Manipulator Dynamics 4

Lagrange Method

Introduction

Lagrangian Formulation of Manipulator Dynamics - Summary

1. Define a set of ***generalized coordinates*** for ***i=1,2,3…N.***

   These coordinates can be chosen arbitrarily as long as they provide a set of independent variables that map the system in a 1-to-1 manner. The usual variable set for serial manipulators is:

$$q_i = \begin{cases} \theta_i & \text{Revolute Joint} \\ d_i & \text{Prismatic Joint} \end{cases}$$

2. Define a set of ***generalized velocities*** $\dot{q}_i$ for ***i=1,2,3…N***

3. Define a set of ***generalized forces (and moments)*** Q_i for ***i=1,2,3…N***

   The generalized forces must satisfy $Q_i \delta q_i = \delta W$

$$Q = \tau + J^T \mathcal{F}_e - \tau_{fr}$$

where δq_i is a small change in the generalized coordinate and δW is the work done corresponding to that small change, τ is the joint torque vector generated by the actuators, J^T is the Jacobian matrix transposed, $\mathcal{F}_e$ is the vector of the external forces and torques applied on the end effector, and τ_{fr} is the friction torque vector generated at the joints

Langrangian Formulation of Manipulator Dynamics 2/

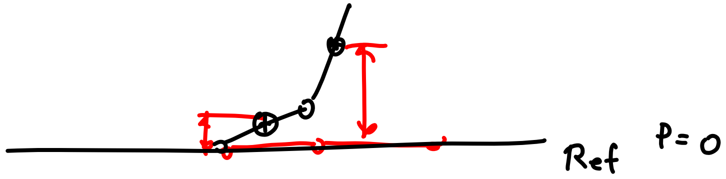
4. Write the equations describing the ***kinetic and potential energies*** as functions of the generalized coordinates as well as the resulting Lagrangian.

Let KE denote the expression describing the kinetic energy. Where $KE = f(q_i,\ \dot{q}_i,\ t)$

$$KE = \frac{1}{2}\sum_{i=1}^{n}\left({}^0V_{c_i}^{T} m_i\, {}^0V_{c_i} + {}^0\omega_i^{T}\, {}^0I_{i_c}\, {}^0\omega_i\right)$$

Let PE denote the expression describing the potential energy, where $P = f(q_i,\ t)$ $P = f(q_i,\ t)$

$$PE = -\sum_{i=1}^{n} m_i g^T P_{c_i}$$



Let L denote the Lagrangian given by:

$$L = KE - PE$$

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5. The equations of motion are given by

$$Q_i = \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} - \boxed{\frac{\partial L}{\partial q_i}}$$

$$L = KE - PE$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{q}_i} - \cancelto{0}{\frac{\partial PE}{\partial \dot{q}_i}}\right)}$$

$$\boxed{\frac{\partial L}{\partial q_i} = \left[\frac{\partial (KE - PE)}{\partial q_i}\right] = \left[\frac{\partial KE}{\partial q_i} - \frac{\partial PE}{\partial q_i}\right]}$$

or, more practically, by

$$Q_i = \frac{d}{dt}\left(\frac{\partial KE}{\partial \dot{q}_i}\right) - \frac{\partial KE}{\partial q_i} + \frac{\partial PE}{\partial q_i}$$

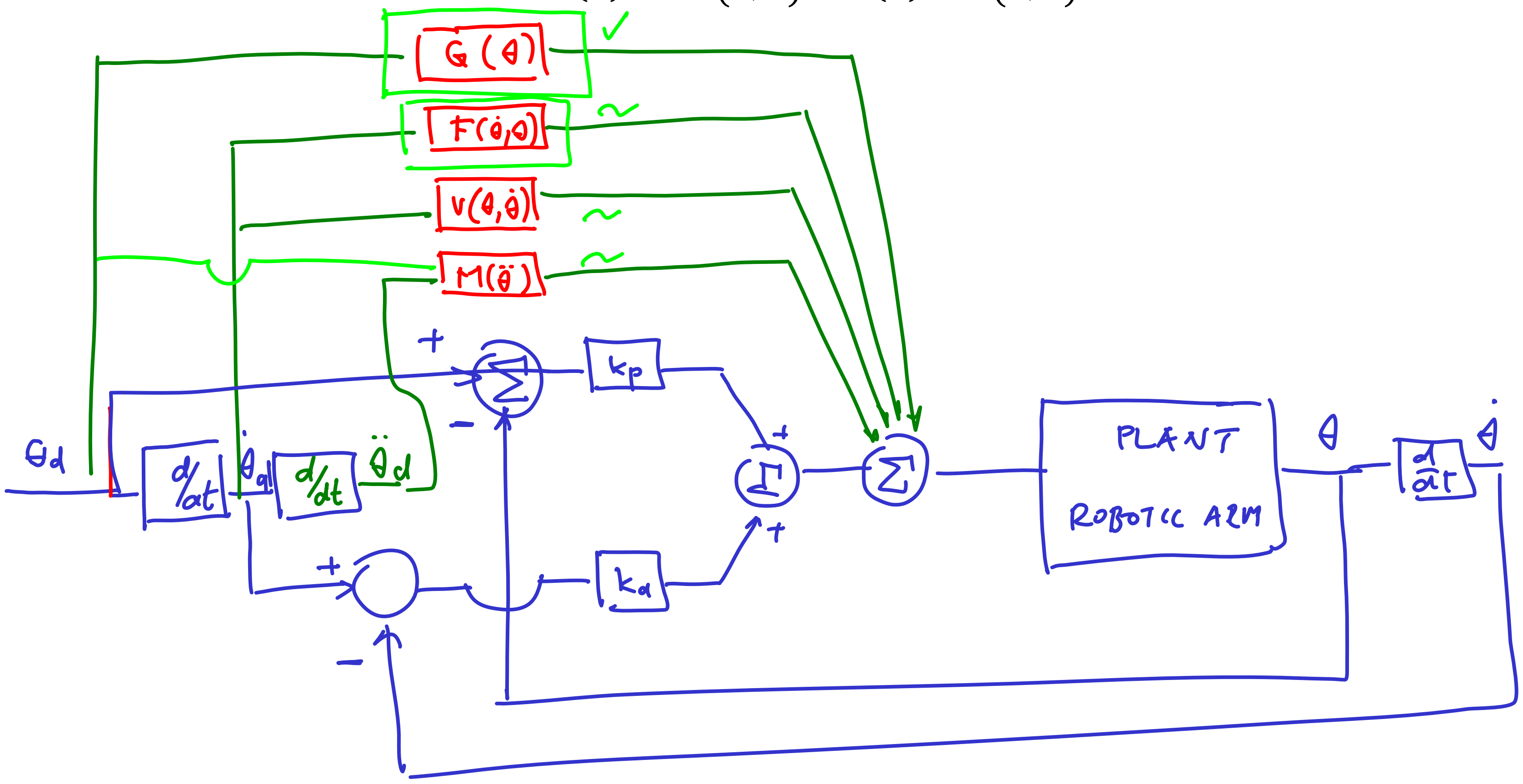
Gravity Effects - Langrangian Formulation

$$\tau_i = \boxed{\frac{d}{dt}\left(\frac{\partial KE(\theta,\dot{\theta})}{\partial \dot{\theta}_i}\right) - \frac{\partial KE(\theta,\dot{\theta})}{\partial \theta_i}} + \boxed{\frac{\partial PE(\theta)}{\partial \theta_i}}$$

$$\tau = \boxed{M(\theta)\ddot{\theta} + V\left(\theta,\dot{\theta}\right)} + \boxed{G(\theta)}$$

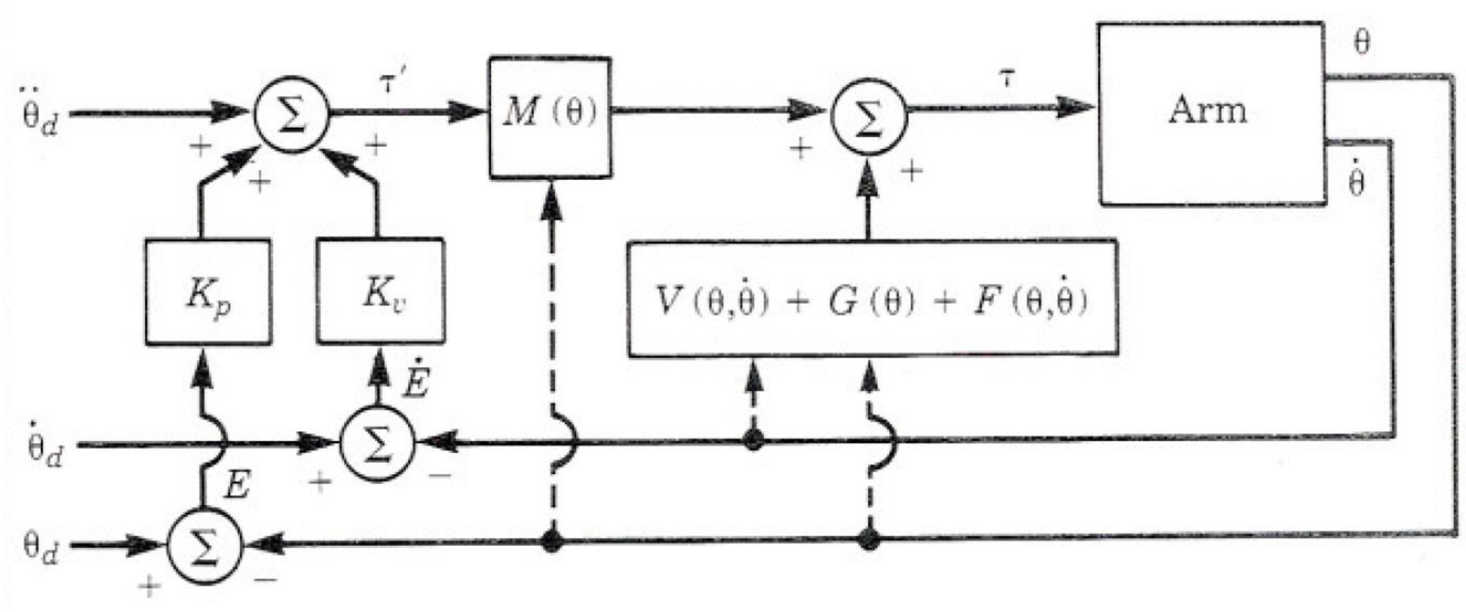
Manipulators – Non Linear Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$



