Manipulator Dynamics 2

Forward Dynamics

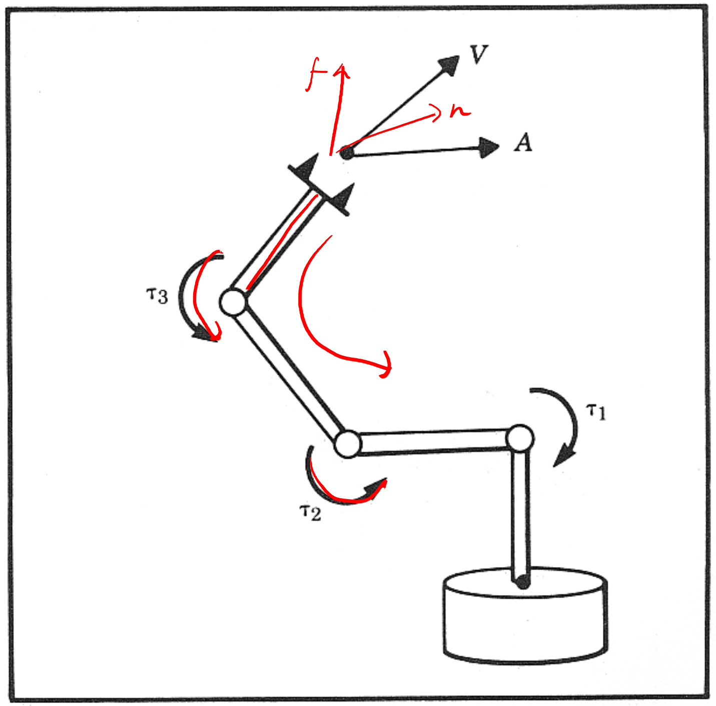
$$\begin{cases} \boldsymbol{\tau}_i \\ Link_i(x,y,z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$

Problem

Given: *Joint torques and links geometry, mass, inertia, friction, joint torques*

Compute: *Angular acceleration of the links (solve differential equations)*

$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \end{cases}$$



Solution

solve a set of differential equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

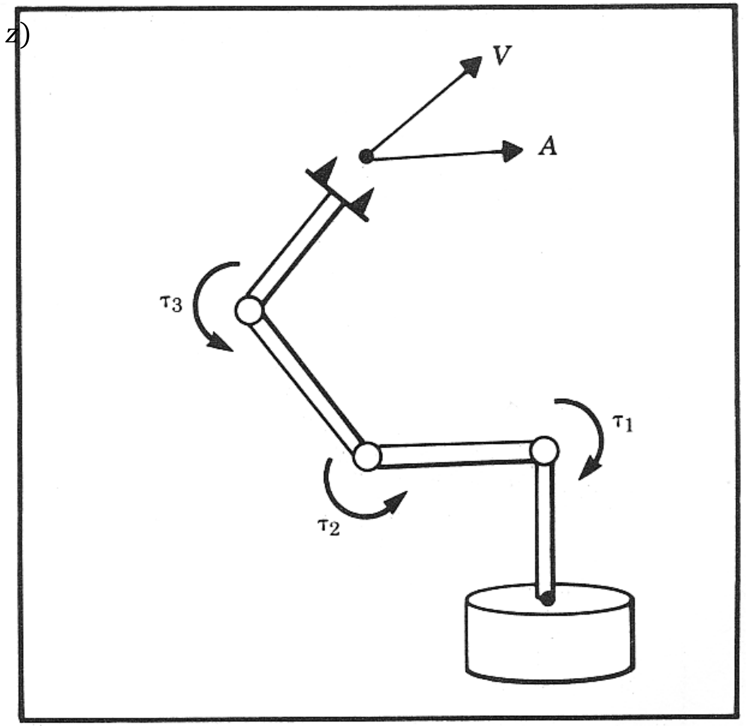
$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

Inverse Dynamics

Problem

Given: *Angular acceleration, velocity and angels of the links in addition to the links geometry, mass, inertia, friction*

$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \\ Link_i(x,y,z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$



Compute: *Joint torques*

$$\{\tau$$

Solution

Solve a set of algebraic equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

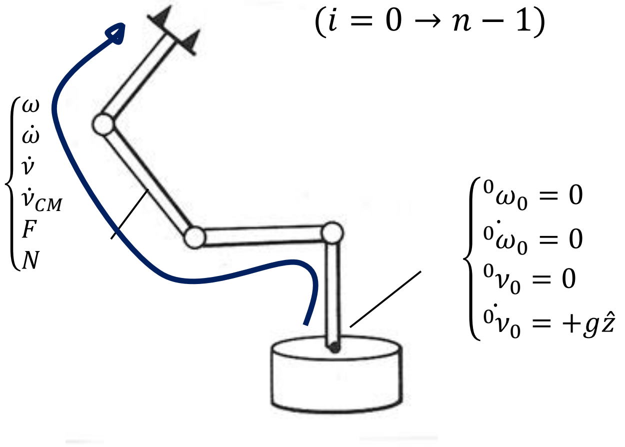
Iterative Newton Euler Equations
Steps of the Algorithm

- (1) Outward Iterations $(i = 0 \rightarrow n-1)$
  - Starting With velocities and accelerations of the base

$$^{0}\omega_0 = 0,\ ^{0}\dot{\omega}_0 = 0,\ ^{0}v_0 = 0,\ ^{0}\dot{v}_0 = +g\hat{z}$$

  - Calculate velocities accelerations, along with forces and torques (at the CM)

$$\omega, \dot{\omega}, \dot{v}, \dot{v}_{CM}, F, N$$

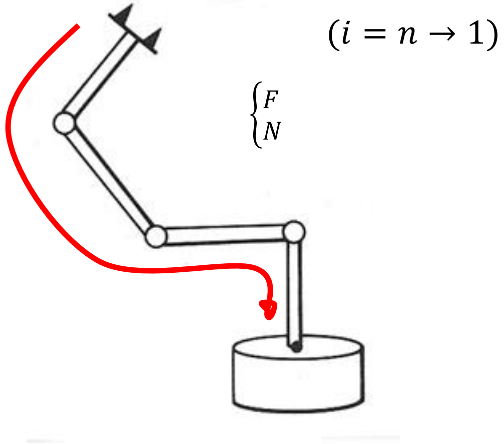


- (2) Inward Iteration $(i = n \rightarrow 1)$
  - Starting with forces and torques (at the CM)

$$F, N$$

  - Calculate forces and torques at the joints

$$f, n$$



Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i : 0 \rightarrow 5$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i) $$

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1} $$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i : 6 \rightarrow 1$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot's base.

$$ {}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ \tau_i = {}^{i+1}n^{T}{}_{i+1}\ {}^{i}\hat{Z}_i $$

Manipulator Dynamics – Newton Euler Equations

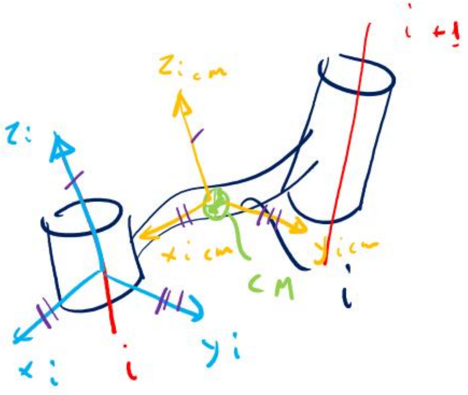
The Inertia Tensor (Moment of Inertia)

$$^{c}I$$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

  $\dot{v}_c$ - The linear acceleration of the center of mass
  $\dot{\omega}$ - The angular acceleration
  ${}^{c}I$ - The Inertia tensor (moment of inertia)
  F - The sum of all the forces applied on the center of mass
  N - The sum of all the moments applied on the center of mass



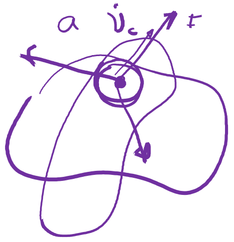
$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

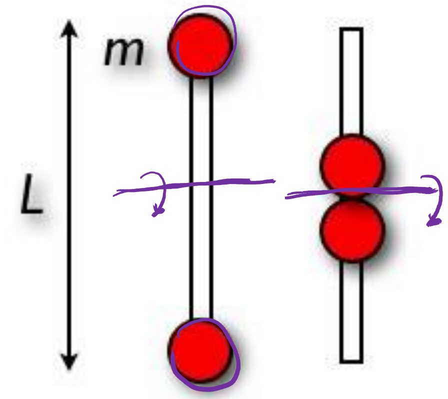
Moment of Inertia – Intuitive Understanding

