Advanced Kinematics

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Invers Kinematics – Two Problem

$${}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T \,\big|\, {}^3_4T \,\big|\, {}^4_5T\,{}^5_6T$$

PROBLEM 1 (${}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T$)

PROBLEM 2 (${}^3_4T\,{}^4_5T\,{}^5_6T$)

$$\{\text{FRAME } 3\} \xrightarrow{\alpha_3} \{\text{FRAME}(R)\} \xrightarrow{a_3} \{\text{FRAME}(Q)\} \xrightarrow{\theta_4} \{\text{FRAME}(P)\} \xrightarrow{d_4} \{\text{FRAME}(4)\}$$

$$R_{x_3}(\alpha_3)\, D_{x_3}(a_3) \,\big|\, R_{z_4}(\theta_4)\, D_{z_4}(d_4)$$

PROBLEM 1

$${}^3_4T \,\big|\, \theta_4 = 0$$

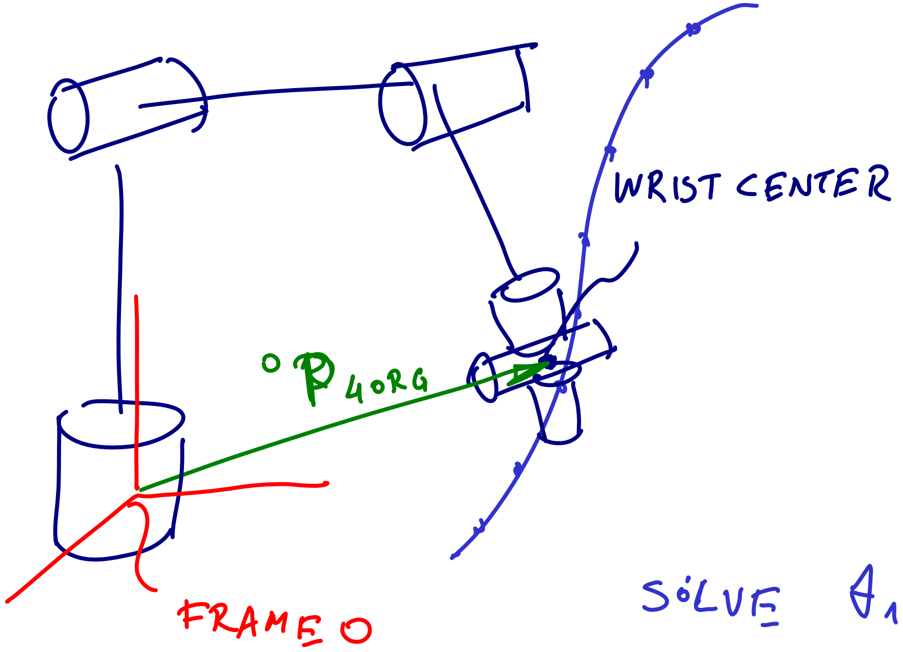
PROBLEM 2

$${}^3_4T \,\big|\, \alpha_3 = 0$$

PROBLEM 1

INVERSE POSITION KIN.

$$^0P_4 = {}^0_1T\,{}^1_2T\,{}^2_3T\;{}^3P_{4ORG}$$



$$^3_4T = \left[\begin{array}{c|c} {}^3_4R & {}^3P_{4ORG} \\ \hline & \end{array}\right]$$

SOLVE $\theta_1, \theta_2, \theta_3$

PROBLEM 2

INVERSE ORI. KIM.

$$^0_6R = {}^0_3R\ {}^3_4R\ {}^4_6R$$

$$^0_3R \left(R_{x_3}(\alpha_3)\ \ I\ \ R_{z_4}(\theta_4)\ \ I \right) {}^4_6R$$

$I \leftarrow D_{x_3}(a_3)$, $\quad I \leftarrow D_{z_4}(d_i)$

PROBLEM 1

$$^0_6R = \left[{}^0_3R\ R_{x_3}\alpha_3\right]\left[R_{z_4}(\theta_4)\ {}^4_6R\right]$$

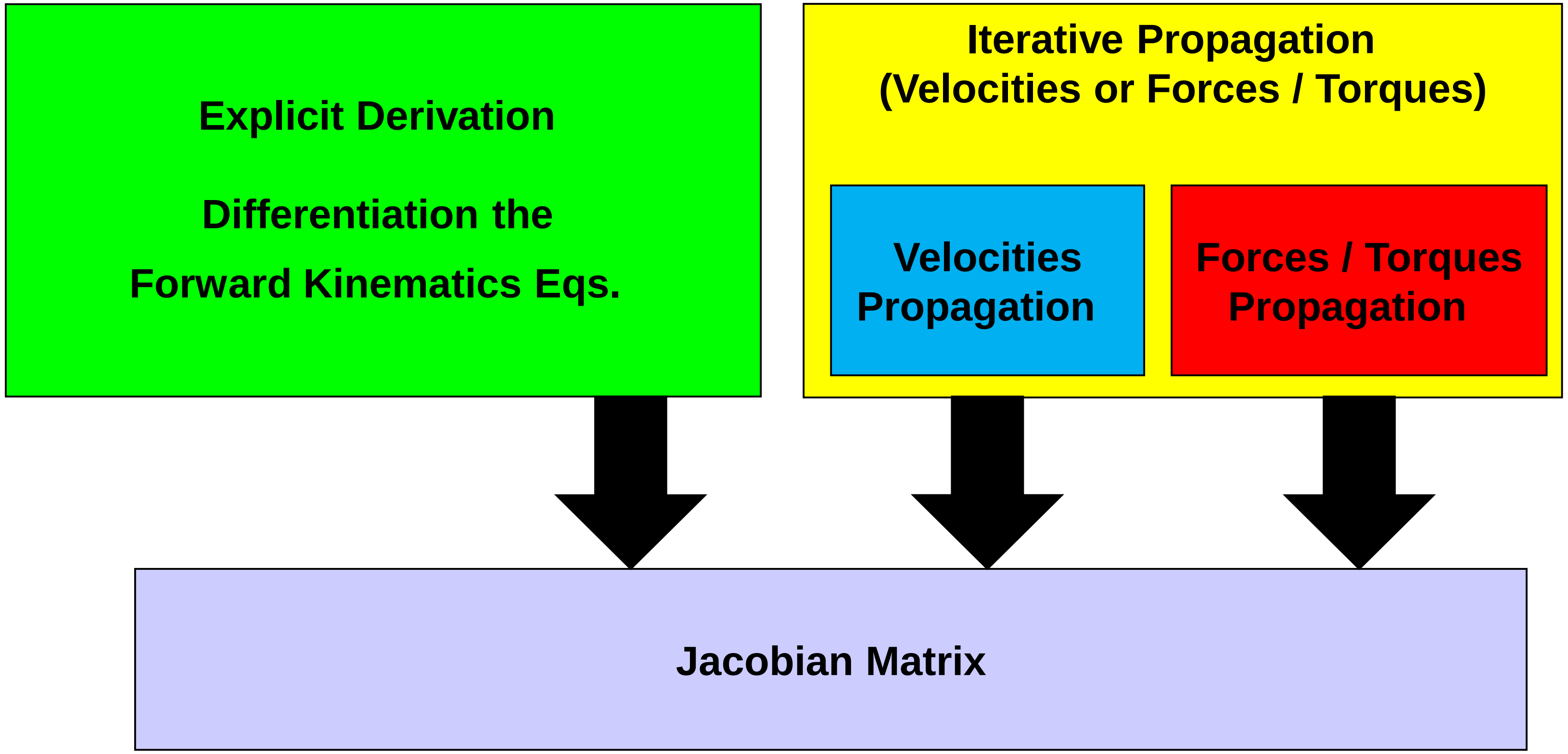
$$R_{z_4}(\theta_4)\ {}^4_6R = \left[{}^0_3R\ R_{x_3}(\alpha_3)\right]^{-1}\ {}^0_6R$$

GIVEN FOR EVERY POINT ALONG THE TRAJECTORY

GIVE

Advanced Kinematics
Linear and Angular Velocities

Jacobian Matrix - Calculation Methods



Jacobian Matrix - Introduction

In the field of robotics the Jacobian matrix describe the relationship between

- The joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$).

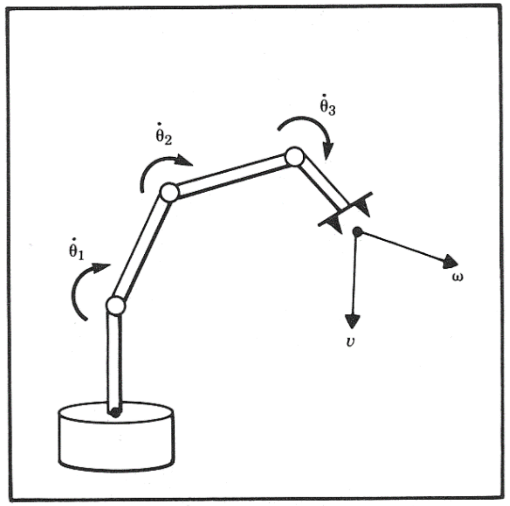
$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

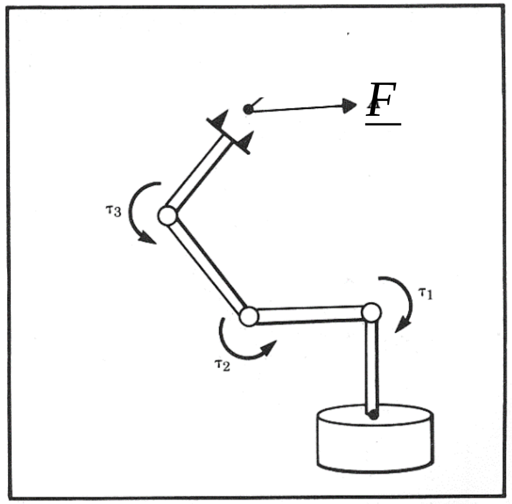
$$\underline{\dot{\theta}} = J^{-1}(\underline{\theta})\underline{\dot{x}}$$