Manipulators – Non Linear Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$



Lagrange Method

General Approach – Formal Derivation

Lagrangian Formulation of Manipulator Dynamics

- Lagrangian function- The difference between kinetic and potential energy of a mechanical system:

$$L = KE - PE$$

- Where:
  - L - Lagrangian
  - KE – kinetic energy of a mechanical system
  - PE – Potential energy of a mechanical system

- The kinetic energy is function of the position and velocity of the link

$$K = f(P, V)$$

- The potential energy is a function of the position of the link

$$P = f(P)$$

Lagrangian Formulation of Manipulator Dynamics

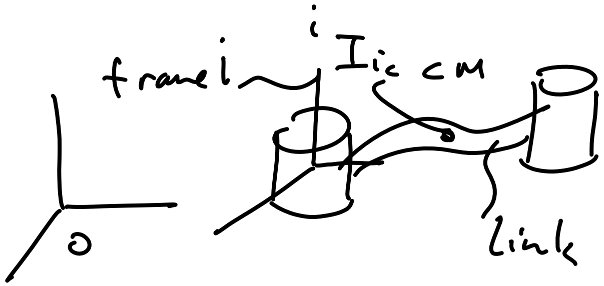
- Lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

- Where:
  - q - vector of generalized coordinates $q = [q_1, q_2, \dots q_n]^T$
  - Q – vector of generalized forces $Q = [Q_1, Q_2, \dots Q_n]^T$

Lagrangian Formulation of Manipulator Dynamics

- **Kinetic Energy**



$$KE = \frac{1}{2}\sum_{i=1}^{n}\left({}^{0}V_{c_i}^{T} m_i \, {}^{0}V_{c_i} + {}^{0}\omega_i^{T} \, {}^{0}I_{i_c} \, {}^{0}\omega_i\right)$$

Where:

- ${}^{0}I_{i_c}$ is the inertia matrix of link i about its CM and expressed in the base frame (Frame 0)
- ${}^{i}I_{i_c}$ is the inertia matrix of link i about its CM and expressed in the link frame
- ${}^{0}V_{c_i}$ is the linear velocity of the CM of link i expressed in the base frame (Frame 0)
- ${}^{0}\omega_i$ is the angular velocity of the CM of link i expressed in the base frame (Frame 0)

- **Note** – The kinetic equation of each individual link is expressed with respect to base frame (Frame 0). The implication is that the velocities and the tensor of inertia should use the reference frame as frame 0

Lagrangian Formulation of Manipulator Dynamics

- **Kinetic Energy – Tensor of Inertia (Expressed in the Base Frame)**

- Rotating the moment of inertia of the link ${}^{i}I_{i_c}$ expressed in the i'th coordinate CM to the base frame ${}^{0}I_{icm}$

$$ {}^{0}I_{i_c} = {}^{0}_{i}R\,{}^{i}I_{i}\left[{}^{0}_{i}R\right]^{T} $$

Where:

- ${}^{0}I_{i_c}$ is the inertia matrix of link i about its CM and expressed in the base frame
- ${}^{i}I_{i_c}$ is the inertia matrix of link i about its CM and expressed in the link frame

- Note:
  - ${}^{i}I_{i_c}$- Time Invariant
  - ${}^{0}I_{i_c}$- Depends on to the robot arm posture because it is expressed in the base frame and the orientation of the link i with respect to the base is a function of joint variables

Lagrangian Formulation of Manipulator Dynamics

- **Kinetic Energy – Linear & Angular Velocity (Expressed in the Base Frame)**

- Methods for expressing the velocity of the CM at the base frame (Frame 0) ${}^{0}V_{c_i}$, ${}^{0}\omega_{\ i}$
  - Method 1: Recursive method
  - Method 2: Partial Jacobian (Instantaneous screw motion) – Developed for each link as opposed to the previously defined Jacobian

$$ {}^{0}\dot{X}_{ci} = {}^{0}J_i \, \dot{q} $$

$$ \begin{bmatrix} {}^{0}V_{c_i} \\ {}^{0}\omega_{\ i} \end{bmatrix} = \begin{bmatrix} {}^{0}J_{vi} \\ {}^{0}J_{\omega i} \end{bmatrix} [\ \dot{q}\ \ ] $$

$$ [{}^{0}V_{c_i}] = [{}^{0}J_{vi}][\ \dot{q}\ \ ] $$

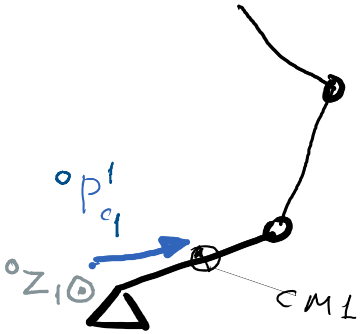
$$ [{}^{0}\omega_{\ i}] = [{}^{0}J_{\omega i}][\ \dot{q}\ \ ] $$

Lagrangian Formulation of Manipulator Dynamics

– Partial Jacobian (Example)

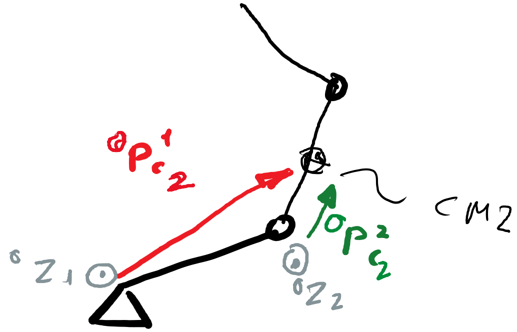
$$[{}^0V_{c_1}] = [{}^0J_{v1}]\left[\dot q\right] = \left[\begin{matrix} {}^0Z_1 \times {}^0P^1_{c_1} \end{matrix} \;\; \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

$$[{}^0\omega_i] = [{}^0J_{\omega1}]\left[\dot q\right] = \left[\begin{matrix} {}^0Z_1 \end{matrix} \;\; \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$



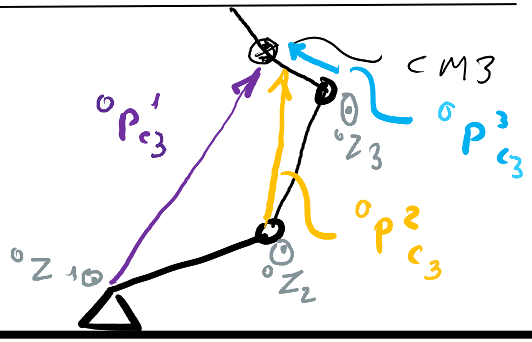
$$[{}^0V_{c_2}] = [{}^0J_{v2}]\left[\dot q\right] = \left[\begin{matrix} {}^0Z_1 \times {}^0P^1_{c_2} & {}^0Z_2 \times {}^0P^2_{c_2} \end{matrix} \;\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

$$[{}^0\omega_2] = [{}^0J_{\omega2}]\left[\dot q\right] = \left[\begin{matrix} {}^0Z_1 & {}^0Z_2 \end{matrix} \;\; \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$



$$[{}^0V_{c_3}] = [{}^0J_{v3}]\left[\dot q\right] = \begin{bmatrix} {}^0Z_1 \times {}^0P^1_{c_3} & {}^0Z_1 \times {}^0P^2_{c_3} & {}^0Z_1 \times {}^0P^3_{c_3} \end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$

$$[{}^0\omega_3] = [{}^0J_{\omega3}]\left[\dot q\right] = \begin{bmatrix} {}^0Z_1 & {}^0Z_2 & {}^0Z_3 \end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\end{bmatrix}$$



Lagrangian Formulation of Manipulator Dynamics

– **Partial Jacobian (Example) – Notes (Observations)**

1. **Vectors Expressed in the Base Frame** - The vectors placed in the partial Jacobian matrixes are expressed in the base frame (Frame 0)
   – In ${}^0J_{vi}$ the position vector ${}^0P_{c_i}^{\,j}$ is expressed with respect to frame 0
   – In ${}^0J_{\omega i}$ The joint rotation axis ${}^0Z_i^{\,j}$ is expressed with respect to frame 0
2. **Population of Vectors** - The columns partial Jacobian $[{}^0J_{vi}]$ $[{}^0J_{\omega i}]$ are populate with values for $j \le i$ and columns for which $j > i$ are populated with zero

