Jacobian Iterative Method - Velocity Propagation (Method No. 2) Part 1 – Method Derivation

Jacobian Matrix - Derivation Methods

Explicit Method

Differentiation the
Forward Kinematics Eqs.
(Method 1)

Iterative Methods
Recursive Equations

Velocity
Propagation –
Base to EE
(Method 2)

Force/Torque
Propagation –
EE to Base
(Method 3)

Jacobian Matrix

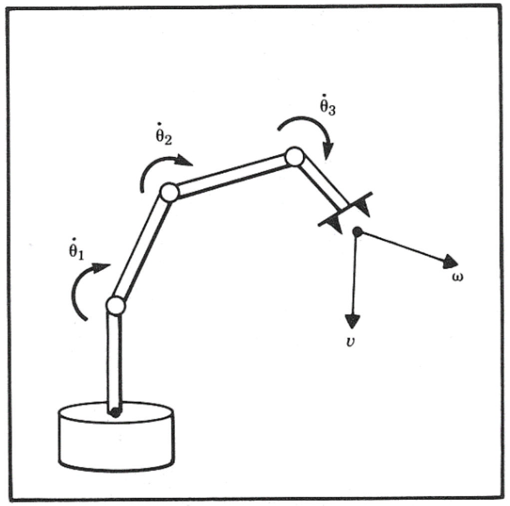
Jacobian Matrix - Introduction

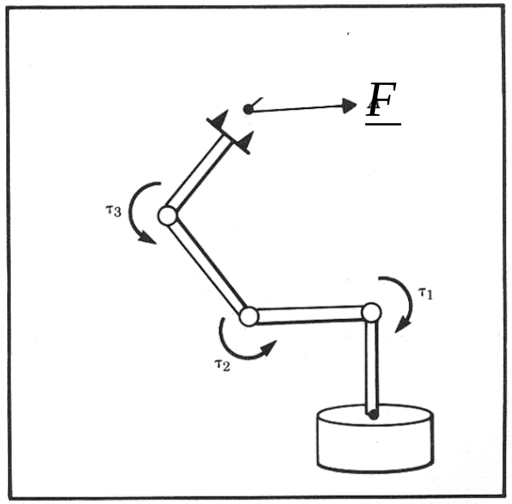
- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} \qquad \underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} \qquad \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_6 \end{Bmatrix}$$

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$





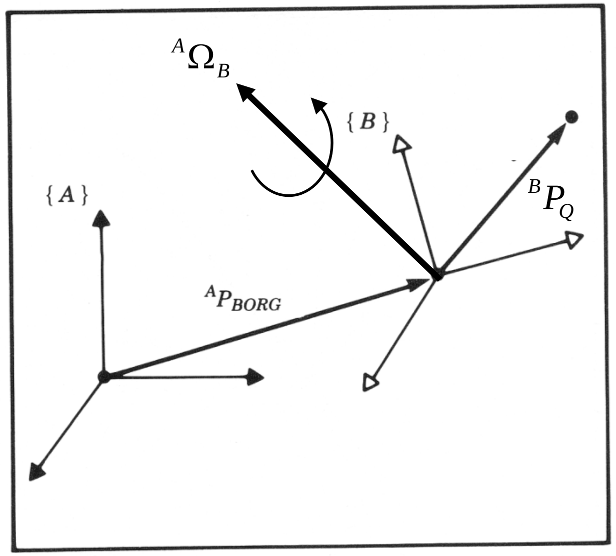
Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q $$

  - Matrix Form

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_\Omega \left({}^{A}_{B}R\,{}^{B}P_Q \right) $$



- Angular Velocity
  - Vector Form

$$ {}^{A}\Omega_C = {}^{A}\Omega_B + {}^{A}_{B}R\,{}^{B}\Omega_C $$

  - Matrix Form

$$ {}^{A}_{C}\dot{R}_\Omega = {}^{A}_{B}\dot{R}_\Omega + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_\Omega\,{}^{A}_{B}R^T $$

Velocity of Adjacent Links - Summary

- Angular Velocity

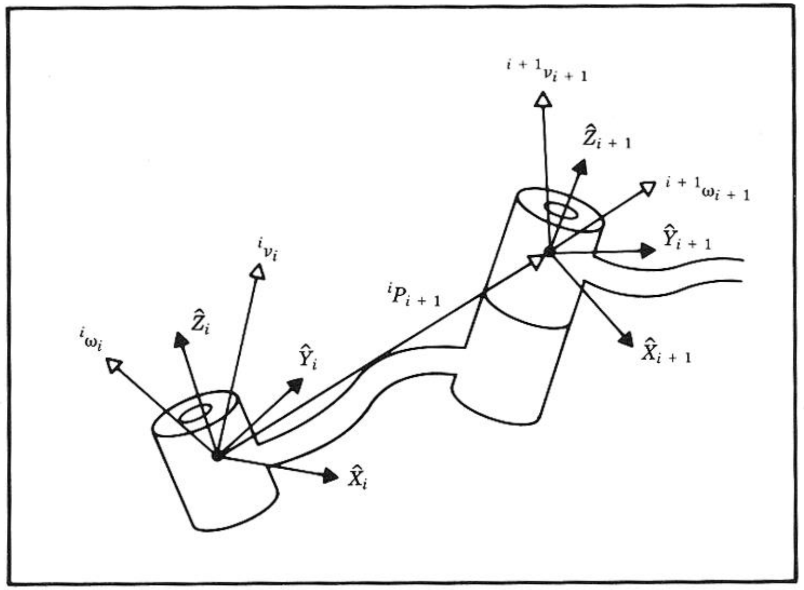
0 - Prismatic Joint

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;\,i}R\,{}^{i}\omega_{i} + \cancel{\begin{bmatrix}0\\0\\\dot{\theta}_{i+1}\end{bmatrix}}}$$

- Linear Velocity

0 - Revolute Joint

$$\boxed{{}^{i+1}v_{i+1} = {}^{i+1}_{\;\;\,i}R\left({}^{i}\omega\times{}^{i}P_{i+1} + {}^{i}v_{i}\right) + \cancel{\begin{bmatrix}0\\0\\\dot{d}_{i+1}\end{bmatrix}}}$$



Representation / Reference Frame
Computed / Measured Frame

Frame Notation

Frame - Velocity

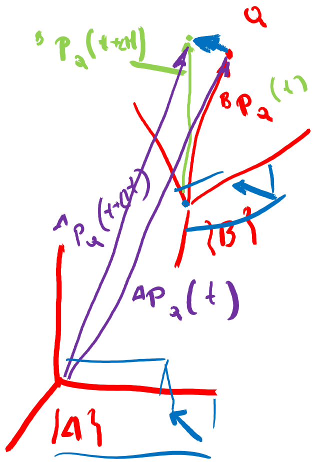
- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.

- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

Represented (Reference Frame)

$${}^{A}\left({}^{B}V_{Q}\right) = \frac{{}^{A}d}{dt}\,{}^{B}P_{Q}$$

Computed (Measured)



Position Propagation

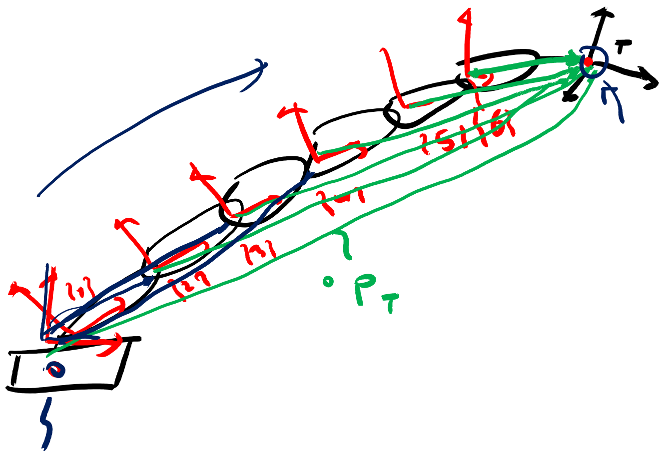
- The homogeneous transform matrix provides a complete description of the linear and angular position relationship between adjacent links.

- These descriptions may be combined together to describe the position of a link relative to the robot base frame {0}.

$$ {}^{o}_{i}T = {}^{o}_{1}T \, {}^{1}_{2}T \cdots {}^{i-1}_{i}T $$

- A similar description of the linear and angular velocities between adjacent links as well as the base frame would also be useful.

Position Propagation



${}^4P_T$

${}^5P_T$

${}^6P_T$

$$ {}^0_T T = {}^0_1 T \; {}^1_2 T \; {}^2_3 T \; {}^3_4 T \; {}^4_5 T \; {}^5_6 T \; {}^6_T T $$

Motion of the Link of a Robot

- In considering the motion of a robot link we will always use link frame {0} as the reference frame (Computed AND Represented). However any frame can be used as the reference (represented) frame including the link's own frame (i)

Where: v_i - is the linear velocity of the origin of link frame (i) with respect to frame {0} (Computed AND Represented)

ω_i - is the angular velocity of the origin of link frame (i) with respect to frame {0} (Computed AND Represented)

- Expressing the velocity of a frame {i} (associated with link i) relative to the robot base (frame {0}) using our previous notation is defined as follows:

$$v_i \equiv {}^0\left[{}^0V_i\right] = \left[{}^0V_i\right]$$

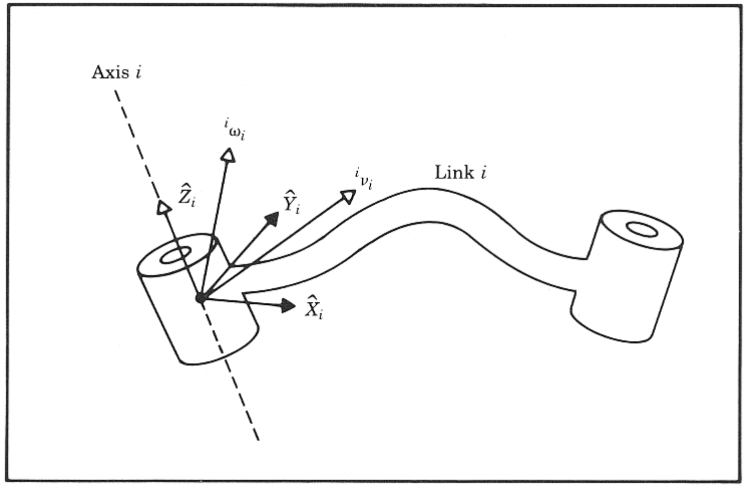
$$\omega_i \equiv {}^0\left[{}^0\Omega_i\right] = \left[{}^0\Omega_i\right]$$

Velocities - Frame & Notation

- The velocities differentiate (computed) relative to the base frame $\{0\}$ are often represented relative to other frames $\{k\}$. The following notation is used for this conditions

$$ {}^{k}v_i \equiv {}^{k}\left[{}^{0}V_i\right] = {}^{k}_{0}R\left[{}^{0}V_i\right] = {}^{k}_{0}R \cdot v_i $$

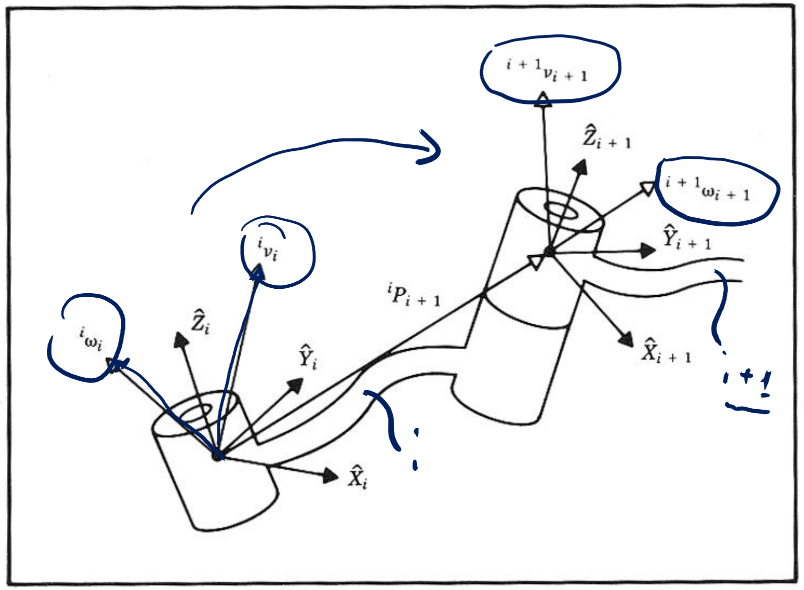
$$ {}^{k}\omega_i \equiv {}^{k}\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R \cdot \omega_i $$



