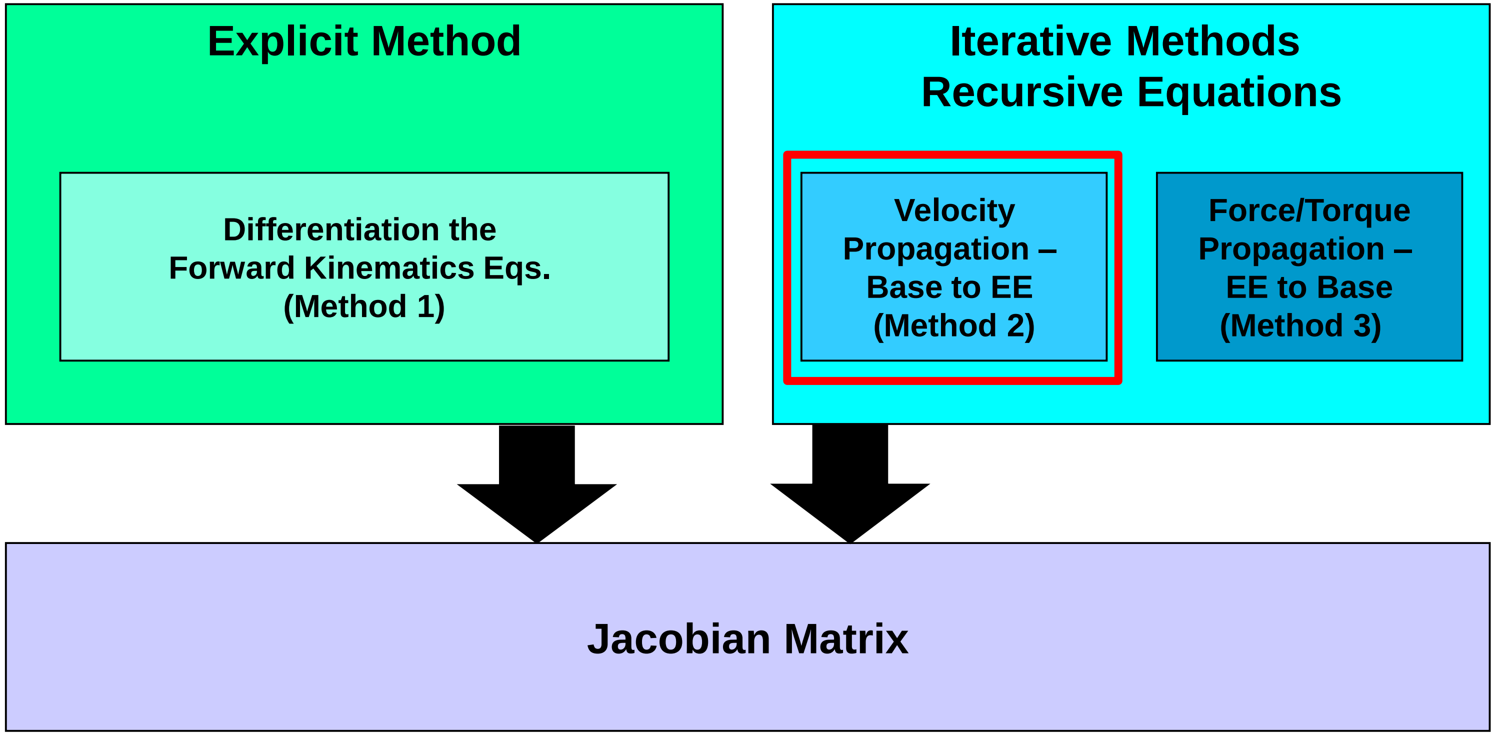
Jacobian Iterative Method - Velocity Propagation (Method No. 2) Part 2 – Reference Frame

