Jacobian

Introduction

Jacobian – Mapping Operator
Joint & Cartesian/Task Spaces

Kinematics Relations - Joint & Cartesian/Task Spaces

- A robot is often used to manipulate object attached to its tip (end effector).

- The location of the robot tip may be specified using one of the following descriptions:

- ***Joint Space***

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \; d \\ \theta_N \end{bmatrix}$$

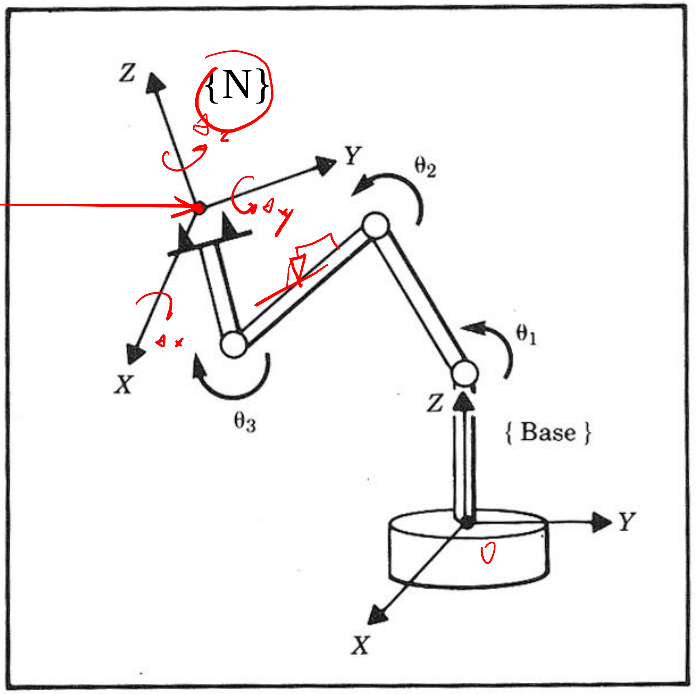
- ***Cartesian Space***

$${}^0_N T = \begin{bmatrix} {}^0_N R & {}^0 P_N \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} [{}^0 P_N] \\ [{}^0 r_N] \end{bmatrix}$$

Euler Angles

θ_x
θ_y
θ_z



Kinematics Relations - Forward & Inverse

- The robot kinematic equations relate the two description of the robot tip location

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

$$X = FK(\theta)$$

$$\theta = IK(X)$$

$$X = \begin{bmatrix} \begin{bmatrix} {}^0P_N \end{bmatrix} \\ \begin{bmatrix} {}^0r_N \end{bmatrix} \end{bmatrix}$$

Tip Location in Joint Space

Tip Location in Cartesian/EE/Task Space

Kinematics Relations - Forward & Inverse

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\dot{\theta} = \frac{d}{dt}[\theta] = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}$$

$$\dot{X} = \frac{d}{dt}[X] = \begin{bmatrix} [\, v_N \,] \\ [\, \omega_N \,] \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\underline{\dot{\theta}} = J^{-1}(\underline{\theta})\underline{\dot{x}}$$

Tip Velocity in Joint Space

Tip velocity in Cartesian/EE/Task Space

Jacobian – Derivation from First Principals
Velocity Maping

Jacobian Matrix - Introduction

- **The Jacobian is a multi dimensional form of the derivative.**
- Suppose that for example we have 6 functions, each of which is a function of 6 independent variables

$$
\begin{cases}
y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\
\vdots \\
y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)
\end{cases}
$$

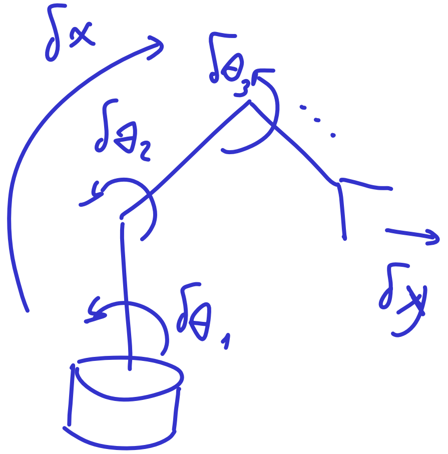
- We may also use a vector notation to write these equations as

$$
Y = F(X)
$$

Jacobian Matrix - Introduction

- If we wish to calculate the differential of y_i as a function of the differential x_i we use the chain rule to get

$$
\begin{aligned}
\delta y_1 &= \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6 \\
\delta y_2 &= \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6 \\
&\vdots \\
\delta y_6 &= \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6
\end{aligned}
$$