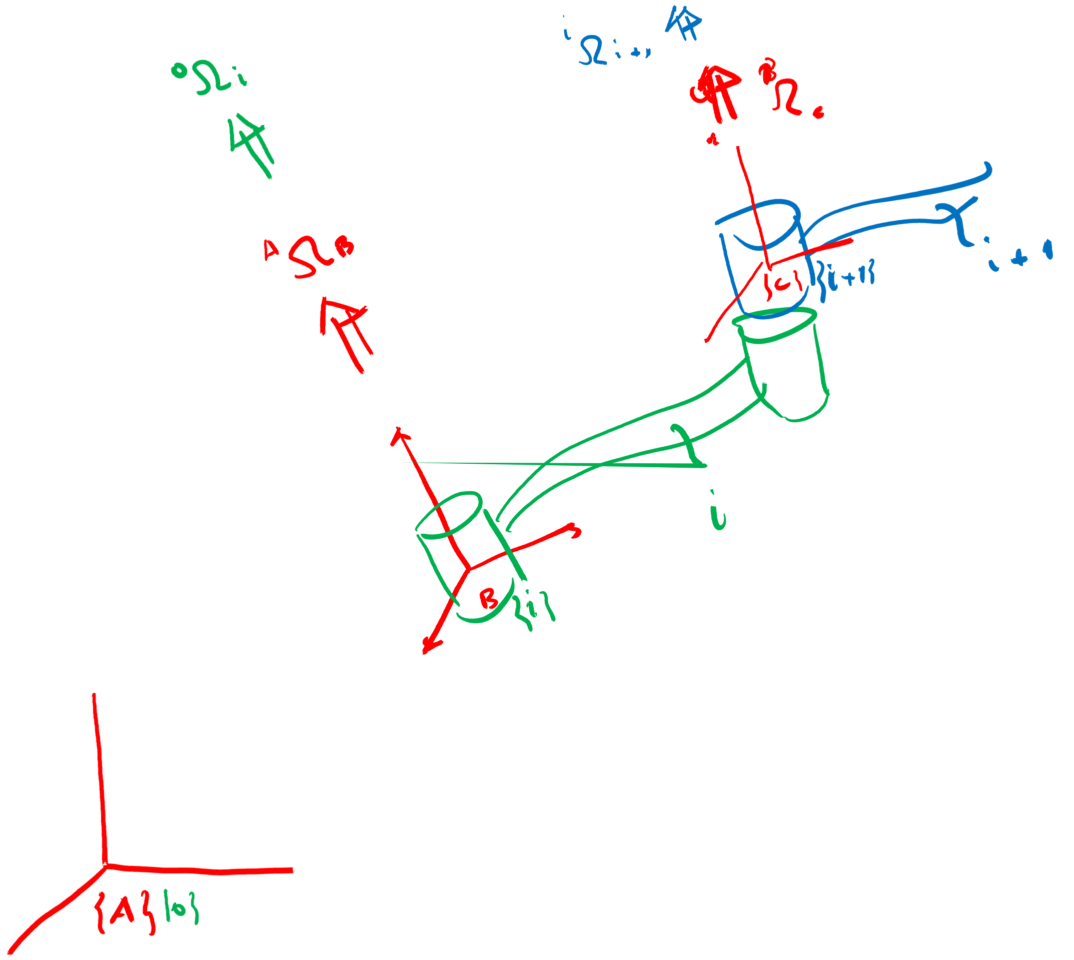
Velocity Propagation

- ***Given:*** A manipulator - A chain of rigid bodies each one capable of moving relative to its neighbor
- ***Problem:*** Calculate the linear and angular velocities of the link of a robot
- ***Solution (Concept):*** Due to the robot structure (serial mechanism) we can compute the velocities of each link in order starting from the base.

The velocity of link i+1 will be that of link i , plus whatever new velocity components were added by joint i+1



Velocity of Adjacent Links - Angular Velocity 0/5



$$\left\{\begin{array}{l} A \to 0 \\ B \to i \\ C \to i+1 \end{array}\right.$$

Angular Velocity Propagation

Velocity of Adjacent Links - Angular Velocity 1/5

- From the relationship developed previously

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

- we can re-assign link names to calculate the velocity of any link i relative to the base frame {0}

$$ \begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases} $$

$$ {}^{0}\Omega_{i+1} = {}^{0}\Omega_{i} + {}^{0}_{i}R\,{}^{i}\Omega_{i+1} $$

- By pre-multiplying both sides of the equation by ${}^{i+1}_{0}R$, we can convert the frame of reference for the base {0} to frame {i+1}

Velocity of Adjacent Links - Angular Velocity 2/5

$${}^{i+1}_{0}R\,{}^{0}\Omega_{i+1} = {}^{i+1}_{0}R\,{}^{0}\Omega_{i} + {}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}\Omega_{i+1}$$

- Using the recently defined notation, we have

$${}^{i+1}\omega_{i+1} = {}^{i+1}\omega_{i} + {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}$$

${}^{i+1}\omega_{i+1}$ - Angular velocity of frame {i+1} measured relative to the robot base, and expressed in frame {i+1} - **Recall the car example** ${}^{c}\left[{}^{w}V_{c}\right] = {}^{c}v_{c}$

${}^{i+1}\omega_{i}$ - Angular velocity of frame {i} measured relative to the robot base, and expressed in frame {i+1}

${}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}$ - Angular velocity of frame {$i+1$} measured relative to frame {i} and expressed in frame {i+1}

Velocity of Adjacent Links - Angular Velocity 3/5

$$^{i+1}\omega_{i+1} = \boxed{^{i+1}\omega_i} + {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}$$

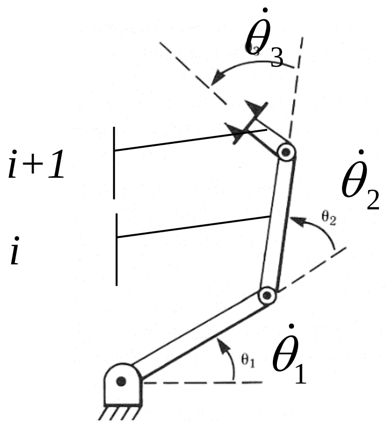
- Angular velocity of frame $\{i\}$ measured relative to the robot base, expressed in frame $\{i+1\}$

$$^{i+1}\omega_i = {}^{i+1}_{i}R\,{}^{i}\omega_i$$

Velocity of Adjacent Links - Angular Velocity 4/5

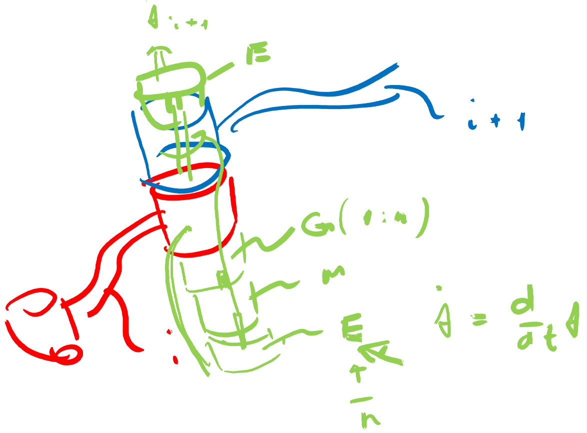
$$ {}^{i+1}\omega_{i+1} = {}^{i+1}\omega_i + \boxed{{}^{i+1}_{i}R\,{}^{i}\Omega_{i+1}} $$

- Angular velocity of frame $\{i+1\}$ measured (differentiate) in frame $\{i\}$ and represented (expressed) in frame $\{i+1\}$
- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (***angular velocity***) or prismatic joint (Linear velocity).
- Based on the frame attachment convention in which we assign the Z axis pointing along the $i+1$ joint axis such that the two are coincide (rotations of a link is preformed only along its Z- axis) we can rewrite this term as follows:



$$ {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

E



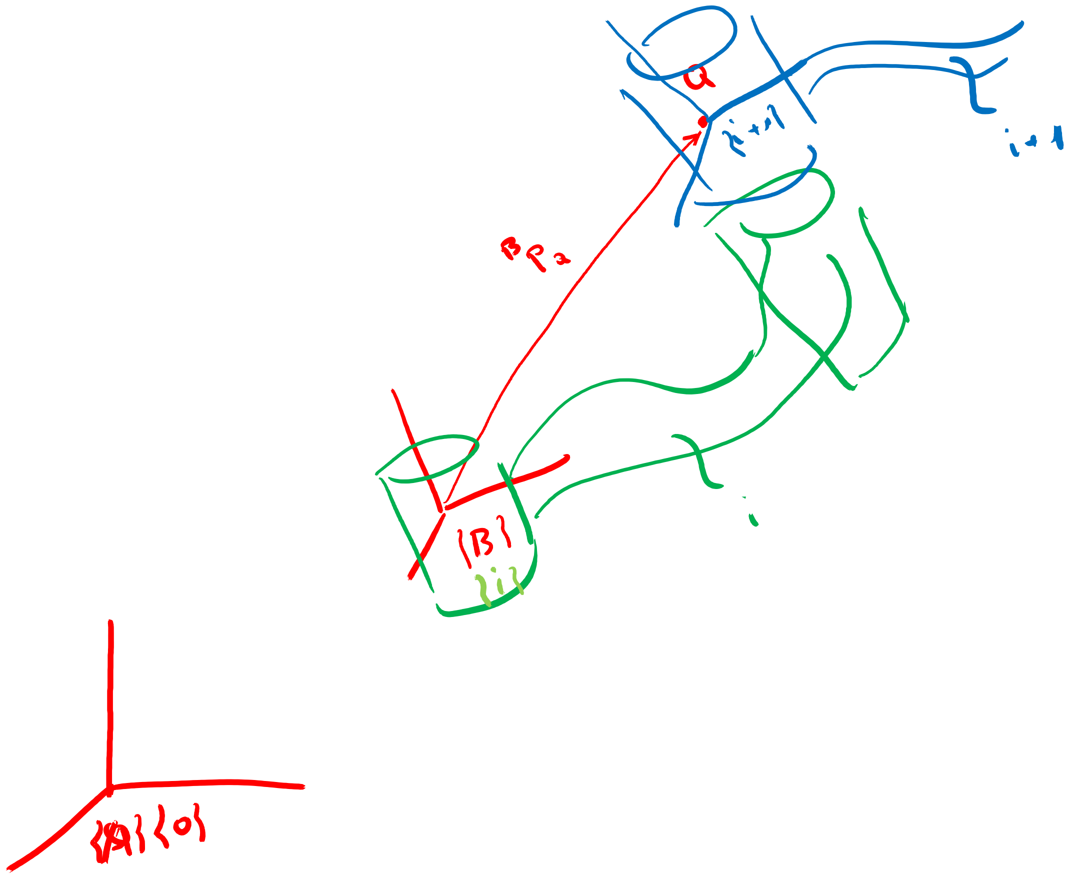
Velocity of Adjacent Links - Angular Velocity 5/5

- The result is a ***recursive equation*** that shows the angular velocity of one link in terms of the angular velocity of the previous link plus the relative motion of the two links.

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}}$$

- Since the term ${}^{i+1}\omega_{i+1}$ depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.