- The robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

$$\underline{F} = \left(J(\underline{\theta})^T\right)^{-1} \underline{\tau}$$



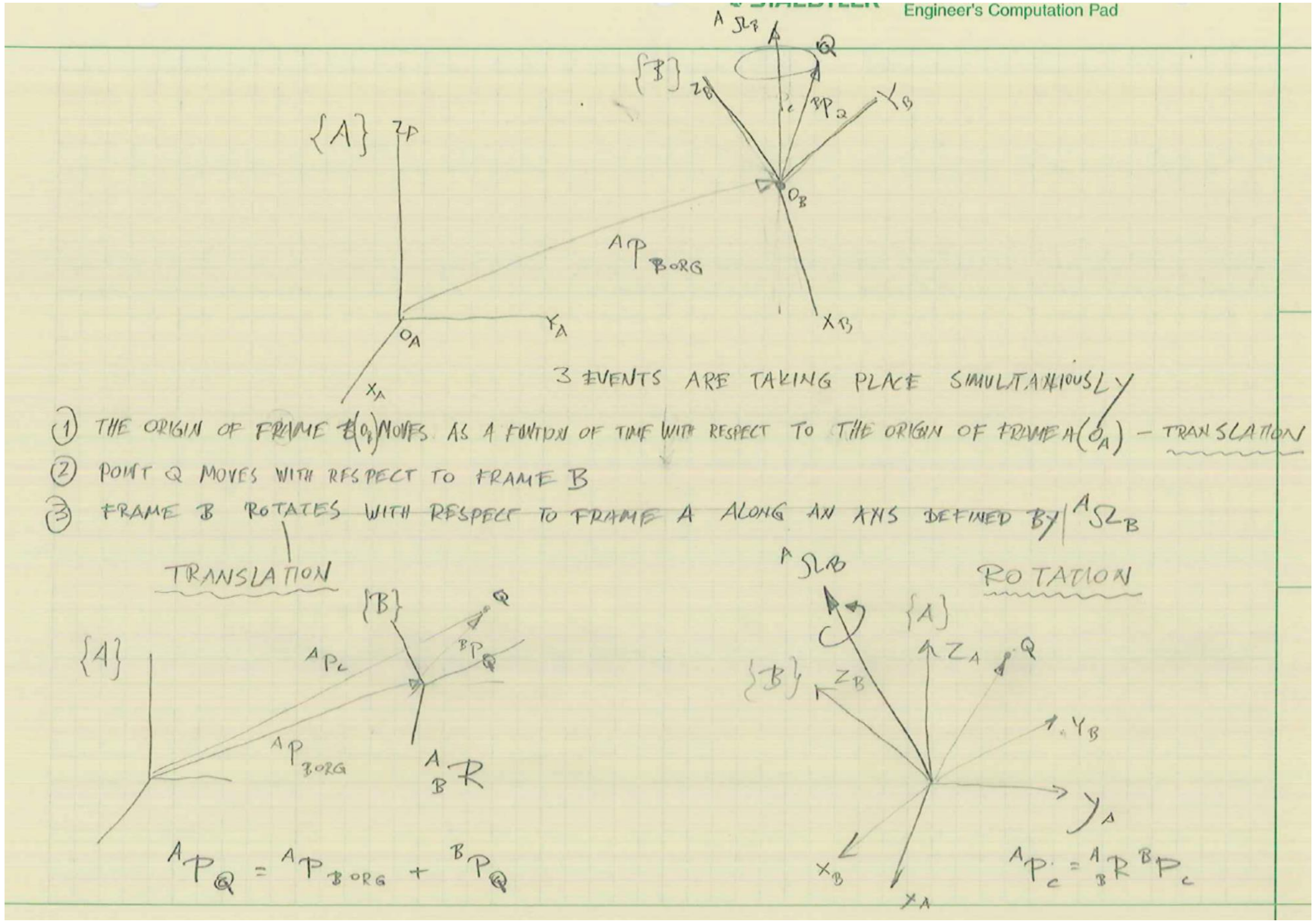


Velocity Propagation – Intuitive Explanation

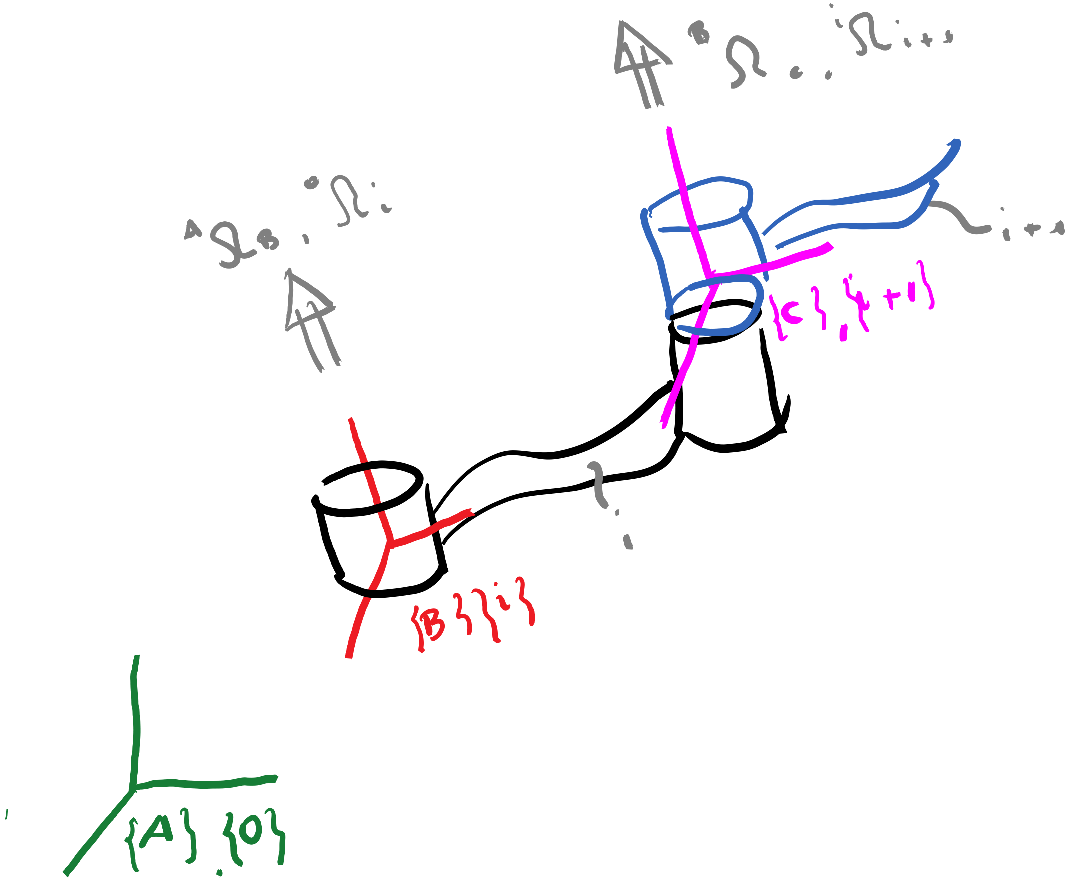
- Show a demo with the stick like frames

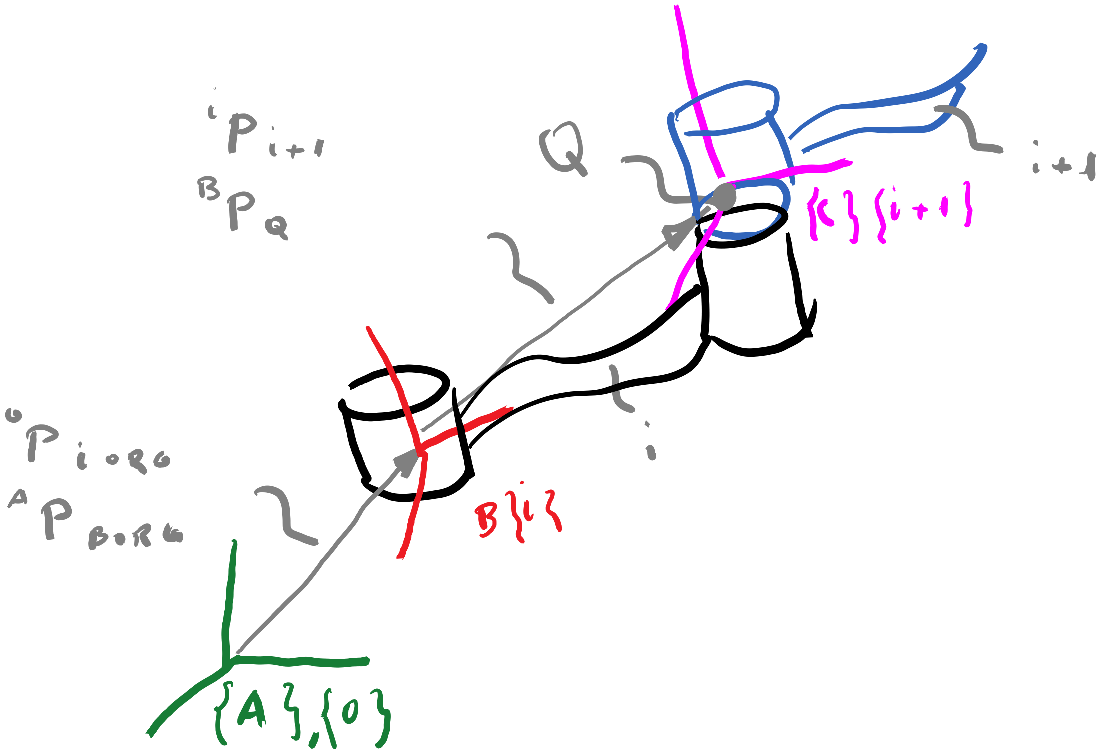
- Three Actions
  - The origin of frame B moves as a function of time with respect to the origin of frame A
  - Point Q moves with respect to frame B
  - Frame B rotates with respect to frame A along an axis defined by ${}^{A}\Omega_{B}$

Velocity Propagation – Intuitive Explanation

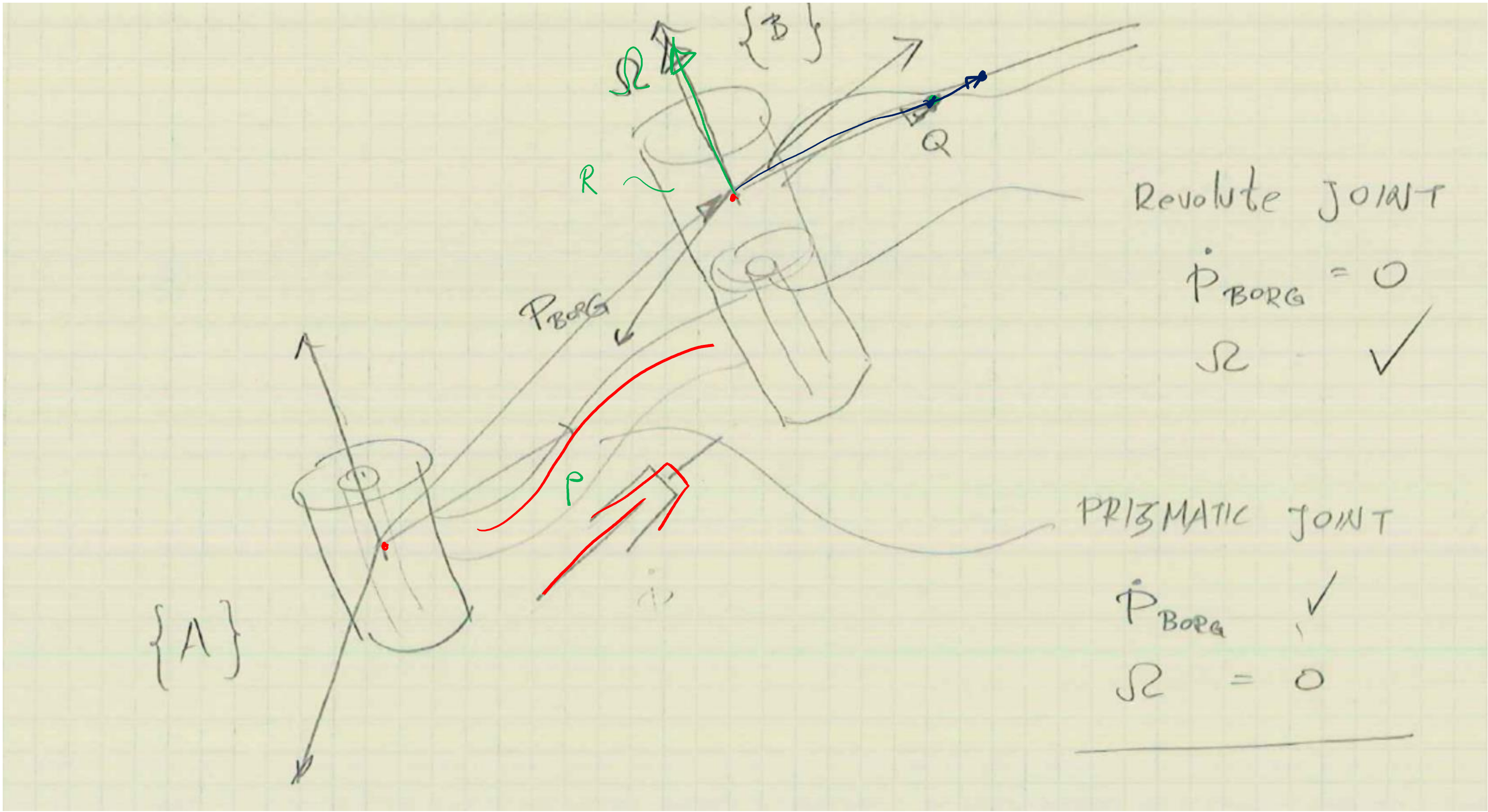


Velocity Propagation – Link / Joint Abstraction





Velocity Propagation – Link / Joint Abstraction



Central Topic - Simultaneous Linear and Rotational Velocity

$$^{A}V_Q = f\left(^{B}P_Q,\ \ ^{B}V_Q,\ \ ^{A}V_{BORG},\ \ ^{A}\Omega_B,\ \ ^{A}_{B}R\right)$$

- Vector Form (Method No. 1)

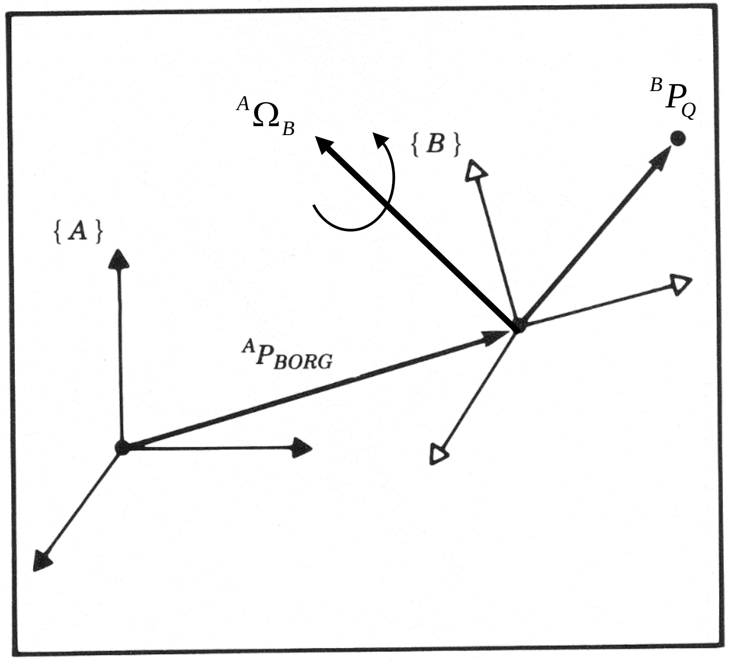
$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

- Matrix Form (Method No. 2)

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

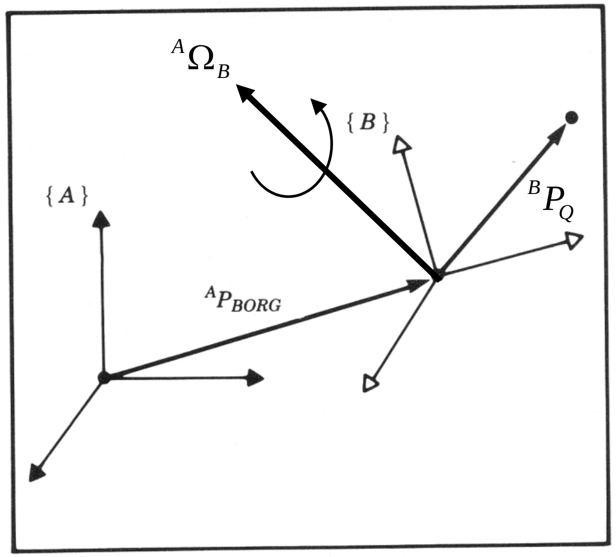
- Matrix Formulation – Homogeneous Transformation Form – Method No. 3

$$\begin{bmatrix} \left[{}^{A}V_Q\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\cdot{}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_Q\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\,org}\right] \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_Q\right] \\ 0 \end{bmatrix}$$



Central Topic - Changing Frame of Representation – Angular Velocity

- Angular Velocity Representation in Various Frames
  - Vector Form $${}^A\Omega_C = {}^A\Omega_B + {}^A_B R\, {}^B\Omega_C$$
  - Matrix Form $${}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_B R\, {}^B_C\dot{R}_\Omega\, {}^A_B R^T$$