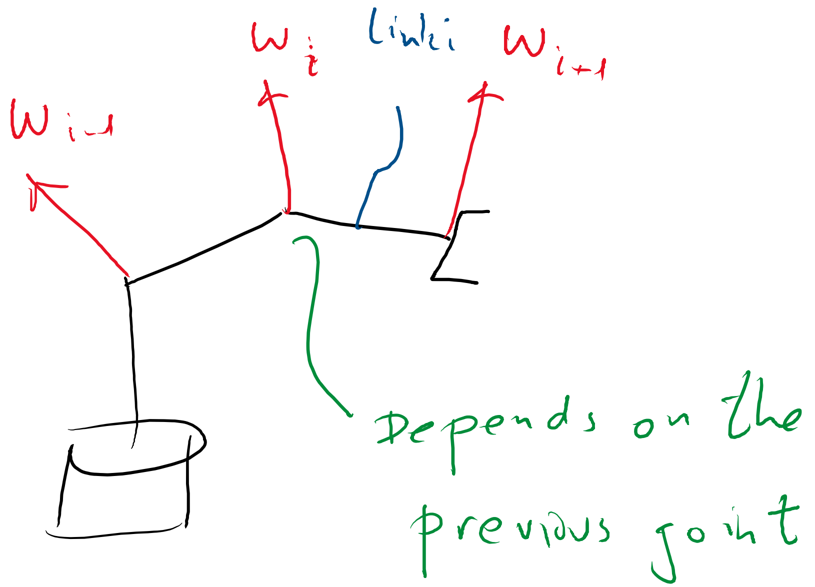
$$
{}^0\dot{X}_{ci} = {}^0J_i\,\dot{q}
$$

$$
[{}^0V_{c_i}] = [{}^0J_{vi}][\,\dot{q}\,]
$$

$$
[{}^0\omega_{\,i}] = [{}^0J_{\omega i}][\,\dot{q}\,]
$$

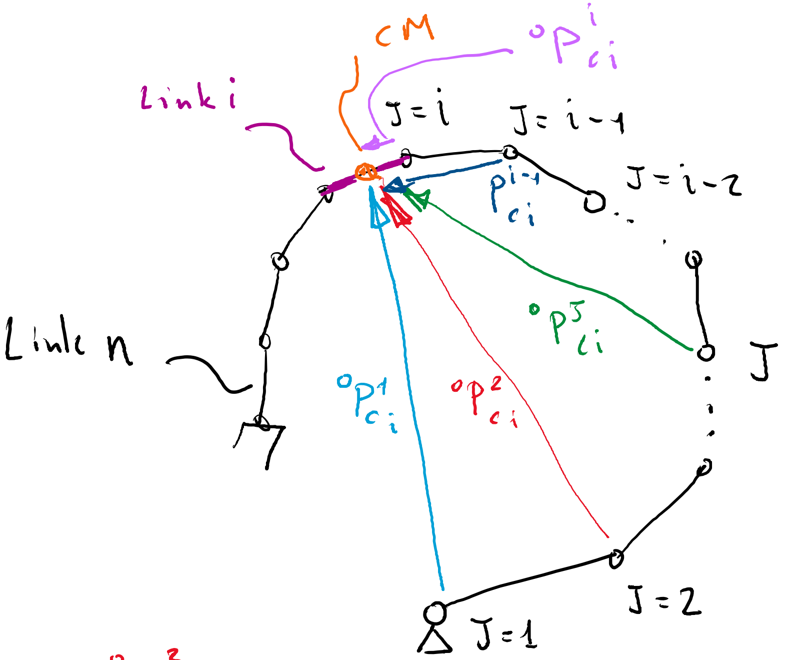
- Since the motion of link i depends only on the joints 1 through i the two vectors J_{vi}^{j} , $J_{\omega i}^{j}$ in the matrix are set to zero for j>i

$$
{}^{0}J_i = \begin{bmatrix} \overbrace{J_{vi}^{1} \; J_{vi}^{2} \; J_{vi}^{3} \; \ldots J_{vi}^{j}}^{j \le i}, \overbrace{0, 0, 0 \; \ldots 0}^{j > i} \\ J_{\omega i}^{1} \; J_{\omega i}^{2} \; J_{\omega i}^{3} \; \ldots J_{\omega i}^{j}, 0, 0, 0 \; \ldots 0 \end{bmatrix}
$$



Lagrangian Formulation of Manipulator Dynamics

– Partial Jacobian – Generalized Interpretation



$$J_{vi}^{j} = \begin{cases} {}^0Z_J \times {}^0P_{c\,i}^{\,j}\,(Reveolute\ Joint) \\ {}^0Z_J\ \ (Prismatic\ Joint) \end{cases}$$

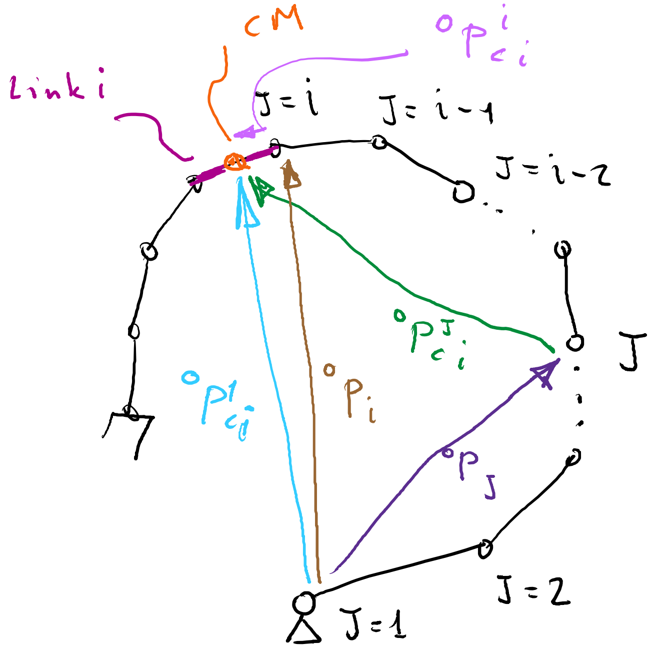
$$
{}^0J_{v_i} = \left[\begin{array}{ccccccccc}
{}^0J^1_{c_i} & {}^0J^2_{c_i} & & {}^0J^i_{c_i} & & {}^0J^{i-1}_{c_i} & {}^0J^i_{c_i} & & \\
\left[{}^0Z_1 \times {}^0P^1_{c_i}\right] & \left[{}^0Z_2 \times {}^0P^2_{c_i}\right] & \cdots & \left[{}^0Z_J \times {}^0P^J_{c_i}\right] & \cdots & \left[{}^0Z_J \times {}^0P^{i-1}_{c_i}\right] & \left[{}^0Z_i \times {}^0P^i_{c_i}\right] & [0]\,[0] & [0] \\
J=1 & J=2 & & J & & J=i-1 & J=i & & n
\end{array}\right]
$$

Lagrangian Formulation of Manipulator Dynamics

– **Partial Jacobian – Generalized Interpretation**

$${}^0P_{c_i}^1 = {}^0P_i + {}^0P_{c_i}^i$$

$${}^0P_{c_i} = {}^0P_i + {}^0_iT\,{}^iP_{c_i}$$



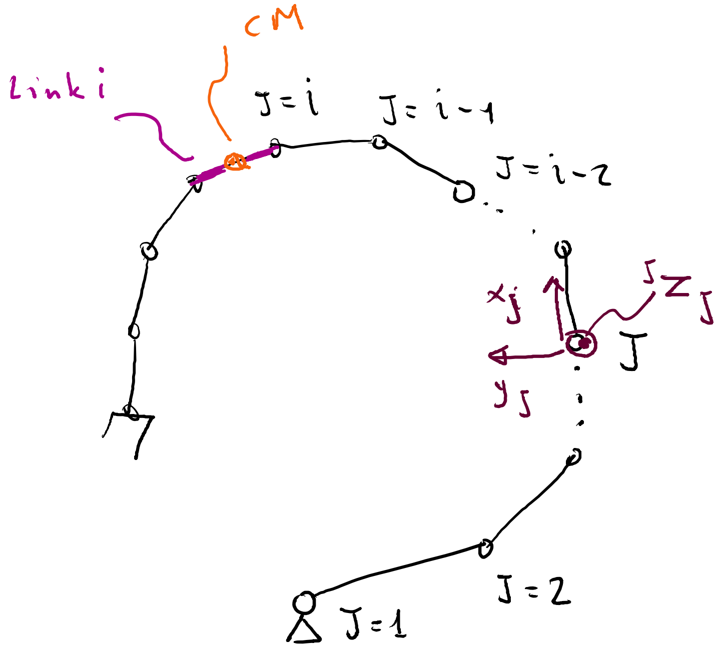
$${}^0P_{c_i}^J = {}^0P_{c_i}^1 - {}^0P_J$$

$$J_{vi}^{j} = \begin{cases} {}^0Z_J \times {}^0P_{c_i}^{\,j}\,(Reveolute\ Joint) \\ {}^0Z_J\ \ (Prismatic\ Joint) \end{cases}$$

$$J_{vi}^{j} = \begin{cases} {}^0Z_J \times ({}^0P_{c_i} - {}^0P_j)(Reveolute\ Joint) \\ {}^0Z_J\ \ (Prismatic\ Joint) \end{cases}$$