Velocity of Adjacent Links - Linear Velocity 0/6



$$\begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases}$$

Linear Velocity Propagation

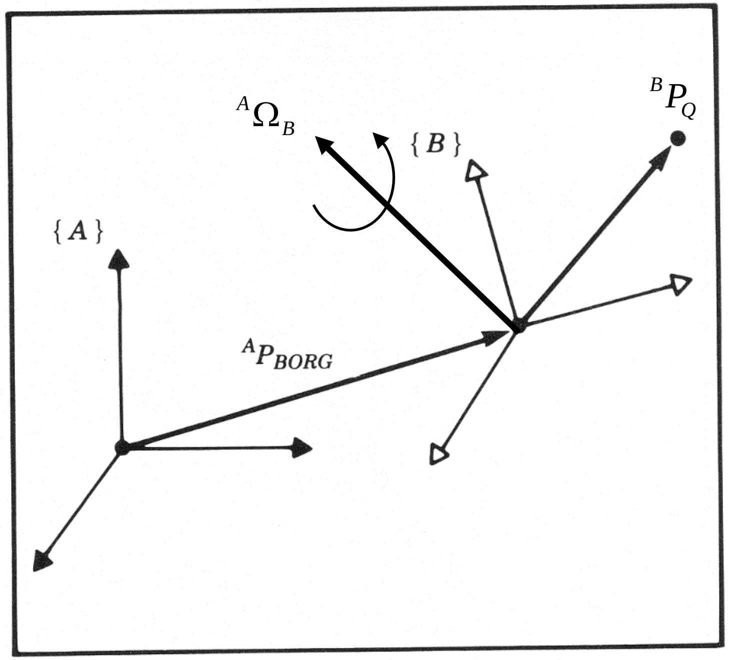
Velocity of Adjacent Links - Linear Velocity 1/6

- Simultaneous Linear and Rotational Velocity
- The derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame
- Vector Form

$$\boxed{{}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}}$$

- Matrix Form

$$\boxed{{}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)}$$



Velocity of Adjacent Links - Linear Velocity 2/6

- From the relationship developed previously (matrix form)

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

- we re-assign link frames for adjacent links (***i*** and ***i*** +1) with the velocity computed relative to the robot base frame {0}

$$ \begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases} $$

$$ {}^{0}V_{i+1} = {}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{0}V_i + {}^{0}_{i}R\,{}^{i}V_{i+1} $$

- By pre-multiplying both sides of the equation by ${}^{i+1}_{0}R$,we can convert the frame of reference for the left side to frame {*i*+1}

Velocity of Adjacent Links - Linear Velocity 3/6

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = {}^{i+1}_{0}R\,{}^{0}_{i}R\,\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{i+1}_{0}R\,{}^{0}V_{i} + \boxed{{}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}V_{i+1}} $$

- Which simplifies to

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = \boxed{{}^{i+1}_{0}R\,{}^{0}_{i}R\,\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{i+1}_{0}R\,{}^{0}V_{i}} + \boxed{{}^{i+1}_{i}R\,{}^{i}V_{i+1}} $$

- Factoring out ${}^{i+1}_{i}R$ from the left side of the first two terms

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = \boxed{{}^{i+1}_{i}R\left({}^{i}_{0}R\,{}^{0}_{i}R\,\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i}_{0}R\,{}^{0}V_{i}\right)} + {}^{i+1}_{i}R\,{}^{i}V_{i+1} $$

Velocity of Adjacent Links - Linear Velocity 4/6

$$
{}^{i+1}_{0}R\,{}^{0}V_{i+1} = {}^{i+1}_{i}R\left({}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i}_{0}R\,{}^{0}V_{i}\right) + \boxed{{}^{i+1}_{i}R\,{}^{i}V_{i+1}}
$$

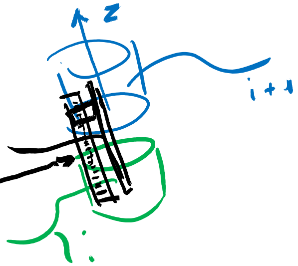
${}^{i+1}_{i}R\,{}^{i}V_{i+1}$ - Linear velocity of frame $\{i+1\}$ measured relative to frame $\{i\}$ and expressed in frame $\{i+1\}$

- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (angular velocity) or prismatic joint (Linear velocity).

- Based on the frame attachment convention in which we assign the Z axis pointing along the $i+1$ joint axis such that the two are coincide (translation of a link is preformed only along its Z- axis) we can rewrite this term as follows:

$$
{}^{i+1}_{i}R\,{}^{i}V_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}
$$

P



Velocity of Adjacent Links - Linear Velocity 5/6

$$\boxed{{}^{i+1}_{0}R\,{}^{0}V_{i+1}} = {}^{i+1}_{i}R\left(\boxed{{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R}\,{}^{i}P_{i+1} + \boxed{{}^{i}_{0}R\,{}^{0}V_{i}}\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

$$\boxed{{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R} = {}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{i}_{0}R^{T} = {}^{i}_{0}R\,{}^{0}\Omega_{i} = {}^{i}_{0}R\,\omega_{i} = {}^{i}\omega_{i}$$

Multiply by Matrix

Definition

$$\boxed{{}^{i+1}_{0}R\,{}^{0}V_{i+1}} = {}^{i+1}v_{i+1} \qquad \boxed{{}^{i}_{0}R\,{}^{0}V_{i}} = {}^{i}v_{i}$$

Definition

Definition

Angular Velocity - Matrix & Vector Forms

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}\Omega_B} \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}_{B}\dot{R}_{\Omega}} \left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

	Matrix Form	Vector Form
Definition	$ {}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} $	$ {}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} $
Multiply by Constant	$ k\left[{}^{A}_{B}\dot{R}_{\Omega}\right] $	$ k\left[{}^{A}\Omega_B\right] $
Multiply by Vector	$ \left[{}^{A}_{B}\dot{R}_{\Omega}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} $	$ {}^{A}\Omega_B \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} $ $[W \times r]$
Multiply by Matrix	$ \left[{}^{s}_{t}R\right]\left[{}^{A}_{B}\dot{R}_{\Omega}\right]\left[{}^{s}_{t}R\right]^T $	$ \left[{}^{s}_{t}R\right]\left[{}^{A}\Omega_B\right] $

Velocity of Adjacent Links - Linear Velocity 6/6

- The result is a ***recursive equation*** that shows the linear velocity of one link in terms of the previous link plus the relative motion of the two links.

$$\boxed{{}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}}$$

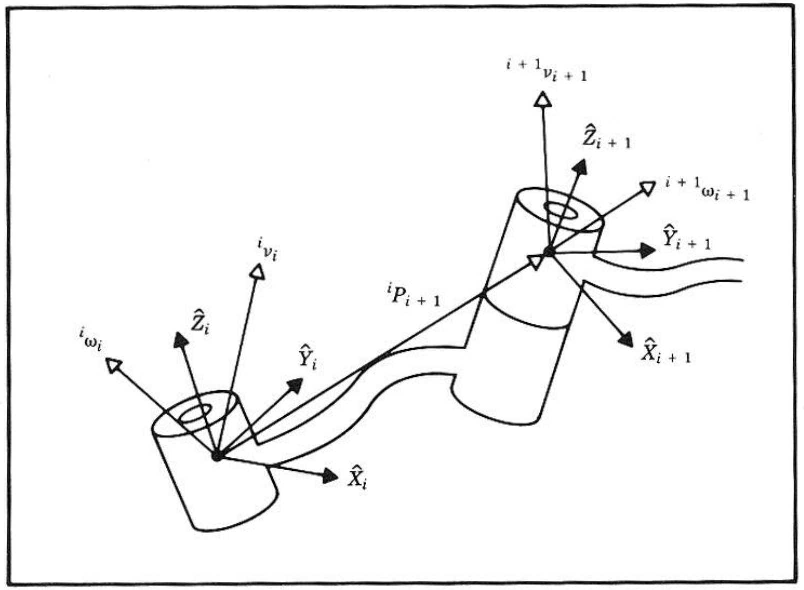
- Since the term ${}^{i+1}v_{i+1}$ depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.

Velocity of Adjacent Links - Summary

- Angular Velocity

0 - Prismatic Joint

$$
{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}
$$



- Linear Velocity

0 - Revolute Joint

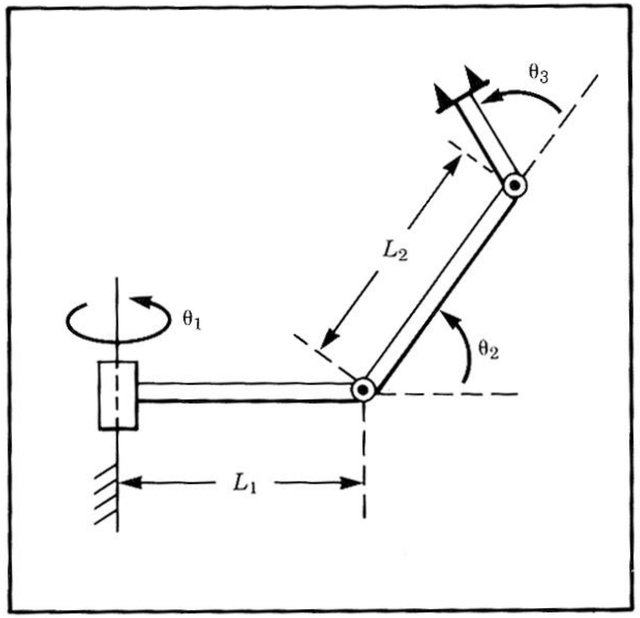
$$
{}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}
$$

3R – Example

Analytical Approach

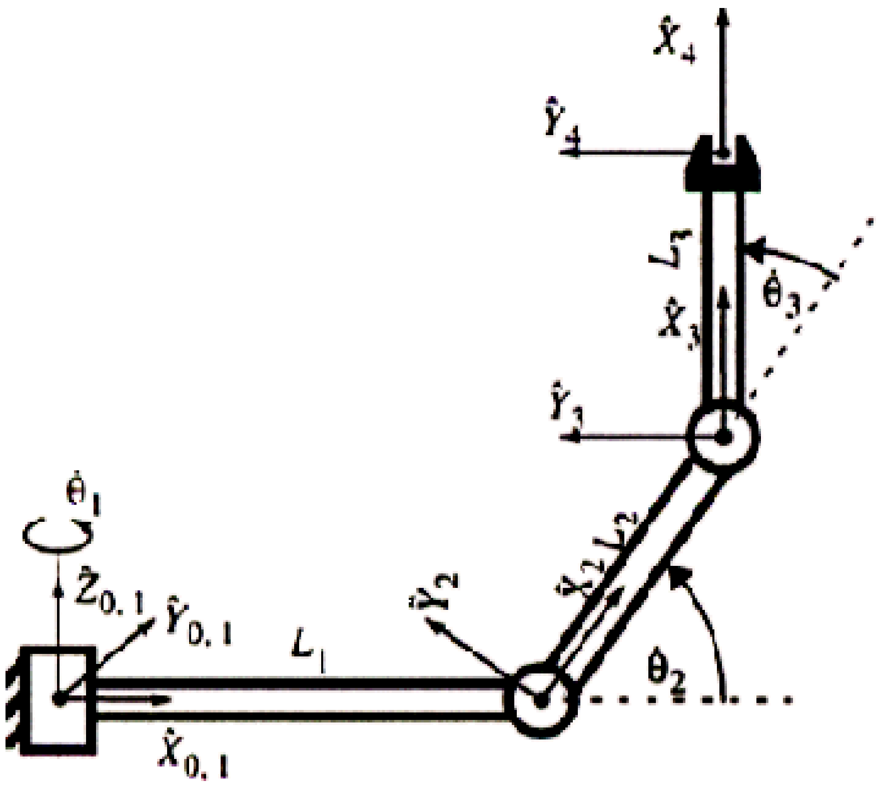
Angular and Linear Velocities - 3R Robot - Example

- For the manipulator shown in the figure, compute the angular and linear velocity of the "tool" frame relative to the base frame expressed in the "tool" frame (that is, calculate ${}^4\omega_4$ and ${}^4v_4$).



