Manipulator Dynamics 3

Iterative Newton – Euler Equations

2R Example

Forward Dynamics

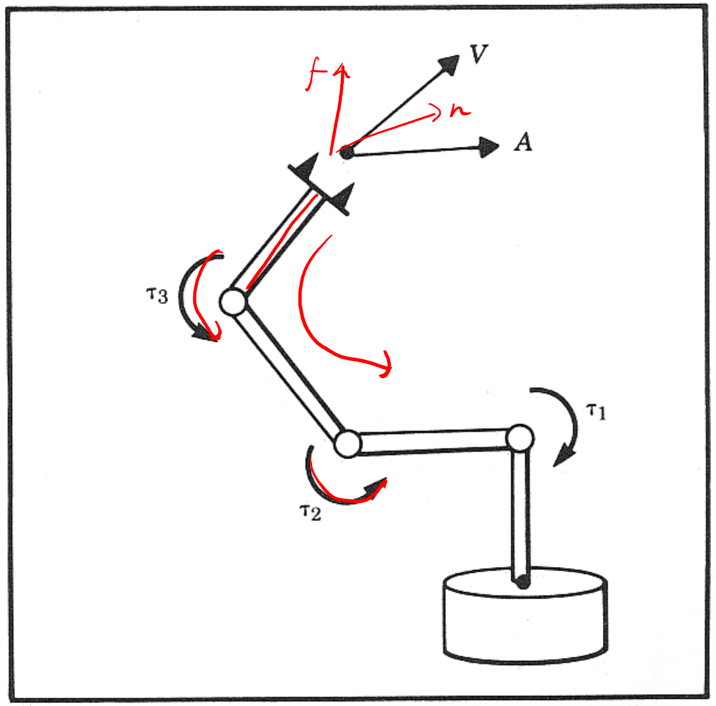
$$\begin{cases} \boldsymbol{\tau}_i \\ Link_i(x,y,z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$

Problem

Given: *Joint torques and links geometry, mass, inertia, friction, joint torques*

Compute: *Angular acceleration of the links (solve differential equations)*

$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \end{cases}$$



Solution

solve a set of differential equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

Inverse Dynamics

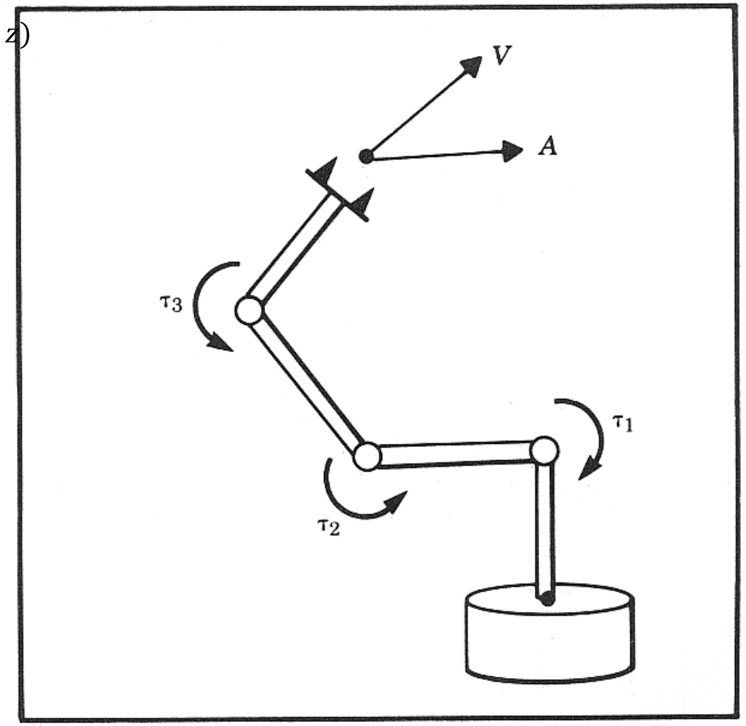
$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \\ Link_i(x,y,z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$