Jacobian Matrix - Derivation Methods



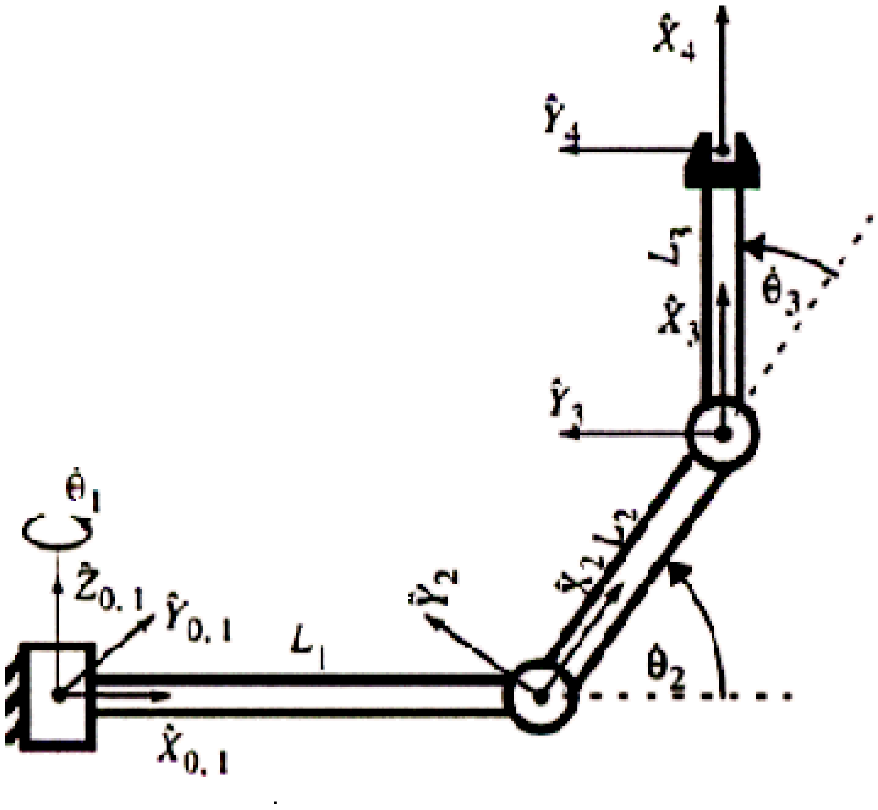
Jacobian: Velocity propagation

- The recursive expressions for the adjacent joint linear and angular velocities describe a relationship between the joint angle rates ($\dot{\theta}$) and the transnational and rotational velocities of the end effector ($\dot{X}$):

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_{i}\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$

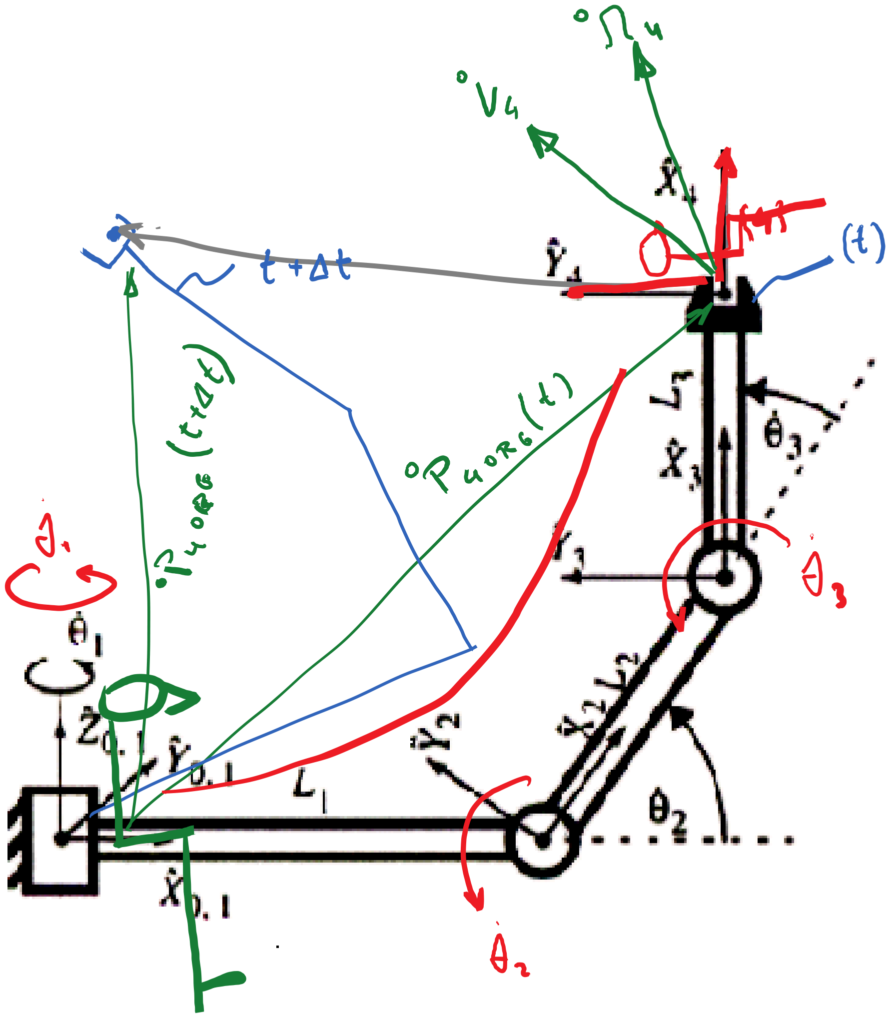
Angular and Linear Velocities - 3R Robot - Example