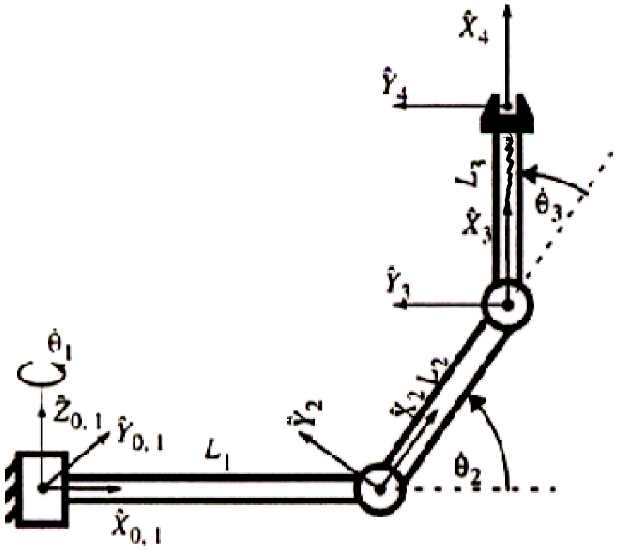
Angular and Linear Velocities - 3R Robot - Example

- Frame attachment



Angular and Linear Velocities - 3R Robot - Example

- DH Parameters



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90	L1	0	θ_2
3	0	L2	0	θ_3
4	0	L3	0	0

Angular and Linear Velocities - 3R Robot - Example

- From the DH parameter table, we can specify the homogeneous transform matrix for each adjacent link pair:

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

${}^0_1R$ ${}^1P_{2ORG}$ ${}^1_2R$

$$
{}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}_{2}T = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

${}^2_3R$ ${}^3_4R$

$$
{}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Angular and Linear Velocities - 3R Robot - Example

- Compute the angular velocity of the end effector frame relative to the base frame expressed at the end effector frame.

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_{i} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

- For ***i=0***

$$^{1}\omega_{1} = {}^{1}_{0}R\, {}^{0}\omega_{0} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=1***

$$
{}^{2}\omega_2 = {}^{2}_{1}R\,{}^{1}\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

- For ***i=2***

$$
{}^{3}\omega_3 = {}^{3}_{2}R\,{}^{2}\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$

- For ***i=3***

$$
{}^{4}\omega_4 = {}^{4}_{3}R\,{}^{3}\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$

- Note

$$
{}^{3}\omega_3 = {}^{4}\omega_4
$$

Angular and Linear Velocities - 3R Robot - Example

- Compute the linear velocity of the end effector frame relative to the base frame expressed at the end effector frame.
-
- Note that the term involving the prismatic joint has been dropped from the equation (it is equal to zero).

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_{i}\right) + \cancelto{0}{\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}}$$

Angular and Linear Velocities - 3R Robot - Example

$^0v_0 = 0$

$^0\Omega_0 = 0$

- For ***i=0***

$$ {}^1v_1 = {}^1_0R\left\{ {}^0\omega_0 \times {}^0P_1 + {}^0v_0 \right\} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- For ***i=1***

$$ {}^2v_2 = {}^2_1R\left\{ {}^1\omega_1 \times {}^1P_2 + {}^1v_1 \right\} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \left(\left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \right) = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=3***

$$
{}^{3}v_{3} = {}^{3}_{2}R\left\{ {}^{2}\omega_{2} \times {}^{2}P_{3} + {}^{2}v_{2} \right\} = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s2\dot{\theta}_{1} \\ c2\dot{\theta}_{1} \\ \dot{\theta}_{2} \end{bmatrix} \times \begin{bmatrix} L2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -L1\dot{\theta}_{1} \end{bmatrix} \right\}
$$

$$
= \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ L2\dot{\theta}_{1} \\ -L2c2\dot{\theta}_{1} - L1\dot{\theta}_{1} \end{bmatrix} \right\} = \begin{bmatrix} L2s3\dot{\theta}_{2} \\ L2c3\dot{\theta}_{2} \\ (-L1 - L2c2)\dot{\theta}_{1} \end{bmatrix}
$$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=4***

$$
{}^4v_4 = {}^4_3R\left\{ {}^3\omega_3 \times {}^3P_4 + {}^3v_3 \right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \times \begin{bmatrix} L3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix} \right\}
$$

$$
= \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \end{bmatrix}
$$

Angular and Linear Velocities - 3R Robot - Example

- Note that the linear and angular velocities (${}^{4}\omega_4, {}^{4}v_4$) of the end effector where differentiate (measured) in frame {0} however represented (expressed) in frame {4}

- In the car example: Observer sitting in the “Car” ${}^{C}\left[{}^{W}V_C\right]$

  Observer sitting in the “World” ${}^{W}\left[{}^{W}V_C\right]$

$$
{}^{k}v_i \equiv {}^{k}\left[{}^{0}V_i\right] = {}^{k}_{0}R\left[{}^{0}V_i\right] = {}^{k}_{0}R \cdot v_i
$$

$$
{}^{k}\omega_i \equiv {}^{k}\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R \cdot \omega_i
$$

Solve for v_4 and ω_4 by multiply both side of the questions from the left by ${}^{4}_{0}R^{-1}$

$$
{}^{4}v_4 = {}^{4}_{0}R \cdot v_4
$$

$$
{}^{4}\omega_4 = {}^{4}_{0}R \cdot \omega_4
$$

Angular and Linear Velocities - 3R Robot - Example

- Multiply both sides of the equation by the inverse transformation matrix, we finally get the linear and angular velocities expressed and measured in the stationary frame {0}

$$v_4 = {}^4_0R^{-1} \cdot {}^4v_4 = {}^4_0R^T \cdot {}^4v_4 = {}^0_4R \cdot {}^4v_4$$

$$\omega_4 = {}^4_0R^{-1} \cdot {}^4\omega_4 = {}^4_0R^T \cdot {}^4\omega_4 = {}^0_4R \cdot {}^4\omega_4$$

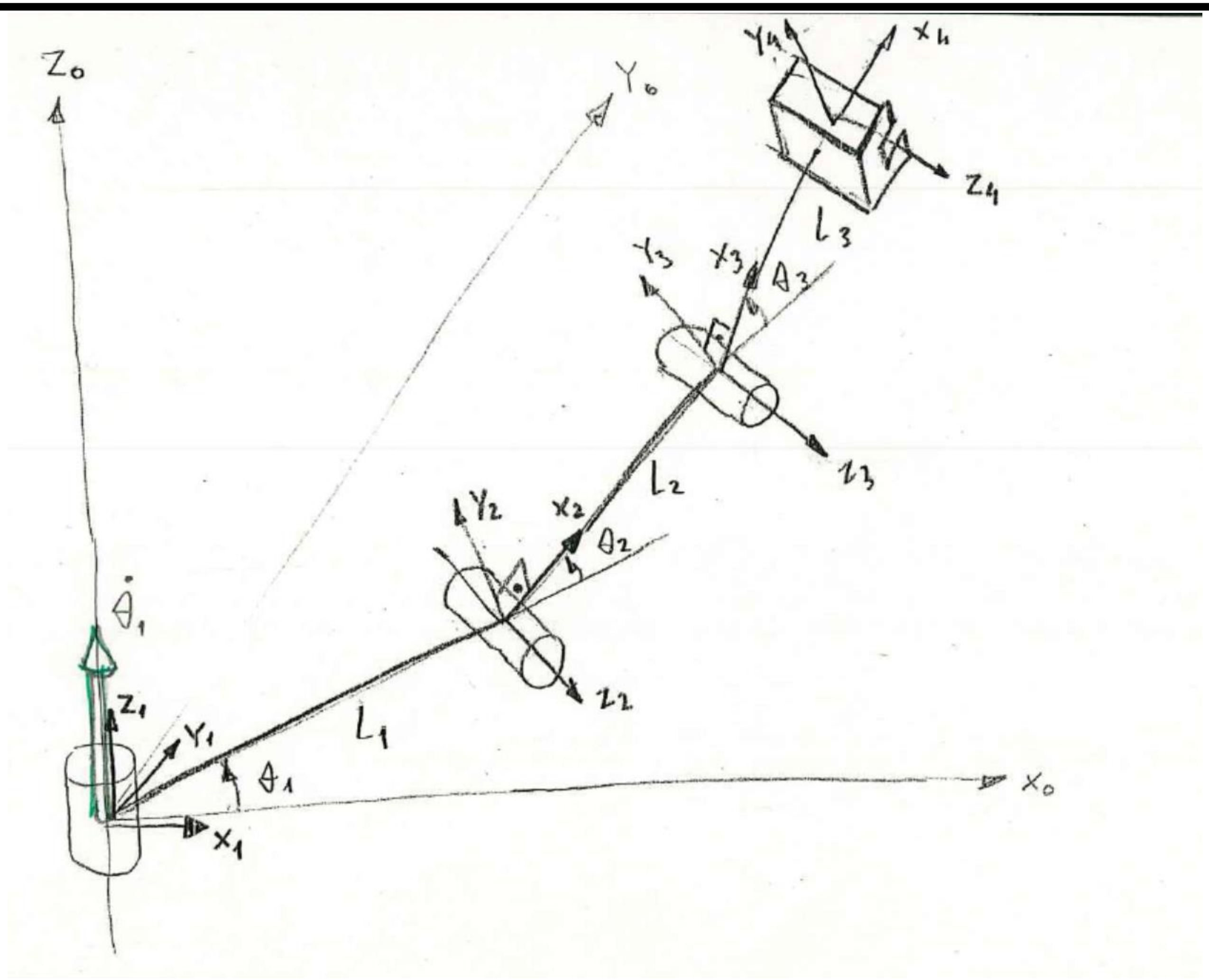
$${}^0_4T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T$$

3R – Example

Analytical Approach – Graphical Interpretation

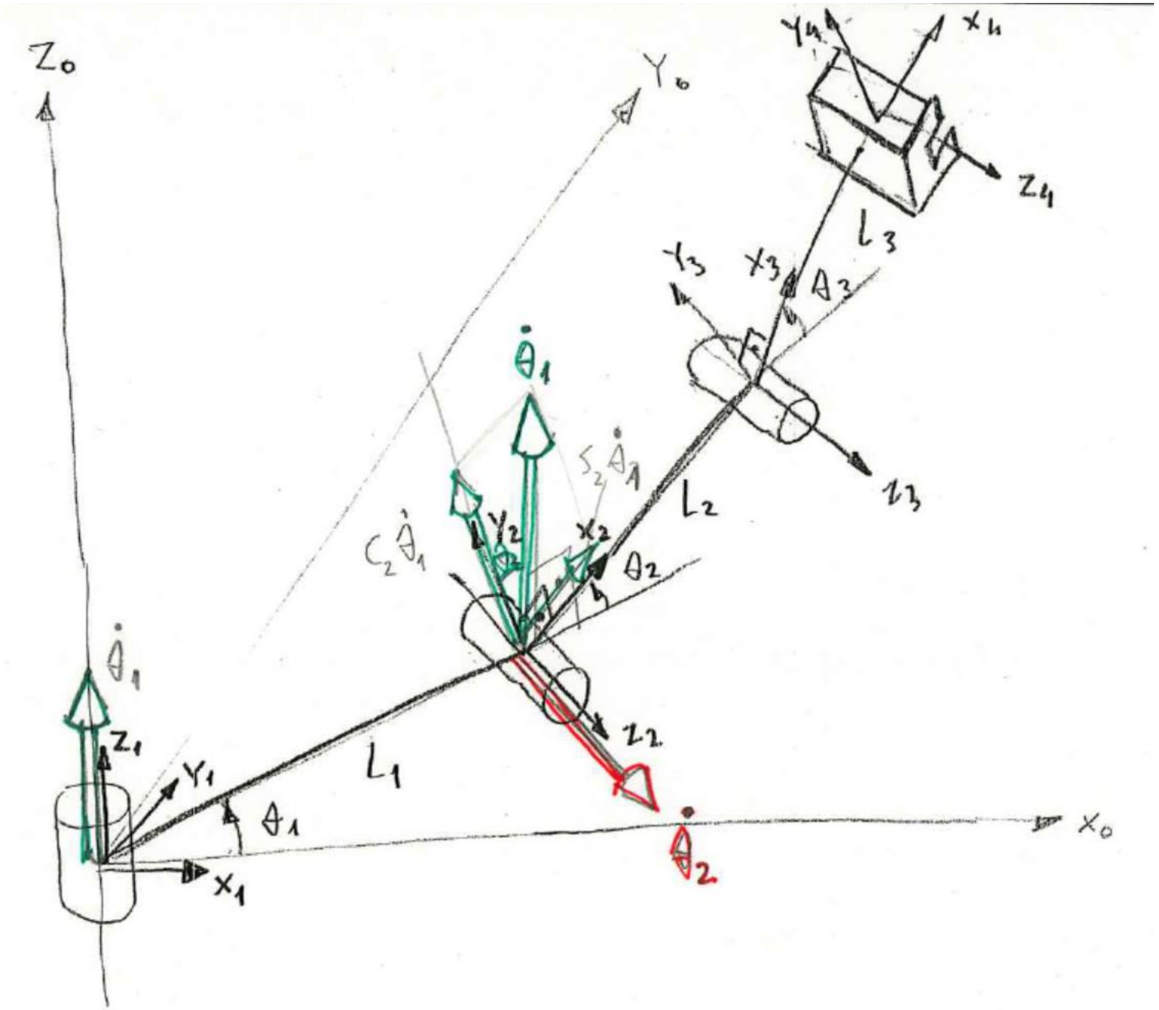
Angular and Linear Velocities - 3R Robot - Example