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

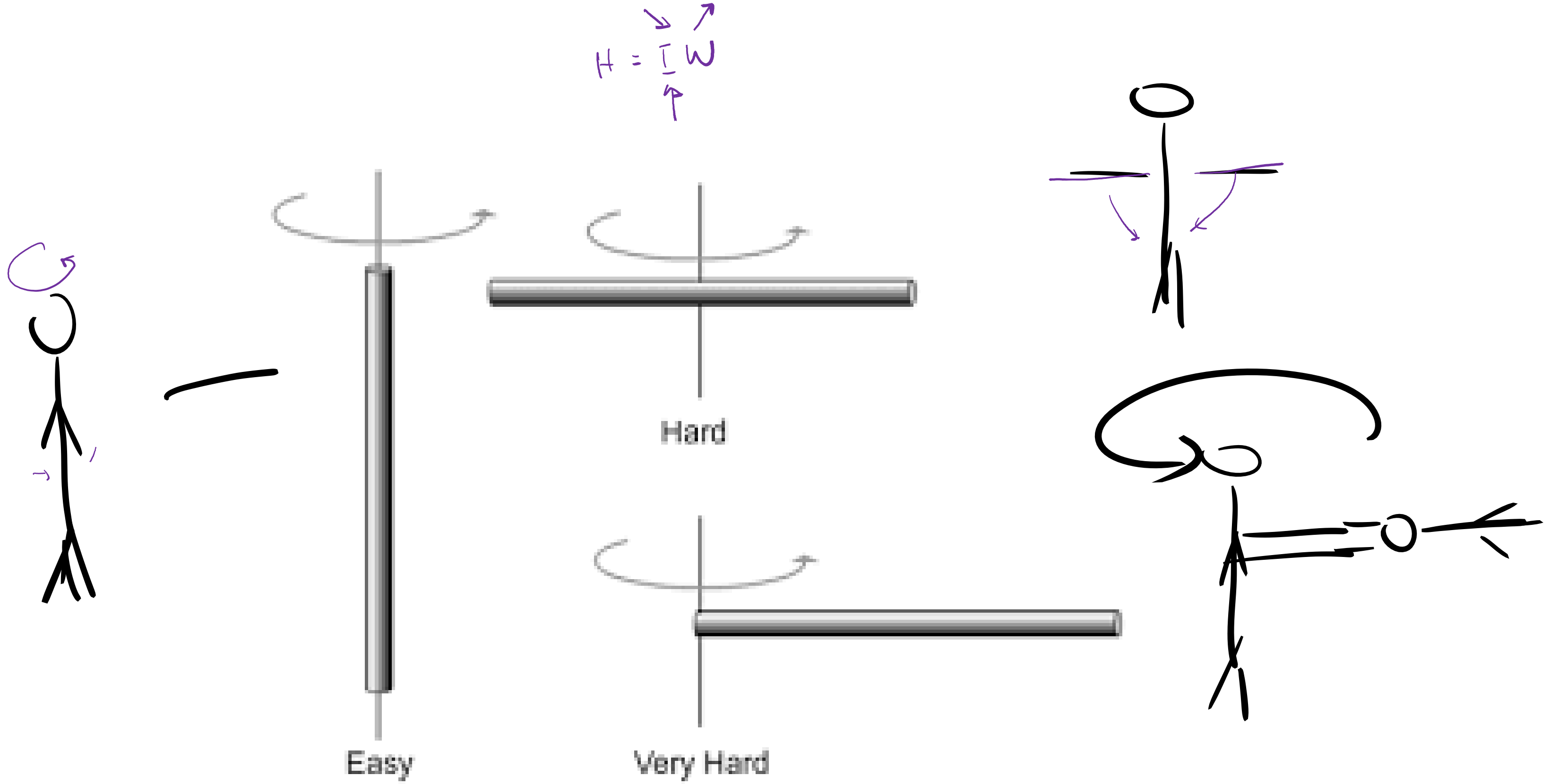


$I_{AA} > I_{BB}$

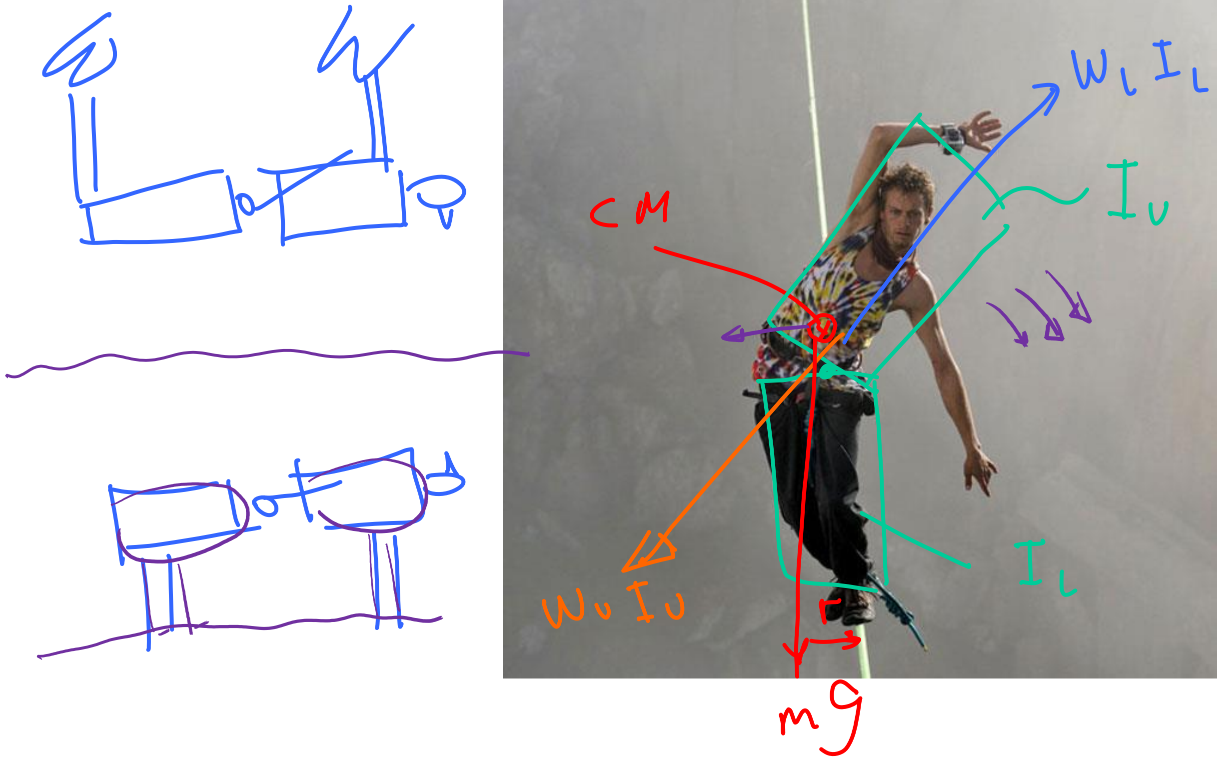


Moment of Inertia – Intuitive Understanding

Moment of Inertia – Intuitive Understanding



Moment of Inertia – Intuitive Understanding

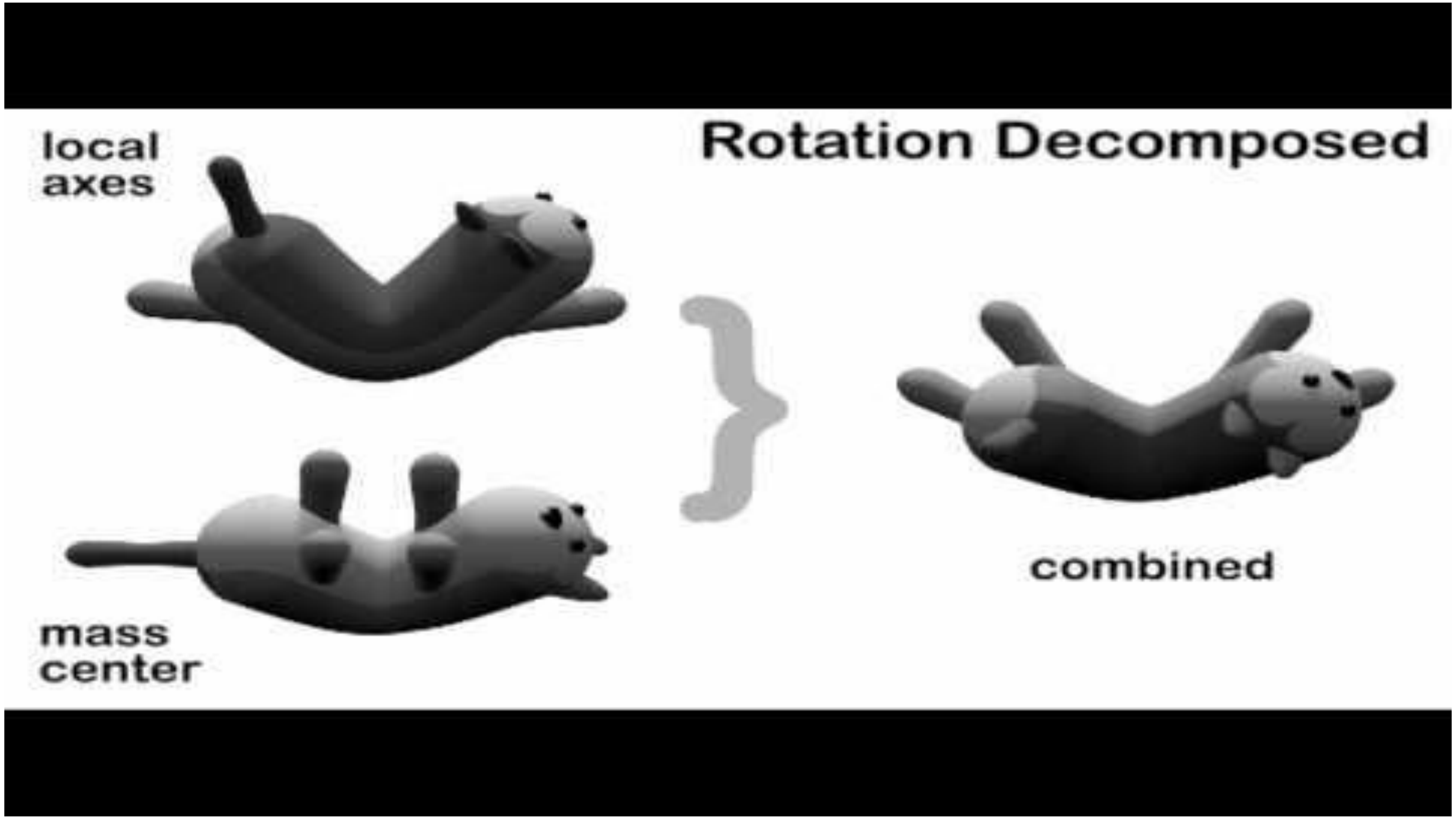


$I_U > I_L$

$W_U < W_L$

Moment of Inertia – Intuitive Understanding

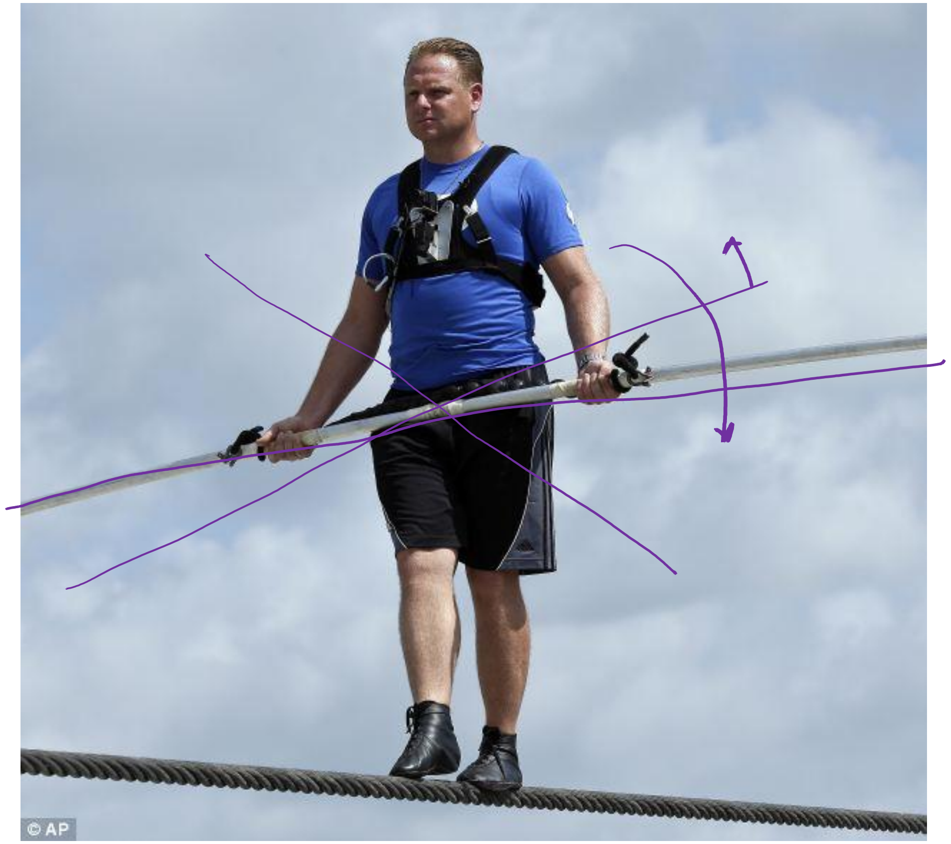
Moment of Inertia – Intuitive Understanding