Lagrangian Formulation of Manipulator Dynamics

– Partial Jacobian – Generalized Interpretation



$$ {}^0Z_J = {}^0_JR\,{}^JZ_J = {}^0_JR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad {}^0_JT = \begin{array}{c} \begin{array}{cccc} & & {}^0Z_J & \end{array} \\ \left[\begin{array}{ccc|c} \square & \square & \square & \square \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array} $$

$$ J^j_{\omega i} = \begin{cases} {}^0Z_J\,(Reveolute\ Joint) \\ 0\ (Prismatic\ Joint) \end{cases} $$

Lagrangian Formulation of Manipulator Dynamics

$$\begin{bmatrix} {}^{i}V_{ci} \\ \omega_i \end{bmatrix} = \begin{bmatrix} {}^{0}J_{vi} \\ {}^{0}J_{\omega i} \end{bmatrix} [\dot{\theta}_i]$$

Column Vector J

$${}^{0}J_i = \begin{bmatrix} {}^{0}J_{vi} \\ {}^{0}J_{\omega i} \end{bmatrix} = \begin{bmatrix} * & * & J_{vi}^{j} & * \\ * & * & J_{\omega i}^{j} & * \end{bmatrix}$$

$$[{}^{0}V_{c_i}] = [{}^{0}J_{vi}][\dot{q}\ ]$$

$$[{}^{0}\omega_{\ i}] = [{}^{0}J_{\omega i}][\dot{q}\ ]$$

$$J_{vi}^{j} = \begin{cases} {}^{0}Z_J \times {}^{0}P_{c_i}^{\ j} (Reveolute\ Joint) \\ {}^{0}Z_J \ (Prismatic\ Joint) \end{cases} \qquad J_{\omega i}^{j} = \begin{cases} {}^{0}Z_J (Reveolute\ Joint) \\ 0\ (Prismatic\ Joint) \end{cases}$$

- ${}^{0}J_i$- The link Jacobian matrix - A 6xN matrix that maps the instantaneous joint rates into the instantaneous velocity at the center of mass
- ${}^{0}J_{vi}$ ${}^{0}J_{\omega i}$ - Two 3xN submatrices of ${}^{0}J_i$
- ${}^{j}P_{cmi}$ - Position vector defined from the origin of the J joint frame to the CM of link i
- J_{vi}^{j} , $J_{\omega i}^{j}$ - the j'th column vector of ${}^{0}J_{vi}$ ${}^{0}J_{\omega i}$ respectivly

Lagrangian Formulation of Manipulator Dynamics

- Rewriting the Kinetic Energy

$$KE = \frac{1}{2}\sum_{i=1}^{n}\left({}^{0}V_{c_i}^{T} m_i\, {}^{0}V_{c_i} + {}^{0}\omega_i^{T}\, {}^{0}I_{i_c}\, {}^{0}\omega_i\right) = \frac{1}{2}\sum_{i=1}^{n}[({}^{0}J_{vi}\dot{q})^{T} m_i ({}^{0}J_{vi}\dot{q}) + ({}^{0}J_{\omega i}\dot{q})\, {}^{0}I_{i_c} ({}^{0}J_{\omega i}\dot{q})]$$

$$= \frac{1}{2}\dot{q}^{T}\underbrace{\boxed{[\sum_{i=1}^{n} {}^{0}J_{vi}^{T} m_i\, {}^{0}J_{vi} + {}^{0}J_{\omega i}^{T}\, {}^{0}I_{i_c}\, {}^{0}J_{\omega i}]}}_{M}\dot{q}$$

Lagrangian Formulation of Manipulator Dynamics

- Define a nxn manipulator inertia matrix as

$$M = \sum_{i=1}^{n} {}^{0}J_{vi}^{T} m_i \, {}^{0}J_{vi} + {}^{0}J_{\omega i}^{T} \, {}^{0}I_{i_c} \, {}^{0}J_{\omega i}$$

- The total kinetic energy of a robot arm can be expressed in terms of the manipulator inertia matrix and the vector of joint rates

$$KE = \frac{1}{2}\dot{q}^{T} M \dot{q}$$

- M is configuration dependent because J_v and J_w are configuration dependent as well

- Properties of the mass matrix M:
  - Symmetric
  - Positive Definite

- The quadratic form of the equation indicates that the kinetic energy is always positive unless the system is at rest

- **Potential Energy**
  - Potential energy stores in a link is defined as the amount of work required to raise the center of mass of link i from the horizontal reference plane to its present position under the influence of gravity
  - With reference to the inertial frame (frame 0), the work required to displace link i to position P_{c_i} is given by

$$PE = -\sum_{i=1}^{n} m_i g^T P_{c_i}$$

Lagrangian Formulation of Manipulator Dynamics

- **Generalized Forces**
  - Grivation forces
  - Inertial forces
  - All the rest

- All the forces acting on a robot arm that consistent with the mechanical constraints

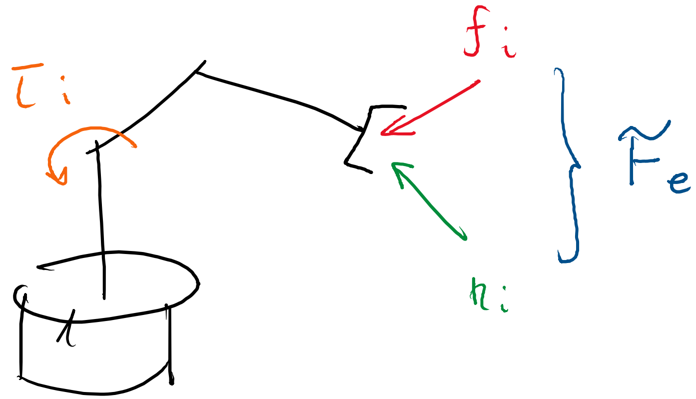
- The vector of generalized forces $Q = [Q_1, Q_2, \dots Q_n]^T$ is defined by the principle of virtual work as

$$\delta W = Q^T \delta q$$

- Actuators -> Force/Torque at the joint
- External Forces/Moment -> End Effector

Lagrangian Formulation of Manipulator Dynamics

$$\delta W = \tau^T \delta q + \mathcal{F}_e^T \delta x$$



- Where
  - $\tau = [\tau_1, \tau_2, \dots \tau_n]^T$ - n dimensional vector of joint torques generated by the actuators
  - $\delta q = [\tau_1, \tau_2, \dots \tau_n]$ - n dimensional vector of joint displacement generated at the joint
  - $\mathcal{F}_e^T = [f_e^T \ \ n_e^T]$ – six dimensional vector of resultant force and moments exerted at the end effector
  - δx – six dimensional virtual displacement vector of the end effector
  - δW – virtual work

Lagrangian Formulation of Manipulator Dynamics

- Substituting

$$\delta x = J\delta q$$

- We have

$$Q^T\delta q = \tau^T\delta q + \mathcal{F}_e^T J\delta q$$
$$Q^T = \tau^T + \mathcal{F}_e^T J$$
$$(Q^T)^T = (\tau^T + \mathcal{F}_e^T J)^T$$
$$Q = \tau + J^T\mathcal{F}_e$$

Joint Torque External Forces

Lagrangian Formulation of Manipulator Dynamics

- **Joint Frictions**
  - Highly Non-linear
  - Grease/oil lubricated bearing – Types of frictions
    - Static friction
    - Boundary lubrication
    - Partial fluid lubrication
    - Full fluid lubrication
  - Full fluid lubrication

$$f_{r_i} = -b_i\,\dot{q}_i$$

  - $f_r = [b_1\dot{q}_1, b_2\dot{q}_2, \dots, b_n\dot{q}_n]$ – The frictional torques or forces in the joints. The minus sign indicates that the direction of the frictional torque or forces is always opposite to the joint velocity.