$$
{}^4v_4 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ -(L1 + L2c2 + L3c23)\dot{\theta}_1 \end{bmatrix}
$$

$$
{}^4\omega_4 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$

Angular and Linear Velocities - 3R Robot - Example



$$
{}^4v_4 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ -(L1 + L2c2 + L3c23)\dot{\theta}_1 \end{bmatrix}
$$

$$
{}^4\omega_4 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$

Diff in frame {0}

$\left. \begin{matrix} {}^0V_4 \\ {}^0\Omega_4 \end{matrix} \right\}$

$^4({}^0V_4) = {}^4v_4$

$^4({}^0\Omega_4) = {}^4\omega_4$

$^0({}^0V_4) = {}^0v_4$

$^0({}^0\Omega_4) = {}^0\omega_4$

Jacobian Expression
Frame of Reference

Frame Notation

Jacobian: Velocity propagation

- The recursive expressions for the adjacent joint linear and angular velocities defines the Jacobian in the end effector frame

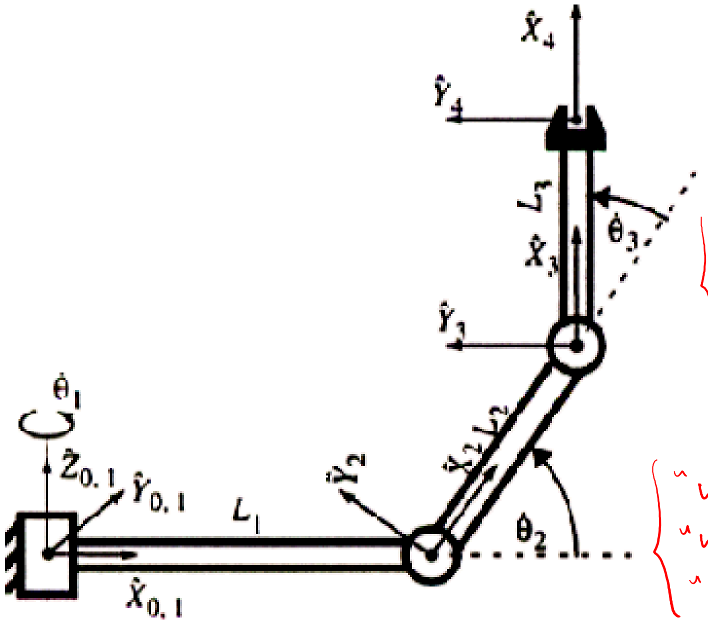
$$ {}^{N}\dot{X} = {}^{N}J(\theta)\dot{\theta} $$

- This equation can be expanded to:

$$ {}^{N}\dot{X} = {}^{N}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = \begin{bmatrix} {}^{N}v_{N} \\ {}^{N}\omega_{N} \end{bmatrix} = {}^{N}\begin{bmatrix} & J(\theta) & \end{bmatrix} \begin{bmatrix} \dot{\theta}1 \\ \dot{\theta}2 \\ \ldots \\ \dot{\theta}_{N} \end{bmatrix} $$

Jacobian - 3R - Example

- The linear angular velocities of the end effector (N=4)



$$
{}^4v_4 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \end{bmatrix}
$$

$$
{}^4\omega_4 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$

Jacobian - 3R - Example

- Re-arranged to previous two terms gives an expression that encapsulates

$$ {}^{4}\dot{X} = {}^{4}J(\theta)\dot{\theta} $$

$$ {}^{4}\dot{X} = {}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} {}^{4}v_4 \\ {}^{4}\omega_4 \end{bmatrix} = \overbrace{\begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \\ s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}}^{{}^{4}J(\theta)\dot{\theta}} $$

- We can now factor out the joint velocities vector $\dot{\theta} = [\dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3]^T$ from the above vector to formulate the Jacobian matrix

$$ {}^{4}J(\theta) $$

Jacobian - 3R - Example

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \overbrace{{}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}^{{}^{4}J(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

$$
{}^{4}\dot{X} = {}^{4}J(\theta)\dot{\theta}
$$

- The equations for ${}^{N}v_N$ and ${}^{N}\omega_N$ are always a linear combination of the joint velocities, so they can always be used to find the 6xN Jacobian matrix (${}^{N}J(\theta)$) for any robot manipulator.