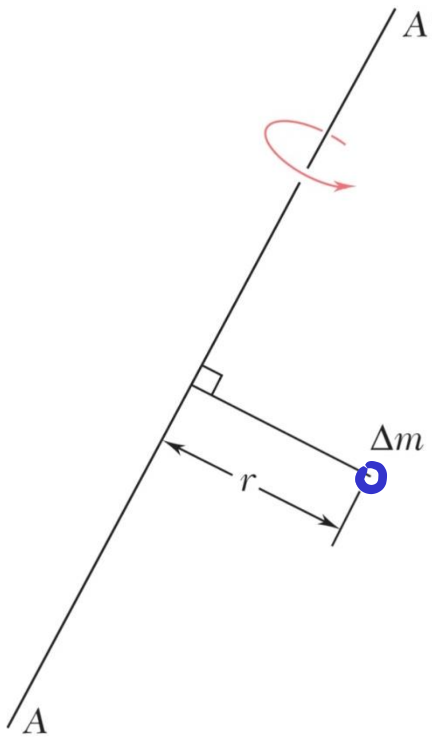
Moment of Inertia – Intuitive Understanding

- https://www.youtube.com/watch?v=9SaShn8OkJI

Moment of Inertia – Intuitive Understanding



Moment of Inertia – Particle – WRT Axis



$$I_{AA'} = \mathrm{r}^2 \Delta m$$

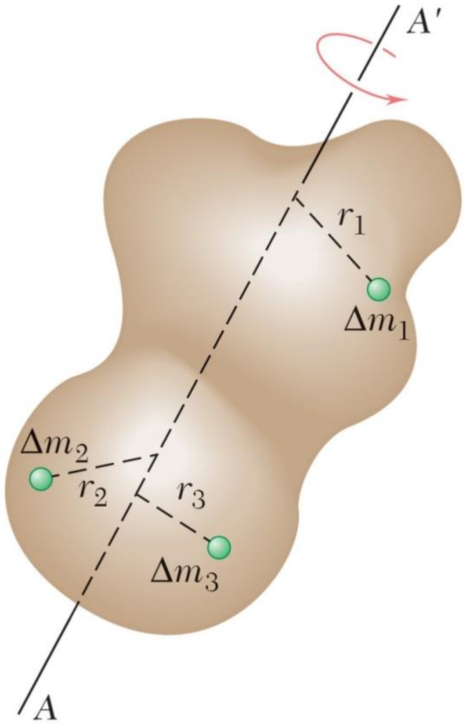
Moment of Inertia – Solid – WRT Axis

$$I_{AA'} = \sum_i r_i^2 \Delta m_i$$

$$I_{AA'} = \int_v r^2 \, dm = \iiint_v r^2 \, \rho dv$$

$\uparrow$

ρdv

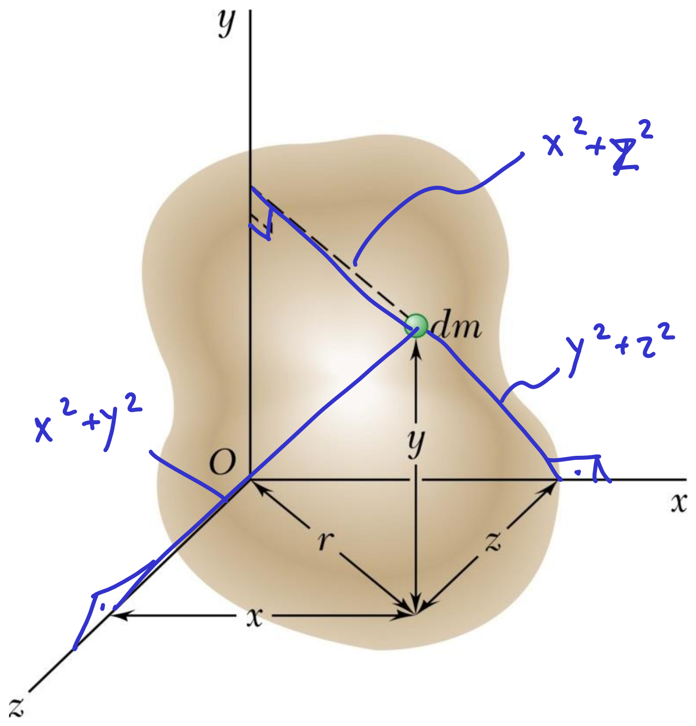


Moment of Inertia – Solid – WRT Coordinate Frame

$$I_{yy} = \int r^2\, dm = \int (z^2 + x^2)\, dm = \iiint_v (z^2 + x^2)\, \rho dv$$

$$I_{xx} = \iiint_v (z^2 + y^2)\, \rho dv$$

$$I_{zz} = \iiint_v (x^2 + y^2)\, \rho dv$$

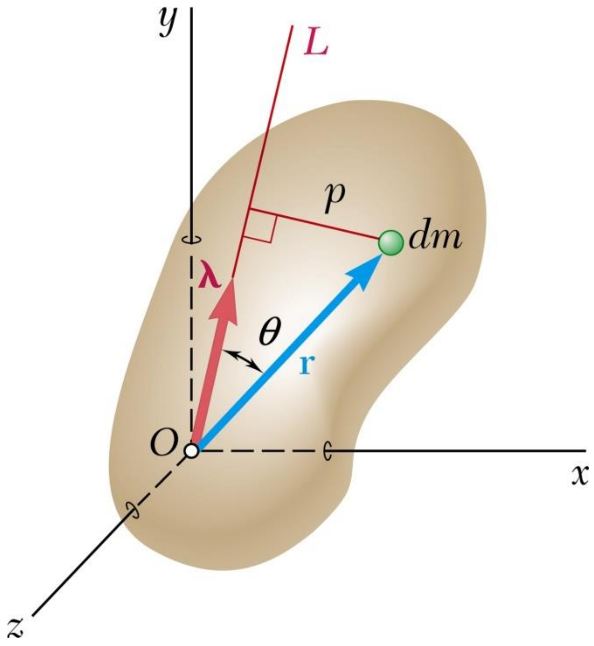


Moment of Inertia – Solid – WRT an Arbitrary Axis

$$p = \mathrm{r}\sin\theta = \lambda \times r$$

$$\mathrm{I_{OL}} = \int p^2\, dm = \int (\lambda \times r)^2\, dm = \int (\lambda \times r)^T (\lambda \times r)\, dm$$

$$\lambda \times r = \begin{vmatrix} i & j & k \\ \lambda_x & \lambda_y & \lambda_z \\ x & y & z \end{vmatrix} = i(\lambda_y z - \lambda_z y) + j(\lambda_z x - \lambda_x z) + k(\lambda_x y - \lambda_y x)$$



$$\mathrm{I_{OL}} = \int (\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2\, dm$$

Moment of Inertia – Solid – WRT an Arbitrary Axis

$$\mathrm{I_{OL}} = \int \left(\lambda_x y - \lambda_y x\right)^2 + \left(\lambda_y z - \lambda_z y\right)^2 + (\lambda_z x - \lambda_x z)^2 \, dm$$

$$\mathrm{I_{OL}} = \lambda_x^2 \overbrace{\int (y^2 + z^2)\, dm}^{I_{xx}} + \lambda_y^2 \overbrace{\int (z^2 + x^2)\, dm}^{I_{yy}} + \lambda_z^2 \overbrace{\int (x^2 + y^2)\, dm}^{I_{zz}}$$

$$-2\lambda_x\lambda_y \underbrace{\int xy\, dm}_{I_{xy}} - 2\lambda_y\lambda_z \underbrace{\int yz\, dm}_{I_{yz}} - 2\lambda_z\lambda_x \underbrace{\int zx\, dm}_{I_{zx}}$$

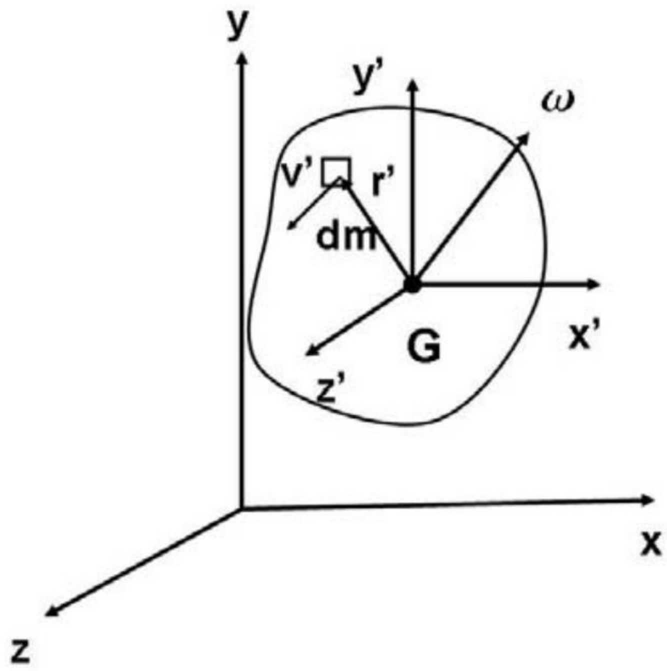
$$\mathrm{I_{OL}} = I_{xx}\lambda_x^2 + I_{yy}\lambda_y^2 + I_{zz}\lambda_z^2 - 2I_{xy}\lambda_x\lambda_y - 2I_{yz}\lambda_y\lambda_z - 2I_{zx}\lambda_z\lambda_x$$

Inertia Tensor – WRT a Coordinate Frame at the CM

- Expression of the angular momentum of a system of particles about the center of mass, the angular momentum $\mathrm{H_G}$ is defined as

$$\boldsymbol{H}_G = \sum_{i=1}^{n} (\boldsymbol{r}'_i \times m_i(\boldsymbol{\omega} \times \boldsymbol{r}'_i)) = \sum_{i=1}^{n} m_i r_i'^2 \boldsymbol{\omega}$$

Where, r' is the position vector relative to the center of mass, v' is the velocity relative to the center of mass.