- Total Generalized forces including the joint's fluid lubrication resulted in

$$Q = \tau + J^T \mathcal{F}_e - f_r$$

Lagrangian Formulation of Manipulator Dynamics

- General Form of Dynamical Equations

$$L = KE - PE$$

$$L = \frac{1}{2}\dot{q}^T M \dot{q} + \sum_{i=1}^{n} m_i g^T P_{c_i}$$

- Expand the term for the kinetic energy into a sum of scalars
- M_{ij} - The (i,j) element of the manipulator inertia matrix M

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i g^T P_{c_i}$$

Lagrangian Formulation of Manipulator Dynamics

$$\boxed{\frac{d}{dt}\boxed{\left(\frac{\partial L}{\partial \dot{q}_i}\right)}} - \boxed{\frac{\partial L}{\partial q_i}} = Q_i \qquad\qquad L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\ \dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i g^T P_{c_i}$$

- Take partial derivative of L with respect to $\dot{q}_i$
- Note that the potential energy does not depend on $\dot{q}_i$

$$\boxed{\frac{\partial L}{\partial \dot{q}_i}} = \frac{1}{2}\sum_{J=1}^{n} M_{ij}\dot{q}_j$$

- Taking the derivative of $\frac{\partial L}{\partial \dot{q}_i}$ with respect to time

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} = \frac{1}{2}\sum_{J=1}^{n} M_{ij}\ddot{q}_j + \frac{1}{2}\sum_{J=1}^{n}\frac{d}{dt}(M_{ij})\dot{q}_j = \frac{1}{2}\sum_{J=1}^{n} M_{ij}\ \ddot{q}_j + \frac{1}{2}\sum_{J=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\right]\dot{q}_J$$

$(gh)' = gh' + g'h$

$\frac{\delta M}{\delta q}\ \frac{\delta q}{\delta t}$

Lagrangian Formulation of Manipulator Dynamics

- Taking the partial derivative $\frac{\partial L}{\partial q_i}$

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\ \dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i g^T P_{c_i}$$

$$\boxed{\frac{\partial L}{\partial q_i}} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{J=1}^{n}\sum_{k=1}^{n} M_{jk}\ \dot{q}_j\dot{q}_k\right) + \sum_{j}^{n} m_j g^T \left(\frac{\partial P_{cj}}{\partial q_i}\right)$$

The ith column vector of the link Jacobian sub matrix ${}^{0}J_{vi}$

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\ \dot{q}_i\dot{q}_j\right) + \sum_{i=1}^{n} m_i g^T \left(\frac{\partial P_{cj}}{\partial q_i}\right)$$

Lagrangian Formulation of Manipulator Dynamics

- Substitute all the equation into the Lagrange Equation

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} - \boxed{\frac{\partial L}{\partial q_i}} = Q_i$$

$$\boxed{\sum_{J=1}^{n} M_{ij}\,\ddot{q}_j + \sum_{J=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\right]\dot{q}_j} - \boxed{\frac{1}{2}\frac{\partial}{\partial \dot{q}_i}\left(\sum_{J=1}^{n}\sum_{k=1}^{n} M_{jk}\,\dot{q}_j\dot{q}_k\right) + \sum_{j}^{n} m_j g^T\left(\frac{\partial P_{cj}}{\partial q_i}\right)} = Q_i$$

$$\sum_{J=1}^{n} M_{ij}\,\ddot{q}_j + V_i + G_i = Q_i$$

Lagrangian Formulation of Manipulator Dynamics

- Substitute all the equation into the Lagrange Equation

$$\sum_{J=1}^{n} M_{ij}\,\ddot{q}_j + V_i + G_i = Q_i$$

- Where

$\sum_{J=1}^{n} M_{ij}\,\ddot{q}_j$ - is the Inertia

$V_i = \sum_{J=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right) \dot{q}_k \dot{q}_j$ - is the Coriolis and centrifugal force

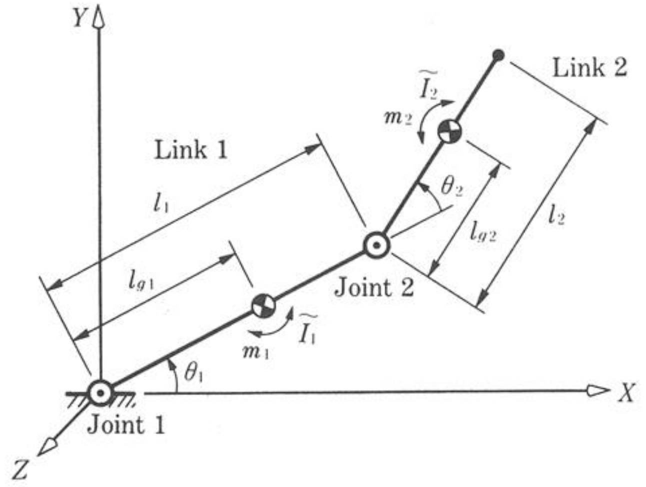
$G_i = -\sum_{j=1}^{n} m_j g^T \left(\frac{\partial P_{cj}}{\partial q_i}\right)$ - is the gravitational effects

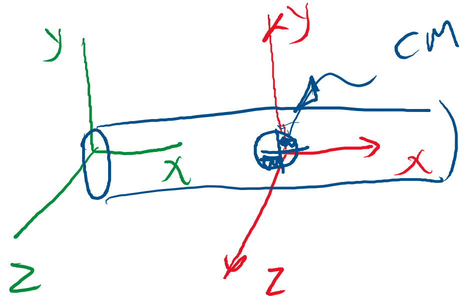
Lagrange Method

2R - Example

Lagrangian Formulation of Manipulator Dynamics

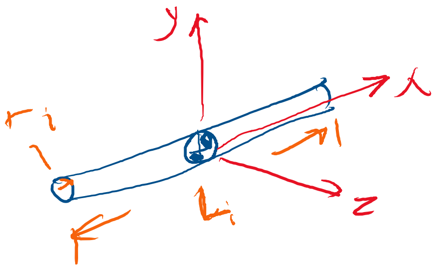
- Example: 2DOF robot
  - Note: The link coordinate axes are aligned with the principal axes at each link





Lagrangian Formulation of Manipulator Dynamics

- **(a) Link Inertia Matrix:** Assume that the length of the link is much longer than the radius of the link L>>r



Assume L>>r

$$ {}^{i}I_{c_i} = \frac{1}{12} m L_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}I_i = {}^{0}_{i}R \, {}^{i}I_{c_i} ({}^{0}_{i}R)^T $$

$i = 1$

$$ {}^{0}I_1 = {}^{0}_{1}R \, {}^{1}I_{c_1} ({}^{0}_{1}R)^T $$

$$ {}^{0}I_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m L_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{12} m L_i^2 \begin{bmatrix} s_1^2 & -c_1 s_1 & 0 \\ -c_1 s_1 & c_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Lagrangian Formulation of Manipulator Dynamics

$i = 2$

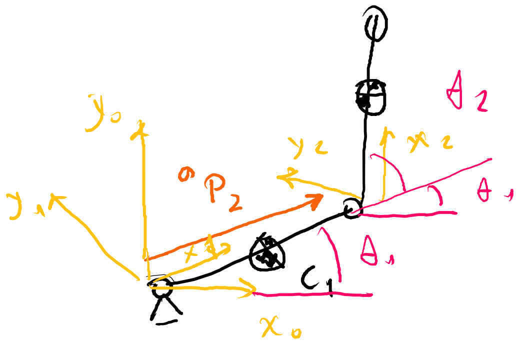
$$ {}^{0}I_2 = {}^{0}_{2}R\, {}^{1}I_{c_1} \left({}^{0}_{2}R\right)^T $$

$$ {}^{0}_{2}R = {}^{0}_{1}R\, {}^{1}_{2}R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 \\ s_1c_2 + c_1s_2 & -s_1s_2 + c_1c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}I_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m L_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{12} m L_i^2 \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{12} m L_i^2 \begin{bmatrix} s_{12}^2 & -s_{12}c_{12} & 0 \\ -s_{12}c_{12} & c_{12}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Lagrangian Formulation of Manipulator Dynamics

- **(b) Link Jacobian Matrix**:

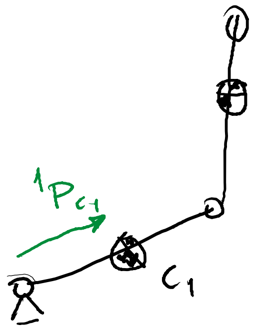


$$^0P_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ^0P_2 = \begin{bmatrix} L_1 c_1 \\ L_1 s_1 \\ 0 \end{bmatrix}$$

$$^0P_2 = {}^0_1T\ {}^1P_2 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} L_1 c_1 \\ L_1 s_1 \\ 0 \\ 1 \end{bmatrix}$$

- **(b) Link Jacobian Matrix**:
  - Position of the CM:

$$ {}^{1}P_{c_1} = \begin{bmatrix} \frac{1}{2} L_1 \\ 0 \\ 0 \end{bmatrix} $$

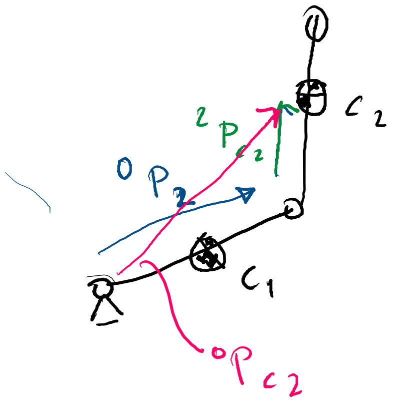


$$ {}^{0}P_{c_1} = {}^{0}_{1}T\,{}^{1}P_{c_1} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} L_1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} L_1 c_1 \\ \frac{1}{2} L_1 s_2 \\ 0 \\ 1 \end{bmatrix} $$