$$^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$



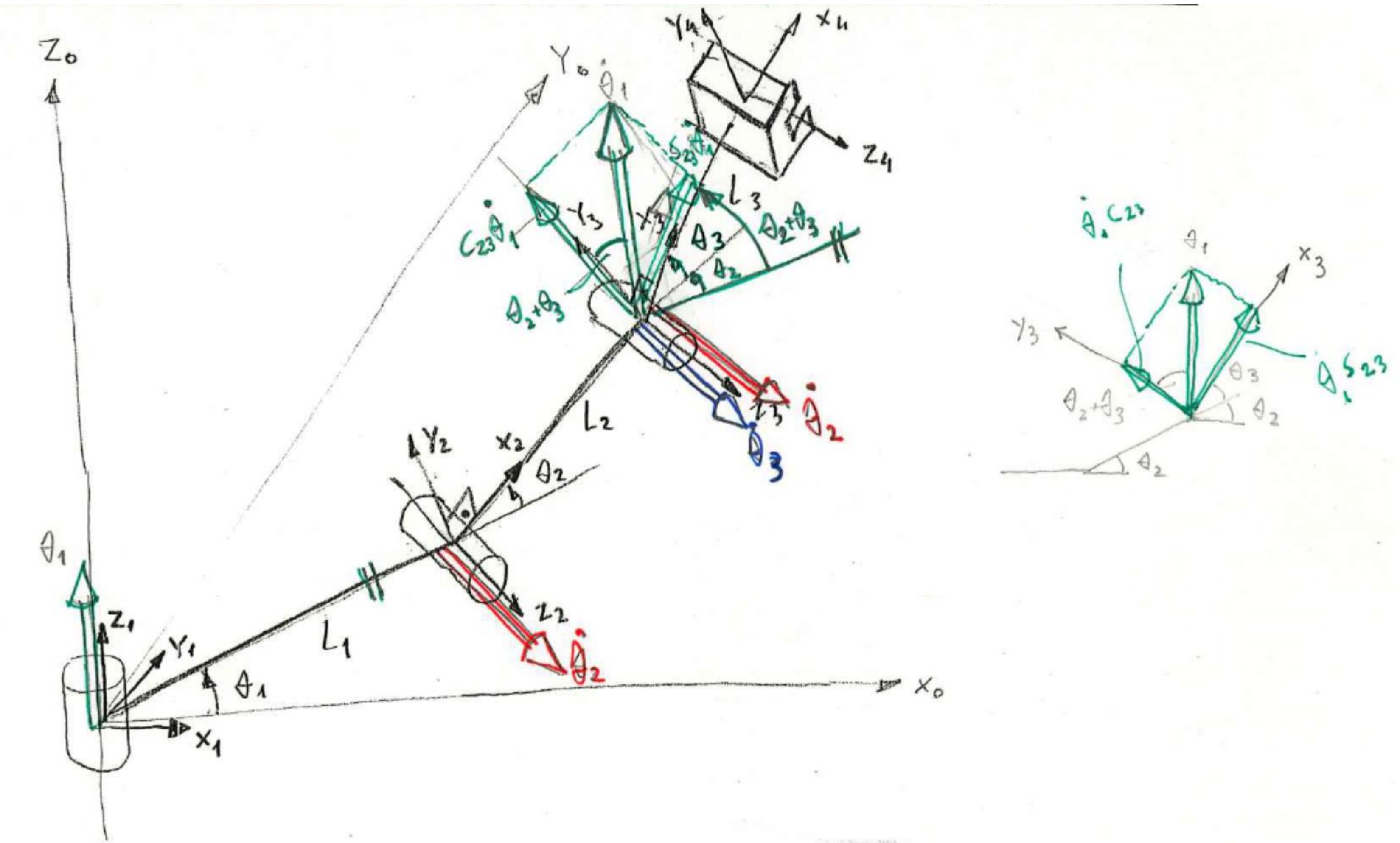
Angular and Linear Velocities - 3R Robot - Example

$$^{2}\omega_2 = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



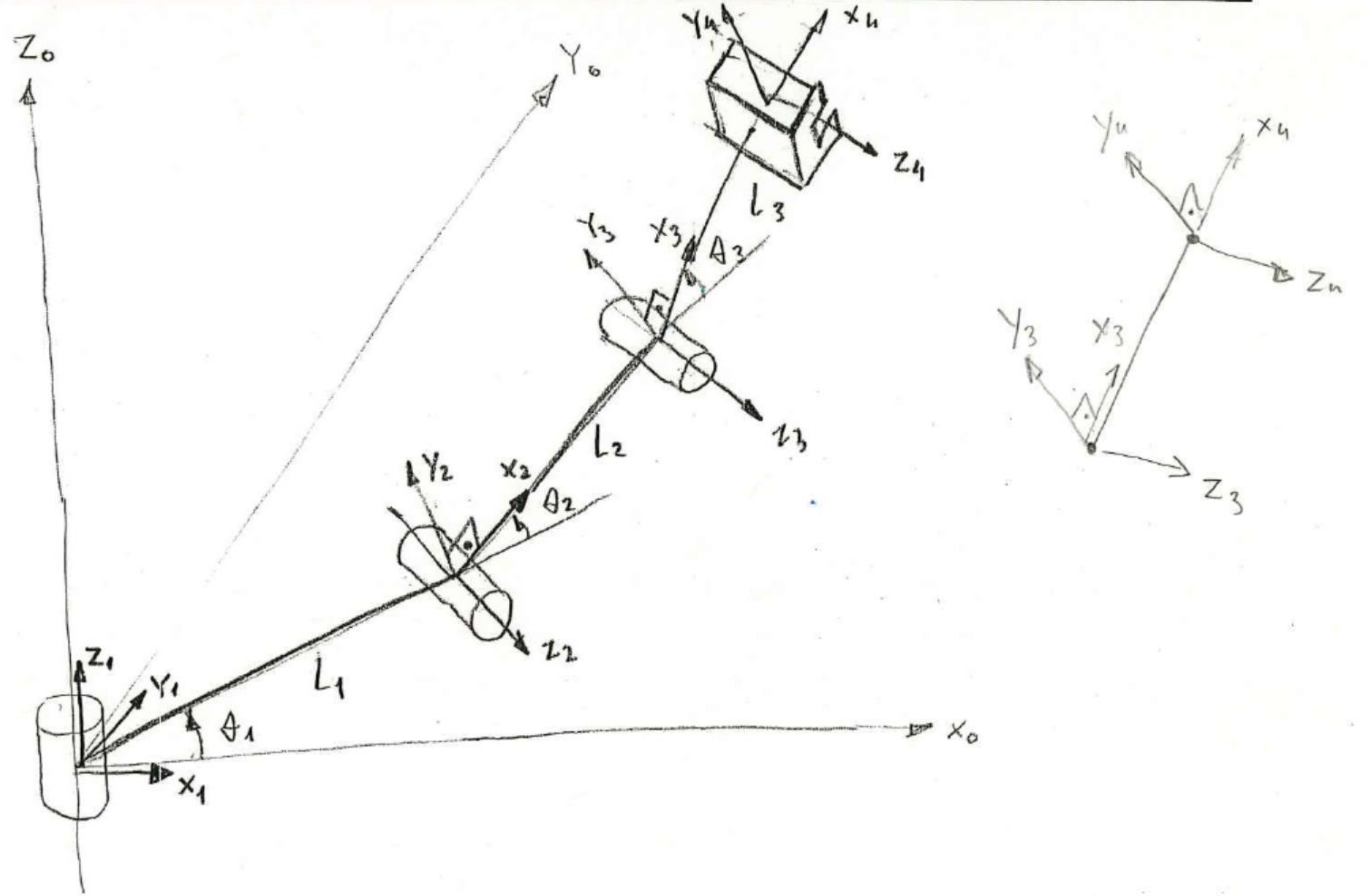
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{3}\omega_3 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$



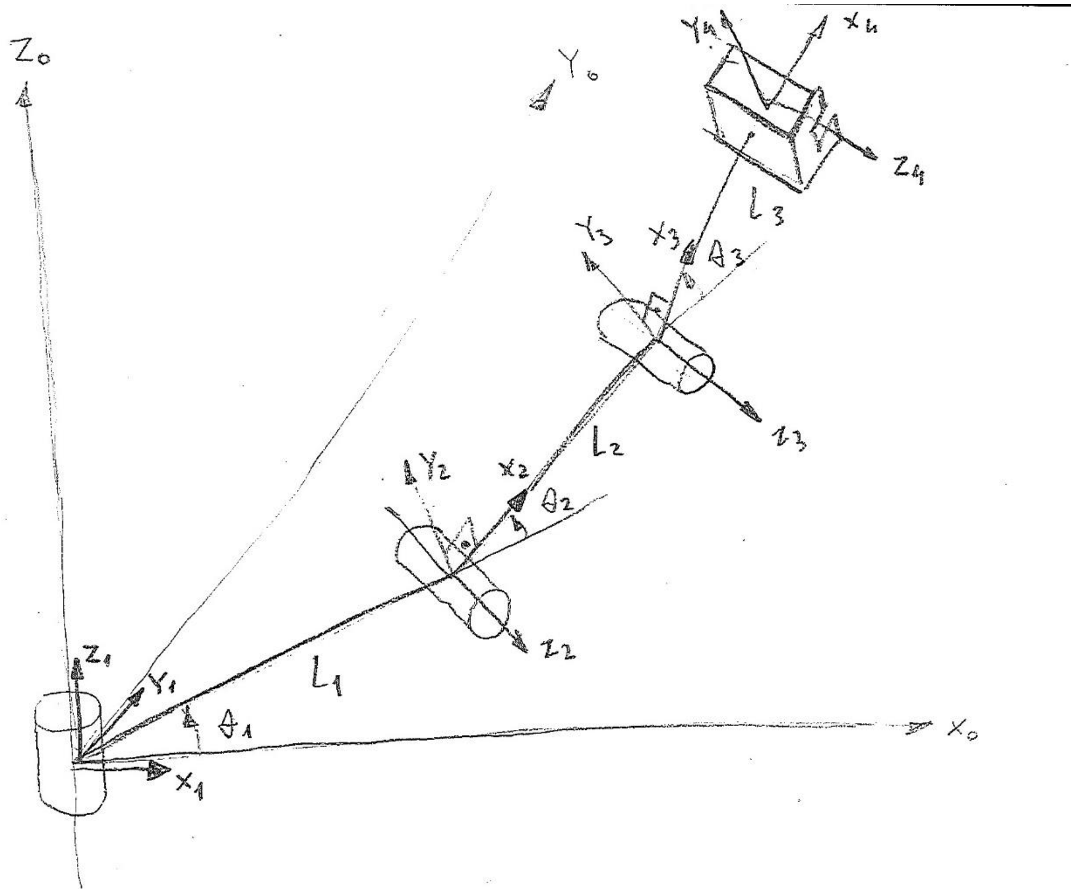
Angular and Linear Velocities - 3R Robot - Example

$$^{3}\omega_{3} = {}^{4}\omega_{4}$$



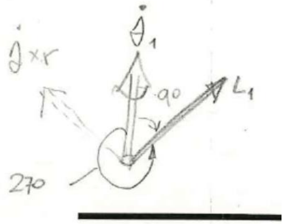
Angular and Linear Velocities - 3R Robot - Example