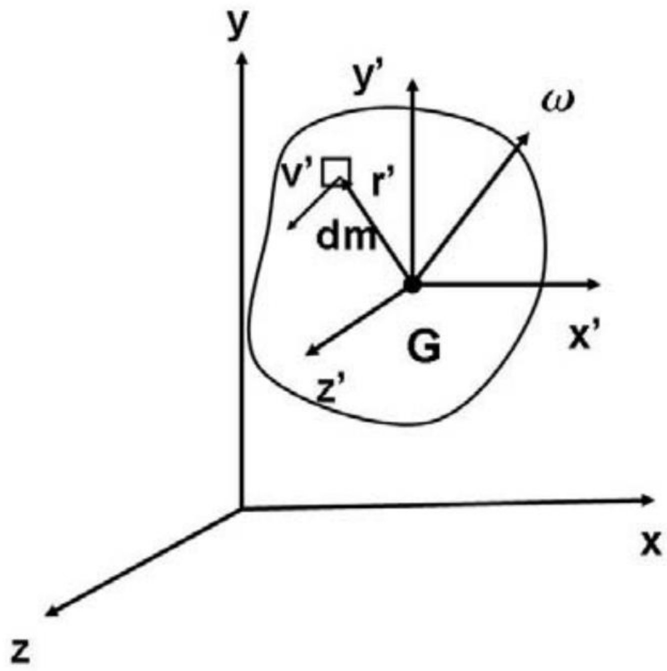
- For a 3D continuum mass of a rigid body, the summation can be replaced by an integration over the entire mass.

$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times \boldsymbol{v}' \, dm$$

Inertia Tensor – WRT a Coordinate Frame at the CM

- For a 3D rigid body, the distance r' between infinitesimal mass dm and the center of mass G remains constant, and the infinitesimal mass velocity v', relative to the center of mass G, due to the rotation of the rigid body by an angular velocity ω is expressed by

$$\boldsymbol{v}' = \boldsymbol{\omega} \times \boldsymbol{r}'$$



- Using the vector identity

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$$

the expression the angular momentum $\mathrm{H_G}$ is rewritten as

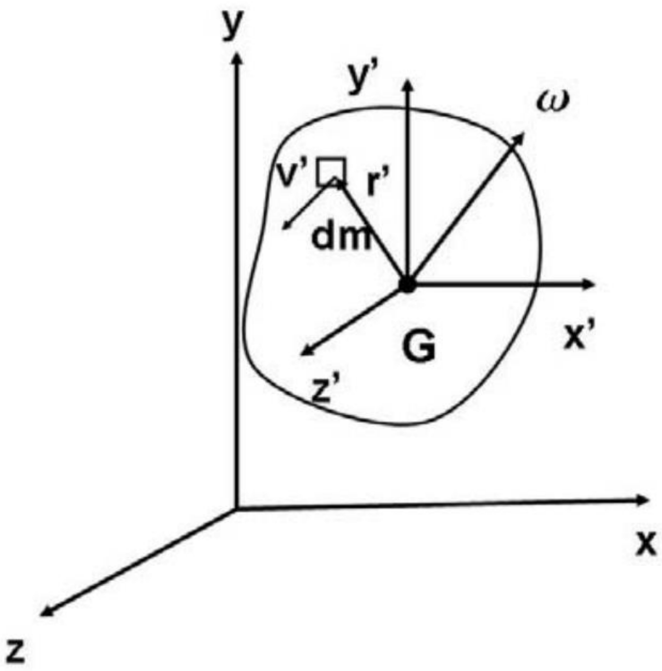
$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times (\boldsymbol{\omega} \times \boldsymbol{r}')\, dm = \int_m [(\boldsymbol{r}' \cdot \boldsymbol{r}')\boldsymbol{\omega} - (\boldsymbol{r}' \cdot \boldsymbol{\omega})\boldsymbol{r}']\, dm$$

Inertia Tensor – WRT a Coordinate Frame at the CM

- For a 2D rigid body, rotating in its own plane the distance r' between infinitesimal mass dm is perpendicular to the angular velocity ω , therefore the term $r' \cdot \omega$ is zero, as a result the angular velocity vector ω is parallel to the angular momentum $\mathrm{H_G}$

$$\boldsymbol{H}_G = \int_m \boldsymbol{r}' \times (\boldsymbol{\omega} \times \boldsymbol{r}')\, dm = \int_m [(\boldsymbol{r}' \cdot \boldsymbol{r}')\boldsymbol{\omega} - \underbrace{(\boldsymbol{r}' \cdot \boldsymbol{\omega})}_{0}\boldsymbol{r}']\, dm$$

$$H_G = \int_m (r' \cdot r')\omega$$



- In the three-dimensional case however, this simplification does not occur, and as a consequence, the angular velocity vector, ω, and the angular momentum vector, $\mathrm{H_G}$, are in general, not parallel.

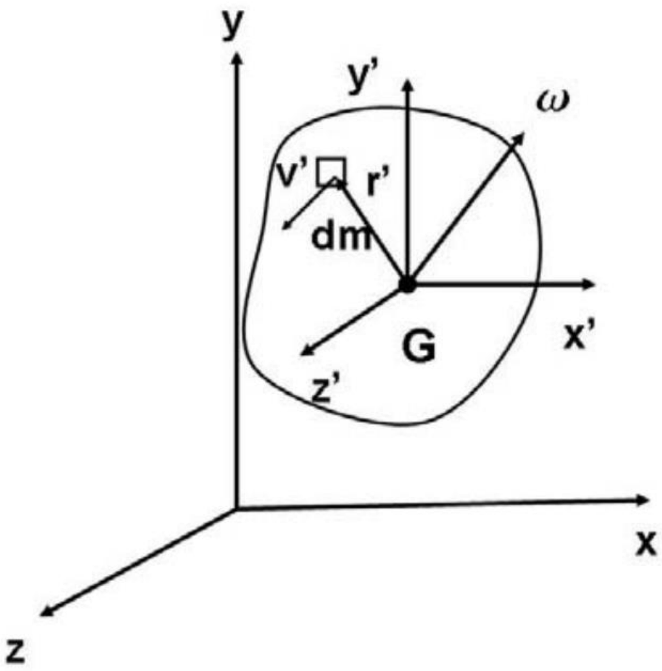
Inertia Tensor – WRT a Coordinate Frame at the CM

- In cartesian coordinates, the distance r' between infinitesimal mass dm and the center of mass G and the angular velocity vector, ω are defined as

$$\boldsymbol{r}' = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}$$
$$\boldsymbol{\omega} = \omega_x\boldsymbol{i} + \omega_y\boldsymbol{j} + \omega_z\boldsymbol{k}$$

- The expression for the the angular momentum $\mathrm{H_G}$ can be expended to

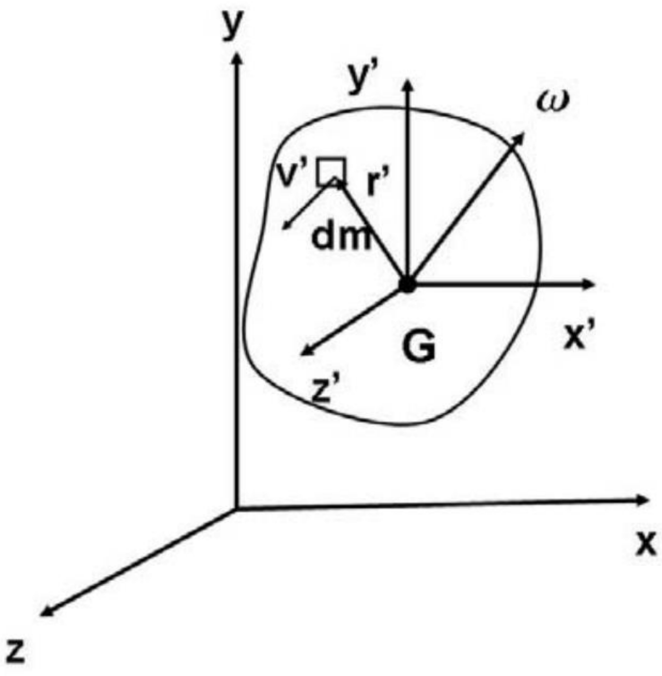
$$\begin{aligned}
\boldsymbol{H}_G &= \left(\omega_x \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')x'\, dm\right)\boldsymbol{i} \\
&+ \left(\omega_y \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')y'\, dm\right)\boldsymbol{j} \\
&+ \left(\omega_z \int_m (x'^2 + y'^2 + z'^2)\, dm - \int_m (\omega_x x' + \omega_y y' + \omega_z z')z'\, dm\right)\boldsymbol{k} \\
&= (\ \ I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\,\boldsymbol{i} \\
&+ (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\,\boldsymbol{j} \\
&+ (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\,\boldsymbol{k}\ .
\end{aligned}$$



Inertia Tensor – WRT a Coordinate Frame at the CM

- The quantities I_{xx}, I_{yy}, and I_{zz} are called the mass moments of inertia with respect to the x, y and z axis, respectively, and are given by

$$\left.\begin{aligned} I_{xx} &= \int_m (y'^2 + z'^2) dm = \iiint_V (y'^2 + z'^2)\, \rho dv \\ I_{yy} &= \int_m (x'^2 + z'^2) dm = \iiint_V (x^2 + z'^2)\, \rho d \\ I_{zz} &= \int_m (x'^2 + y'^2) dm = \iiint_V (x'^2 + y'^2)\, \rho d \end{aligned}\right\} \text{Mass moments of inertia}$$



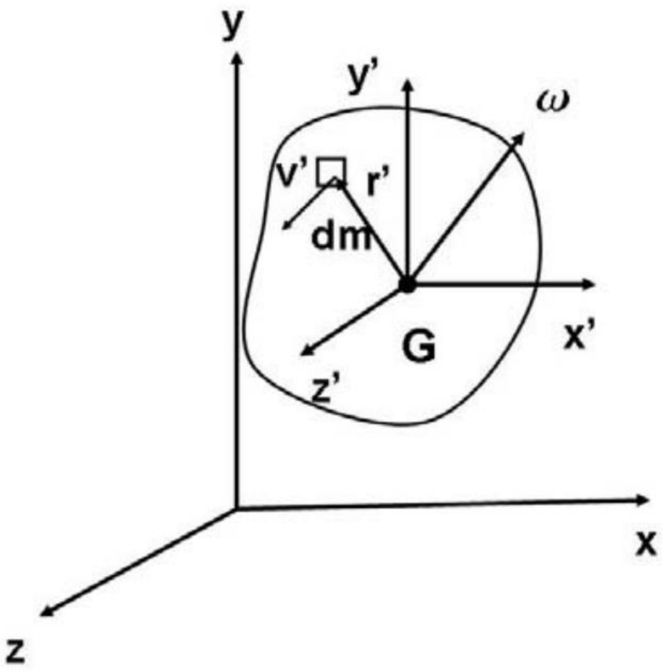
- We observe that the quantity in the integrand is precisely the square of the distance to the x, y and z axis, respectively.