Problem

Given: Angular acceleration, velocity and angels of the links in addition to the links geometry, mass, inertia, friction

Compute: Joint torques

$$\{\tau$$



Solution

Solve a set of algebraic equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

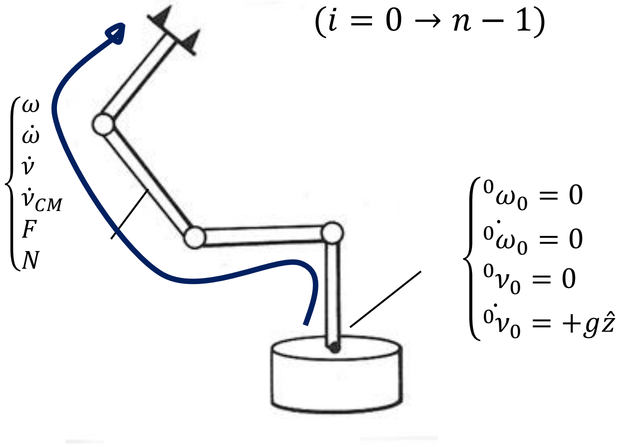
Iterative Newton Euler Equations
Steps of the Algorithm

- (1) Outward Iterations $(i = 0 \rightarrow n-1)$
  - Starting With velocities and accelerations of the base

$$ {}^{0}\omega_0 = 0,\ {}^{0}\dot{\omega}_0 = 0,\ {}^{0}v_0 = 0,\ {}^{0}\dot{v}_0 = +g\hat{z} $$

  - Calculate velocities accelerations, along with forces and torques (at the CM)

$$ \omega, \dot{\omega}, \dot{v}, \dot{v}_{CM}, F, N $$

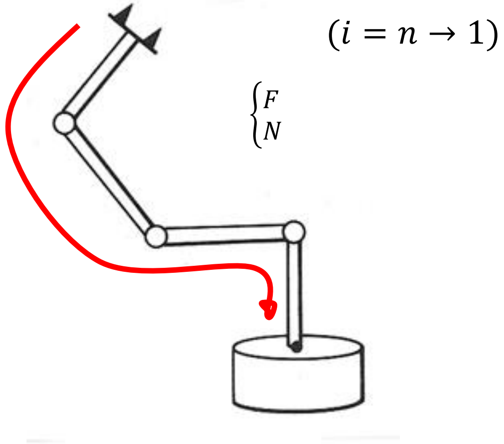


- (2) Inward Iteration $(i = n \rightarrow 1)$
  - Starting with forces and torques (at the CM)

$$ F, N $$

  - Calculate forces and torques at the joints

$$ f, n $$



Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i : 0 \rightarrow 5$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i) $$

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1} $$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i : 6 \rightarrow 1$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot’s base.

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

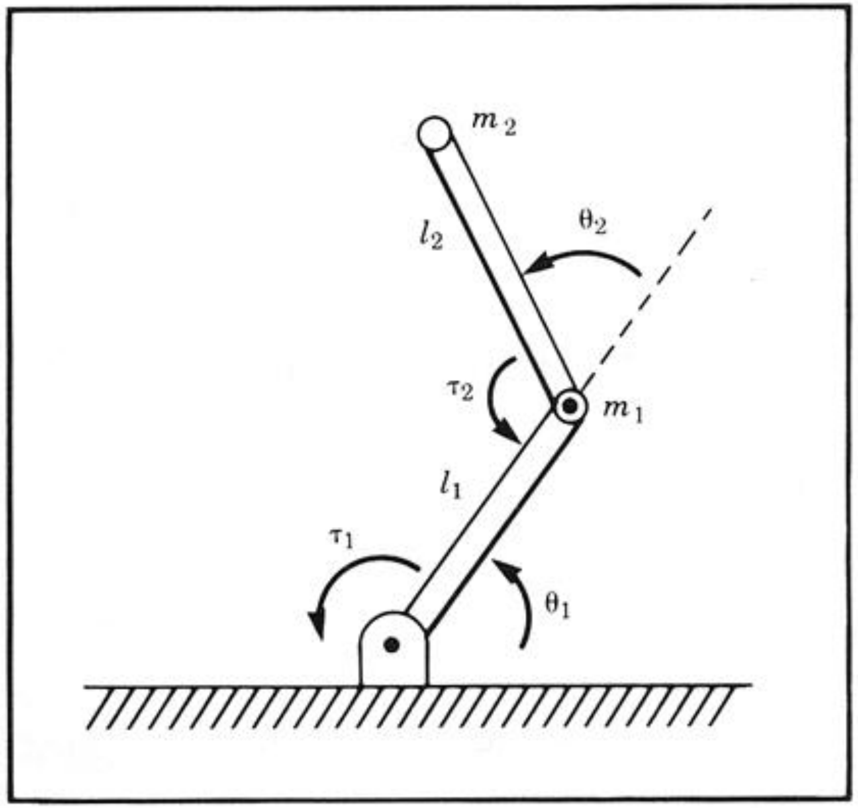
$$ \tau_i = {}^{i+1}n^{T}{}_{i+1}\;{}^{i}\hat{Z}_i $$