- Which again might be written more simply using a vector notation as

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

Jacobian Matrix - Introduction

- The 6x6 matrix of partial derivative is defined as the Jacobian matrix

$\dot{Y}$ $\quad J \quad$ $\dot{x}$

$$\frac{\delta Y}{dt} = \frac{\partial F}{\partial X}\frac{\delta X}{dt} = J(X)\frac{\delta X}{dt}$$

- By dividing both sides by the differential time element, we can think of the Jacobian as mapping velocities in X to those in Y

$$\dot{Y} = J(X)\dot{X}$$

- Note that the Jacobian is time varying linear transformation

$$\rightarrow \frac{\delta x}{\cancel{dt}} = J\frac{\delta \theta}{\cancel{\delta t}}$$

Jacobian Matrix - Introduction

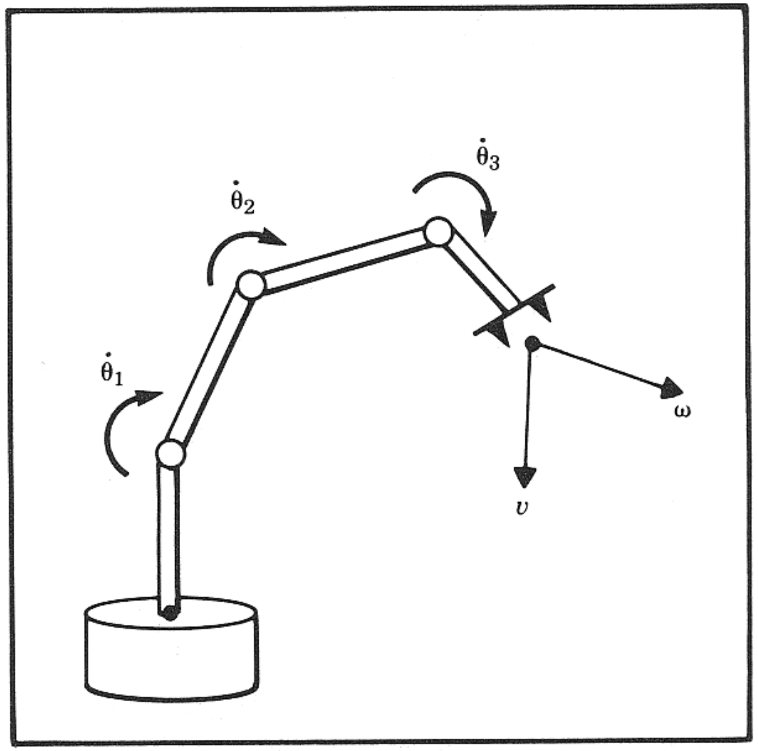
- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

$$\begin{Bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{Bmatrix} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix} \quad \begin{Bmatrix} v \\ w \end{Bmatrix}$$