- It is also clear, from their expressions, that the moments of inertia are always positive

Inertia Tensor – WRT a Coordinate Frame at the CM

- The quantities I_{xy}, I_{yx}, I_{xz}, I_{zx}, I_{yz}, and I_{zy} are called mass products of inertia and they can be positive, negative, or zero, and are given by,

$$\left.\begin{aligned} I_{xy} = I_{yx} &= \int_m (x'y')dm = \iiint_V x'y'\,\rho d \\ I_{xz} = I_{zx} &= \int_m (x'z')dm = \iiint_V x'z'\,\rho d \\ I_{yz} = I_{zy} &= \int_m (y'z')dm = \iiint_V y'z'\,\rho d \end{aligned}\right\} \text{Mass products of inertia}$$

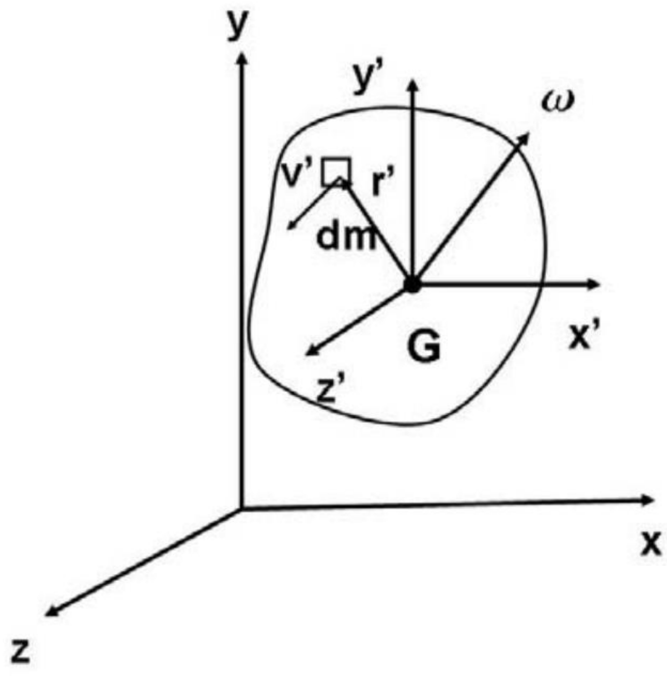


- They are a measure of the imbalance in the mass distribution.

Inertia Tensor – WRT a Coordinate Frame at the CM

- The angular momentum with respect to the center of mass G can be expressed in a matrix form as

$$\boldsymbol{H}_G = [I_G]\boldsymbol{\omega}$$

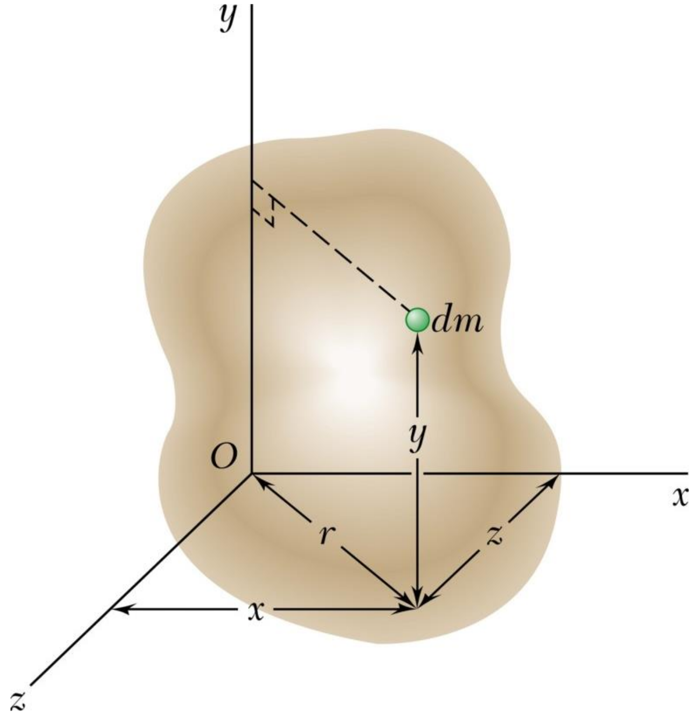


$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Inertia Tensor – WRT an Arbitrary Coordinate Frame

- For a rigid body that is free to move in a 3D space there are infinite possible rotation axes
- The intertie tensor characterizes the mass distribution of the rigid body with respect to a specific coordinate system
- The intertie Tensor relative to frame {A} is express as a matrix

$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$



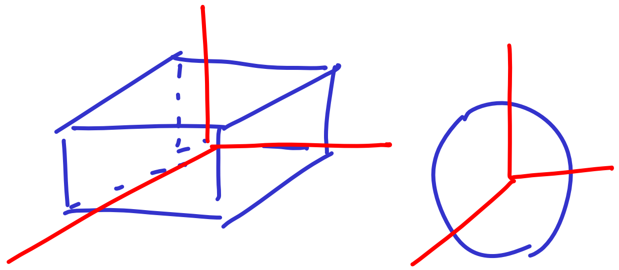
Inertia Tensor

$$ {}^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} $$

$$ \left.\begin{aligned} I_{xx} &= \iiint_V (y^2 + z^2)\,\rho dv \\ I_{yy} &= \iiint_V (x^2 + z^2)\,\rho d \\ I_{zz} &= \iiint_V (x^2 + y^2)\,\rho d \end{aligned}\right\} \text{Mass moments of inertia} $$

$$ \left.\begin{aligned} I_{xy} &= \iiint_V xy\,\rho d \\ I_{xz} &= \iiint_V xz\,\rho d \\ I_{yz} &= \iiint_V yz\,\rho d \end{aligned}\right\} \text{Mass products of inertia} $$

Tensor of Inertia – Example



$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

- This set of six independent quantities for a given body, depend on the **position and orientation** of the frame in which they are defined
- We are free to choose the orientation of the reference frame. It is possible to cause the product of inertia to be zero

$$\left. \begin{aligned} I_{xy} &= 0 \\ I_{xz} &= 0 \\ I_{yz} &= 0 \end{aligned} \right\} \text{Mass products of inertia}$$

$$^{A}I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

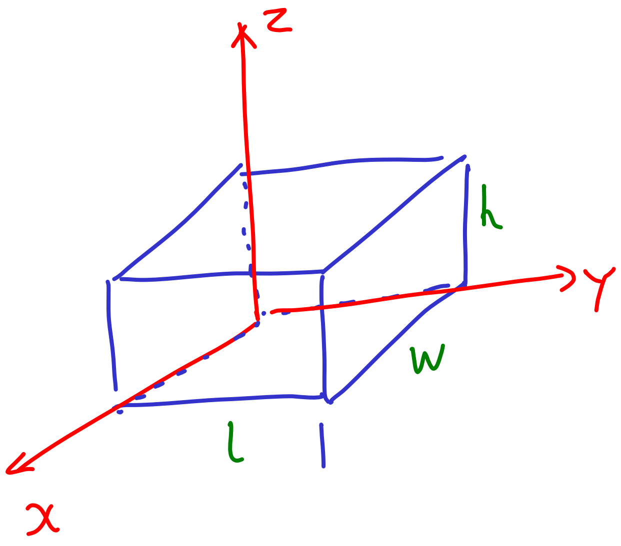
- The axes of the reference frame when so aligned are called the **principle axes** and the corresponding mass moments are called the principle **moments of intertie**

Tensor of Inertia – Example

$$x: 0 \rightarrow w$$

$$y: 0 \rightarrow l$$

$$z: 0 \rightarrow h$$



$$I_{xx} = \int_0^h \int_0^l \int_0^w (y^2 + z^2)\, \rho dxdydz = \int_0^h \int_0^l (y^2 + z^2)\, w\rho dydz$$

$$= \left(\frac{hl^3w}{3} + \frac{h^3lw}{3}\right)\rho = \rho hlw \frac{l^2}{3} phlw \frac{h^2}{3} = \int_0^h \left(\frac{l^3}{3} + z^2 l\right) w\rho dz = \frac{m}{3}(l^2 + h^2)$$

$$I_{yy} = \frac{m}{3}(w^2 + h^2)$$

$$I_{zz} = \frac{m}{3}(l^2 + w^2)$$

Tensor of Inertia – Example

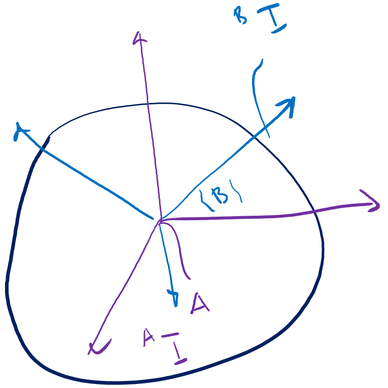
$$I_{xy} = \int_0^h \int_0^l \int_0^w xy\,\rho dx dy dz = \int_0^h \int_0^l \frac{w^2}{2} y\,\rho dy dz = \int_0^h \frac{w^2 l^2}{4} \rho dz = \frac{w^2 l^2 h}{4} \rho = (wlh\rho)\frac{wl}{4} = \frac{m}{4} wl$$

$$I_{xz} = \frac{m}{4} hw$$

$$I_{yz} = \frac{m}{4} hl$$

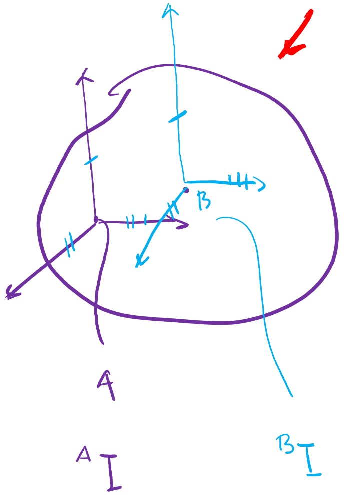
$${}^{A}I = \begin{bmatrix} \frac{m}{3}(l^2 + h^2) & -\frac{m}{4} wl & -\frac{m}{4} hw \\ -\frac{m}{4} wl & \frac{m}{3}(w^2 + h^2) & -\frac{m}{4} hl \\ -\frac{m}{4} hw & -\frac{m}{4} hl & \frac{m}{3}(l^2 + w^2) \end{bmatrix}$$

Tensor of Inertia – Operations



Translations of the Inertia Tensor

Parallel Axis Theorem



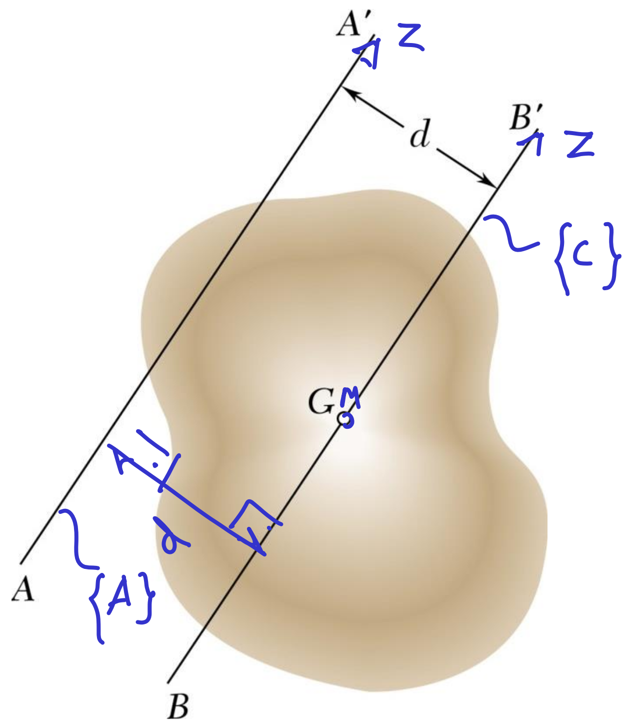
Parallel Axis Theorem – 1D

- The inertia tensor is a function of the position and orientation of the reference frame
- **Parallel Axis Theorem** – How the inertia tensor changes under translation of the reference coordinate system