Iterative Newton-Euler Equations - Solution Procedure

- **Error Checking -** Check the units of each term in the resulting equations

- **Gravity Effect** - The effect of gravity can be included by setting ${}^{0}\dot{v}_{0} = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.

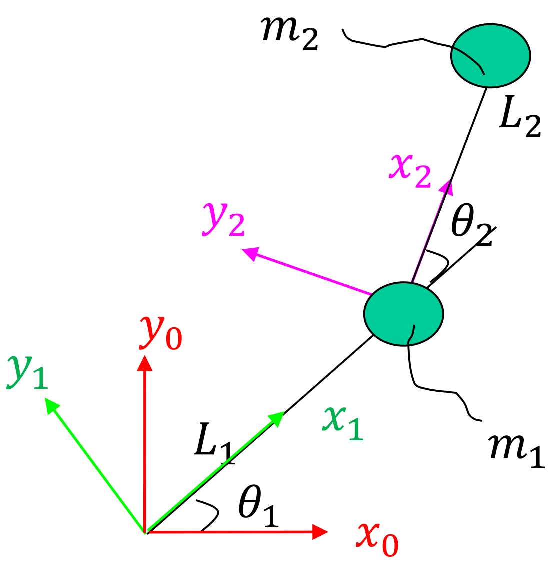
Iterative Newton-Euler Equations - 2R Robot Example



Iterative Newton-Euler Equations - 2R Robot Example

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & L1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



- Vectors locates the center of mass for each link

$$
{}^{1}P_{c1} = l_1\hat{x}_1 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{2}P_{c2} = l_2\hat{x}_2 = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} {}^{c_1}I_1 = 0 \\ {}^{c_2}I_2 = 0 \end{cases}
$$

Iterative Newton-Euler Equations - 2R Robot Example

- No External force/torque on the end effector

$$\begin{cases} f_3 = 0 \\ n_3 = 0 \end{cases}$$

- The base of the robot is not rotating

$$\begin{cases} \omega_0 = 0 \\ \dot{\omega}_0 = 0 \end{cases}$$

- To include gravity

$${}^0\dot{V}_0 = g\,\hat{Y}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i = 0$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} $$

$$ {}^{i+1}\dot\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\dot\omega_i + {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i \times \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} + \ddot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} $$

$$ {}^{i+1}\dot v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}\dot\omega_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot v_i\right) $$

$$ {}^{i+1}\dot v_{C_{i+1}} = {}^{i+1}\dot\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot v_{i+1} $$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot v_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot\omega_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{1}\omega_{1} = {}^{1}_{0}R\,\overset{0}{\cancel{{}^{0}\omega_{0}}} + \dot{\theta}_{1}\hat{Z}_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{1}\dot{\omega}_1 = {}^{1}_{0}R\,\cancelto{0}{{}^{0}\dot{\omega}_0} + {}^{1}_{0}R\,\cancelto{0}{{}^{0}\omega_0} \times \dot{\theta}_1\,{}^{1}\hat{Z}_1 + \ddot{\theta}_1\,{}^{1}\hat{Z}_1 = \ddot{\theta}_1 \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right) $$