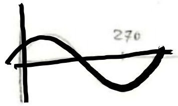
$$
{}^{1}v_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$



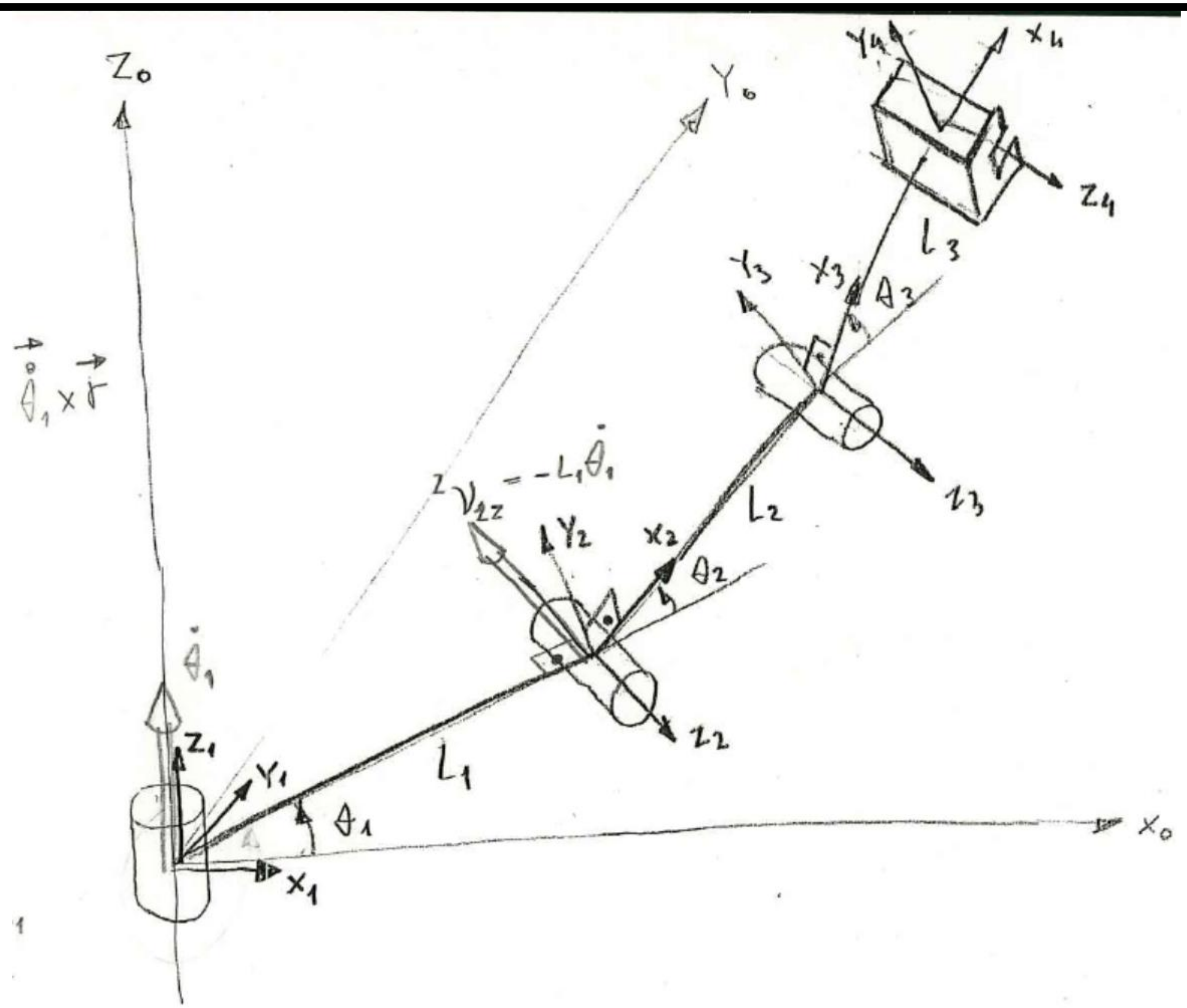
Angular and Linear Velocities - 3R Robot - Example

$$^2v_2 = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}$$



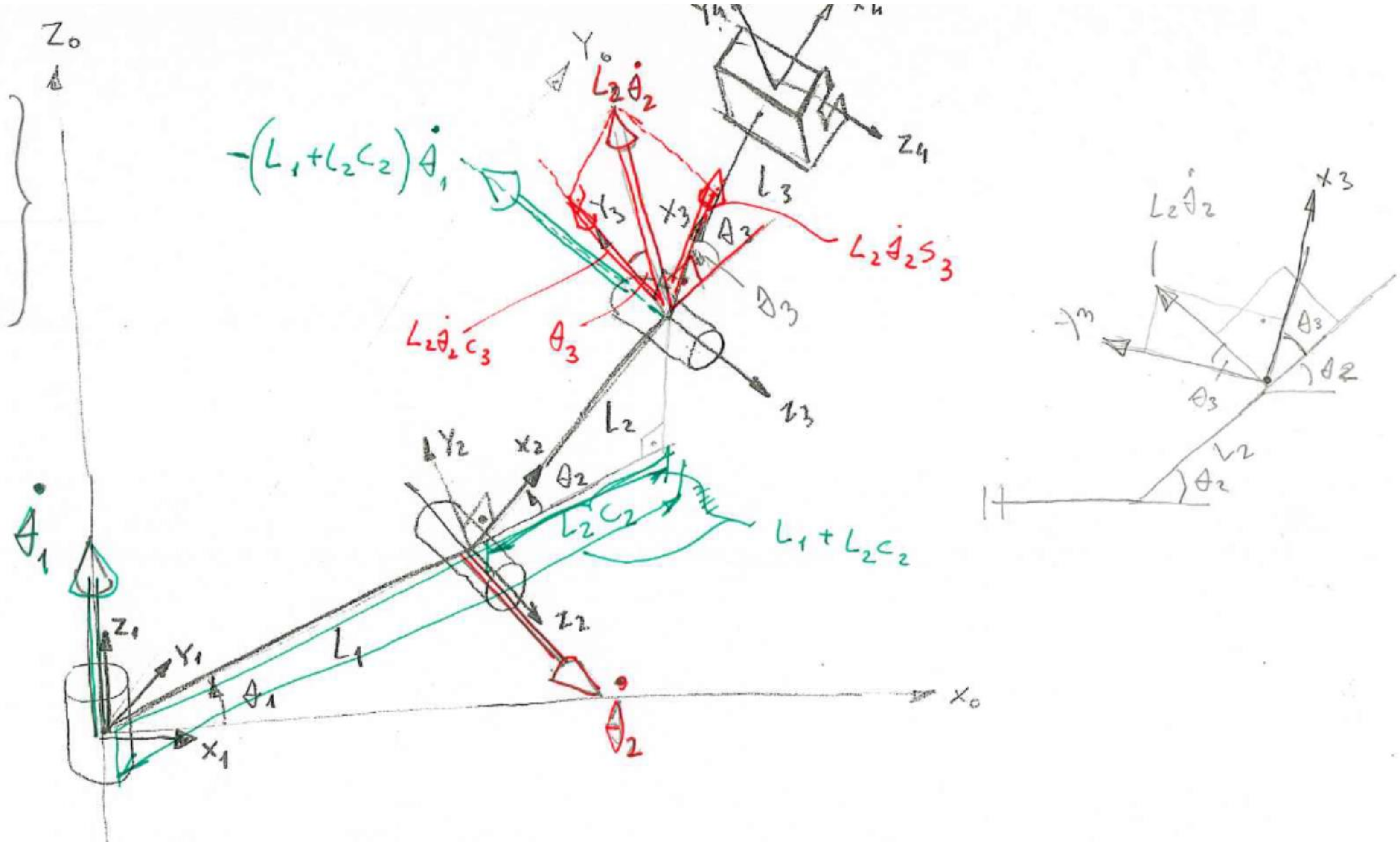


sin(270°) = -1



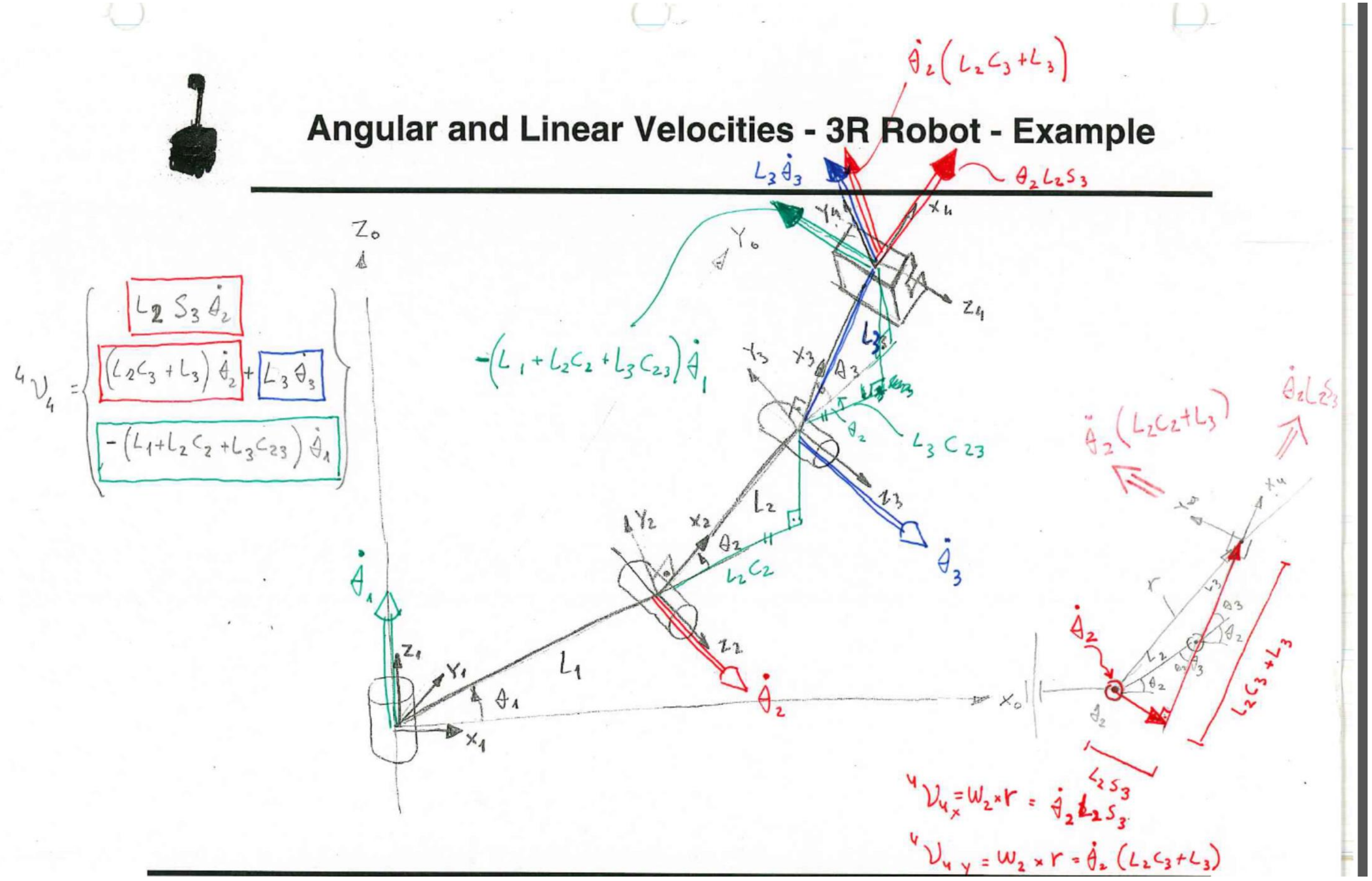
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{3}v_3 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix}
$$