UCLA

Lagrangian Formulation of Manipulator Dynamics

- **(b) Link Jacobian Matrix**:
  - Position of the CM:



$$^2P_{c_2} = \begin{bmatrix} \frac{1}{2}L_2 \\ 0 \\ 0 \end{bmatrix} \qquad ^0P_{c_2} = \begin{bmatrix} \frac{1}{2}L_2c_{12} + L_1c_1 \\ \frac{1}{2}L_2s_{12} + L_1s_1 \\ 0 \\ 1 \end{bmatrix}$$

$$^0P_{c_2} = {}^0_1T\,{}^1_2T\;{}^2P_{c_2} = {}^0_2T\;{}^2P_{c_2} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}L_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} & 0 & L_1c_1 \\ s_{12} & c_{12} & 0 & L_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}L_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}L_2c_{12} + L_1c_1 \\ \frac{1}{2}L_2s_{12} + L_1s_1 \\ 0 \\ 1 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$$\begin{bmatrix} {}^iV_{ci} \\ \omega_i \end{bmatrix} = \begin{bmatrix} {}^0J_{vi} \\ {}^0J_{\omega i} \end{bmatrix} [\dot{\theta}_i]$$

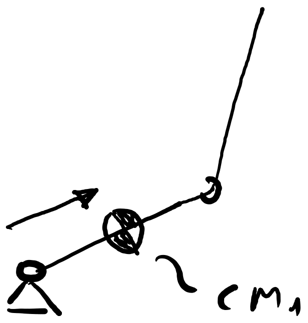
$${}^0J_i = \begin{bmatrix} {}^0J_{vi} \\ {}^0J_{\omega i} \end{bmatrix} = \begin{bmatrix} * & * & J_{vi}^j & * \\ * & * & J_{\omega i}^j & * \end{bmatrix}$$

$$J_{vi}^j = \begin{cases} \boxed{{}^0Z_J \times ({}^0P_{c_i} - {}^0P_j)(Reveolute\ Joint)} \\ {}^0Z_J\ \ (Prismatic\ Joint) \end{cases}$$

$$J_{\omega i}^j = \begin{cases} \boxed{{}^0Z_J(Reveolute\ Joint)} \\ 0\ (Prismatic\ Joint) \end{cases}$$

$i = 1$

$$J_{vi}^{j} = \begin{cases} {}^{0}Z_{J} \times ({}^{0}P_{c_i} - {}^{0}P_{j}) (Reveolute\ Joint) \\ \end{cases}$$

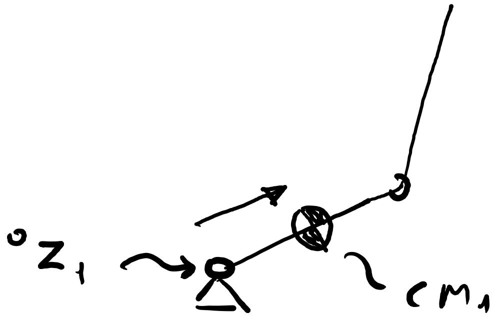


$${}^{0}J_{v1} = \left[{}^{0}Z_{1} \times ({}^{0}P_{c1} - {}^{0}P_{1}) \quad 0\right] = \left[\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{2}L_1c_1 & \frac{1}{2}L_1s_1 & 0 \end{vmatrix} \quad 0\right] = \begin{bmatrix} -\frac{1}{2}L_1s_1 & 0 \\ \frac{1}{2}L_1c_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} \frac{1}{2}L_1c_1 \\ \frac{1}{2}L_1s_1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$i = 1$

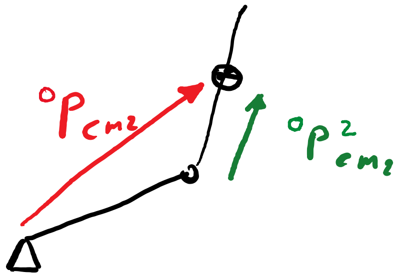


$$J_{\omega i}^{j} = \begin{cases} {}^{0}Z_J (Reveolute\ Joint) \\ \end{cases}$$

$${}^{0}J_{\omega 1} = \begin{bmatrix} {}^{0}Z_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$i = 2$



$$J^j_{vi} = \begin{cases} {}^0Z_J \times ({}^0P_{c_i} - {}^0P_j) (Reveolute\ Joint) \end{cases}$$

J = 1 J = 2

$${}^0J_{v2} = \left[{}^0Z_1 \times ({}^0P_{c2} - {}^0P_1) \mid {}^0Z_2 \times ({}^0P_{c2} - {}^0P_2)\right] = \left[\begin{bmatrix}0\\0\\1\end{bmatrix} \times \left[\begin{bmatrix}\frac{1}{2}L_2c_{12} + L_1c_1\\ \frac{1}{2}L_2s_{12} + L_1s_1\\ 0\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right] \quad \begin{bmatrix}0\\0\\1\end{bmatrix} \times \left[\begin{bmatrix}\frac{1}{2}L_2c_{12} + L_1c_1\\ \frac{1}{2}L_2s_{12} + L_1s_1\\ 0\end{bmatrix} - \begin{bmatrix}L_1c_1\\L_1s_1\\0\end{bmatrix}\right]\right]$$

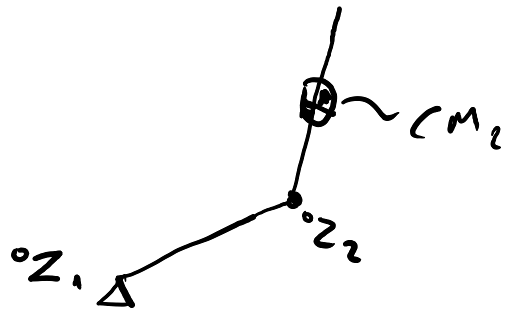
$$= \left[\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{2}L_2c_{12} + L_1c_1 & \frac{1}{2}L_2s_{12} + L_1s_1 & 0 \end{vmatrix} \quad \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{2}L_2c_{12} & \frac{1}{2}L_2s_{12} & 0 \end{vmatrix}\right] = \left[\begin{array}{c|c} -L_1s_1 - \frac{1}{2}L_2s_{12} & -\frac{1}{2}L_2s_2 \\ \hline L_1c_1 + \frac{1}{2}L_2c_{12} & \frac{1}{2}L_2c_2 \\ 0 & 0 \end{array}\right]$$

Lagrangian Formulation of Manipulator Dynamics

$i = 2$

$$J_{\omega i}^{j} = \begin{cases} {}^{0}Z_{J}(Reveolute\ Joint) \\ \end{cases}$$

$${}^{0}J_{\omega 1} = [{}^{0}Z_{1} \quad {}^{0}Z_{2}] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



Lagrangian Formulation of Manipulator Dynamics

- (c) Manipulator Inertia Matrix:

$$M = J_{v1}^T m_1 J_{v1} + J_{\omega 1}^T\, {}^1I_{1_c} J_{\omega 1} + J_{v2}^T m_2 J_{v2} + J_{\omega 2}^T\, {}^2I_{2_c} J_{\omega 2} =$$

$$\begin{bmatrix} -\frac{1}{2}L_1 s_1 & \frac{1}{2}L_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} m_1 \begin{bmatrix} -\frac{1}{2}L_1 s_1 & 0 \\ \frac{1}{2}L_1 c_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m_1 L_1^2 \begin{bmatrix} s_{12}^2 & -s_{12}c_{12} & 0 \\ -s_{12}c_{12} & c_{12}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} -L_1 s_1 - \frac{1}{2}L_2 s_{12} & L_1 c_1 + \frac{1}{2}L_2 c_{12} & 0 \\ -\frac{1}{2}L_2 s_2 & \frac{1}{2}L_2 c_2 & 0 \end{bmatrix} m_2 \begin{bmatrix} -L_1 s_1 - \frac{1}{2}L_2 s_{12} & -\frac{1}{2}L_2 s_2 \\ L_1 c_1 + \frac{1}{2}L_2 c_{12} & \frac{1}{2}L_2 c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m_2 L_2^2 \begin{bmatrix} s_{12}^2 & -s_{12}c_{12} & 0 \\ -s_{12}c_{12} & c_{12}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$$\frac{1}{4}l_1^2\underbrace{\left(s_1^2+c_1^2\right)}_{1}$$

$$= m_1\begin{bmatrix} \boxed{\frac{1}{4}l_1^2 s_1^2 + \frac{1}{4}l_1^2 c_1^2} & 0 \\ 0 & 0 \end{bmatrix}$$

$$+ \frac{1}{12}m_1 l_1^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(*)

$$+ m_2\left[\begin{array}{c} \left(-l_1 s_1 - \frac{1}{2}l_2 s_{12}\right)\left(-l_1 s_1 - \frac{1}{2}l_2 s_{12}\right) + \left(l_1 c_1 + \frac{1}{2}l_2 c_{12}\right)\left(l_1 c_1 + \frac{1}{2}l_2 c_{12}\right) + 0\cdot 0 \\ \hline \left(-\frac{1}{2}l_2 s_{12}\right)\left(-l_1 s_1 - \frac{1}{2}l_2 s_{12}\right) + \left(\frac{1}{2}l_2 c_{12}\right)\left(l_1 c_1 + \frac{1}{2}l_2 c_{12}\right) + 0\cdot 0 \end{array}\right] \ldots$$

$$\left.\begin{array}{c} \left(-l_1 s_1 - \frac{1}{2}l_2 s_{12}\right)\left(-\frac{1}{2}l_2 s_{12}\right) + \left(l_1 c_1 + \frac{1}{2}l_2 c_{12}\right)\left(\frac{1}{2}l_2 c_{12}\right) + 0\cdot 0 \\ \hline \left(-\frac{1}{2}l_2 s_{12}\right)\left(-\frac{1}{2}l_2 s_{12}\right) + \left(\frac{1}{2}l_2 c_{12}\right)\left(\frac{1}{2}l_2 c_{12}\right) \end{array}\right]$$