- Note that the Jacobian matrix is expressed in frame {4}

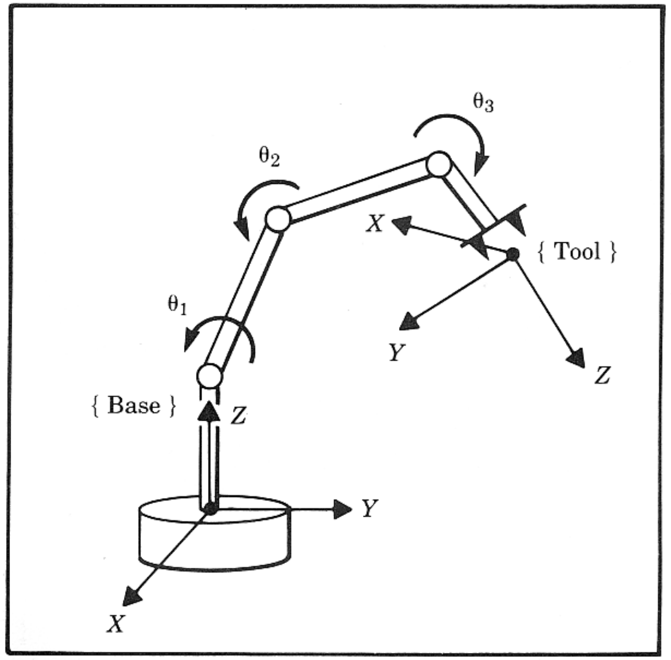
Jacobian: Frame of Representation

- Using the velocity propagation method we expressed the relationship between the velocity of the robot end effector measured relative to the robot base frame {0} and expressed in the end effector frame {N}.

$$ {}^{N}\dot{X} = {}^{N}J(\theta)\dot{\theta} $$

- Occasionally, it may be desirable to express (represent) the end effector velocities in another frame (e.g. frame {0}, in which case we will need a method to provide the transformation.

$$ {}^{0}\dot{X} = {}^{0}J(\theta)\dot{\theta} $$



Jacobian: Frame of Representation

- There are two methods to change the references frame (frame of representation) of the Jacobian Matrix
  - Method 1: Transforming the linear and angular velocities to the new frame prior to formulating the Jabobian matrix.
  - Method 2: Transforming the Jacobian matrix from it existing frame to the new frame after it was formulated.

Jacobian Expression
Frame of Reference

Method No. 1
Transform the Velocity Vectors

Jacobian: Frame of Representation – Method 1

- Consider the velocities in a different frame {B}

$$ {}^{B}\dot{X} = \begin{bmatrix} {}^{B}v_N \\ {}^{B}\omega_N \end{bmatrix} = {}^{B}J(\theta)\dot{\theta} $$

- We may use the rotation matrix to find the velocities in frame {A}:

$$ {}^{A}\dot{X} = \begin{bmatrix} {}^{A}v_N \\ {}^{A}\omega_N \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}v_N \\ {}^{A}_{B}R\,{}^{B}\omega_N \end{bmatrix} $$

Jacobian: Frame of Representation – Method 1

- Example: Analyzing a 6 DOF manipulator while utilizing velocity propagation method results in an expressing the end effector (frame 6) linear and angular velocities.

$$ {}^{6}\dot{X} = \begin{bmatrix} {}^{6}v_{6} \\ {}^{6}\omega_{6} \end{bmatrix} $$

- Using the forward kinematics formulation the rotation matrix from frame 0 to frame 6 can be defined as

$$ {}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T = \begin{bmatrix} & {}^{0}_{6}R & & {}^{0}P_{6ORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- The linear and angular velocities can than be expressed in frame 0 prior to extracting the Jacobian in frame 0

$$ {}^{0}\dot{X} = \begin{bmatrix} {}^{0}v_{6} \\ {}^{0}\omega_{6} \end{bmatrix} = \begin{bmatrix} {}^{0}_{6}R\,{}^{6}v_{6} \\ {}^{0}_{6}R\,{}^{6}\omega_{6} \end{bmatrix} = {}^{0}J\dot{\theta} $$