$$ {}^{1}\dot{v}_1 = {}^{1}_{0}R\left(\cancelto{0}{{}^{0}\dot{\omega}_0} \times {}^{0}P_1 + \cancelto{0}{{}^{0}\omega_0} \times \left(\cancelto{0}{{}^{0}\omega_0} \times {}^{0}P_1\right) + {}^{0}\dot{v}_0\right) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}$$

$$^{1}\dot{v}_{C_1} = {}^{1}\dot{\omega}_1 \times {}^{1}P_{C_1} + {}^{1}\omega_1 \times \left({}^{1}\omega_1 \times {}^{1}P_{C_1}\right) + {}^{1}\dot{v}_1 =$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + {}^{1}\omega_1 \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + {}^{1}\omega_1 \times \begin{bmatrix} 0 \\ L_1\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & L_1\dot{\theta}_1 & 0 \end{vmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\dot{\theta}_1^2 + gs_1 \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i : 0$

- Calculate the link velocities and accelerations iteratively from the robot’s base to the end effector

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i)$$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}$$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$$^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$$^{1}F_{1} = m_1\,{}^{1}\dot{v}_{C_1} = \begin{bmatrix} -L_1\dot{\theta}_1^2 + gs_1 \\ L_1\ddot{\theta}_1 + gs_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -m_1L_1\dot{\theta}_1^2 + m_1gs_1 \\ m_1L_1\ddot{\theta}_1 + m_1gs_1 \\ 0 \end{bmatrix} \begin{matrix} {}^{1}F_{1x} \\ {}^{1}F_{1y} \\ \end{matrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

$$^{1}N_{1} = \overset{0}{\cancel{{}^{C_1}I_1}}\,{}^{1}\dot{\omega}_{1} + {}^{1}\omega_{1} \times \overset{0}{\cancel{{}^{C_1}I_1}}\,{}^{1}\omega_{1}$$

Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i = 1$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right)$$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}$$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$$^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{2}\omega_2 = {}^{2}_{2}R\,{}^{1}\omega_1 + \dot{\theta}_2\,{}^{2}\hat{Z}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{2}\dot{\omega}_2 = {}^{2}_{1}R\,{}^{1}\dot{\omega}_1 + {}^{2}_{1}R\,{}^{i}\omega_i \times \dot{\theta}_2\,{}^{2}\hat{Z}_2 + \ddot{\theta}_2\,{}^{2}\hat{Z}_2$$

$$\begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & 0 & \dot{\theta}_2 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right) $$

$$ {}^{2}\dot{v}_{2} = {}^{2}_{1}R\left({}^{1}\dot{\omega}_1 \times {}^{1}P_{2} + {}^{1}\omega_1 \times \left({}^{1}\omega_1 \times {}^{1}P_{2}\right) + {}^{1}\dot{v}_1\right) = $$

$$ \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{vmatrix} I & J & K \\ 0 & 0 & \ddot{\theta}_1 \\ L_2 & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{vmatrix} I & J & K \\ 0 & 0 & \dot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} \right) = $$

$$ \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} I & J & K \\ 0 & 0 & \dot{\theta}_1 \\ 0 & L_1\dot{\theta}_2 & 0 \end{vmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} \right) = $$

$$ \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ L_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} L_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L_1\dot{\theta}_1^2 + g s_1 \\ L_1\ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 + g s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}$$

$$^{2}\dot{v}_{C_2} = {}^{2}\dot{\omega}_{2} \times {}^{2}P_{C_2} + {}^{2}\omega_{2} \times \left({}^{2}\omega_{2} \times {}^{2}P_{C_2}\right) + {}^{2}\dot{v}_{2} =$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 + \ddot{\theta}_2 \\ L_2 & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ L_2 & 0 & 0 \end{vmatrix} + \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 + g s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ L_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ 0 & L_2(\dot{\theta}_1 + \dot{\theta}_2) & 0 \end{vmatrix} + \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 + g s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ L_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 + g s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + g s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g c_{12} \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i : 1$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i)$$