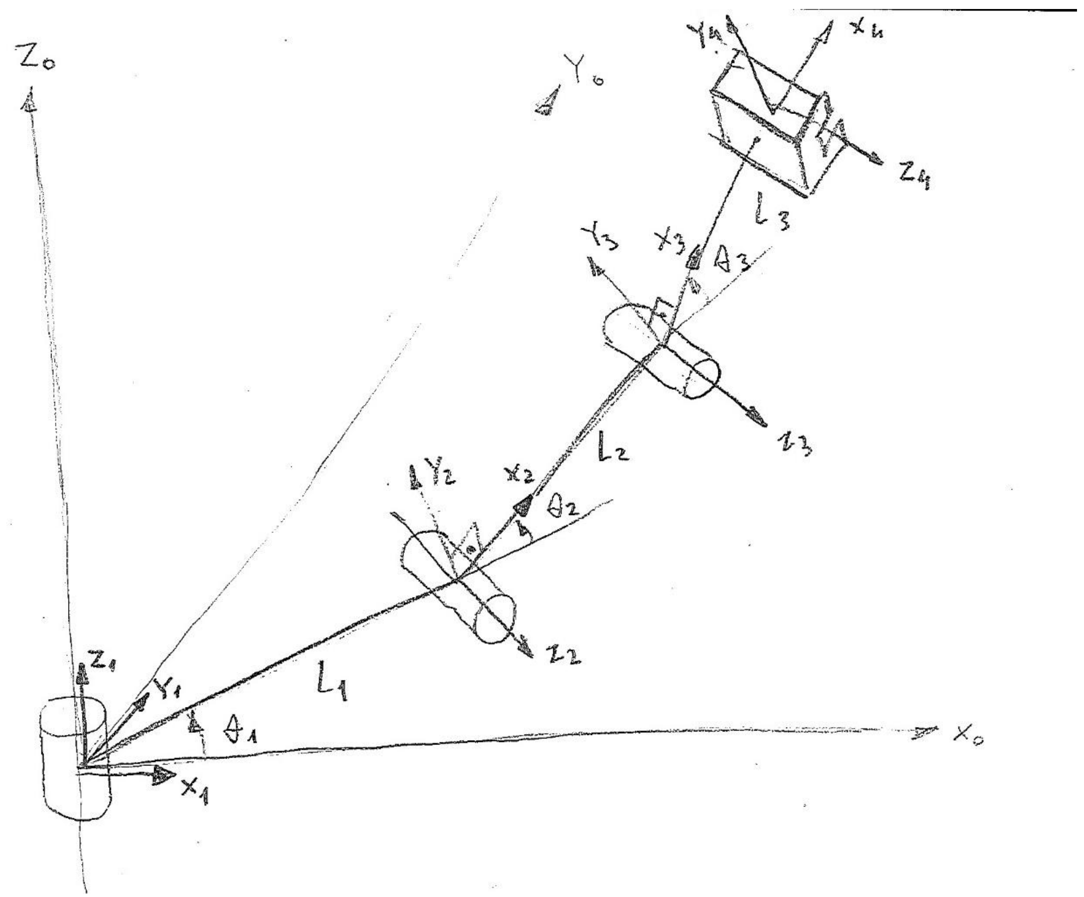
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{4}v_{4} = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ -(L1 + L2c2 + L3c23)\dot{\theta}_1 \end{bmatrix}
$$



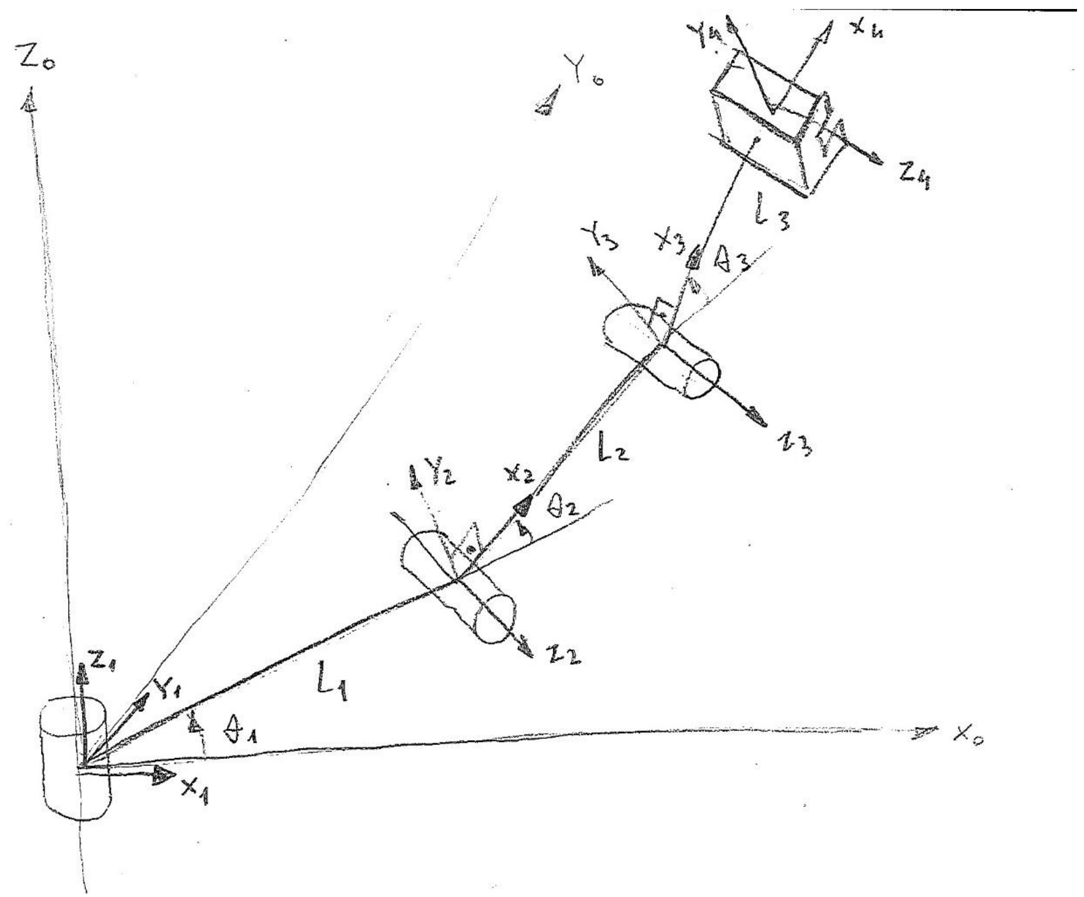
Angular and Linear Velocities - 3R Robot - Example

$$ {}^{1}\omega_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} $$



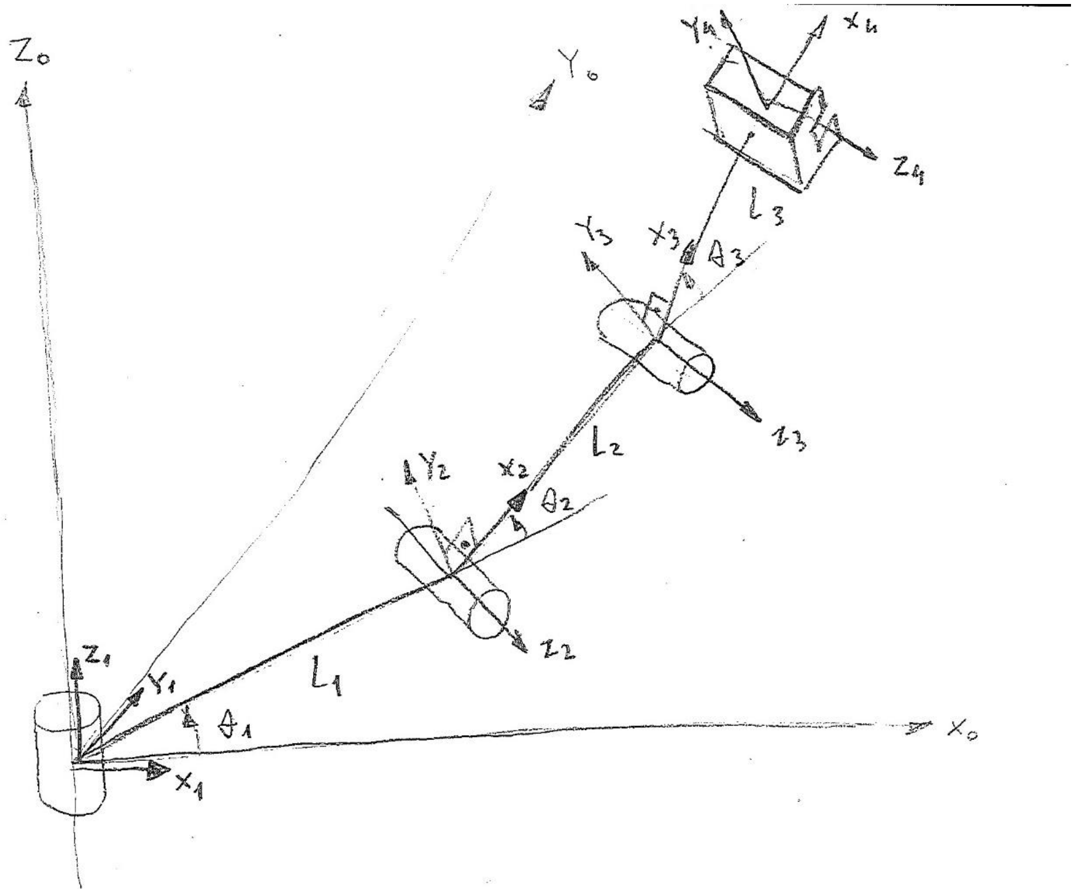
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{2}\omega_{2} = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$