Velocity – Derivation Method No. 1 & 2

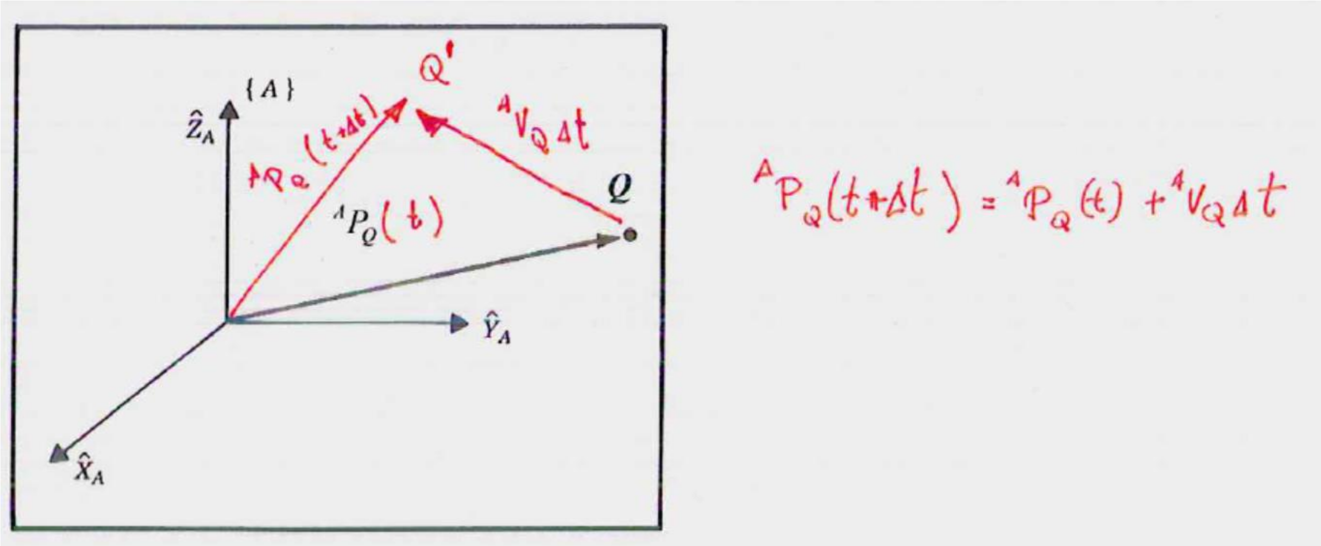
Vector Form

Matrix Form

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Definitions - Linear Velocity

- ***Linear velocity*** - The instantaneous rate of change in linear position of a point relative to some frame.



$$ {}^{A}V_Q = \frac{d}{dt}\,{}^{A}P_Q \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_Q(t+\Delta t) - {}^{A}P_Q(t)}{\Delta t} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

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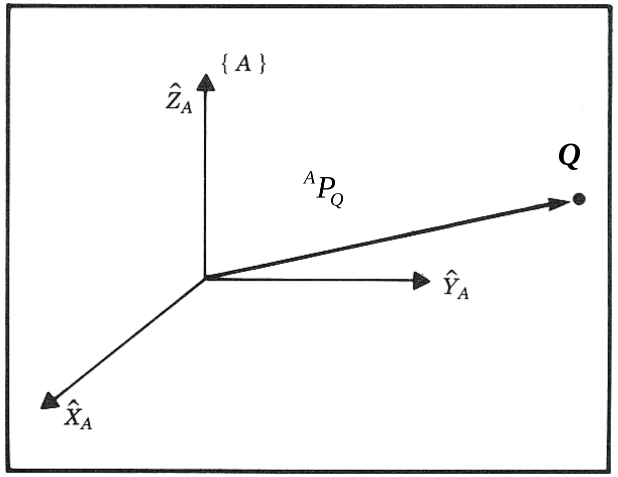

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Definitions - Linear Velocity

- The position of point Q in frame {A} is represented by the ***linear position vector***

$$ {}^{A}P_{Q} = \begin{bmatrix} {}^{A}P_{Qx} \\ {}^{A}P_{Qy} \\ {}^{A}P_{Qz} \end{bmatrix} $$

- The velocity of a point Q relative to frame {A} is represented by the ***linear velocity vector***

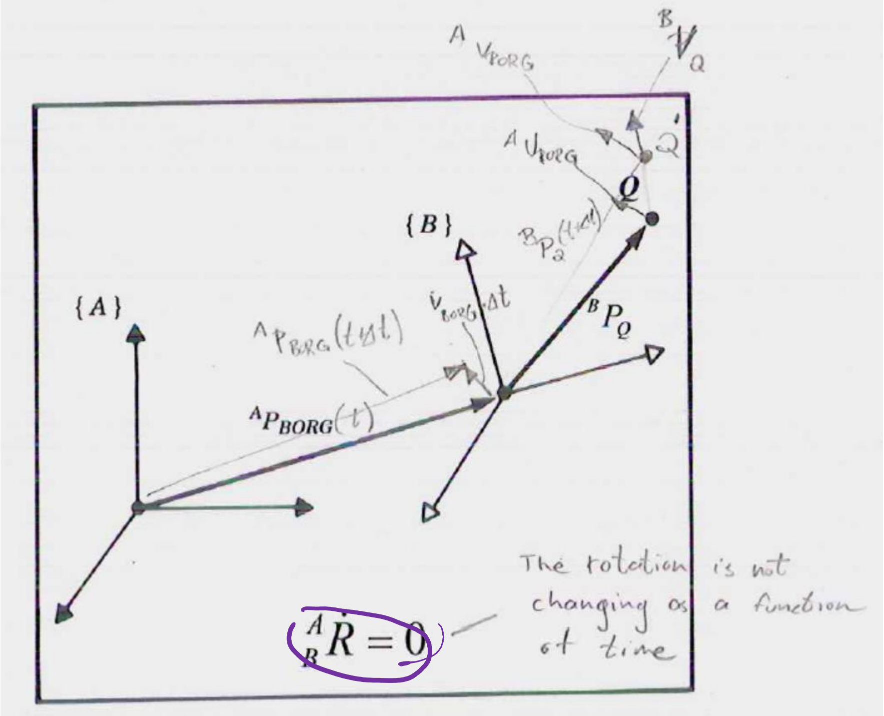
$$ {}^{A}V_{Q} = \frac{{}^{A}d}{dt} \begin{bmatrix} {}^{A}P_{Qx} \\ {}^{A}P_{Qy} \\ {}^{A}P_{Qz} \end{bmatrix} = \begin{bmatrix} {}^{A}\dot{P}_{Qx} \\ {}^{A}\dot{P}_{Qy} \\ {}^{A}\dot{P}_{Qz} \end{bmatrix} $$

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Linear Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The orientation of frame {A} with respect to frame {B} is not changing as a function of time $^{A}_{B}\dot{R} = 0$
- ***Problem:*** describe the motion of of the vector $^{B}P_{Q}$ relative to frame {A}
- ***Solution:*** Frame {B} is located relative to frame {A} by a position vector $^{A}P_{BORG}$ and the rotation matrix $^{A}_{B}R$ (assume that the orientation is not changing in time $^{A}_{B}\dot{R} = 0$) expressing both components of the velocity in terms of frame {A} gives



$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}({}^{B}V_{Q}) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

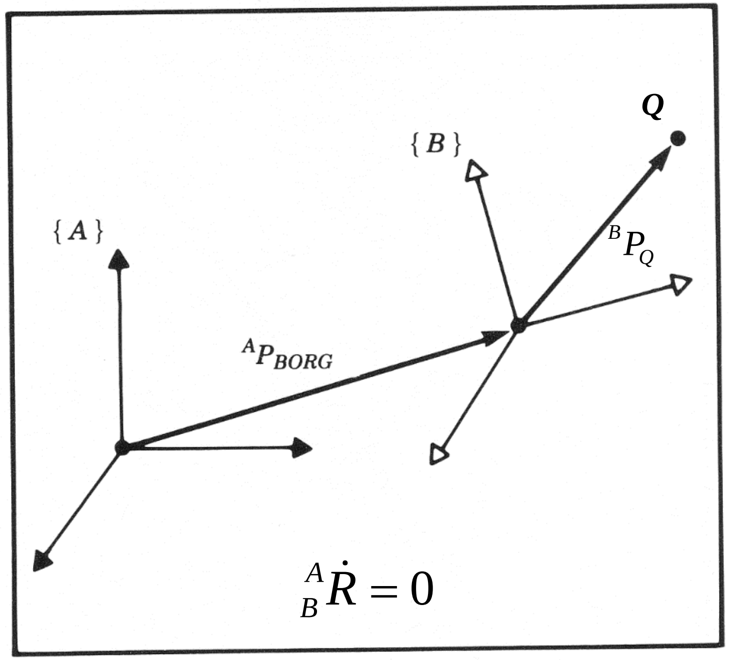
$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Linear Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The orientation of frame {A} with respect to frame {B} is not changing as a function of time $^{A}_{B}\dot{R} = 0$

- ***Problem:*** describe the motion of of the vector $^{B}P_Q$ relative to frame {A}

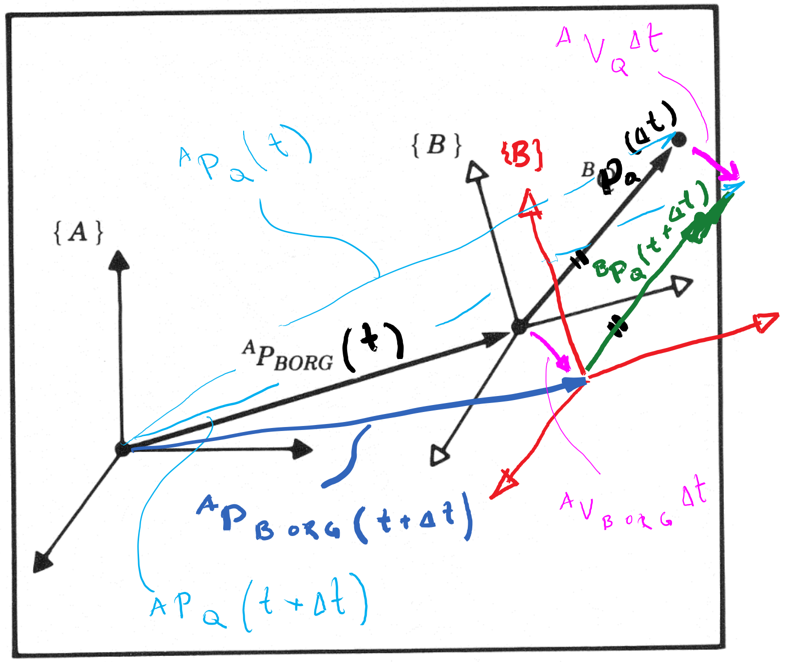
- ***Solution:*** Frame {B} is located relative to frame {A} by a position vector $^{A}P_{BORG}$ and the rotation matrix $^{A}_{B}R$ (assume that the orientation is not changing in time $^{A}_{B}\dot{R} = 0$) expressing both components of the velocity in terms of frame {A} gives



$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}({}^{B}V_Q) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q$$

Linear Velocity – Translation (No Rotation)- Problem 1 Derivation

- **Problem No. 1** – Change in a position of Point Q
- Conditions
  - Point Q is fixed in frame {B}
  - Frame {B} translates with respect to Frame {A}

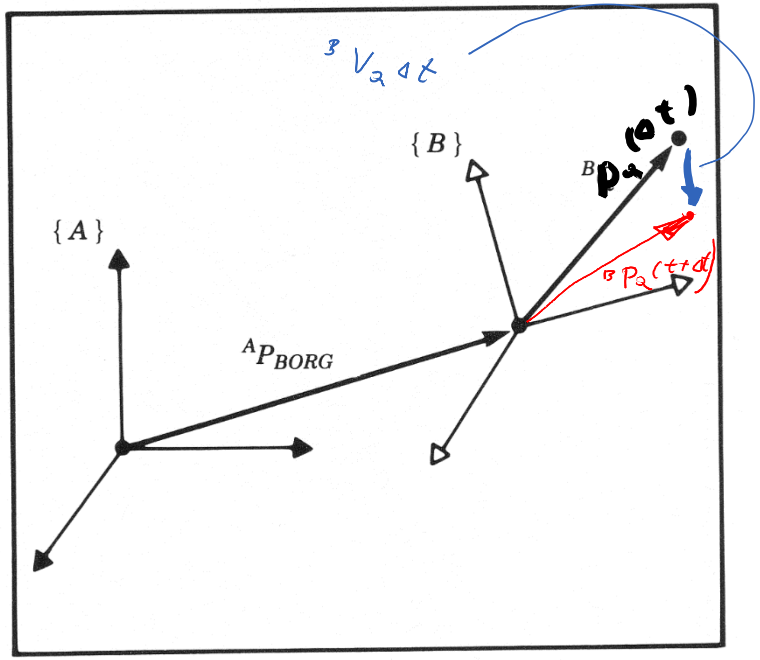


$$\frac{^Bd}{dt}\left(^BP_Q\right) \approx \lim_{\Delta t\to 0} {}^B\left(\frac{\overbrace{^AP_Q(t+\Delta t) - {}^AP_Q(t)}^{=0}}{\Delta t}\right) = {}^B\left(^BV_Q\right) = 0$$

$$\frac{^Ad}{dt}\left(^AP_Q\right) \approx \lim_{\Delta t\to 0} {}^A\left(\frac{^AP_Q(t+\Delta t) - {}^AP_Q(t)}{\Delta t}\right) = {}^A\left(^AV_Q\right) = {}^AV_Q = {}^AV_{BORG}$$

Linear Velocity – Translation (No Rotation) – Problem 2 Derivation

- **Problem No. 2** – Translation of frame {B}
- Conditions
  - Point Q is fixed in frame {B}
  - Frame {B} translates with respect to Frame {A}



$$\frac{{}^Ad}{dt}\left(\overbrace{{}^AP_{B\,ORG}}^{Const}\right) \approx \lim_{\Delta t\to 0} {}^A\left(\frac{\overbrace{{}^AP_{B\,ORG}(t+\Delta t) - {}^AP_{B\,ORG}(t)}^{=0}}{\Delta t}\right) = {}^A\left({}^AV_{B\,ORG}\right) = {}^AV_{B\,ORG} = 0$$