$${}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}$$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$${}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$${}^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}F_{i+1} = m_{i+1}\,^{i+1}\dot{v}_{C_{i+1}}$$

$$^{2}F_{2} = m_{2}\,^{2}\dot{v}_{C_{2}} = \begin{bmatrix} m_2(L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 - L_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + g s_{12}) \\ m_2(L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + \ g c_{12}) \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$
-

$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1} \, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} \, {}^{i+1}\omega_{i+1} $$

$$ {}^{2}N_{2} = {}^{C_2}\cancelto{0}{I_2} \, {}^{2}\dot{\omega}_{2} + {}^{2}\omega_{2} \times {}^{C_2}\cancelto{0}{I_2} \, {}^{2}\omega_{2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i = 2$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot's base.

$${}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$${}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i+1}n^{T}_{i+1}\,{}^{i}\hat{Z}_i$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 2$

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{2}f_2 = {}^{2}_{3}R\,{}^{3}\cancelto{0}{f_3} + {}^{2}F_2 = {}^{2}F_2$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 2$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{2}n_2 = \underbrace{{}^{2}N_2}_{0} + \underbrace{{}^{2}_{3}R\,{}^{3}n_3}_{0} + {}^{2}P_{C_2} \times {}^{2}F_2 + {}^{2}P_3 \times \underbrace{{}^{2}_{3}R\,{}^{3}f_3}_{0} $$

$$ = \begin{vmatrix} i & j & k \\ L_2 & 0 & 0 \\ {}^{2}F_{2x} & {}^{2}F_{2y} & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_2\,{}^{2}F_{2y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ m_2 L_2 (L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g c_{12}) \end{bmatrix} $$

$$ {}^{2}n_2 \hat{Z} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i = 1$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot's base.

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i+1}n^{T}_{i+1}\,{}^{i}\hat{Z}_i$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 1$

$$^{i}f_{i} = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_{i}$$

$$^{1}f_{1} = {}^{1}_{2}R\,{}^{2}f_{2} + {}^{1}F_{1} =$$

$$\begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2(L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + g s_{12}) \\ m_2(L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g c_{12}) \\ 0 \end{bmatrix} + \begin{bmatrix} -m_1 L_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 L_1 \ddot{\theta}_1 + m_1 g s_1 \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 1$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

0

$$ {}^{1}n_1 = {}^{1}N_1 + {}^{1}_{2}R\,{}^{2}n_2 + {}^{1}P_{C_1} \times {}^{1}F_1 + {}^{1}P_2 \times {}^{1}_{2}R\,{}^{2}f_2 = $$

$$ \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ {}^{2}n_2\hat{Z} \end{bmatrix} + \begin{vmatrix} I & J & K \\ L_1 & 0 & 0 \\ {}^{1}F_{1x} & {}^{1}F_{1y} & 0 \end{vmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{2}f_{2x} \\ {}^{2}f_{2x} \\ 0 \end{bmatrix} = $$

$$ \begin{bmatrix} 0 \\ 0 \\ {}^{2}n_2\hat{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_1\,{}^{1}F_{1y} \end{bmatrix} + \begin{vmatrix} i & j & k \\ L_1 & 0 & 0 \\ c_2\,{}^{2}f_2x - s_2\,{}^{2}f_2y & s_2\,{}^{2}f_2x + c_2\,{}^{2}f_2y & 0 \end{vmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration

$$
{}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ {}^{2}n_2\hat{Z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_1\,{}^{1}F_{1y} \end{bmatrix} + \begin{vmatrix} i & j & k \\ L_1 & 0 & 0 \\ c_2\,{}^{2}f_2x - s_2\,{}^{2}f_2y & s_2\,{}^{2}f_2x + c_2\,{}^{2}f_2y & 0 \end{vmatrix} =
$$