Frame {C} – is located at the CM

Frame {A} – an arbitrarily translated frame

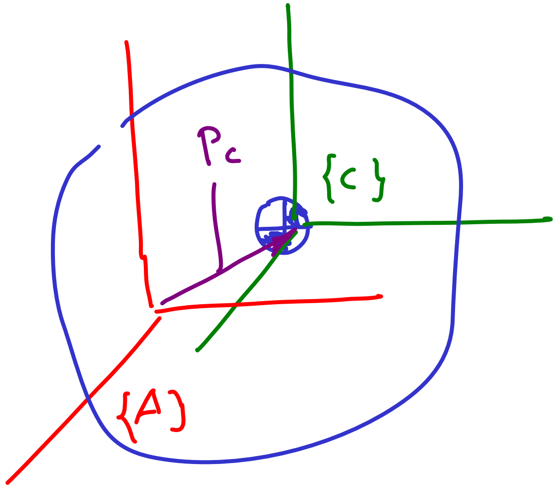
$$ {}^{A}I_{zz} = {}^{C}I_{zz} + md^2 $$



Parallel Axis Theorem – 3D

$$^{A}I_{xx} = {}^{C}I_{xx} + m(z_c^2 + y_c^2)$$
$$^{A}I_{yy} = {}^{C}I_{yy} + m(x_c^2 + z_c^2)$$
$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$$

$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$
$$^{A}I_{xz} = {}^{C}I_{xz} - m x_c z_c$$
$$^{A}I_{yz} = {}^{C}I_{yz} - m y_c z_c$$



$P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$ - Location of the CM (origin of C) relative to frame [A]

Parallel Axis Theorem – 3D

$$^{A}I = {}^{C}I + m[(P \cdot P) I_3 - P \otimes P]$$

$$(x_c^2 + y_c^2 + z_c^2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Outer Product

$$\begin{bmatrix} P_x P_x & P_x P_y & P_x P_z \\ P_y P_x & P_y P_y & P_y P_z \\ P_z P_x & P_z P_y & P_z P_z \end{bmatrix}$$

$$\begin{bmatrix} x_c^2 + y_c^2 + z_c^2 & 0 & 0 \\ 0 & x_c^2 + y_c^2 + z_c^2 & 0 \\ 0 & 0 & x_c^2 + y_c^2 + z_c^2 \end{bmatrix}$$

$$^{A}I = \begin{bmatrix} {}^{C}I_{xx} + m(y_c^2 + z_c^2) & {}^{C}I_{xy} - m x_c y_c & {}^{C}I_{xz} - m x_c z_c \\ {}^{C}I_{xy} - m x_c y_c & {}^{C}I_{yy} + m(x_c^2 + z_c^2) & {}^{C}I_{yz} - m y_c z_c \\ {}^{C}I_{xz} - m x_c z_c & {}^{C}I_{yz} - m y_c z_c & {}^{C}I_{zz} + m(x_c^2 + y_c^2) \end{bmatrix}$$

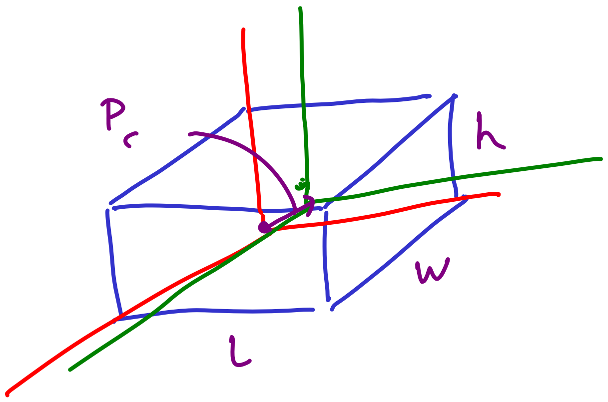
Tensor of Inertia – Example

$$P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} w \\ l \\ h \end{bmatrix}$$

$${}^{C}I_{zz} = {}^{A}I_{zz} - m(x_c^2 + y_c^2) = \frac{m}{3}(l^2 + w^2) - \frac{m}{4}(w^2 + l^2) = \frac{m}{12}(w^2 + l^2)$$

$${}^{C}I_{xy} = {}^{A}I_{xy} + m x_c y_c = -\frac{mwl}{4} + m\frac{1}{2}w\frac{1}{2}l = 0$$

$${}^{C}I = \frac{m}{12}\begin{bmatrix} h^2 + l^2 & 0 & 0 \\ 0 & w^2 + h^2 & 0 \\ 0 & 0 & l^2 + w^2 \end{bmatrix}$$



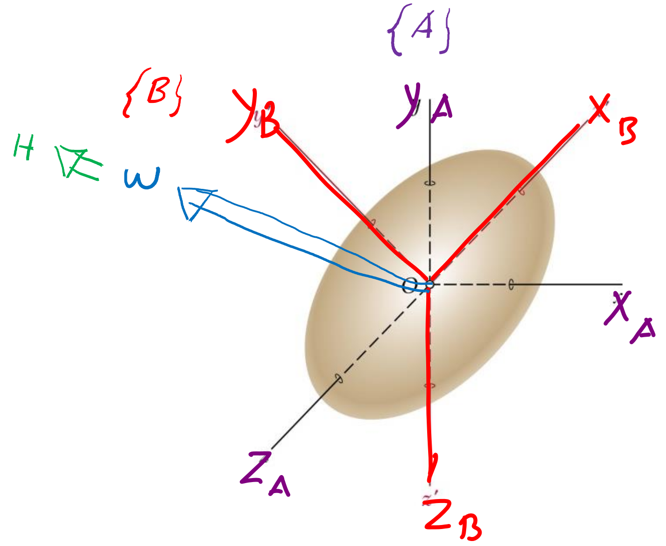
Tensor of Inertia – Operations

Rotation of the Inertia Tensor

Rotation of the Inertia Tensor

- Given:
  - The inertia tensor of the a body expressed in frame A
  - Frame B is rotated with respect to frame A
  - Note: Both frames are stationary in space

- Calculate
  - The inertia tensor of the body expressed in frame B



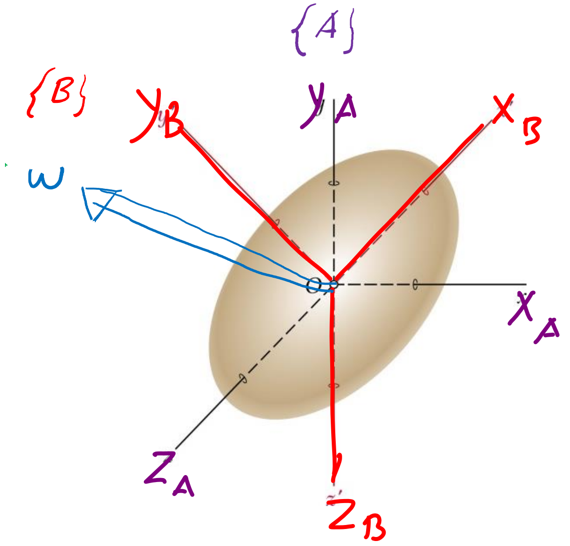
Rotation of the Inertia Tensor

$$^{A}H = {^{A}I}\,{^{A}\omega}$$

$$^{B}H = {^{B}I}\,{^{B}\omega}$$

$(*)\,{^{A}\omega}, {^{A}H}$ - angular velocity and momentum expressed in frame A

$(*)\,{^{B}\omega}, {^{B}H}$ - angular velocity and momentum expressed in frame B



$$^{A}H = {^{A}_{B}R}\,{^{B}H} \qquad {^{A}\omega} = {^{A}_{B}R}\,{^{B}\omega}$$

$$^{A}H = {^{A}_{B}R}\,{^{B}I}\,{^{B}\omega}$$

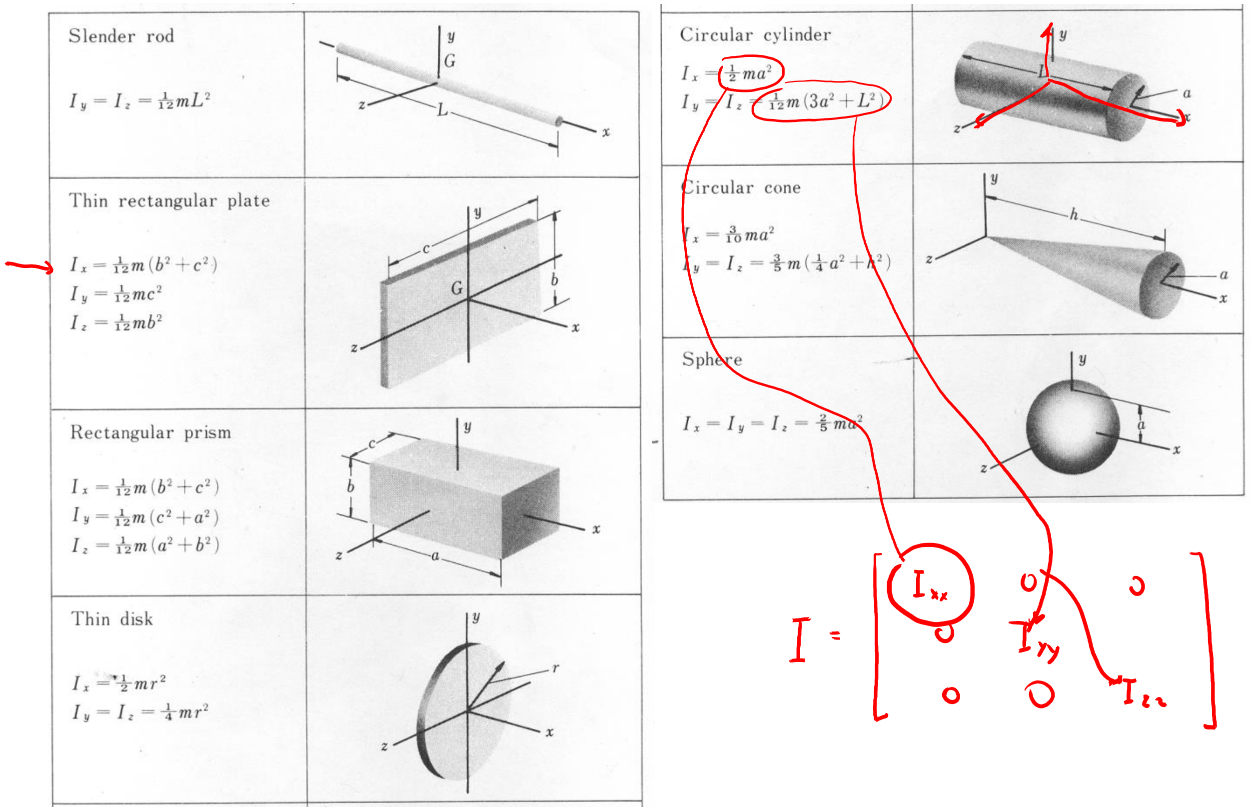
$$^{A}H = {^{A}_{B}R}\,{^{B}I}\,({^{A}_{B}R^{-1}}\,{^{A}_{B}R})\,{^{B}\omega}$$