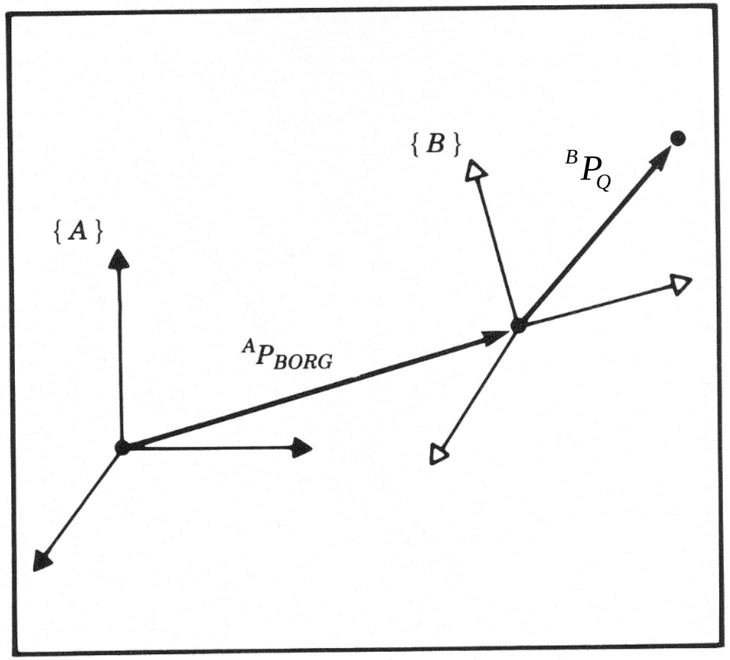
$$\frac{{}^Ad}{dt}\left({}^BP_Q\right) \approx \lim_{\Delta t\to 0} {}^A\left(\frac{{}^BP_Q(t+\Delta t) - {}^BP_Q(t)}{\Delta t}\right) = {}^A\left({}^BV_Q\right)$$

$${}^AV_Q = {}^A_BR\,{}^BV_Q$$

Linear Velocity – Translation (No Rotation) – Problem 1&2 -Derivation Summary

- **Problem No. 1** – Change in a position of Point Q
- **Problem No. 2** – Translation of frame {B}

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}({}^{B}V_{Q}) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} $$



Linear Velocity – Translation – Simultaneous Derivation

$${}^{A}P_{Q} = {}^{A}P_{BORG} + {}^{B}P_{Q}$$

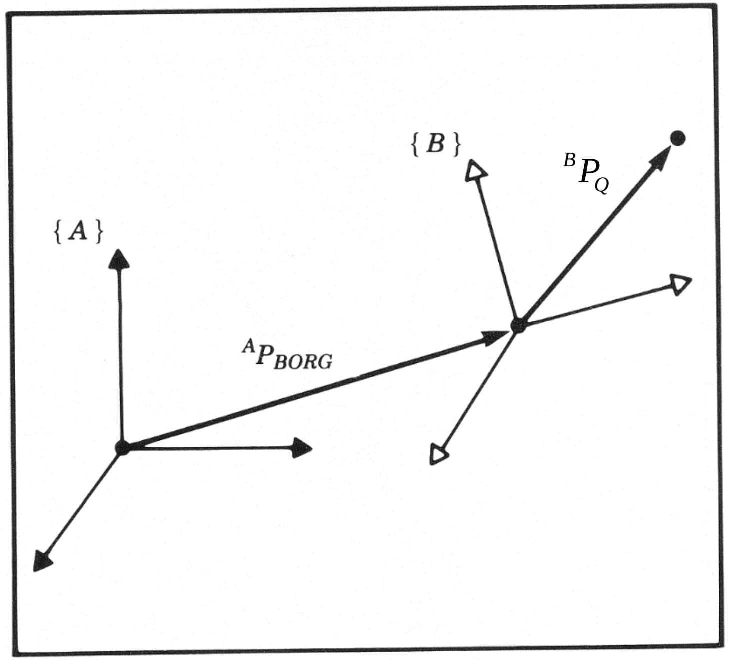
- Differentiate with respect to coordinate system {A}

$$\frac{{}^{A}d}{dt}\left({}^{A}P_{Q}\right) = \frac{{}^{A}d}{dt}\left({}^{A}P_{BORG}\right) + \frac{{}^{A}d}{dt}\left({}^{B}P_{Q}\right)$$

$${}^{A}\left({}^{A}\dot{P}_{Q}\right) = {}^{A}\left({}^{A}\dot{P}_{BORG}\right) + {}^{A}\left({}^{B}\dot{P}_{Q}\right)$$

$${}^{A}\left({}^{A}V_{Q}\right) = {}^{A}\left({}^{A}V_{BORG}\right) + {}^{A}\left({}^{B}V_{Q}\right)$$

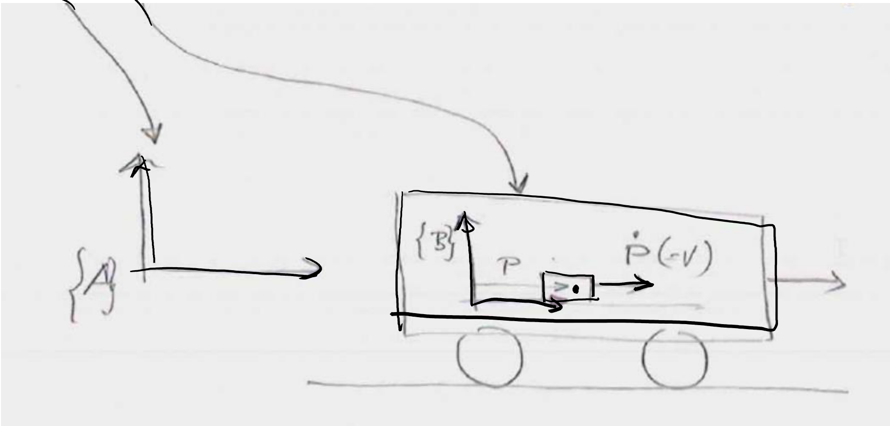
$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}\left({}^{B}V_{Q}\right) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$



$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Linear & Angular Velocities - Frames

- When describing the velocity (linear or angular) of an object, there are two important frames that are being used:
  - ***Represented Frame (Reference Frame)*** : e.g. [A]
    This is the frame used to represent **represent (express)** the object's velocity.
  - ***Computed Frame***: e.g. [B]
    This is the frame in which the velocity is **measured** (differentiate the position).



$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Frame - Velocity

- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.
- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

Represented (projected) in fram {A}

Represented (projected) in frame {B}

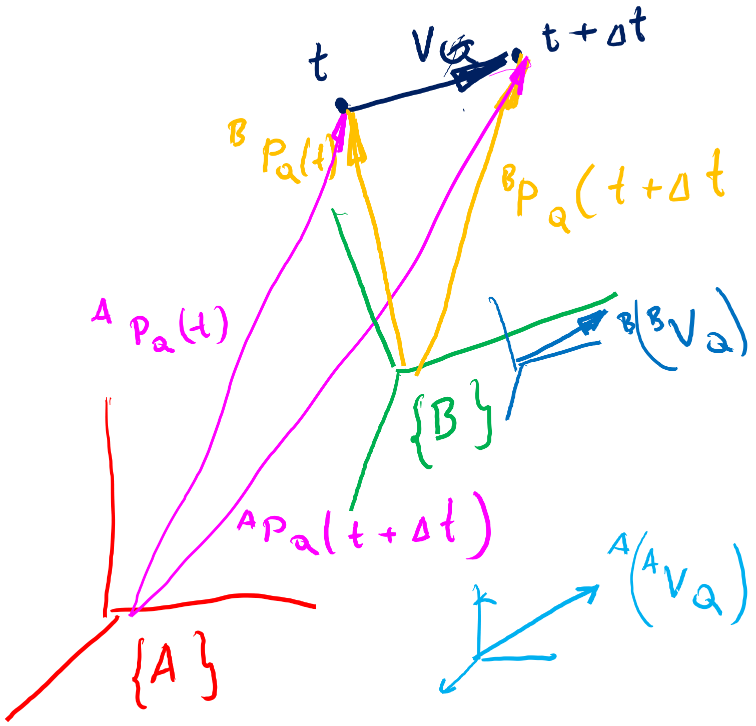
Represented
(Reference Frame) -– Projected on

$$^{Represented}\left(^{Computed}V_{Q}\right) = \frac{^{Represented}d}{dt}\,{}^{Computed}P_{Q}$$

Computed
(Measured) - Differentiate with respect to

computed in frame {B}

computed in fram {A}



$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Frame - Linear Velocity

- We can always remove the outer, leading superscript by explicitly including the rotation matrix which accomplishes the change in the reference frame

$$ {}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

- Note that in the general case ${}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q \neq {}^{A}V_Q$ because ${}^{A}_{B}R$ may be time-verging ${}^{A}_{B}\dot{R} \neq 0$

- If the calculated velocity is written in terms of of the frame of differentiation the result could be indicated by a single leading superscript.

$$ {}^{A}({}^{A}V_Q) = {}^{A}V_Q $$

- In a similar fashion when the angular velocity is expresses and measured as a vector

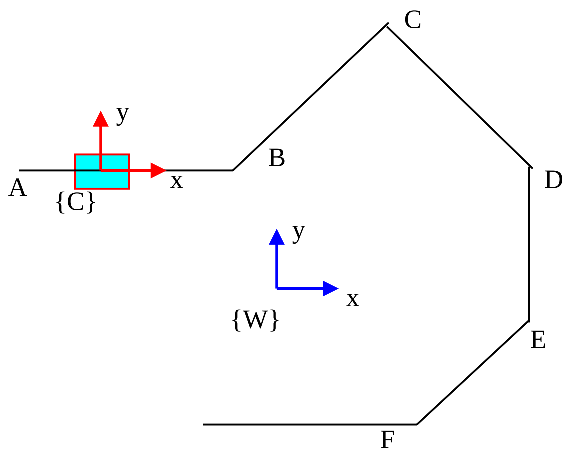
$$ {}^{A}({}^{B}\Omega_C) = {}^{A}_{B}R\,{}^{B}\Omega_C $$

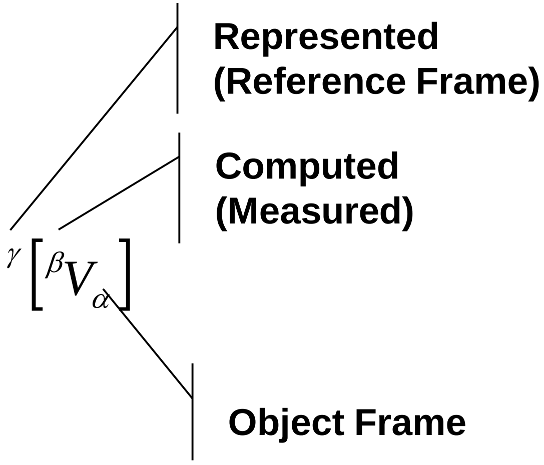
$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Frames - Linear Velocity - Example

- ***Given:*** The driver of the car maintains a ***speed*** of 100 km/h (as shown to the driver by the car's speedometer).

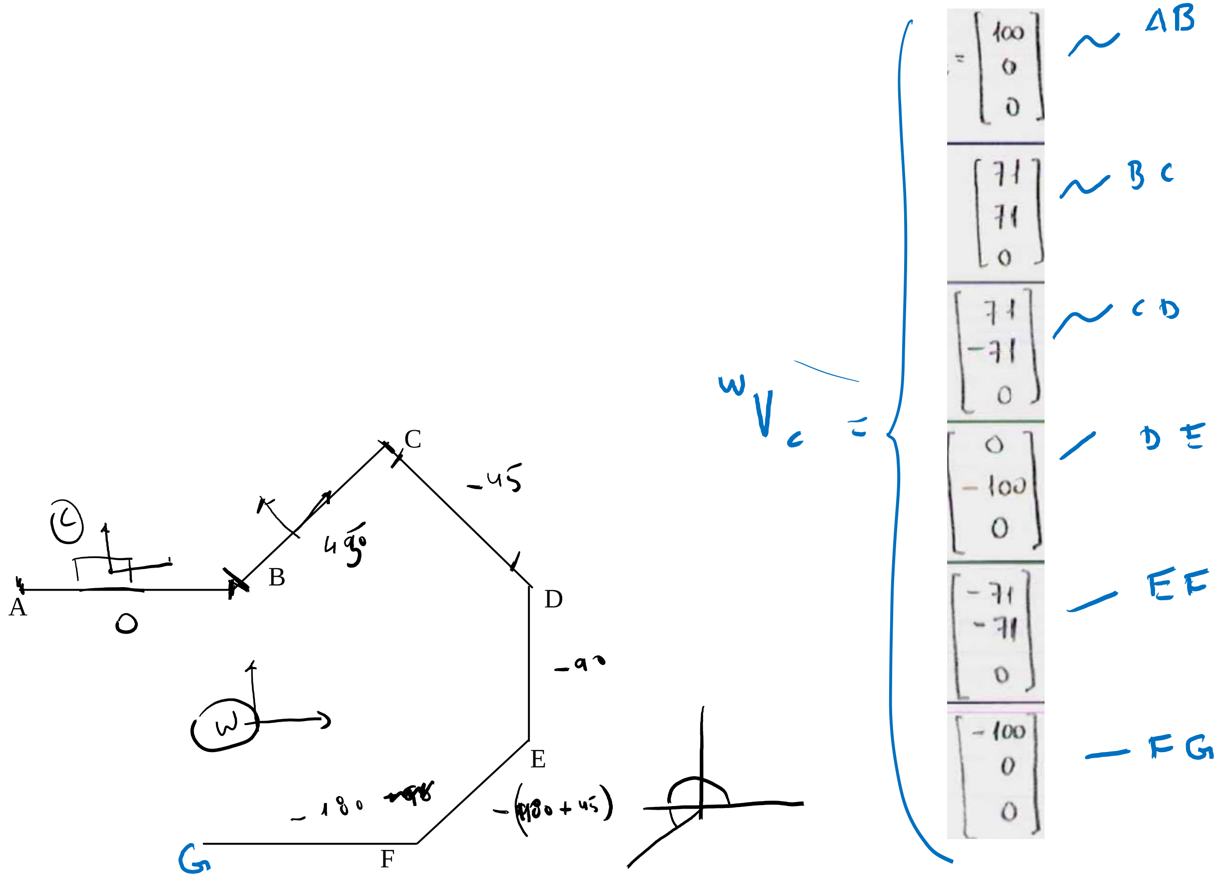
- ***Problem:*** Express the velocities $^{C}\left[{}^{C}V_C\right]$ $^{W}\left[{}^{W}V_C\right]$ $^{W}\left[{}^{C}V_C\right]$ $^{C}\left[{}^{W}V_C\right]$ in each section of the road A, B, C, D, E, F where {C} - Car frame, and {W} - World frame