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{Bmatrix}$$



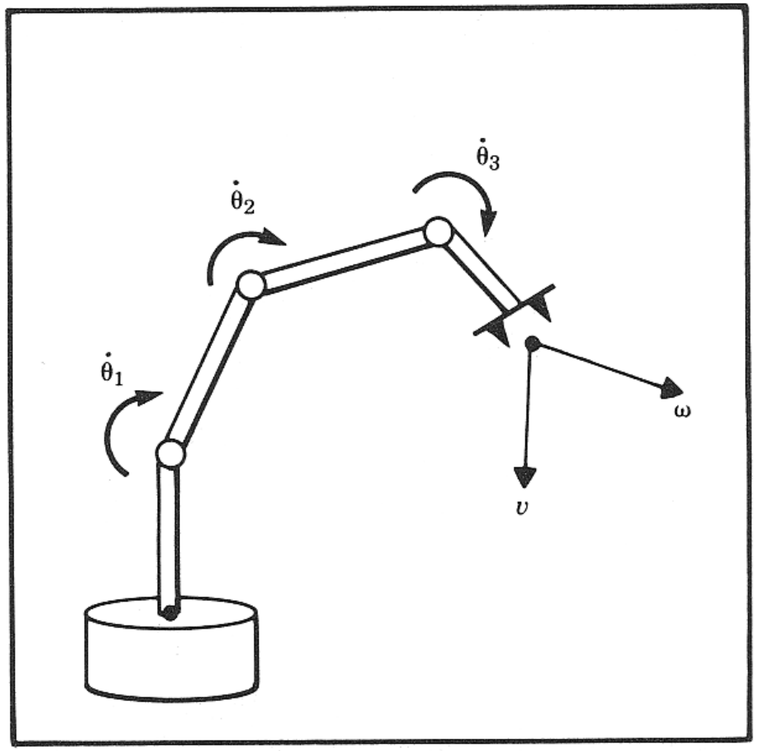
Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \\ \\ & J(\underline{\theta}) & \\ \\ \\ \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \\ \dots \\ \\ \dot{\theta}_N \end{bmatrix}$$

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$



- Note: The Jacobian is a function of joint angle (θ) meaning that the Jacobian varies as the configuration of the arm changes

Jacobian Matrix - Introduction

- This expression can be expanded to:

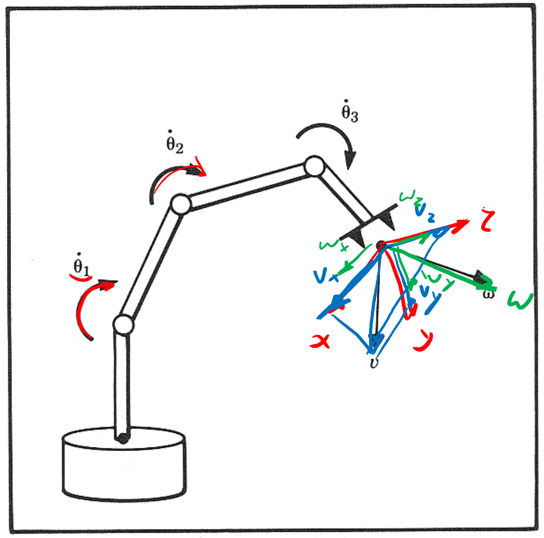
$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\textbf{6x1}} = \underbrace{\begin{bmatrix} J_\upsilon(\underline{\theta}) \\ J_\omega(\underline{\theta}) \end{bmatrix}}_{\textbf{6xN}} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \dot{\theta}_N \end{bmatrix}}_{\textbf{Nx1}}$$

- Where:
  - $\underline{\dot{x}}$ is a 6x1 vector of the end effector linear and angular velocities
  - $J(\underline{\theta})$ is a 6xN Jacobian matrix
  - $\underline{\dot{\theta}}_N$ is a Nx1 vector of the manipulator joint velocities
  - N is the number of joints

Jacobian Matrix - Introduction

- The meaning of **<u>each line</u>** (e.g. the first line) of the Jacobian matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} J_{11} \quad J_{12} \quad J_{13} \quad J_{14} \quad J_{15} \quad J_{16} \\ J_{v}(\underline{\theta}) \\ \\ J_{\omega}(\underline{\theta}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \dot{\theta}_N \end{bmatrix}$$



- The first line maps the contribution of the angular velocity of each joint to the linear velocity of the end effector along the x-axis