$$+ \frac{1}{12}m_2 l_2^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$$
(*)\quad m_2 \begin{bmatrix} \overbrace{\boxed{l_1^2 s_1^2} + \boxed{l_1^2 c_1^2}}^{l_1^2} + \overbrace{\boxed{\tfrac{1}{4} l_2^2 s_{12}^2} + \boxed{\tfrac{1}{4} l_2^2 c_{12}^2}}^{\frac{1}{4} l_2^2} + l_1 l_2 s_1 s_{12} + l_1 l_2 c_1 c_{12} & \tfrac{1}{2} l_1 l_2 s_1 s_{12} + \overbrace{\boxed{\tfrac{1}{4} l_2^2 s_{12}^2} + \boxed{\tfrac{1}{4} l_2^2 c_{12}^2}}^{\frac{1}{4} l_2^2} + \tfrac{1}{2} l_1 l_2 c_1 c_{12} \\ \tfrac{1}{2} l_1 l_2 s_1 s_{12} + \underbrace{\boxed{\tfrac{1}{4} l_2^2 s_{12}^2} + \boxed{\tfrac{1}{4} l_2^2 c_{12}^2}}_{\frac{1}{4} l_2^2} + \tfrac{1}{2} l_1 l_2 c_1 c_{12} & \underbrace{\boxed{\tfrac{1}{4} l_2^2 s_{12}^2} + \boxed{\tfrac{1}{4} l_2^2 c_{12}^2}}_{\frac{1}{4} l_2^2} \end{bmatrix}
$$

$$
= \begin{bmatrix} l_1^2 + \tfrac{1}{4} l_2^2 + \overbrace{\boxed{l_1 l_2 s_1 s_{12} + l_1 l_2 c_1 c_{12}}}^{l_1 l_2 c_2} & \tfrac{1}{4} l_2^2 + \tfrac{1}{2} l_1 l_2 \overbrace{\boxed{(s_1 s_{12} + c_1 c_{12})}}^{c_2} \\ \tfrac{1}{4} l_2^2 + \tfrac{1}{2} l_1 l_2 \underbrace{\boxed{(s_1 s_{12} + c_1 c_{12})}}_{c_2} & \tfrac{1}{4} l_2^2 \end{bmatrix}
$$

Lagrangian Formulation of Manipulator Dynamics

$$M = m_1 \begin{bmatrix} \frac{1}{4} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + m_1 \begin{bmatrix} \frac{1}{12} l_1^2 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} l_1^2 + \frac{1}{4} l_2^2 + l_1 l_2 c_2 & \frac{1}{4} l_2^2 + \frac{1}{2} l_1 l_2 c_2 \\ \frac{1}{4} l_2^2 + \frac{1}{2} l_1 l_2 c_2 & \frac{1}{4} l_2^2 \end{bmatrix} + m_2 \begin{bmatrix} \frac{1}{12} l_2^2 & \frac{1}{12} l_2^2 \\ \frac{1}{12} l_2^2 & \frac{1}{12} l_2^2 \end{bmatrix}$$

Lagangian Formulation of Manipulator Dynamics

$$M = \begin{bmatrix} m_1\left(\overbrace{\tfrac{1}{4} l_1^2 + \tfrac{1}{12} l_1^2}^{\frac{1}{3} l_1^2}\right) + m_2\left(l_1^2 + \overbrace{\boxed{\tfrac{1}{4} l_2^2 + \tfrac{1}{12} l_2^2}}^{\frac{1}{3} l_2^2} + l_1 l_2 c_2\right) & m_2\left(\overbrace{\tfrac{1}{4} l_2^2 + \tfrac{1}{12} l_2^2}^{\frac{1}{3} l_2^2}\right) + \tfrac{1}{2} l_1 l_2 c_2 \\ m_2\left(\underbrace{\boxed{\tfrac{1}{4} l_2^2 + \tfrac{1}{12} l_2^2}}_{\frac{1}{3} l_2^2} + \tfrac{1}{2} l_1 l_2 c_2\right) & m_2\left(\underbrace{\tfrac{1}{12} l_2^2 + \tfrac{1}{4} l_2^2}_{1/3\, l_2^2}\right) \end{bmatrix}$$

Lagrangian Formulation of Manipulator Dynamics

$$M = J_{v1}^T m_1 J_{v1} + J_{\omega 1}^T \, {}^{1}I_{1c} J_{\omega 1} + J_{v2}^T m_2 J_{v2} + J_{\omega 2}^T \, {}^{2}I_{2c} J_{\omega 2} =$$

$$\begin{bmatrix} \frac{1}{3} m_1 L_1^2 + m_2 (L_1^2 + L_1 L_2 c_2 + \frac{1}{3} L_2^2) & m_2 (\frac{1}{2} L_1 L_2 c_2 + \frac{1}{3} L_2^2) \\ m_2 (\frac{1}{2} L_1 L_2 c_2 + \frac{1}{3} L_2^2) & \frac{1}{3} m_2 L_2^2 \end{bmatrix}$$

- (d) Velocity Coupling Vector:

$$V_i = \sum_{J=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right) \dot{q}_k \dot{q}_j$$

$$V_1 = \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial M_{1j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_1}\right) = \overset{J=1,\ k=1}{\left(\frac{\partial M_{11}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_1}\right)} \dot{\theta}_1\dot{\theta}_1 + \overset{J=1,\ k=2}{\left(\frac{\partial M_{21}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{12}}{\partial \theta_1}\right)} \dot{\theta}_1\dot{\theta}_2 + \overset{J=2,\ k=1}{\left(\frac{\partial M_{22}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{21}}{\partial \theta_1}\right)} \dot{\theta}_2\dot{\theta}_1 + \overset{J=2,\ k=2}{\left(\frac{\partial M_{22}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1}\right)} \dot{\theta}_2\dot{\theta}_2 =$$

$$\left(0 \cancel{-\tfrac{1}{2}} 0\right)\dot{\theta}_1\dot{\theta}_1 + \left(-m_2 L_1 L_2 s_2 \cancel{-\tfrac{1}{2}} 0\right)\dot{\theta}_1\dot{\theta}_2 + (0 \cancel{-\tfrac{1}{2}} 0)\dot{\theta}_2\dot{\theta}_1) + \left(-m_2 \tfrac{1}{2} L_1 L_2 s_2 \cancel{-\tfrac{1}{2}} 0\right)\dot{\theta}_2\dot{\theta}_2 = -m_2 L_1 L_2 s_2 (\dot{\theta}_1\dot{\theta}_2 - 1/2\ \dot{\theta}_2^2)$$

$$V_2 = \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial M_{2j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_2}\right) = \overset{J=1,\ k=1}{\left(\frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_1}\right)} \dot{\theta}_1\dot{\theta}_1 + \overset{J=1,\ k=2}{\left(\frac{\partial M_{11}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{12}}{\partial \theta_1}\right)} \dot{\theta}_1\dot{\theta}_2 + \overset{J=2,\ k=1}{\left(\frac{\partial M_{12}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{21}}{\partial \theta_1}\right)} \dot{\theta}_2\dot{\theta}_1 + \overset{J=2,\ k=2}{\left(\frac{\partial M_{12}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1}\right)} \dot{\theta}_2\dot{\theta}_2 =$$

$$[0 - \tfrac{1}{2}(-m_2 L_1 L_2 s_2)]\dot{\theta}_1\dot{\theta}_1 + \left(-\tfrac{1}{2} m_2 \cancel{L_1} L_2 s_2 + \tfrac{1}{2} m_2 \tfrac{1}{2} L_1 L_2 s_2\right)\dot{\theta}_1\dot{\theta}_2 + (0 + \tfrac{1}{2}(m_2 \tfrac{1}{2} L_1 \cancel{L_2} s_2))\dot{\theta}_2\dot{\theta}_1) + \left(0 \cancel{-\tfrac{1}{2}} 0\right)\dot{\theta}_2\dot{\theta}_2 = \tfrac{1}{2} m_2 L_1 L_2 s_2 \dot{\theta}_1^{\ 2}$$

Cancel

Lagrangian Formulation of Manipulator Dynamics

- (e) Gravitational Vector

$$G_i = -\sum_{j=1}^{n} m_j g^T J_{vj}^{i}$$

$i = 1$

FIRST COLUMN of J_{v1}

FIRST COLUMN of J_{v2}

$j=1$ $j=2$

$$G_1 = -m_1 g^T J_{v1}^{1} - m_2 g^T J_{v2}^{1} = -m_1 [0 \quad -g \quad 0] \begin{bmatrix} -\frac{1}{2} L_1 s_1 \\ \frac{1}{2} L_1 c_1 \\ 0 \end{bmatrix} - m_2 [0 \quad -g \quad 0] \begin{bmatrix} -L_1 s_1 - \frac{1}{2} L_2 s_{12} \\ L_1 c_1 + \frac{1}{2} L_2 c_{12} \\ 0 \end{bmatrix}$$

$$G_1 = \frac{1}{2} m_1 g L_1 c_1 + m_2 g L_1 c_1 + \frac{1}{2} m_2 g L_2 s_{12}$$