$$
= \begin{bmatrix} 0 \\ 0 \\ m_2L_2(L_1\ddot{\theta}_1c_2 + L_1\dot{\theta}_1^2s_2 + L_2(\dot{\theta}_1 + \dot{\theta}_2) + gc_{12}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_1(m_1L_1\ddot{\theta}_1 + m_1gs_1) \end{bmatrix} +
$$

$$
\begin{bmatrix} 0 \\ 0 \\ L_1(s_2(m_2(L_1\ddot{\theta}_1s_2 - L_1\dot{\theta}_1^2c_2 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + gs_{12})) + c_2(m_2(L_1\ddot{\theta}_1c_2 + L_1\dot{\theta}_1^2s_2 - L_1\,(\ddot{\theta}_1 + \ddot{\theta}_2) + gc_{12})) \end{bmatrix}
$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration

$$ {}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ L_1 s_2\left[m_2 L_1 \ddot{\theta}_2 - m_2 L_1 \dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right] + L_1 c_2\left[m_2 L_1 \ddot{\theta}_2 + m_2 L_1 \dot{\theta}_1^2 s_2 + m_2 g c_{12} - m_2 L_1\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\right] \end{bmatrix} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i = 1,2$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot’s base.

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i}n^{T}_{i}\,{}^{i}\hat{Z}_i$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 1,2$

$$ {}^1n_1 = \begin{bmatrix} 0 \\ 0 \\ L_1 s_2\left[m_2 L_1 \ddot{\theta}_2 - m_2 L_1 \dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right] + L_1 c_2\left[m_2 L_1 \ddot{\theta}_2 + m_2 L_1 \dot{\theta}_1^2 s_2 + m_2 g c_{12} - m_2 L_1\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)\right] \end{bmatrix} $$

$$ {}^2n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2 L_2 (L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g c_{12}) \end{bmatrix} $$

$$ \tau_i = {}^i n^T{}_i \; {}^i\hat{Z}_i $$

$$ \begin{cases} \tau_1 = L_1(s_2(m_2(L_1 \ddot{\theta}_1 s_2 - L_1 \dot{\theta}_1^2 c_2 - L_2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + g s_{12})) + c_2(m_2(L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + L_2(\dot{\theta}_1 + \dot{\theta}_2) + g c_{12})) \\ \tau_2 = m_2 L_2 (L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + L_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g c_{12}) \end{cases} $$

Iterative Newton-Euler Equations - 2R Robot Example

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Mass/Inertia Matrix

Gravity

$$C(\theta)[\dot{\theta}^2] + B(\theta)[\dot{\theta}]$$

Centrifugal

Coriolis

Iterative Newton-Euler Equations - 2R Robot Example

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix} m_2L_2^2 + 2m_2L_1L_2c_2 + L_1^2(m_1+m_2) & m_2L_2^2 + m_2L_1L_2c_2 \\ m_2L_2^2 + m_2L_1L_2c_2 + m_2L_2^2 & 2m_2L_2^2 \end{bmatrix} \begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} +$$

$$\begin{bmatrix} 0 & -m_2L_1L_2s_2 \\ m_2L_1L_2s_2 & 0 \end{bmatrix} \begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix} -2m_2L_1L_2s_2 \\ 0 \end{bmatrix} [\dot{\theta}_1\dot{\theta}_2] +$$

$$\begin{bmatrix} m_2L_2gc_{12} + (m_1+m_2)L_1gc_1 \\ m_2L_2gc_{12} \end{bmatrix}$$

Equation of Motion – Non Rigid Body Effects

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

- Viscous Friction $\tau_{friction} = v\dot{\theta}$
- Coulomb Friction $\tau_{friction} = c\ \text{sgn}(\dot{\theta})$
- Model of Friction $\tau_{friction} = v\dot{\theta} + c\ \text{sgn}(\dot{\theta}) = f(\theta,\dot{\theta})$