$$^{\gamma}\left[{}^{\beta}V_{\alpha}\right]$$

Frames - Linear Velocity - Example



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

$$ {}^{A}_{B}R = Rot(\hat{z},\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ Rot(\hat{z},+45^{o}) = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},-45^{o}) = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},+90^{o}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Rot(\hat{z},-90^{o}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\, {}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\, {}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\, {}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

$$ {}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\, {}^{B}V_{Q} $$

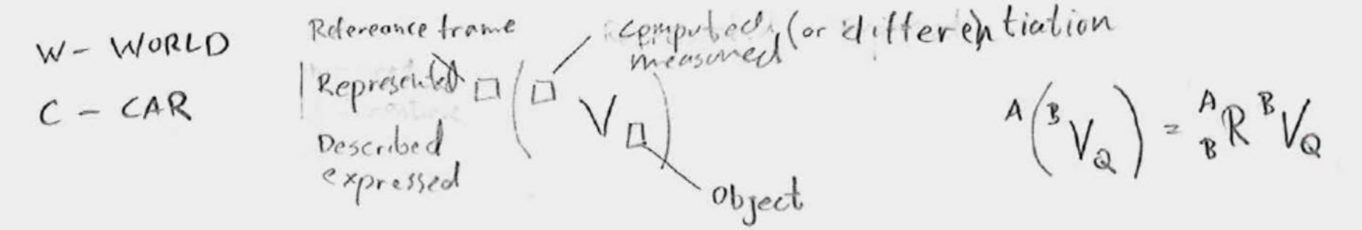
- ${}^{A}_{B}\dot{R} = 0$ is not time-varying (in this example)

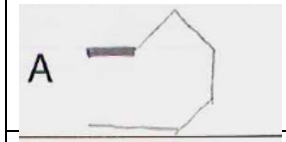
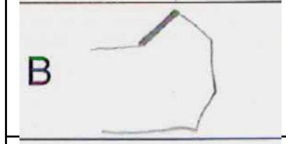
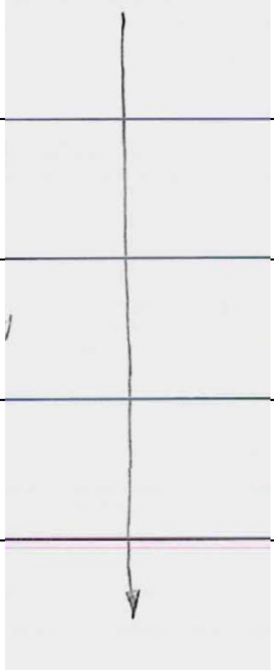
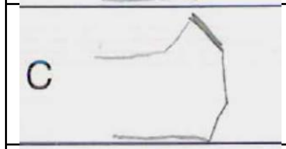
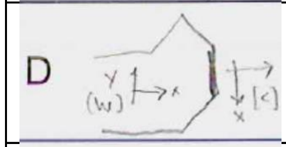
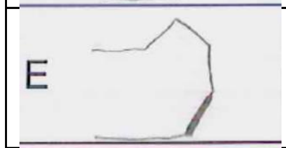
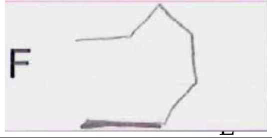
$$ {}^{C}({}^{C}V_{C}) = {}^{C}_{C}R\, {}^{C}V_{C} = I[0] = [0] $$

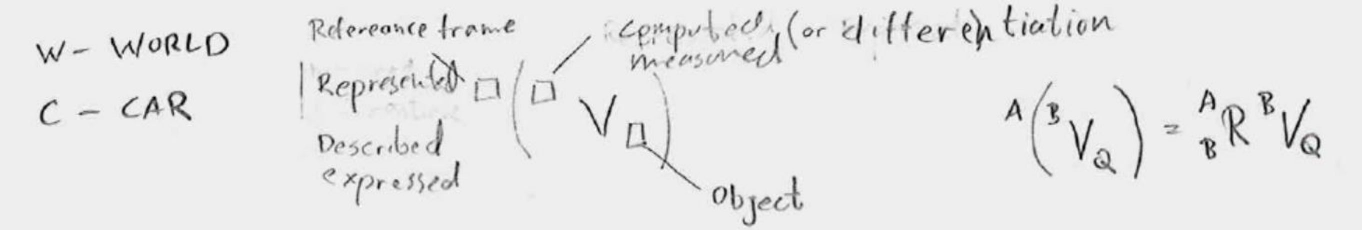
$$ {}^{W}({}^{W}V_{C}) = {}^{W}_{W}R\, {}^{W}V_{C} = I\, {}^{W}V_{C} $$

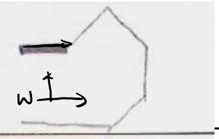
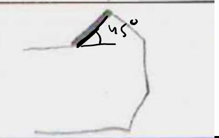
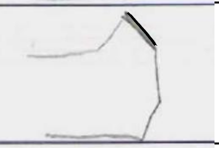
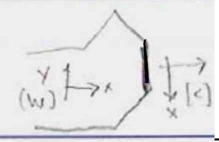
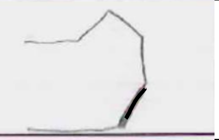
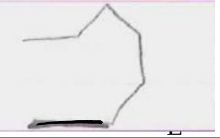
$$ {}^{W}({}^{C}V_{C}) = {}^{W}_{C}R\, {}^{C}V_{C} = {}^{W}_{C}R[0] = [0] $$

$$ {}^{C}({}^{W}V_{C}) = {}^{C}_{W}R\, {}^{W}V_{C} $$



Road Section	Velocity			
	$^C[^C V_C]$	$^W[^W V_C]$	$^W[^C V_C]$	$^C[^W V_C]$
A 	${}^C_C R\, {}^C V_C = I\, {}^C V_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$			
B 				
C 				
D 				
E 				
F 				

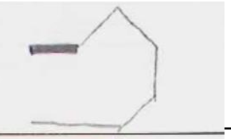
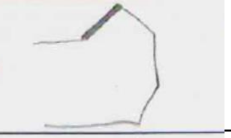
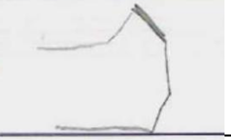
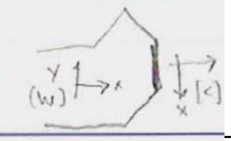
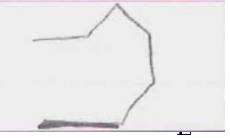


Road Section	Velocity			
	${}^C[{}^C V_C]$	${}^W[{}^W V_C]$	${}^W[{}^C V_C]$	${}^C[{}^W V_C]$
A 	${}^C_C R\, {}^C V_c = I\, {}^C V_c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	${}^W_W R\, {}^W V_c = I\, {}^W V_c = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$		
B 	↓	$\begin{bmatrix} 71 \\ 71 \\ 0 \end{bmatrix}$		
C 	↓	$\begin{bmatrix} 71 \\ -71 \\ 0 \end{bmatrix}$		
D 	↓	$\begin{bmatrix} 0 \\ -100 \\ 0 \end{bmatrix}$		
E 	↓	$\begin{bmatrix} -71 \\ -71 \\ 0 \end{bmatrix}$		
F 	↓	$\begin{bmatrix} -100 \\ 0 \\ 0 \end{bmatrix}$		

W - WORLD
C - CAR

Reference frame
Represented / Described / expressed $\square\,(\square\, V_{\square})$ — computed / measured (or differentiation) — Object

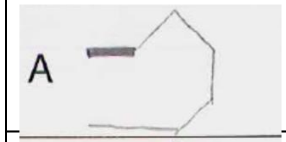
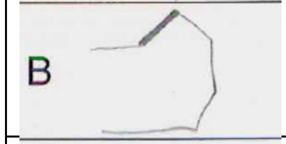
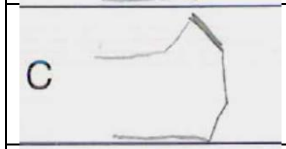
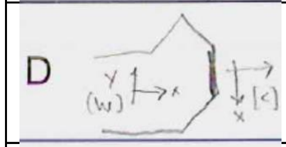
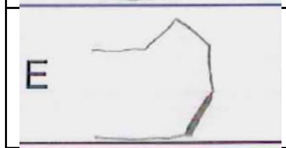
$$^A(^B V_Q) = {}^A_B R\,{}^B V_Q$$

Road Section	Velocity			
	${}^C[{}^C V_C]$	${}^W[{}^W V_C]$	${}^W[{}^C V_C]$	${}^C[{}^W V_C]$
A 	${}^C_C R\,{}^C V_C = I\,{}^C V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^W_W R\,{}^W V_C = I\,{}^W V_C = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^W_C R\,{}^C V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
B 	↓	$\begin{bmatrix}71\\71\\0\end{bmatrix}$	↓	
C 	↓	$\begin{bmatrix}71\\-71\\0\end{bmatrix}$	↓	
D 	↓	$\begin{bmatrix}0\\-100\\0\end{bmatrix}$	↓	
E 	↓	$\begin{bmatrix}-71\\-71\\0\end{bmatrix}$	↓	
F 	↓	$\begin{bmatrix}-100\\0\\0\end{bmatrix}$	↓	

W - WORLD
C - CAR

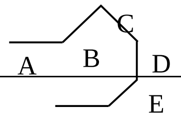
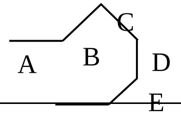
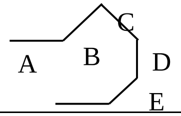
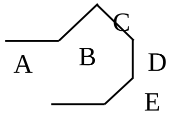
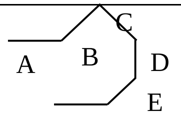
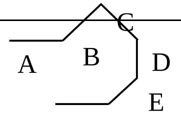
Refereance frame
Represented / Described / expressed $\square \left({}^{\square} V_{\square} \right)$ — computed / measured (or differentiation); Object

$${}^{A}\left({}^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

Road Section	Velocity			
	${}^{C}\left[{}^{C}V_{C}\right]$	${}^{W}\left[{}^{W}V_{C}\right]$	${}^{W}\left[{}^{C}V_{C}\right]$	${}^{C}\left[{}^{W}V_{C}\right]$
A 	${}^{C}_{C}R\,{}^{C}V_{C} = I\,{}^{C}V_{C} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_{C} = I\,{}^{W}V_{C} = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_{C} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},0)\,[I]\begin{bmatrix}100\\0\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
B 	↓	$\begin{bmatrix}71\\71\\0\end{bmatrix}$	↓	$R(\hat{z},-45)\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}.707 & .707 & 0\\-.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
C 	↓	$\begin{bmatrix}71\\-71\\0\end{bmatrix}$	↓	$R(\hat{z},+45)\begin{bmatrix}71\\-71\\0\end{bmatrix} = \begin{bmatrix}.707 & -.707 & 0\\.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\-71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
D 	↓	$\begin{bmatrix}0\\-100\\0\end{bmatrix}$	↓	$R(\hat{z},+90)\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
E 	↓	$\begin{bmatrix}-71\\-71\\0\end{bmatrix}$	↓	↓
F 	↓	$\begin{bmatrix}-100\\0\\0\end{bmatrix}$	↓	↓

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

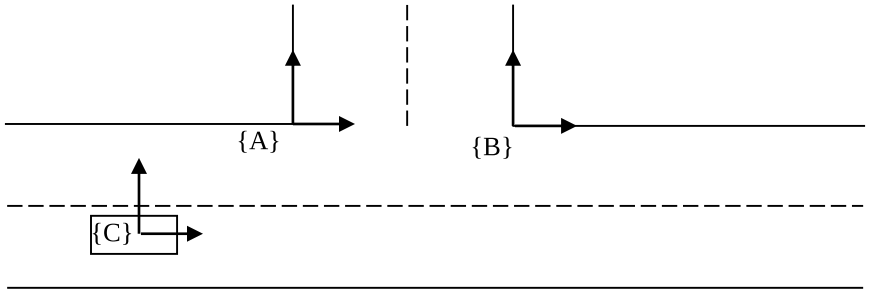
Frames - Linear Velocity - Example

Road Section	**Velocity**			
	${}^{C}[{}^{C}V_{C}]$	${}^{W}[{}^{W}V_{C}]$	${}^{W}[{}^{C}V_{C}]$	${}^{C}[{}^{W}V_{C}]$
A				
B				
C				
D				
E				
F				

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Linear Velocity - Free Vector

- Linear velocity vectors are insensitive to shifts in origin.
- Consider the following example:



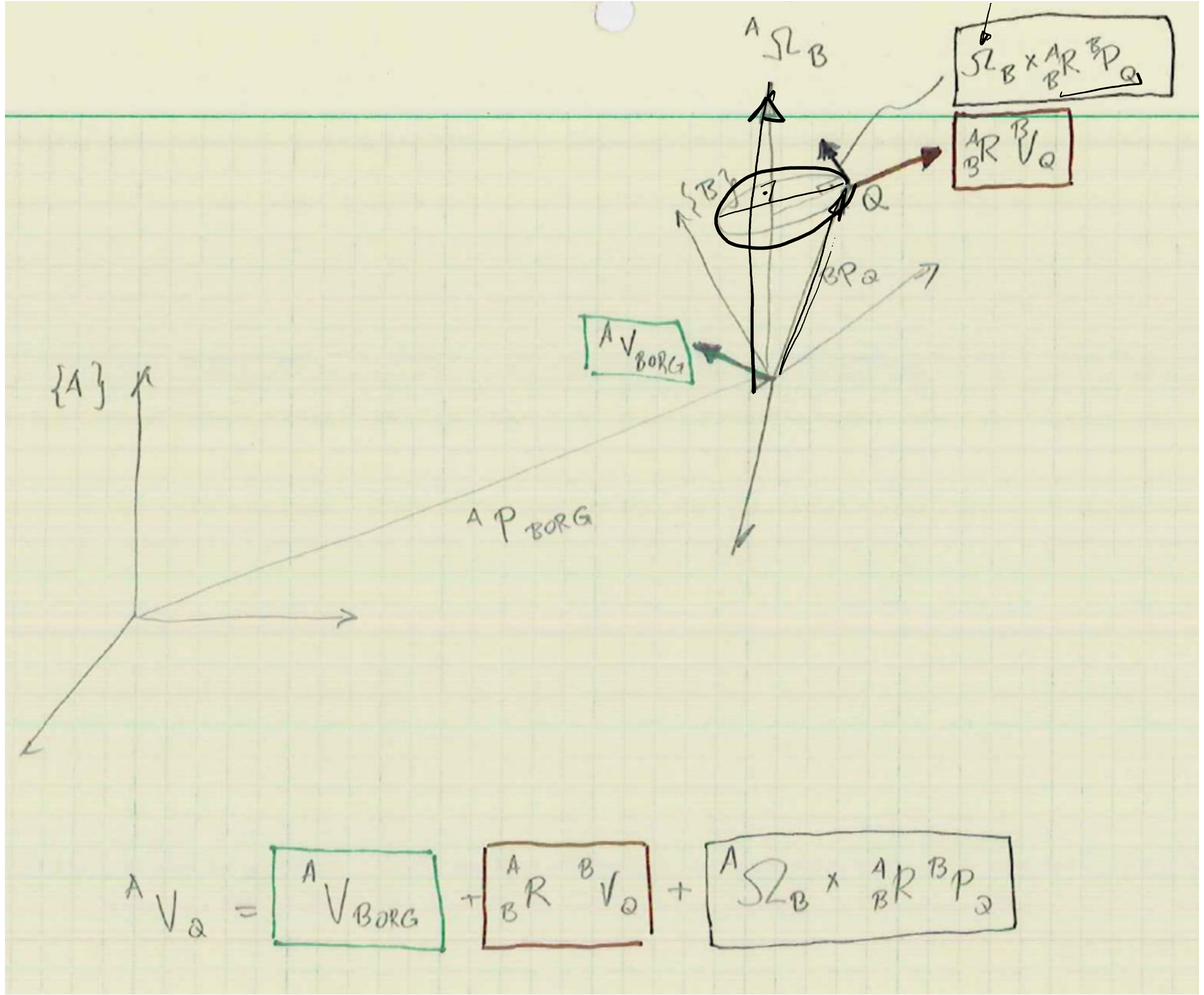
- The velocity of the object in {C} relative to both {A} and {B} is the same, that is

$$ {}^{A}V_{C} = {}^{B}V_{C} $$

- As long as {A} and {B} remain fixed relative to each other (translational but not rotational), then the velocity vector remains unchanged (that is, a ***free vector***).

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach



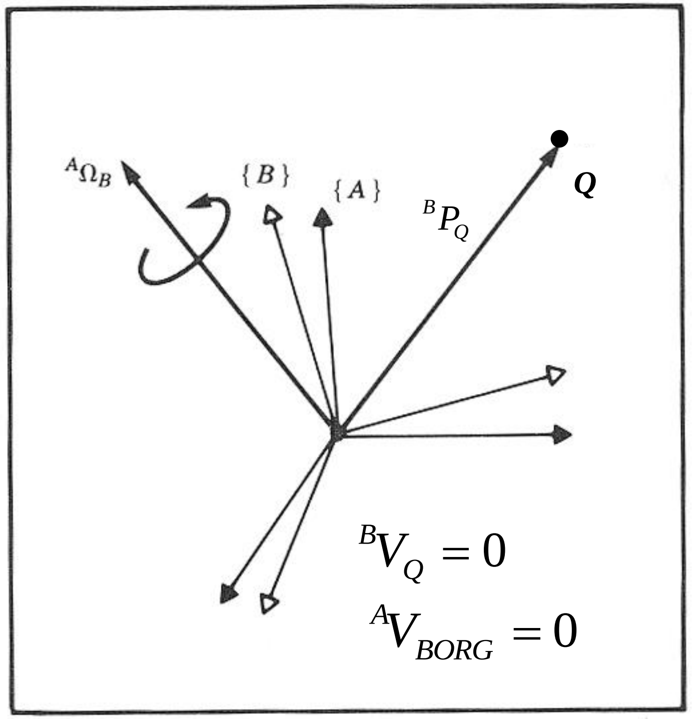
$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The vector ${}^{B}P_Q$ is constant as view from frame {B} ${}^{B}V_Q = 0$

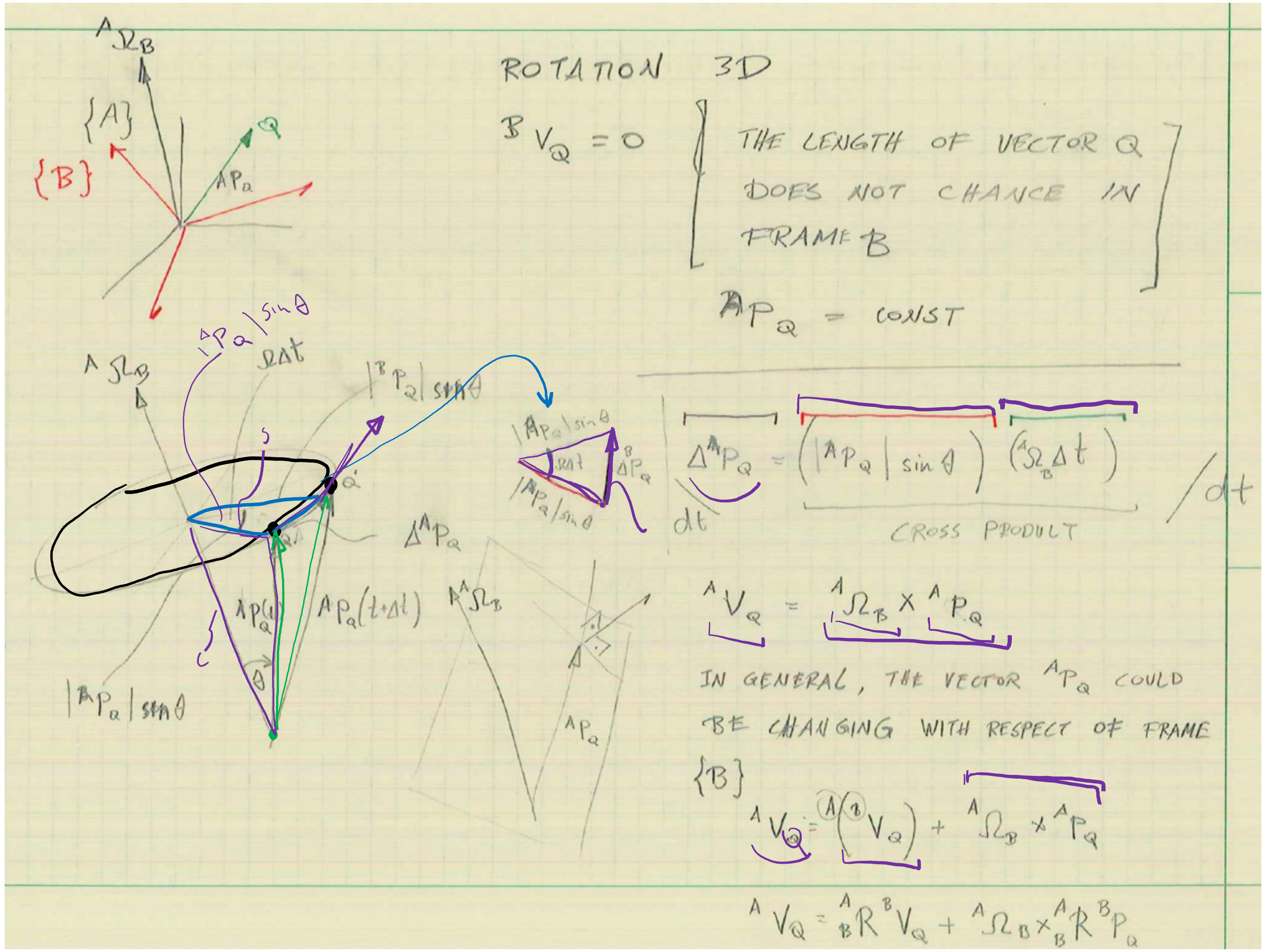
- ***Problem:*** describe the velocity of the vector ${}^{B}P_Q$ representing the the point **Q** relative to frame {A}

- ***Solution:*** Even though the vector ${}^{B}P_Q$ is constant as view from frame {B} it is clear that point **Q** will have a velocity as seen from frame {A} due to the rotational velocity ${}^{A}\Omega_B$



$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach



$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

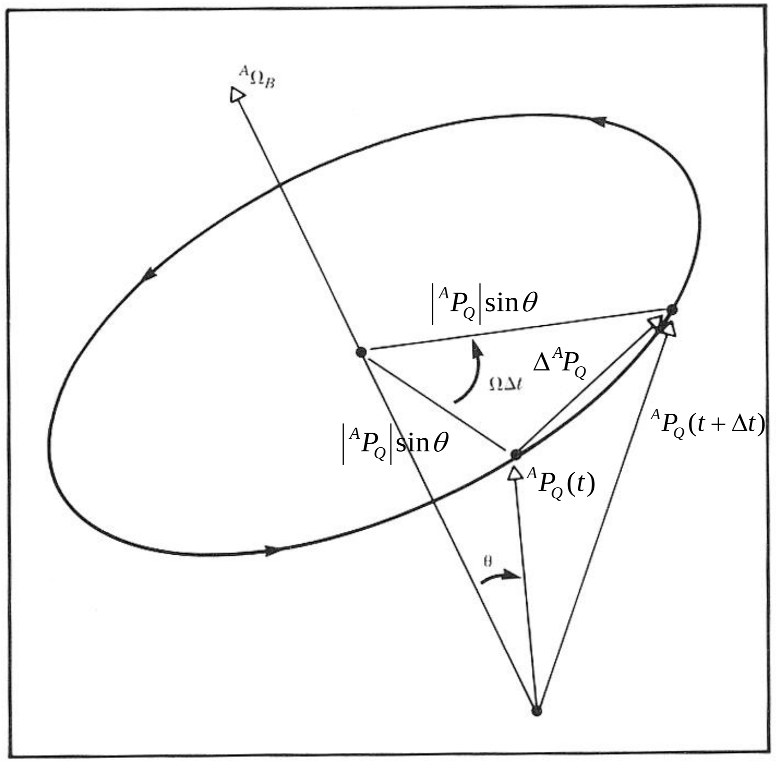
- The figure shows to instants of time as the vector ${}^{A}P_Q$ rotates around ${}^{A}\Omega_B$ This is what an observer in frame {A} would observe.