Jacobian Matrix - Introduction

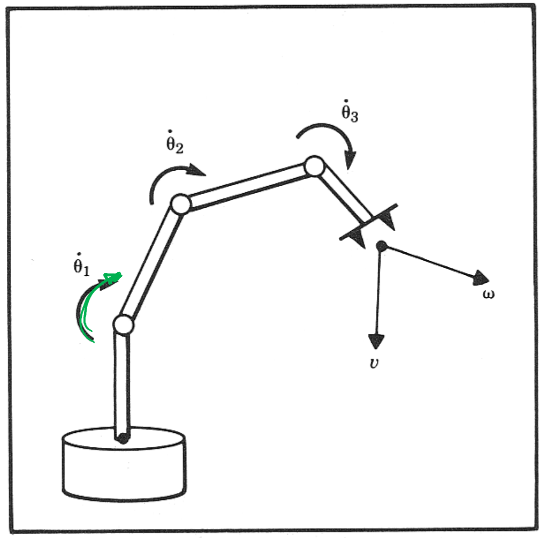
- The meaning of **each column** (e.g. the first column) of the Jacobian matrix:

m/s

RAD/s

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} J_{11} & & \\ J_{21} & J_{\upsilon}(\underline{\theta}) & \\ J_{31} & & \\ J_{41} & & \\ J_{51} & J_{\omega}(\underline{\theta}) & \\ J_{61} & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \dot{\theta}_N \end{bmatrix}$$

RAD/sec



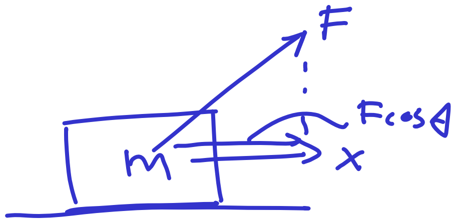
- The first column maps the contribution of the angular velocity of the first joint to the linear and angular velocities of the end effector along all the axis (x,y,z)

Jacobian – Derivation from First Principels

Jacobian Matrix - Introduction

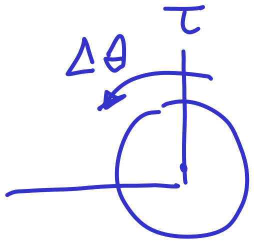
Work

Dot product

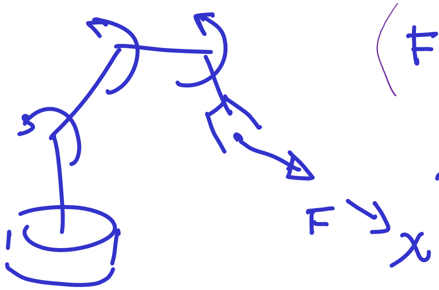


$$W = F \cdot \Delta x$$

$$= F\cos\theta \, x$$



$$W = \tau \cdot \Delta\theta$$



$$(F \cdot \delta x = \tau \cdot \delta\theta)^T$$

|| EE space

|| Joint space

$$F_x \, x = \tau_1 \, \theta_1$$

$\vdots$

$$M_x \, \theta_x = \tau_4 \, \theta_4$$

$\vdots$

$$F^T \delta x = \tau^T \delta\theta$$

$$\delta x = J \, \delta\theta$$

Dot

$$F^T \underbrace{J \, \cancel{\delta\theta}}_{\delta x} = \tau^T \cancel{\delta\theta}$$

Jacobian Matrix - Introduction

$$\left[\tau^T = F^T J\right]^T$$

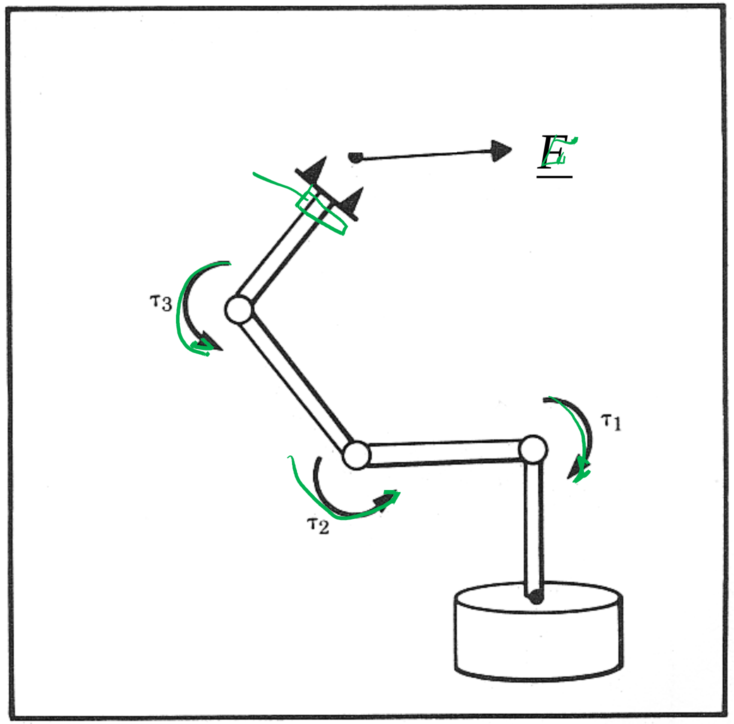
$$\left[\tau^T\right]^T = \left[F^T J\right]^T$$