$$^{A}H = {^{A}_{B}R}\,{^{B}I}\,{^{A}_{B}R^{-1}}\,{^{A}\omega}$$

$$I_A = {^{A}_{B}R}\,{^{B}I}\,{^{A}_{B}R^{-1}} = {^{A}_{B}R}\,{^{B}I}\,{^{A}_{B}R^{T}}$$

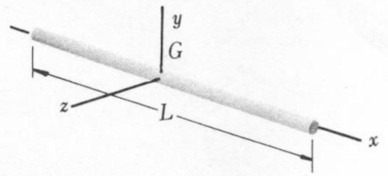
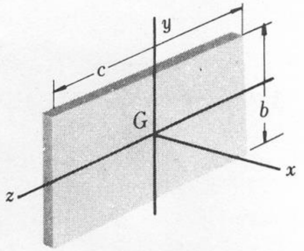
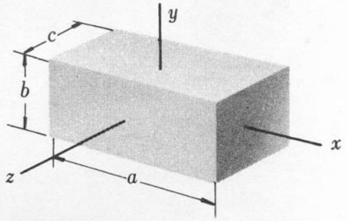
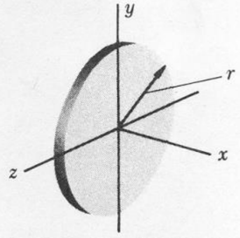
Inertia Tensor 2/

- The elements for relatively simple shapes can be solved from the equations describing the shape of the links and their density. However, most robot arms are far from simple shapes and as a result, these terms are simply measured in practice.

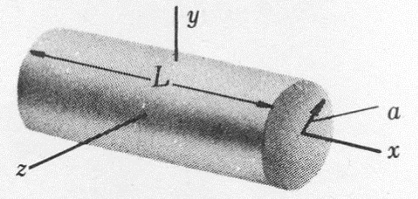
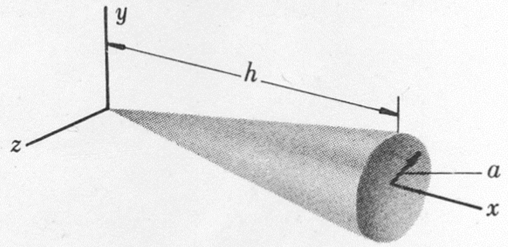
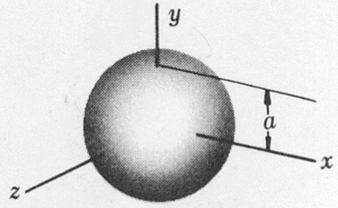


Slender rod	$I_y = I_z = \frac{1}{12} mL^2$
Thin rectangular plate	$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} mc^2$ $I_z = \frac{1}{12} mb^2$
Rectangular prism	$I_x = \frac{1}{12} m(b^2 + c^2)$ $I_y = \frac{1}{12} m(c^2 + a^2)$ $I_z = \frac{1}{12} m(a^2 + b^2)$
Thin disk	$I_x = \frac{1}{2} mr^2$ $I_y = I_z = \frac{1}{4} mr^2$
Circular cylinder	$I_x = \frac{1}{2} ma^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$
Circular cone	$I_x = \frac{3}{10} ma^2$ $I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$
Sphere	$I_x = I_y = I_z = \frac{2}{5} ma^2$

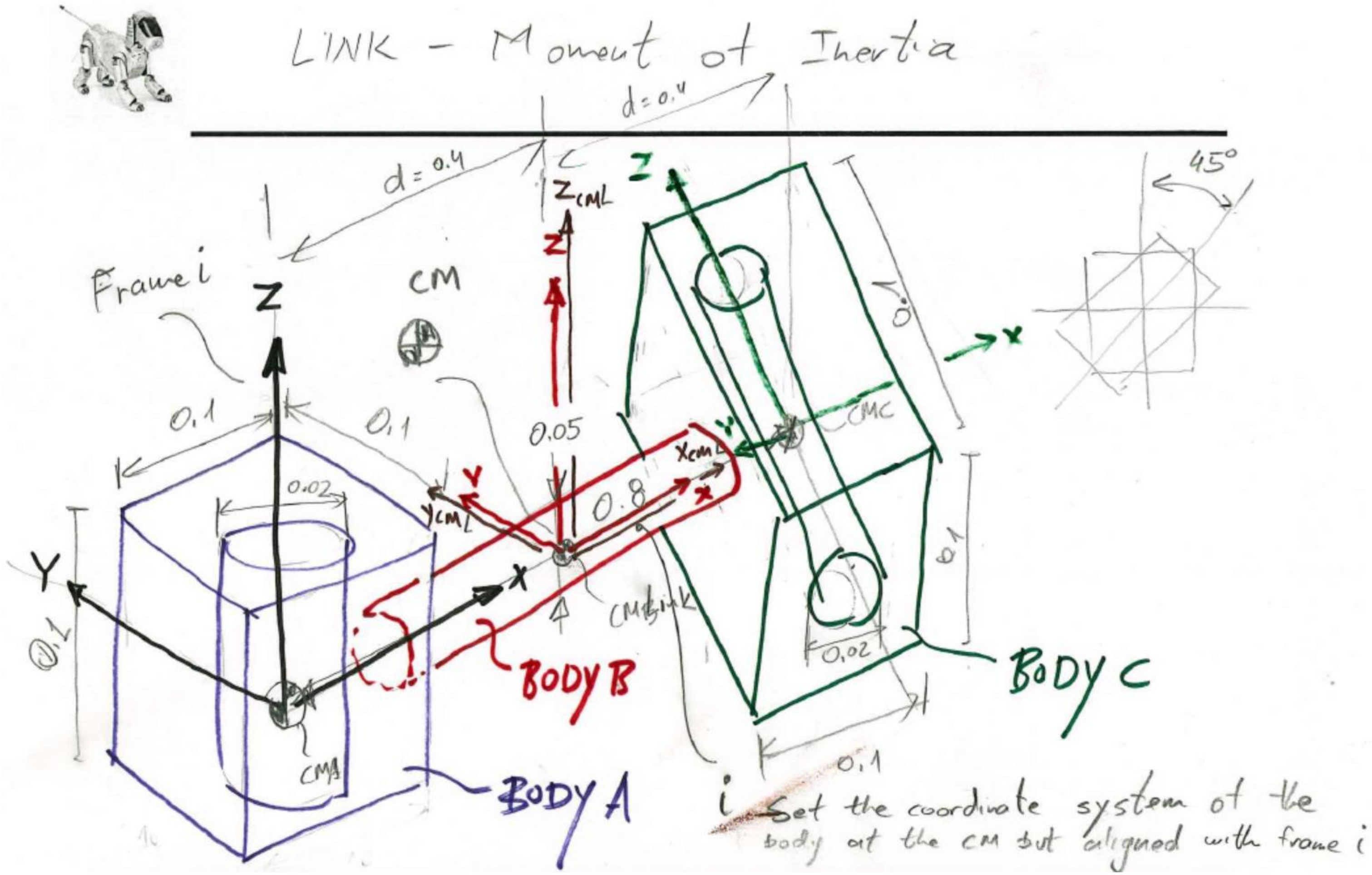
Inertia Tensor 2/

Slender rod $I_y = I_z = \frac{1}{12} m L^2$	
Thin rectangular plate $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m c^2$ $I_z = \frac{1}{12} m b^2$	
Rectangular prism $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m (c^2 + a^2)$ $I_z = \frac{1}{12} m (a^2 + b^2)$	
Thin disk $I_x = \frac{1}{2} m r^2$ $I_y = I_z = \frac{1}{4} m r^2$	

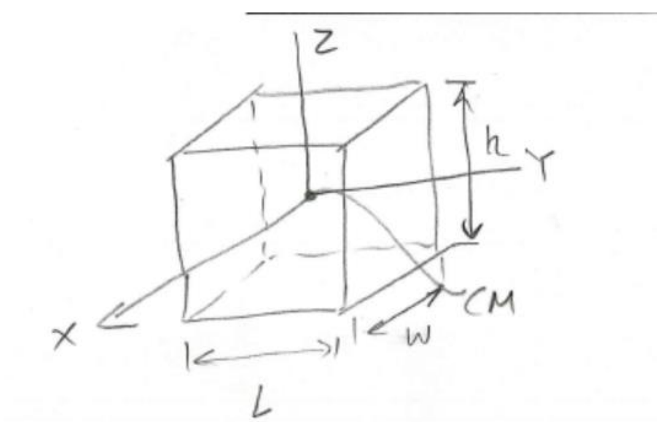
Inertia Tensor 2/

Circular cylinder $I_x = \frac{1}{2} ma^2$ $I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$	
Circular cone $I_x = \frac{3}{10} ma^2$ $I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$	
Sphere $I_x = I_y = I_z = \frac{2}{5} ma^2$	

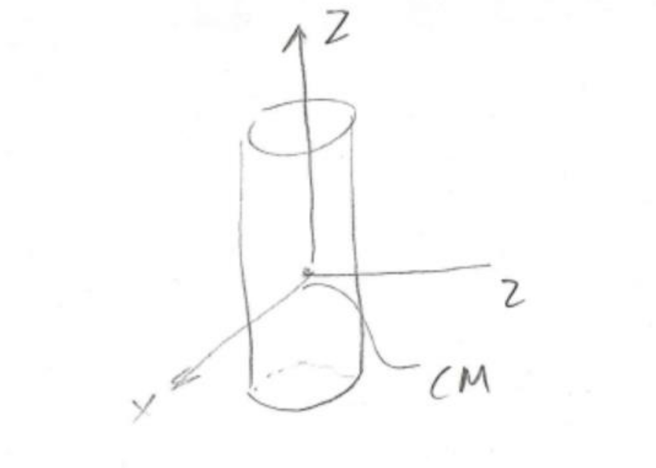
Inertia Tensor – Robotic Links



Inertia Tensor – Robotic Links



$$^{CM}I = \frac{m}{12}\begin{bmatrix} h^2+l^2 & 0 & 0 \\ 0 & w^2+h^2 & 0 \\ 0 & 0 & l^2+h^2 \end{bmatrix}$$



$$^{CM}I = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{mr^2}{2} \end{bmatrix}$$