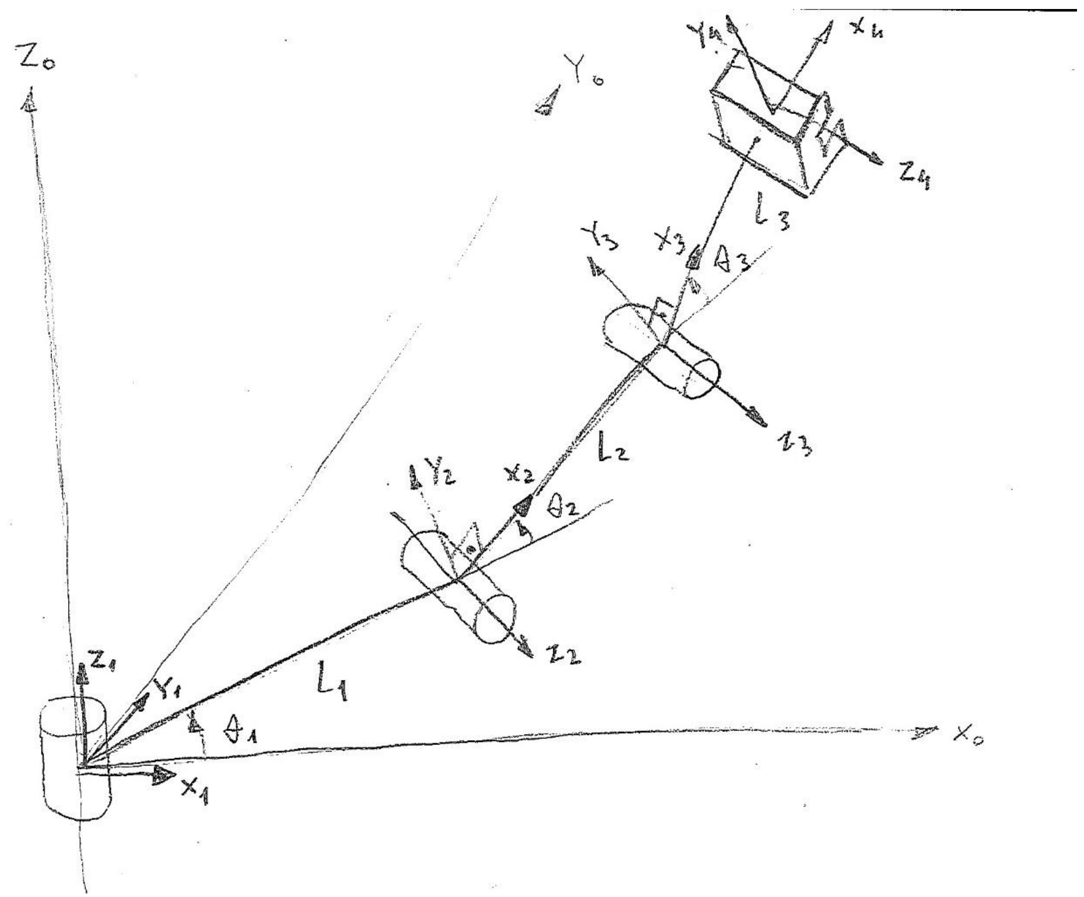
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{3}\omega_{3} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$



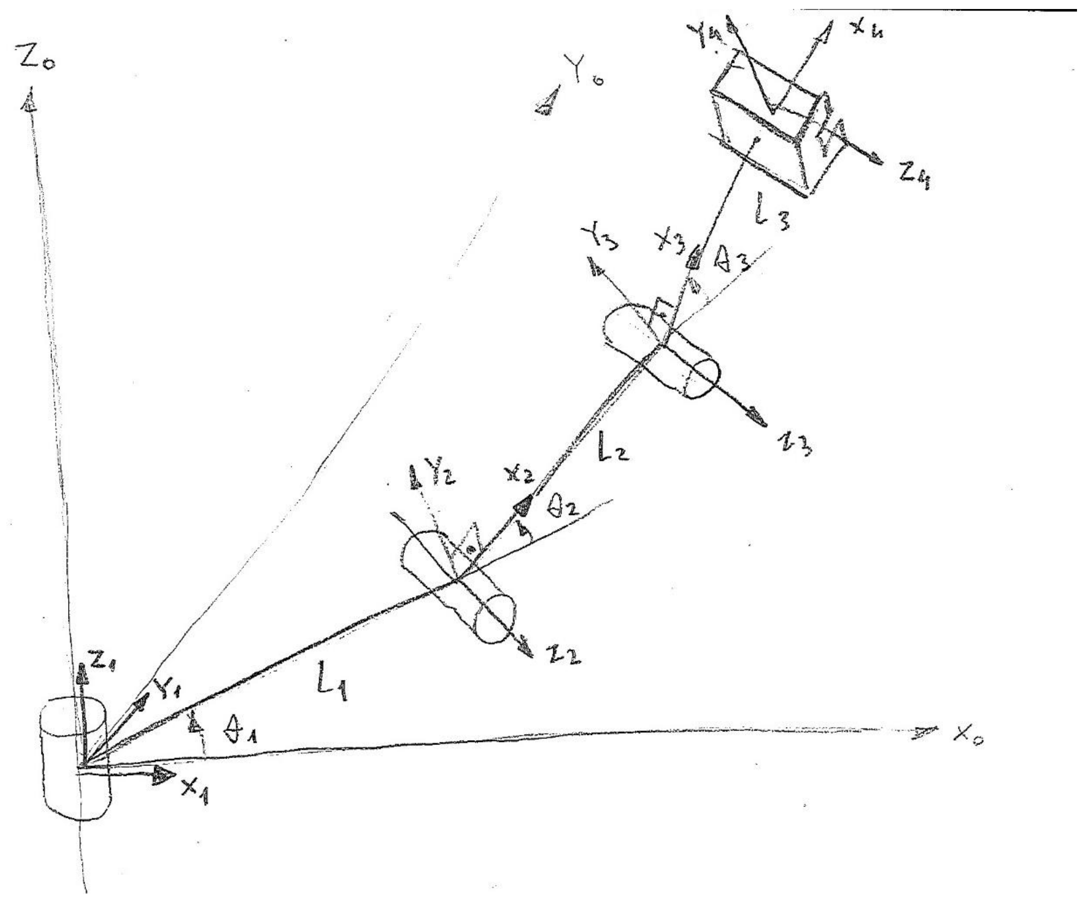
Angular and Linear Velocities - 3R Robot - Example

$$^{3}\omega_{3} = {^{4}\omega_{4}}$$



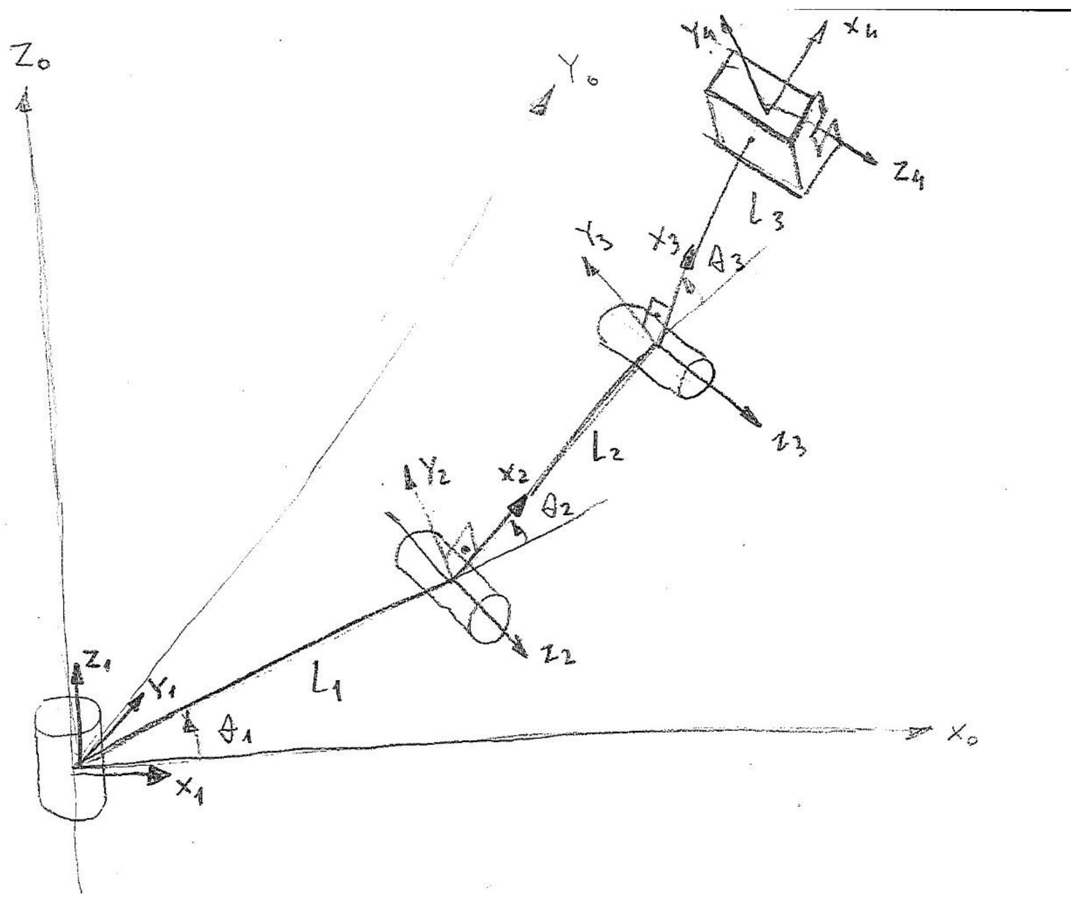
Angular and Linear Velocities - 3R Robot - Example

$$ {}^{1}v_{1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$



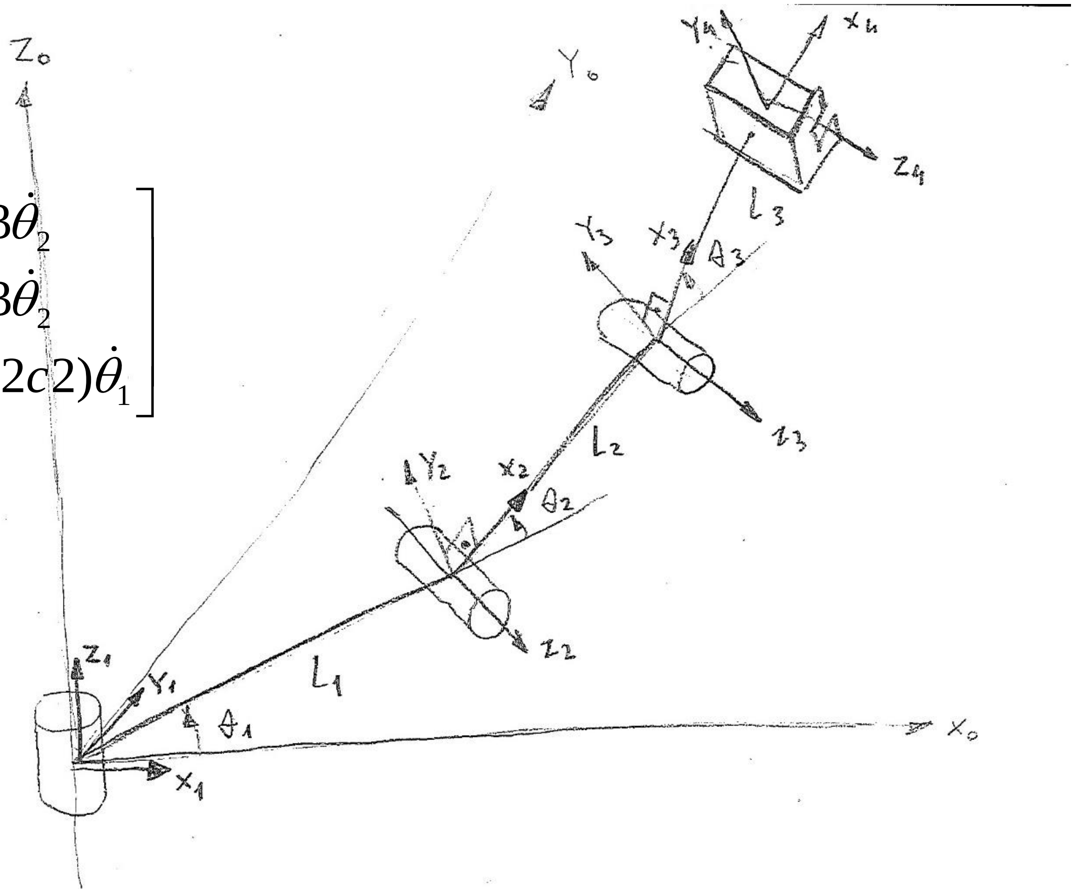
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{2}v_{2} = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}
$$



Angular and Linear Velocities - 3R Robot - Example

$$
{}^{3}v_{3} = \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1-L2c2)\dot{\theta}_1 \end{bmatrix}
$$



Angular and Linear Velocities - 3R Robot - Example

$$
{}^{4}v_{4} = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ -(L1 + L2c2 + L3c23)\dot{\theta}_1 \end{bmatrix}
$$