	$[-20^{\circ}, 20^{\circ}]$	$-20^{\circ}$	10°	
$b_x$	[50, 200]  (mm)	100 (mm)	50 (mm)	
$b_y$	[50, 200]  (mm)	50 (mm)	50 (mm)	
Isomin	[0.1, 0.9]	0.5	0.2	
Result	$C_{max} = 526.3$ for $Iso_{min} = 0.5$			



# **Optimization of Raven IV - Conclusion**











 $\dot{X} = J\dot{\theta}$  is a linear mapping of the joint space velocities  $\dot{\theta}$  which is a n - dimensional vector space  $\dot{\theta} \in \Re^n$  to the end effector velocities  $\vec{X}$  which is a m – dimensional vector space  $\dot{X} \in \Re^m$ 







The subset of all the end effector velocities  $\dot{X}$  resulting from the mapping  $\dot{X} = J\dot{\theta}$  represents all the possible end effector velocities that can be generated by the n joints given the arm configuration







If the rank of the Jacobian matrix J is at full of row rank (square matrix) the joint space  $\dot{\theta}$ covers the entire end effector vector  $\mathbf{x}$ otherwise there is at least one direction in which the end effector can not be moved •









The subset N(J) is the null space of the linear mapping. Any element in this subspace is mapped into a zero vector in  $\Re^m$  such that  $\dot{X} = J\dot{\theta} = 0$  therefore any joint velocity vector  $\dot{\theta}$  that belongs to the null space does not produce any velocity at the end effector







If the Jacobian of a manipulator is full rank the dimension of the null space  $\underline{\dim(N(J))}$  is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the hand may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)





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If the Jacobian of a manipulator is full rank (i.e. for redundant manipulator n>m full row rank where the rows are linearly independent) the dimension of the null space  $\dim(N(J))$  is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the end effector (hand) may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)





When the Jacobian matrix degenerates (i.e. not full rank e.g. due to singularity) the dimension of the range space  $\dim(R(J))$  decreases at the same time as the dimension of the null space increases  $\dim(N(J))$  by the same amount. The sum of the two is always equal to n

 $\dim(R(J)) + \dim(N(J)) = n$ 







If the null space is not empty set, the instantaneous kinematic equation has an infinite number of solutions that cause the same end effector velocities (recall the 3 axis end effector)



















endpoint force is borne entirely by the structure and not by the joint torque.