$$\boxed{\tau = J^T F}$$

Jacobian Matrix - Introduction

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\begin{Bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\\ \tau_5\\ \tau_6\end{Bmatrix} \quad \underline{\tau} = \underline{J(\underline{\theta})^T}\,\underline{F} \quad \begin{Bmatrix}F_x\\ F_y\\ F_z\\ M_x\\ M_y\\ M_z\end{Bmatrix}$$



Jacobian Matrix - Introduction

- This expression can be expanded to:

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \cdots \\ \tau_N\end{bmatrix} = \begin{bmatrix} J_f(\underline{\theta}) & J_\tau(\underline{\theta})\end{bmatrix}^T \begin{bmatrix}F_x\\ F_y\\ F_z\\ M_x\\ M_y\\ M_z\end{bmatrix}$$

Nx1 **6xN** **6x1**

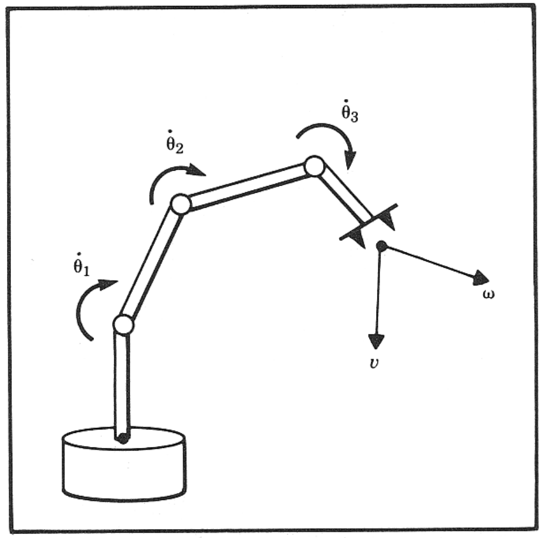
- Where:
  - $\underline{\tau}$ is a 6x1 vector of the robot joint torques
  - $J(\underline{\theta})^T$ is a 6xN Transposed Jacobian matrix
  - $\underline{F}$ is a Nx1 vector of the forces and moments at the robot end effector
  - N is the number of joints

Jacobian Matrix - Introduction

- The meaning of **<u>each line</u>** (e.g. the first line) of the Jacobian matrix:

$$
\begin{bmatrix} \tau_1 \\ \tau_2 \\ \cdots \\ \tau_N \end{bmatrix}
=
\begin{bmatrix} J_{11} & J_{21} & J_{31} & J_{41} & J_{51} & J_{61} \\ \multicolumn{3}{c}{J_f(\underline{\theta})} & \multicolumn{3}{c}{J_\tau(\underline{\theta})} \end{bmatrix}^T
\begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}
$$