- The Magnitude of the differential change is

$$ \left|\Delta\, {}^{A}P_Q\right| = \left(\left|{}^{A}\Omega_B\right|\Delta t\right)\left(\left|{}^{A}P_Q\right|\sin\theta\right) $$

- Using a vector cross product we get

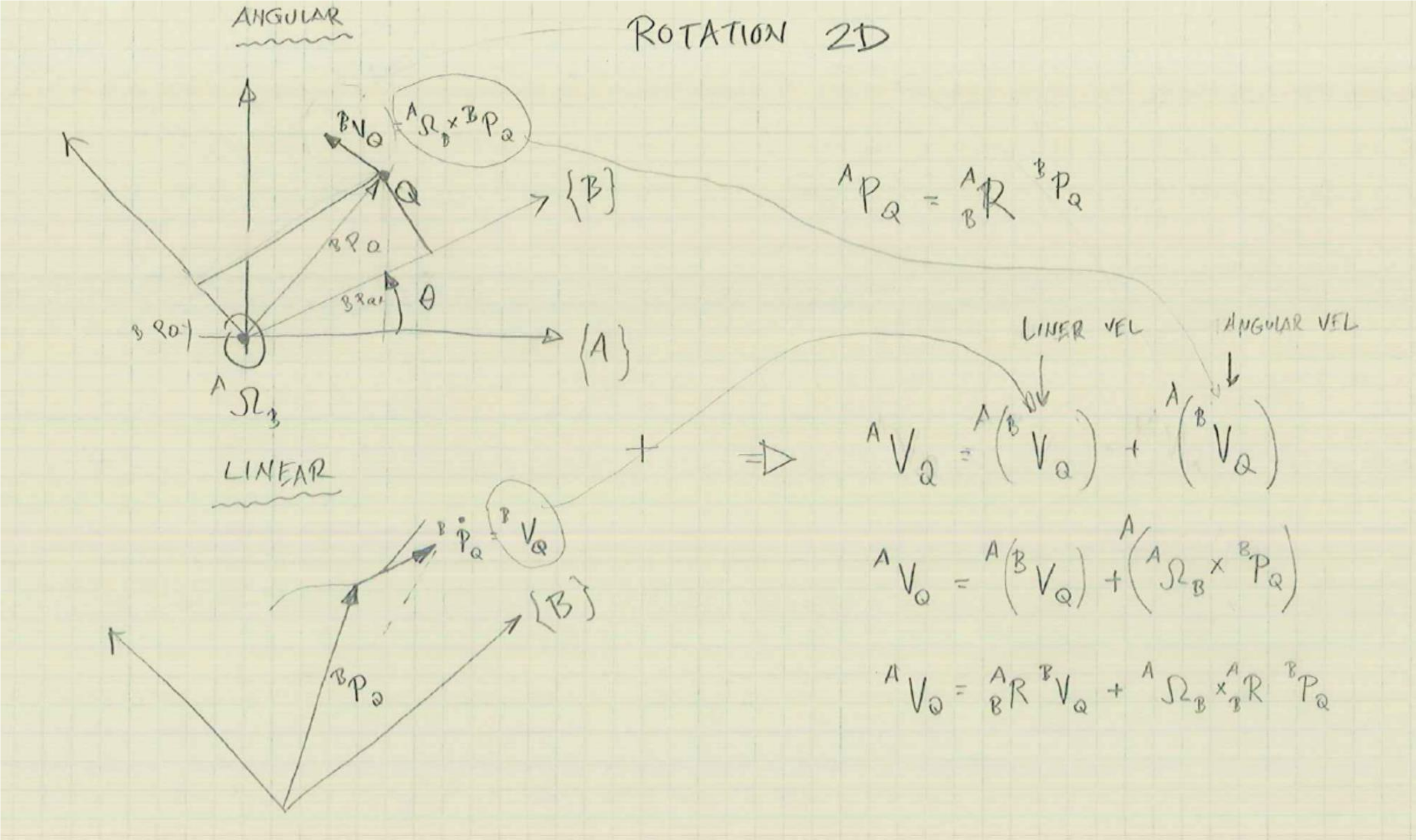
$$ \frac{\Delta\, {}^{A}P_Q}{\Delta t} = {}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}P_Q $$



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

- Rotation in 2D



$${}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach

- In the general case, the vector Q may also be changing with respect to the frame {B}. Adding this component we get.

$${}^AV_Q = {}^A\left({}^BV_Q\right) + {}^A\Omega_B \times {}^AP_Q$$

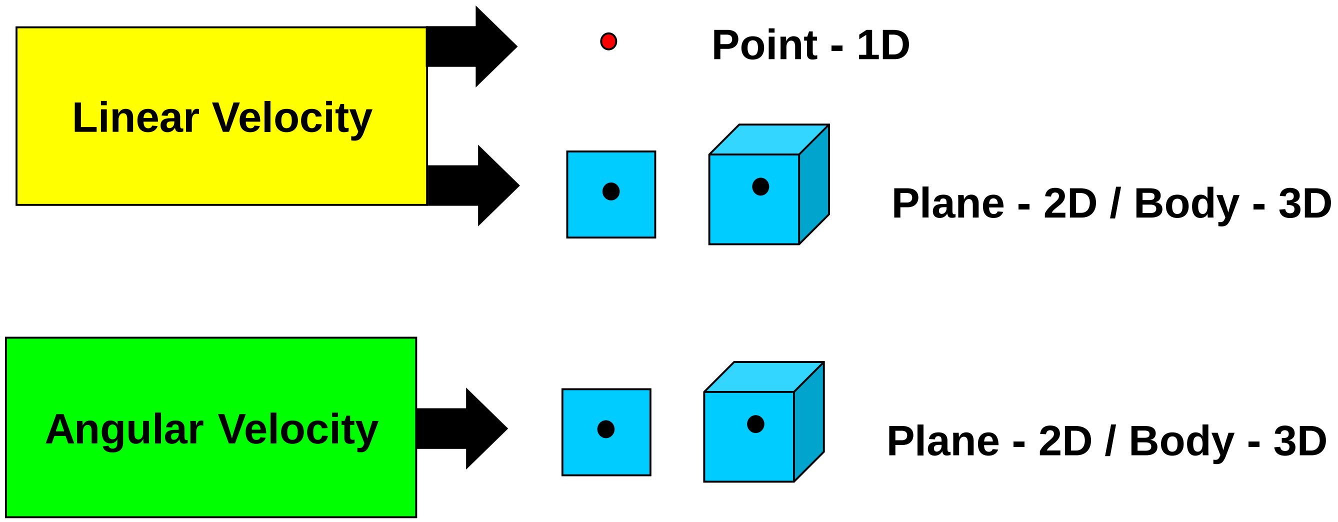
- Using the rotation matrix to remove the dual-superscript, and since the description of ${}^AP_Q$ at any instance is ${}^A_BR\,{}^BP_Q$ we get

$${}^AV_Q = {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q$$

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity

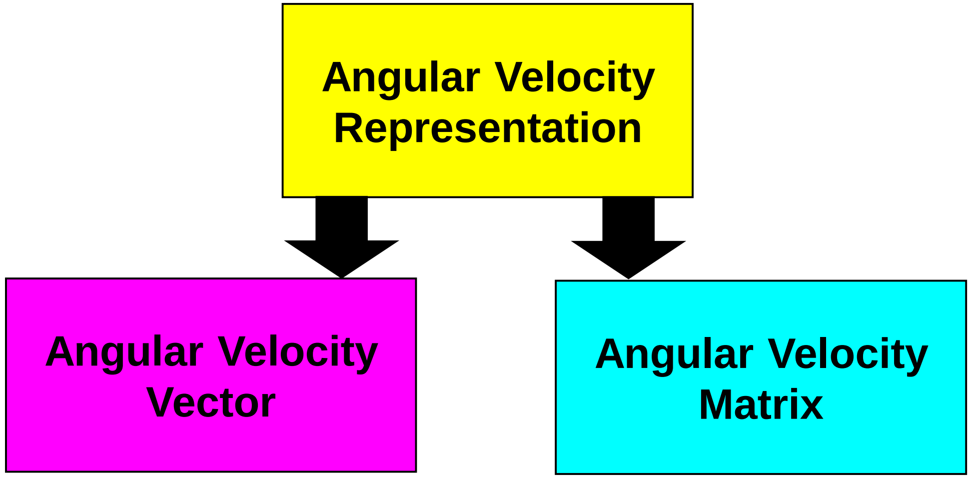
- ***Angular Velocity:*** The instantaneous rate of change in the orientation of one frame relative to another.



$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity

- Just as there are many ways to represent orientation (Euler Angles, Roll-Pitch-Yaw Angles, Rotation Matrices, etc.) there are also many ways to represent the rate of change in orientation.



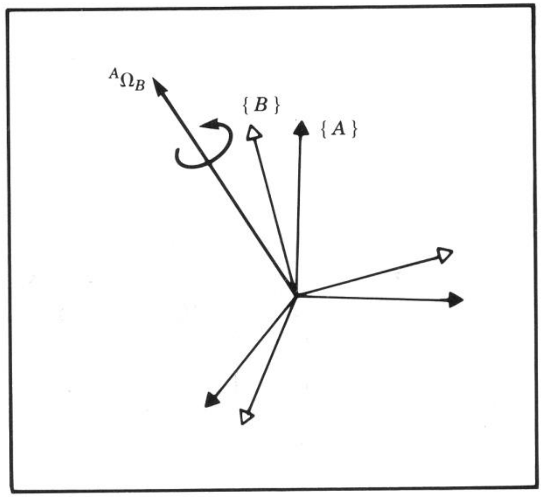
- The angular velocity vector is convenient to use because it has an easy to grasp physical meaning. However, the matrix form is useful when performing algebraic manipulations.

$${}^{A}V_Q = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}_{B}^{A}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_Q + {}_{B}^{A}\dot{R}_{\Omega}\left({}_{B}^{A}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity - Vector

- ***Angular Velocity Vector:*** A vector whose direction is the instantaneous axis of rotation of one frame relative to another and whose magnitude is the rate of rotation about that axis.

$${}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$



- The angular velocity vector ${}^{A}\Omega_B$ describes the instantaneous change of rotation of frame {B} relative to frame {A}

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + \boxed{{}^{A}\Omega_{B} \times} {}_{B}^{A}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + \boxed{{}_{B}^{A}\dot{R}_{\Omega}} \left({}_{B}^{A}R\,{}^{B}P_{Q} \right) $$

Definitions - Angular Velocity - Matrix

- ***Angular Velocity Matrix:***

$$ \left[{}_{B}^{A}\dot{R}_{\Omega} \right] \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{matrix} -\Omega_z y + \Omega_y z \\ \Omega_z x - \Omega_x z \\ -\Omega_y x + \Omega_x y \end{matrix} $$

$$ {}^{A}\Omega_{B} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} i & j & w \\ \Omega_x & \Omega_y & \Omega_z \\ x & y & z \end{vmatrix} = \begin{Bmatrix} \Omega_y z - \Omega_z y \\ -\Omega_x z + \Omega_z x \\ \Omega_x y - \Omega_y x \end{Bmatrix} $$

$$^{A}V_{Q}={}^{A}V_{BORG}+{}^{A}_{B}R\,{}^{B}V_{Q}+{}^{A}\Omega_{B}\times{}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q}={}^{A}V_{BORG}+{}^{A}_{B}R\,{}^{B}V_{Q}+{}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Definitions - Angular Velocity - Matrix

- The rotation matrix ($^{A}_{B}R$) defines the orientation of frame {B} relative to frame {A}. Specifically, the columns of $^{A}_{B}R$ are the unit vectors of {B} represented in {A}.

$$^{A}_{B}R={}^{A}\begin{bmatrix}\left[{}^{B}P_{x}\right] & \left[{}^{B}P_{y}\right] & \left[{}^{B}P_{z}\right]\end{bmatrix}$$

- If we look at the derivative of the rotation matrix, the columns will be the velocity of each unit vector of {B} relative to {A}:

$$^{A}_{B}\dot{R}=\frac{d}{dt}\left[{}^{A}_{B}R\right]={}^{A}\begin{bmatrix}\left[{}^{B}V_{x}\right] & \left[{}^{B}V_{y}\right] & \left[{}^{B}V_{z}\right]\end{bmatrix}$$

$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_\Omega \left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity - Matrix

- The relationship between the rotation matrix ${}^{A}_{B}R$ and the derivative of the rotation matrix ${}^{A}_{B}\dot{R}$ can be expressed as follows:

$V = w \times R$

$${}^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_\Omega \, {}^{A}_{B}R$$

$${}^{A}\left[\left[{}^{B}V_x\right] \; \left[{}^{B}V_y\right] \; \left[{}^{B}V_z\right] \right] = {}^{A}_{B}\dot{R}_\Omega \; {}^{A}\left[\left[{}^{B}P_x\right] \; \left[{}^{B}P_y\right] \; \left[{}^{B}P_z\right] \right]$$

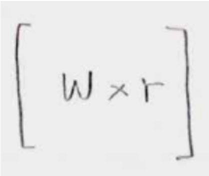
- where ${}^{A}_{B}\dot{R}_\Omega$ is defined as the ***angular velocity matrix***

$${}^{A}_{B}\dot{R}_\Omega \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \qquad {}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Angular Velocity - Matrix & Vector Forms

	Matrix Form	Vector Form
Definition	${}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	${}^{A}\Omega_{B} \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$
Multiply by Constant	$k\left[{}^{A}_{B}\dot{R}_{\Omega}\right]$	$k\left[{}^{A}\Omega_{B}\right]$
Multiply by Vector	$\left[{}^{A}_{B}\dot{R}_{\Omega}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	${}^{A}\Omega_{B} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Multiply by Matrix	$\left[{}^{S}_{t}R\right]\left[{}^{A}_{B}\dot{R}_{\Omega}\right]\left[{}^{S}_{t}R\right]^{T}$	$\left[{}^{S}_{t}R\right]\left[{}^{A}\Omega_{B}\right]$



$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Simultaneous Linear and Rotational Velocity - Vector Versus Matrix Representation

Vector Form

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

Matrix Form

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

$$\Omega \times P = \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ P_x & P_y & P_z \end{vmatrix} = i\,(\Omega_y P_z - \Omega_z P_y) - j\,(\Omega_x P_z - \Omega_z P_x) + k\,(\Omega_x P_y - \Omega_y P_x)$$

$$\dot{R}_{\Omega} P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} -\Omega_z P_y + \Omega_y P_z \\ \Omega_z P_x - \Omega_x P_z \\ -\Omega_y P_x + \Omega_x P_y \end{bmatrix}$$

Simultaneous Linear and Rotational Velocity

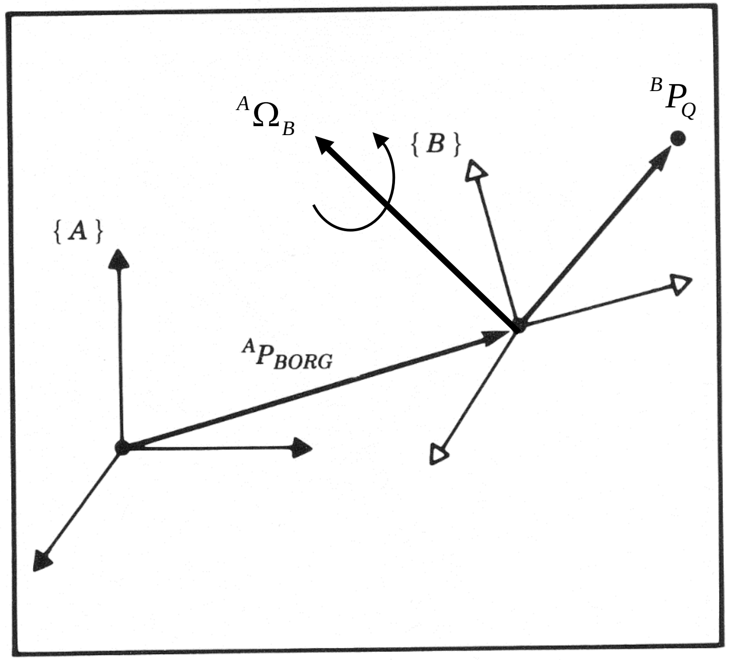
- The final results for the derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame

- Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



Velocity – Derivation Method No. 3

Homogeneous Transformation Form

Changing Frame of Representation - Linear Velocity

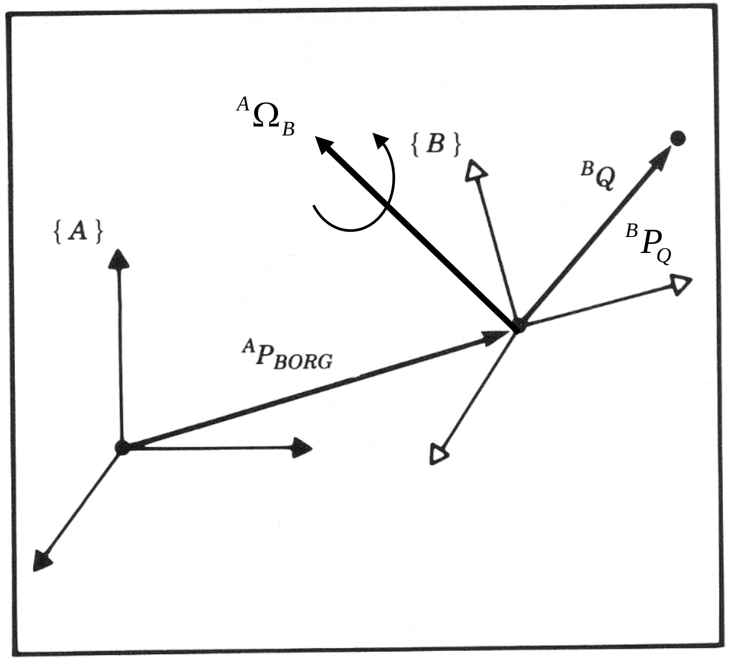
- We have already used the homogeneous transform matrix to compute the location of position vectors in other frames:

$$^{A}P_{Q} = {}^{A}_{B}T\, {}^{B}P_{Q}$$

- To compute the relationship between velocity vectors in different frames, we will take the derivative:

$$\frac{d}{dt}\left[{}^{A}P_{Q}\right] = \frac{d}{dt}\left[{}^{A}_{B}T\, {}^{B}P_{Q}\right]$$

$$^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\, {}^{B}P_{Q} + {}^{A}_{B}T\, {}^{B}\dot{P}_{Q}$$



Changing Frame of Representation - Linear Velocity

- Recall that

$$
{}^{A}_{B}T = \begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- so that the derivative is

$$
{}^{A}_{B}\dot{T} = \frac{d}{dt}\begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}\right] & & & \left[{}^{A}\dot{P}_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}R_{\Omega}\,{}^{A}_{B}R\right] & & & \left[{}^{A}V_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$ {}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q} $$

Changing Frame of Representation - Linear Velocity

$$ {}^{A}_{B}\dot{T} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0 \quad 0 \quad 0 & 0 \end{bmatrix} $$

- Substitute the previous results into the original equation ${}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q}$ we get

$$ \begin{bmatrix} \left[{}^{A}V_{Q}\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\cdot{}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_{Q}\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\,org}\right] \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_{Q}\right] \\ 0 \end{bmatrix} $$

- This expression is equivalent to the following three-part expression:

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

Changing Frame of Representation - Linear Velocity

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

- Converting from matrix to vector form yields

$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times \left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

Angular Velocity –
Changing Frame of Representation

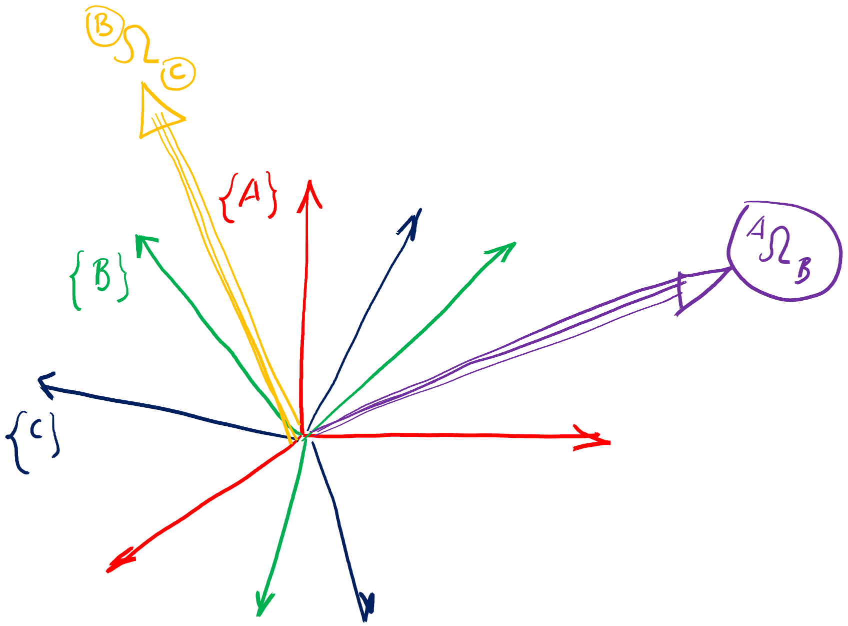
$$ {}^A\Omega_C = {}^A\Omega_B + {}^A_B R\, {}^B\Omega_C \qquad {}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_B R\, {}^B_C\dot{R}_\Omega\, {}^A_B R^T $$

Angular Velocity

- Frame {C} is rotated around frame {B} by ${}^B\Omega_C$
- Frame {B} is rotated around frame {A} by ${}^A\Omega_B$

- Given ${}^B\Omega_C$ ${}^A\Omega_B$

- Find ${}^A\Omega_C$



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$$ {}^A\Omega_C = {}^A\Omega_B + {}^A_BR\,{}^B\Omega_C \qquad {}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_BR\,{}^B_C\dot{R}_\Omega\,{}^A_BR^T $$

Changing Frame of Representation - Angular Velocity

- We use rotation matrices to represent angular position so that we can compute the angular position of {C} in {A} if we know the angular position of {C} in {B} and {B} in {A} by

$$ {}^A_CR = {}^A_BR\,{}^B_CR $$

- To derive the relationship describing how angular velocity propagates between frames, we will take the derivative

$$ {}^A_C\dot{R} = {}^A_B\dot{R}\,{}^B_CR + {}^A_BR\,{}^B_C\dot{R} $$

- Substituting the angular velocity matrixes

$$ {}^A_B\dot{R} = {}^A_B\dot{R}_\Omega\,{}^A_BR \qquad {}^B_C\dot{R} = {}^B_C\dot{R}_\Omega\,{}^B_CR \qquad {}^A_C\dot{R} = {}^A_C\dot{R}_\Omega\,{}^A_CR $$

- we find

$$ {}^A_C\dot{R}_\Omega\,{}^A_CR = {}^A_B\dot{R}_\Omega\,{}^A_BR\,{}^B_CR + {}^A_BR\,{}^B_C\dot{R}_\Omega\,{}^B_CR $$

$$ {}^A_C\dot{R}_\Omega\,{}^A_CR = {}^A_B\dot{R}_\Omega\,{}^A_CR + {}^A_BR\,{}^B_C\dot{R}_\Omega\,{}^B_CR $$

Changing Frame of Representation - Angular Velocity

- Post-multiplying both sides by ${}^A_C R^T$,which for rotation matrices, is equivalent to ${}^A_C R^{-1}$

$$ {}^A_C\dot{R}_\Omega \, {}^A_C R \, {}^A_C R^T = {}^A_B\dot{R}_\Omega \, {}^A_C R \, {}^A_C R^T + {}^A_B R \, {}^B_C\dot{R}_\Omega \, {}^B_C R \, {}^A_C R^T $$

$$ {}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_B R \, {}^B_C\dot{R}_\Omega \, {}^A_B R^T $$

- The above equation provides the relationship for changing the frame of representation of angular velocity matrices.

- The vector form is given by

$$ {}^A\Omega_C = {}^A\Omega_B + {}^A_B R \, {}^B\Omega_C $$

- To summarize, the angular velocities of frames may be added as long as they are expressed in the same frame.

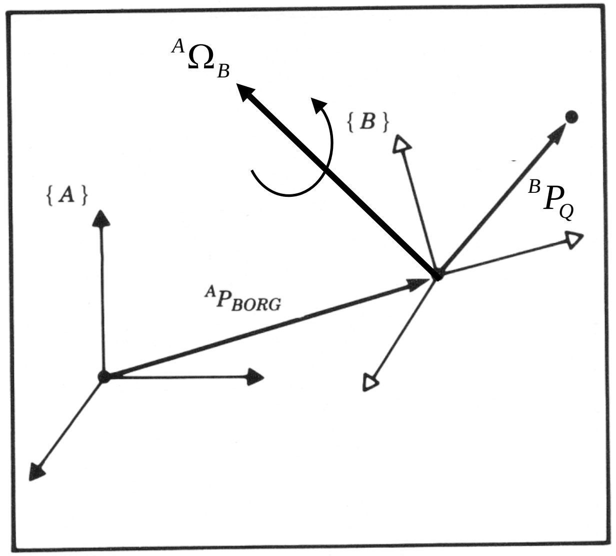
Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

  - Matrix Form

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$



- Angular Velocity
  - Vector Form

$$^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$

  - Matrix Form

$${}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}$$

Linear Algebra - Review

Brief Linear Algebra Review - 1/

- Inverse of Matrix A exists ***if and only if*** the determinant of A is non-zero.

A^{-1} Exists ***if and only if***

$$Det(A) = |A| \neq 0$$

- If the determinant of A is equal to zero, then the matrix A is a singular matrix

$$Det(A) = |A| = 0$$

A Singular

Brief Linear Algebra Review - 2/

- The rank of the matrix A is the size of the largest squared Matrix S for which

$$Det(S) \neq 0$$

- Example 1 -

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad A = S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad |A| = |S| = 3 \quad Rank(A) = 2$$

- Example 2 -

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad S = [1] \quad |S| = 1 \quad Rank(A) = 1$$

Brief Linear Algebra Review - 3/

- If two rows or columns of matrix A are equal or related by a constant, then

$$Det(A) = 0$$

- Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & -3 \\ 10 & -6 & -5 \end{bmatrix}$$

$$\det(A) = |A| = 2\begin{vmatrix} -3 & -3 \\ -6 & -5 \end{vmatrix} - 0\begin{vmatrix} 6 & -3 \\ 10 & -5 \end{vmatrix} - 1\begin{vmatrix} 6 & -3 \\ 10 & -6 \end{vmatrix} = 6 + 0 - 6 = 0$$

Brief Linear Algebra Review - 4/

- ***Eigenvalues***

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

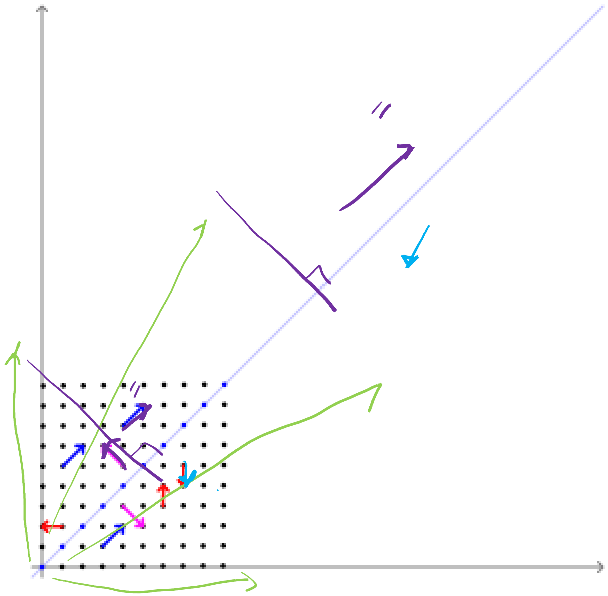
- Eigenvalues are the roots of the polynomial

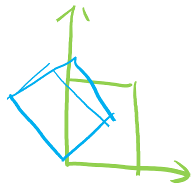
$$Det(A - \lambda I)$$

- If $X \neq 0$ each solution to the characteristic equation (Eigenvalue) λ has a corresponding Eigenvector

Brief Linear Algebra Review - 4/

- Wikipedai - https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors





Brief Linear Algebra Review - 4/

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A - \lambda I)X = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

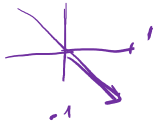
$$Det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

Brief Linear Algebra Review - 4/

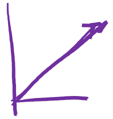
$$\lambda_1 = 1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$\lambda_2 = 3$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Brief Linear Algebra Review - 5/

- Any singular matrix ($Det(A) = 0$) has at least one Eigenvalue equal to zero

Brief Linear Algebra Review - 6/

- If A is non-singular ($Det(A) \neq 0$), and λ is an eigenvalue of A with corresponding to eigenvector X, then

$$A^{-1}X = \lambda^{-1}X$$

Brief Linear Algebra Review - 7/

- If the $n \times n$ matrix A is of full rank (that is, $\mathbf{Rank\ (A) = n}$), then the only solution to

$$AX = 0$$

is the trivial one

$$X = 0$$

- If A is of less than full rank (that is $\mathbf{Rank\ (A) < n}$), then there are $n\text{-}r$ linearly independent (orthogonal) solutions

$$x_j \qquad 0 \le j \le n - r$$

for which

$$Ax_j = 0$$

Brief Linear Algebra Review - 8/

- If A is square, then A and A^T have the same eigenvalues