Lagrangian Formulation of Manipulator Dynamics

- (e) Gravitational Vector

$$G_i = \sum_{j=1}^{n} m_j g^T J_{vj}^i$$

$i = 2$

J = 1   J = 2

SECOND column of J^2_{v1}

SECOND column of J^2_{v2}

$$G_2 = -m_1 G^T J_{v1}^2 - m_2 g^T J_{v2}^2 = -m_1 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - m_2 \begin{bmatrix} 0 & -g & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} L_2 s_{12} \\ \frac{1}{2} L_2 c_{12} \\ 0 \end{bmatrix}$$

$$G_1 = \frac{1}{2} m_2 L_2 c_{12}$$

Lagrangian Formulation of Manipulator Dynamics

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \left[\begin{array}{c|c} \frac{1}{3}m_1L_1^2 + m_2(L_1^2 + L_1L_2c_2 + \frac{1}{3}L_2^2) & \frac{1}{3}m_2L_2^2 + \frac{1}{2}m_2L_1L_2c_2 \\ \hline m_2(\frac{1}{2}L_1L_2c_2 + \frac{1}{3}L_2^2) & \frac{1}{3}m_2L_2^2 \end{array}\right]\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} +$$

$$\left[\begin{array}{c|c} 0 & \frac{1}{2}L_1L_2s_2 \\ \hline \frac{1}{2}m_2L_1L_2s_2 & 0 \end{array}\right]\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix}-m_2L_1L_2s_2\\ 0\end{bmatrix}\theta_1\theta_2 +$$

$$g\begin{bmatrix}\frac{1}{2}m_1L_1c_1 + m_2(L_1c_1 + \frac{1}{2}L_2s_{12})\\ \frac{1}{2}m_2L_2c_{12}\end{bmatrix}$$

Lagrange Method

Summary

Lagrangian Formulation of Manipulator Dynamics – Summary

- **Step 1:** Define a set of *generalized coordinates* for $i=1,2,3\ldots N$. The usual variable set for serial manipulators is:

$$q_i = \begin{cases} \theta_i & \text{Revolute Joint} \\ d_i & \text{Prismatic Joint} \end{cases}$$

- **Step 2:** Define a set of *generalized velocities* $\dot{q}_i$ for $i=1,2,3\ldots N$

- **Step 3:** Define a set of *generalized forces (and moments)* Q_i

$$Q = \tau + J^T \mathcal{F}_e - f_r$$

- **Step 4:** Define the new tensor of inertia at the base frame (frame 0) by transforming the tensor of inertia of all the links form the links' coordinate systems to the base coordinate system for $i=1,2,3\ldots N$

$${}^{0}I_i = {}^{0}_{i}R\,{}^{i}I_{c_i}\left({}^{0}_{i}R\right)^T$$

Lagrangian Formulation of Manipulator Dynamics – Summary

- **Step 5:** Define the frame positions of the links with respect to the base frame ${}^{0}P_i$ as well as the positions of of the center of mass in their own coordinate systems ${}^{i}P_{c_i}$ for $i=1,2,3 \ldots N.$ manipulators is:

$$ {}^{0}P_i = {}^{0}P_{ORG\ i} \qquad\qquad {}^{i}P_{c_i} $$

- **Step 6:** Define the element in the partial Jacobean matrix for $i=1,2,3 \ldots N$

$$ J_{vi}^{j} = \begin{cases} {}^{0}Z_J \times ({}^{0}P_{c_i} - {}^{0}P_{j})(Reveolute\ Joint) \\ {}^{0}Z_J\ (Prismatic\ Joint) \end{cases} $$

$$ J_{\omega i}^{j} = \begin{cases} {}^{0}Z_J(Reveolute\ Joint) \\ 0\ (Prismatic\ Joint) \end{cases} $$

- **Step 7:** Define the inertia matrix

$$ M = \sum_{i=1}^{n} {}^{0}J_{vi}^{T} m_i {}^{0}J_{vi} + {}^{0}J_{\omega i}^{T} {}^{0}I_{i_C} {}^{0}J_{\omega i} = J_{v1}^{T} m_1 J_{v1} + J_{\omega 1}^{T} {}^{1}I_{1_C} J_{\omega 1} + J_{v2}^{T} m_2 J_{v2} + J_{\omega 2}^{T} {}^{2}I_{2_C} J_{\omega 2} + \cdots $$

Lagrangian Formulation of Manipulator Dynamics – Summary

- **Step 8:** Calculate the velocity vector for *i=1,2,3…N.* as:

$$V_i = \sum_{J=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_k}\right)\dot{q}_k\dot{q}_j$$

$$V_1 = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{1j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_1}\right) = \left(\frac{\partial M_{11}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_1}\right)\dot{\theta}_1\dot{\theta}_1 + \left(\frac{\partial M_{21}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{12}}{\partial \theta_1}\right)\dot{\theta}_1\dot{\theta}_2 + \left(\frac{\partial M_{22}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{21}}{\partial \theta_1}\right)\dot{\theta}_2\dot{\theta}_1 + \left(\frac{\partial M_{22}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1}\right)\dot{\theta}_2\dot{\theta}_2 + \cdots$$

$$V_2 = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{2j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_2}\right) = \left(\frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{11}}{\partial \theta_1}\right)\dot{\theta}_1\dot{\theta}_1 + \left(\frac{\partial M_{11}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{12}}{\partial \theta_1}\right)\dot{\theta}_1\dot{\theta}_2 + \left(\frac{\partial M_{12}}{\partial \theta_1} - \frac{1}{2}\frac{\partial M_{21}}{\partial \theta_1}\right)\dot{\theta}_2\dot{\theta}_1 + \left(\frac{\partial M_{12}}{\partial \theta_2} - \frac{1}{2}\frac{\partial M_{22}}{\partial \theta_1}\right)\dot{\theta}_2\dot{\theta}_2 + \ldots$$

Lagrangian Formulation of Manipulator Dynamics – Summary

- **Step 9:** Calculate the gravity vector for $i=1,2,3\ldots N.$ as:

$$G_i = \sum_{j=1}^{n} m_j g^T J_{vj}^i$$

$$G_1 = -m_1 g^T J_{v1}^1 - m_2 g^T J_{v2}^1 + \cdots$$

$$G_2 = -m_1 g^T J_{v1}^2 - m_2 g^T J_{v2}^2 + \cdots$$

Note that the gravity is defined as

$$g^T = \begin{bmatrix} 0 & -g & 0 \end{bmatrix}$$

- **Step 10:** Define the equation of motion for $i=1,2,3\ldots N.$ as:

$$\sum_{J=1}^{n} M_{ij}\,\ddot{q}_j + V_i + G_i = Q_i$$

Equations of Motion in Various Spaces

Joint Space versus Task Space

Dynamic Equations - State Space Equation

- It is often convenient to express the dynamic equations of a manipulator in a single equation

$$\boxed{\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)}$$

where

$M(\theta)$ - Mass matrix (includes inertia terms) - ***nxn Matrix***

$V(\theta,\dot{\theta})$ - Centrifugal (square of joint velocity) and Coriolis (product of two different joint velocities) terms - ***nx1 Vector***

$G(\theta)$ - gravitational terms - ***nx1 Vector***.

Dynamic Equations - Configuration Space Equation

- By rewriting the velocity dependent term $V(\theta, \dot{\theta})$ in a different form, we can write the dynamic equations as

$$\boxed{\tau = M(\theta)\ddot{\theta} + B(\theta)[\dot{\theta}\ \dot{\theta}] + C(\theta)[\dot{\theta}^2] + G(\theta)}$$

where

$C(\theta)$ - Centrifugal coefficients(square of joint velocity)
$B(\theta)$ - Coriolis coefficients (product of two different joint velocities)

- We may refer to this formulation as the **configuration space equation** or the **joint space**

Dynamic Equations - Cartesian State Space Equation

- It can sometimes be desirable to have a relationship between the end effector's Cartesian accelerations and the joint torques.

$$F = M_x(\theta)\ddot{x} + V_x(\theta,\dot{\theta}) + G_x(\theta)$$

- Beginning from the Configuration Space equation

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

- we can substitute the joint moments using our definition of the Jacobian matrix:

$$\tau = J^T(\theta)F \qquad\qquad F = J^{-T}(\theta)\tau$$

$$\dot{x} = J(\theta)\dot{\theta}$$

- By differentiation, we find

$$\ddot{x} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta}$$

Dynamic Equations - Cartesian State Space Equation

- Solving for joint acceleration gives

$$\ddot{\theta} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{\theta}$$

- Substitution yields

$$F = J^{-T}\tau = J^{-T}M(\theta)J^{-1}\ddot{x} - J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta)$$

$$\boxed{F = M_x(\theta)\ddot{x} + V_x(\theta,\dot{\theta}) + G_x(\theta)}$$

Where

$$M_x(\theta) = J^{-T}M(\theta)J^{-1}$$

$$V_x(\theta,\dot{\theta}) = J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta})$$

$$G_x(\theta) = J^{-T}G(\theta)$$

- This equation relates the forces and moments at the end effector to the Cartesian accelerations of the end effector and the manipulator joint positions and velocities.