Nx1 **6xN** **6x1**



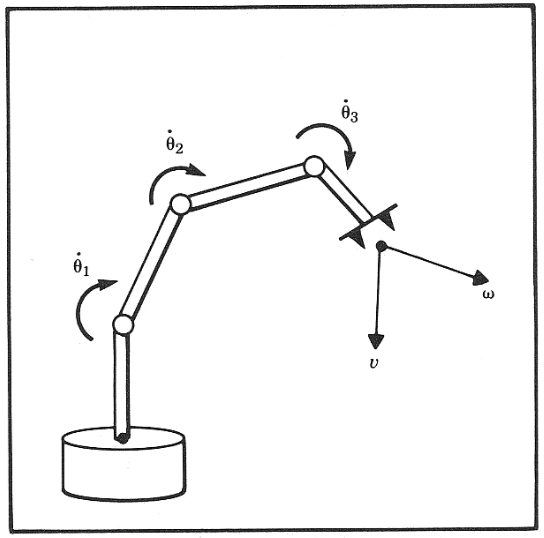
- **Action**: The first line represent how the torque applied at the first joint contributes to the forces and torques applied by the end effector
- **Reaction**: The first line maps the contribution of the partial external loads applied on the end effector to the join torque that needs to be applied to maintain static equilibriums
-

Jacobian Matrix - Introduction

- The meaning of **each column** (e.g. the first column) of the Jacobian matrix:

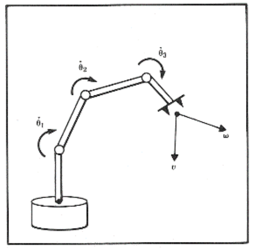
$$\begin{bmatrix}\tau_1\\ \tau_2\\ \\ \cdots\\ \\ \tau_N\end{bmatrix} = \begin{bmatrix} J_{11} & & \\ J_{12} & & \\ J_{13} & J_f(\underline{\theta}) & J_\tau(\underline{\theta}) \\ J_{14} & & \\ J_{15} & & \\ J_{16} & & \end{bmatrix}^T \begin{bmatrix} F_x\\ F_y\\ F_z\\ M_x\\ M_y\\ M_z\end{bmatrix}$$

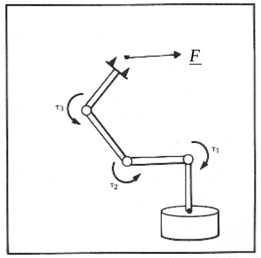
Nx1 **6xN** **6x1**



- **Action**: The first column represent what partial torque applied by each joint is required to create an equilibrium of the force aloning the X- Axis
- **Reaction**: The first column maps the contribution of the partial external loads of the force along the X-axis applied on the end effector to the join torques that are needed to be applied to maintain static equilibriums

Jacobian Matrix - Derivation Methods





Explicit Method

Differentiation the Forward Kinematics Eqs. (Method 1)

Iterative Methods Recursive Equations

Velocity Propagation – Base to EE (Method 2)

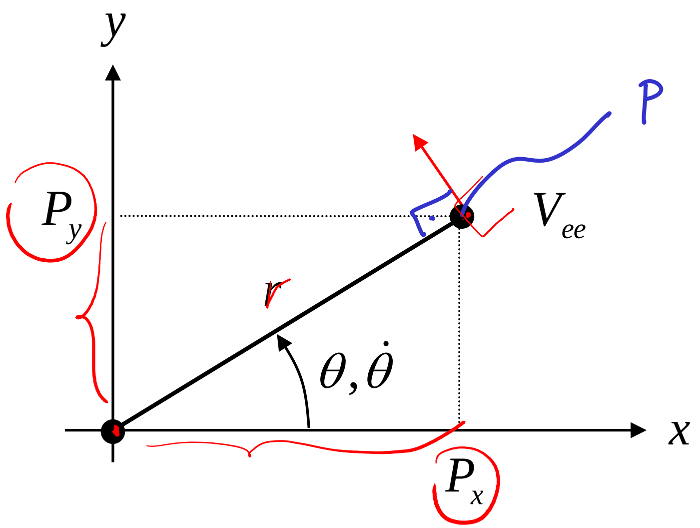
Force/Torque Propagation – EE to Base (Method 3)

Jacobian Matrix

Jacobian – R Robot (1 DOF) - Example

Jacobian Matrix by Differentiation - 1R - 1/4

- Consider a simple planar 1R robot



- The end effector position is given by

$$\begin{cases} {}^0P_x = x = r\cos\theta & /dt \\ {}^0P_y = y = r\sin\theta & /dt \end{cases}$$