Jacobian Expression
Frame of Reference

Method No. 2
Transform the Jacobian Matrix

Jacobian: Frame of Representation – Method 2

- It is possible to define a Jacobian transformation matrix ${}^{A}_{B}R_J$ that can transform the Jacobian from frame A to frame B

$$ {}^{A}\dot{X} = {}^{A}J(\theta)\dot{\theta} = {}^{A}_{B}R_J \, {}^{B}J(\theta)\dot{\theta} $$

- The Jacobian rotation matrix ${}^{A}_{B}R_J$ is given by

$$ {}^{A}_{B}R_J = \begin{bmatrix} \left[{}^{A}_{B}R\right] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \left[{}^{A}_{B}R\right] \end{bmatrix} $$

Jacobian: Frame of Representation

- or equivalently,

$$
{}^{A}J(\theta) = \begin{bmatrix} \left[{}^{A}_{B}R\right] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \left[{}^{A}_{B}R\right] \end{bmatrix} {}^{B}J(\theta)
$$

Jacobian: Frame of Representation - 3R Example

$$
{}^{0}J(\theta)=\begin{bmatrix}\begin{bmatrix}{}^{0}_{4}R\end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix}{}^{0}_{4}R\end{bmatrix}\end{bmatrix}{}^{4}J(\theta)=\begin{bmatrix}\begin{bmatrix}c1c23 & -c1s23 & s1\\ s1c23 & -s1s23 & -c1\\ s23 & c23 & 0\end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix}c1c23 & -c1s23 & s1\\ s1c23 & -s1s23 & -c1\\ s23 & c23 & 0\end{bmatrix}\end{bmatrix}{}^{4}\begin{bmatrix}0 & L2s3 & 0\\ 0 & L2c3+L3 & L3\\ -L1-L2c2-L3c23 & 0 & 0\\ s23 & 0 & 0\\ c23 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}
$$

- The rotation matrix (${}^{0}_{4}R$) can be calculated base on the direct kinematics given by

$$
{}^{0}_{4}T={}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T=\begin{bmatrix} & {}^{0}_{4}R & & {}^{0}P_{4ORG}\\ 0 & 0 & 0 & 1\end{bmatrix}
$$

Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

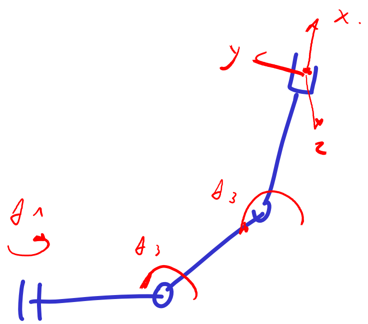
Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)
Explicit (Diff. the Forward Kinematic Eq.)	${}^{0}J_{N}$	None
Iterative Velocity Eq.	${}^{N}J_{N}$	Transform Method 1: ${}^{0}v_{N} = {}^{0}_{N}R\,{}^{N}v_{N}$; ${}^{0}\omega_{N} = {}^{0}_{N}R\,{}^{N}\omega_{N}$ Transform Method 2: ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$
Iterative Force Eq.	${}^{N}J_{N}^{T}$	Transpose: ${}^{N}J_{N} = [{}^{N}J_{N}^{T}]^{T}$ Transform: ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$

Inverse Jacobian - Reduced Jacobian

- ***Problem***
  - When the number of joints (N) is less than 6, the manipulator does not have the necessary degrees of freedom to achieve independent control of all six velocities components.

- ***Solution***
  - We can reduce the number of rows in the original Jacobian to describe a reduced Cartesian vector. For example, the full Cartesian velocity vector is given by

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix}$$



Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 1

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Column of zeroes
- The determinate is equal to zero
- Only two out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 2

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Two columns of zeroes
- The determinate is equal to zero
- Only one out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 3

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- The resulting reduced Jacobian will be square (the number of independent rows in the Jacobian are equal to the number of unknown variables) and can be inverted unless in a singular configuration.

Jacobian: Singular Configuration - 3R Example

$$
{}^{4}J_r(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}
$$