Jacobian Matrix by Differentiation - 1R - 2/4

- The velocity of the end effector is defined by

$$ {}^{0}V_x = {}^{0}\dot{P}_x = \dot{x} = -\dot{\theta}\, r \sin\theta = \left(-\omega\, r \sin\theta\right)\dot{\theta} $$
$$ {}^{0}V_y = {}^{0}\dot{P}_y = \dot{y} = \dot{\theta}\, r \cos\theta = \left(\omega\, r \cos\theta\right)\dot{\theta} $$

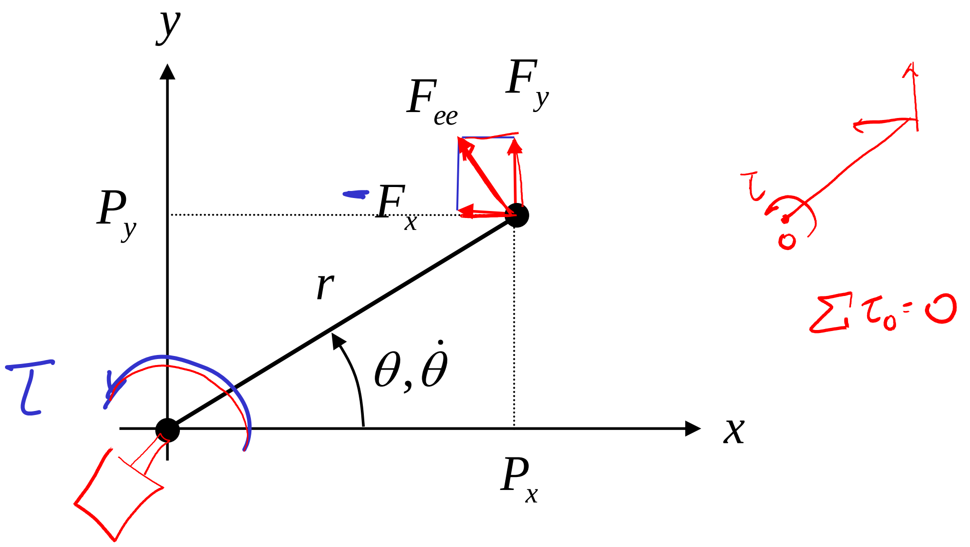
- Expressed in matrix form we have

$$ \underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} $$

J

$$ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \left[\dot{\theta}\right] $$

2x1 2x1 1x1



- The moment about the joint generated by the force acting on the end effector is given by

$$\tau = -rF_x \sin\theta + rF_y \cos\theta$$

Jacobian Matrix by Differentiation - 1R - 4/4

- Expressed in matrix form we have

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

$$[\tau] = \begin{bmatrix} -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

1x1 **1x2** **2x1**

J^T

T

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

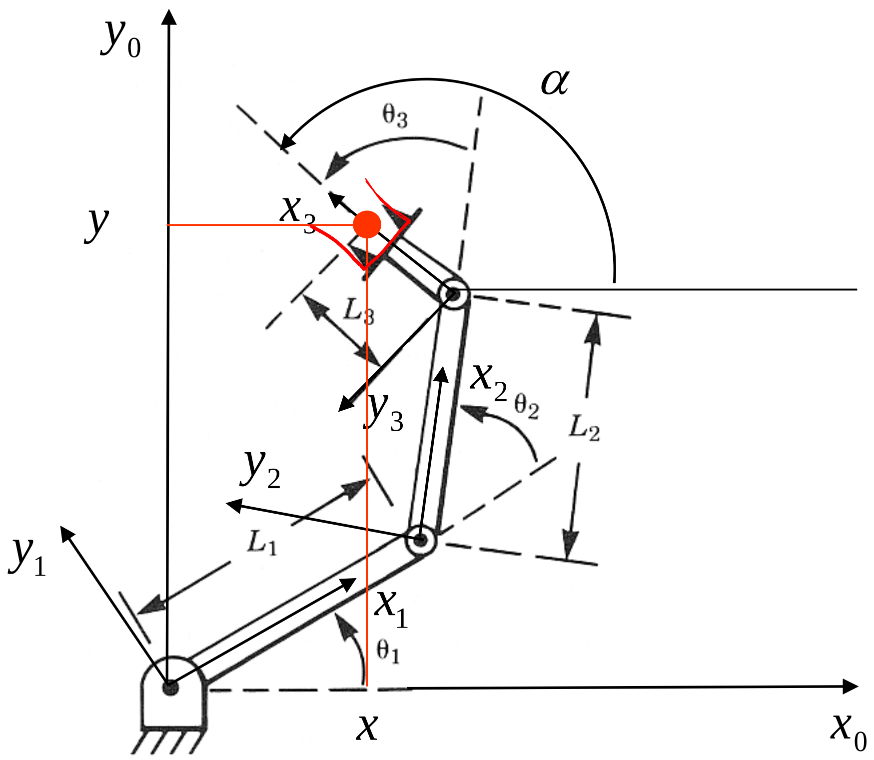
J

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} [\dot{\theta}]$$

Jacobian – RR Robot (3 DOF) - Example

Jacobian Matrix by Differanciation - 3R - 1/4

- Consider the following 3 DOF Planar manipulator



Jacobian Matrix by Differanciation - 3R - 2/4

- **Problem:** Compute the Jacobian matrix that describes the relationship

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} \qquad \underline{\tau} = J(\underline{\theta})^T \underline{F}$$

- **Solution:**

- The end effector position and orientation is defined in the base frame by

$$\underline{x} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix}$$

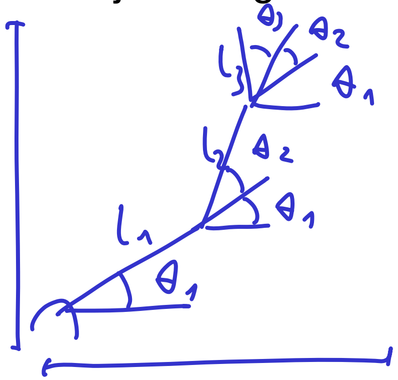
Jacobian Matrix by Differanciation - 3R - 3/4

- The forward kinematics gives us relationship of the end effector to the joint angles:

$$ {}^{0}P_{3org,x} = x = L_1c_1 + L_2c_{12} + L_3c_{123} $$

$$ {}^{0}P_{3org,y} = y = L_1s_1 + L_2s_{12} + L_3s_{123} $$

$$ {}^{0}P_{3org,\alpha} = \alpha = \theta_1 + \theta_2 + \theta_3 $$



- Differentiating the three expressions gives

J

$\dot\theta_1, \dot\theta_2, \dot\theta_3$

$$ \dot{x} = -L_1s_1\dot\theta_1 - L_2s_{12}\left(\dot\theta_1 + \dot\theta_2\right) - L_3s_{123}\left(\dot\theta_1 + \dot\theta_2 + \dot\theta_3\right) $$

$$ = -\left(L_1s_1 + L_2s_{12} + L_3s_{123}\right)\dot\theta_1 - \left(L_2s_{12} + L_3s_{123}\right)\dot\theta_2 - \left(L_3s_{123}\right)\dot\theta_3 $$

$$ \dot{y} = L_1c_1\dot\theta_1 + L_2c_{12}\left(\dot\theta_1 + \dot\theta_2\right) + L_3c_{123}\left(\dot\theta_1 + \dot\theta_2 + \dot\theta_3\right) $$

$$ = \left(L_1c_1 + L_2c_{12} + L_3c_{123}\right)\dot\theta_1 + \left(L_2c_{12} + L_3c_{123}\right)\dot\theta_2 + \left(L_3c_{123}\right)\dot\theta_3 $$

$$ \dot\alpha = \dot\theta_1 + \dot\theta_2 + \dot\theta_3 $$

Jacobian Matrix by Differanciation - 3R - 4/4

- Using a matrix form we get

FRAME 0

$$\underline{\dot{x}} = {}^{0}J(\underline{\theta})\underline{\dot{\theta}}$$

J

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees a freedom. Before tackling these problems, consider this brief review of linear algebra.