Inertia Tensor – Robotic Links

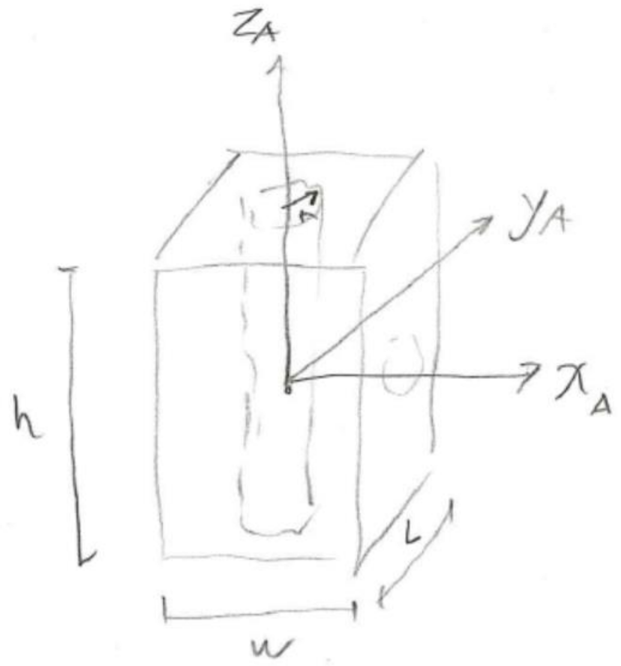
- Body A

STEP 1

$$^{CMA}I = I_{box} - I_{cyl}$$

$$I_{box} = \frac{m}{12}\begin{bmatrix} h^2 + l^2 & 0 & 0 \\ 0 & w^2 + h^2 & 0 \\ 0 & 0 & l^2 + h^2 \end{bmatrix}$$

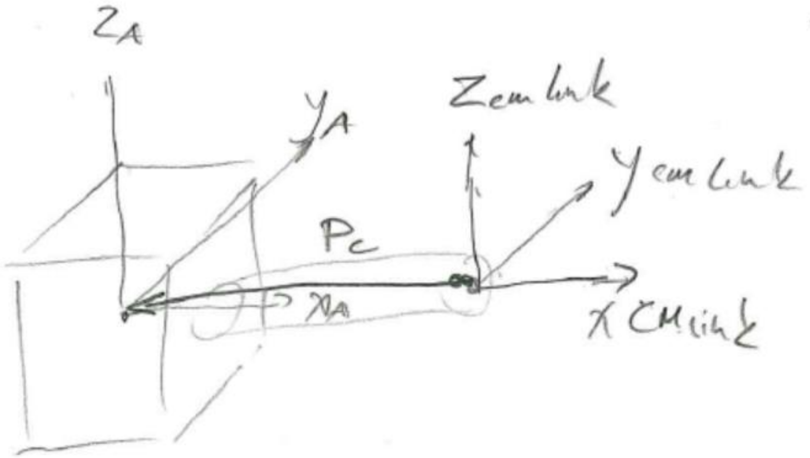
$$I_{cyl} = \begin{bmatrix} \frac{1}{12}m(3r^2 + h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2 + h^2) & 0 \\ 0 & 0 & \frac{mr^2}{2} \end{bmatrix}$$



Inertia Tensor – Robotic Links

- Body A

STEP 2 – Translate from frame A to the frame at the CoM of the link



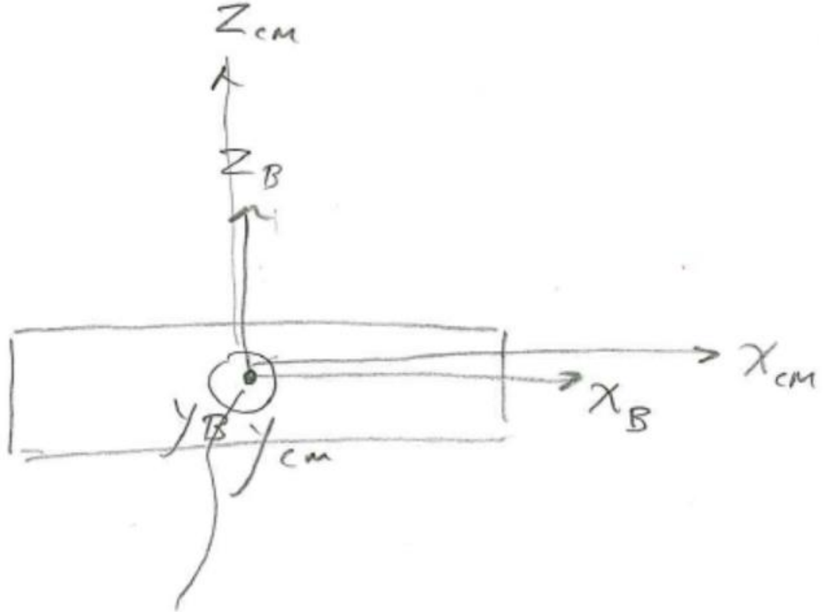
$$^{CM,link}I = {}^{CM,A}I + m[P_c^T P_c I_3 - P \otimes P]$$

$$= {}^{CM,A}I + m\left[\begin{bmatrix}-d & 0 & 0\end{bmatrix}\begin{bmatrix}-d \\ 0 \\ 0\end{bmatrix} I_3 - \begin{bmatrix}d^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\right]$$

Inertia Tensor – Robotic Links

- Body B

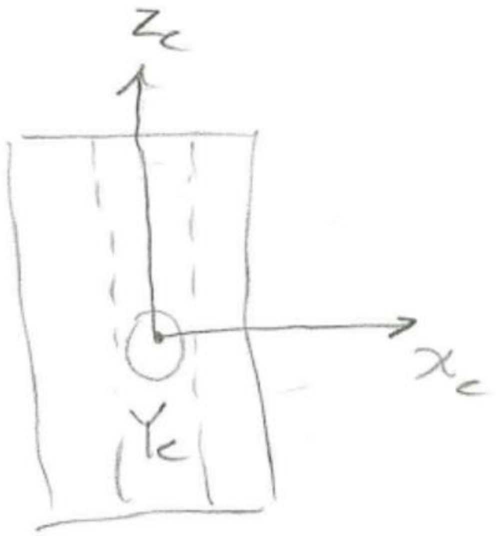
$$
{}^{B}I = \begin{bmatrix} \frac{mr^2}{2} & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2 + h^2) & 0 \\ 0 & 0 & \frac{1}{12}m(3r^2 + h^2) \end{bmatrix}
$$



The frame of body B is aligned with the frame of the CM of the entire body

Inertia Tensor – Robotic Links

- Body C



STEP 1

$$^{C}I = I_{box} - I_{cyl}$$

See body A

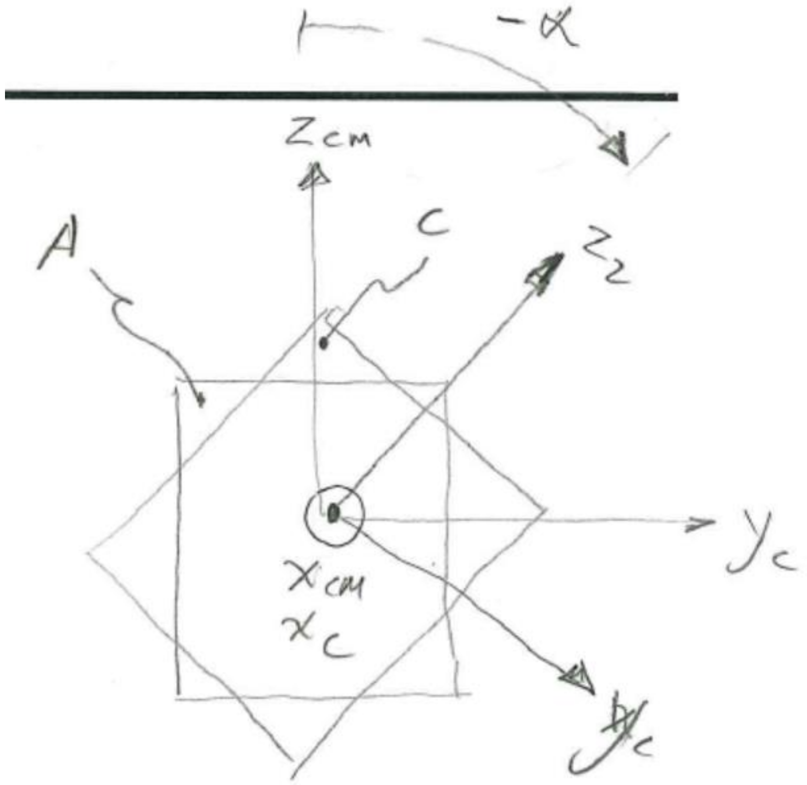
Inertia Tensor – Robotic Links

- Body C

<u>STEP 2</u> – Rotate about x_c by $-\alpha$

$$^{CM,link}_{CMC}R = Rot(\hat{x}_c, -\alpha)$$

$$^{CM,link}I = {}^{CM,link}_{CMC}R \, {}^{CMC}I \, {}^{CM,link}_{CMC}R^T$$



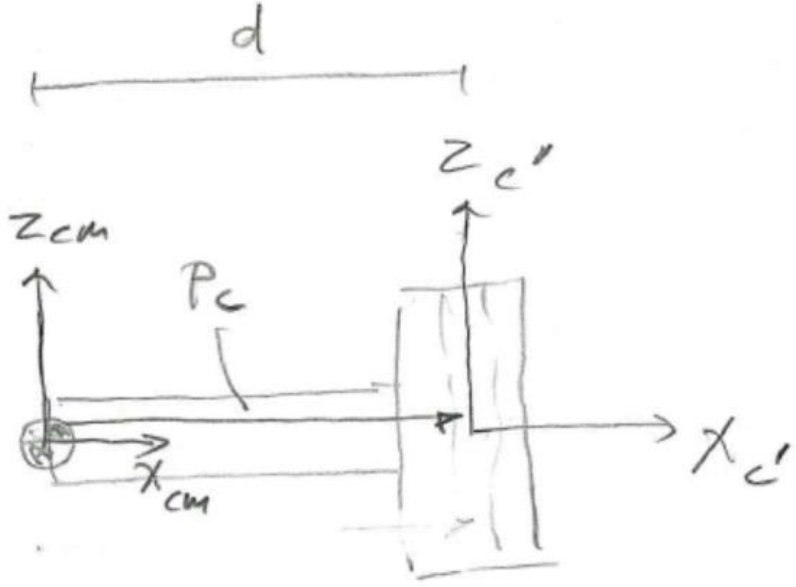
Inertia Tensor – Robotic Links

- Body C

STEP 3 – translate to the CM of the link

$$P_c = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$$

See body A



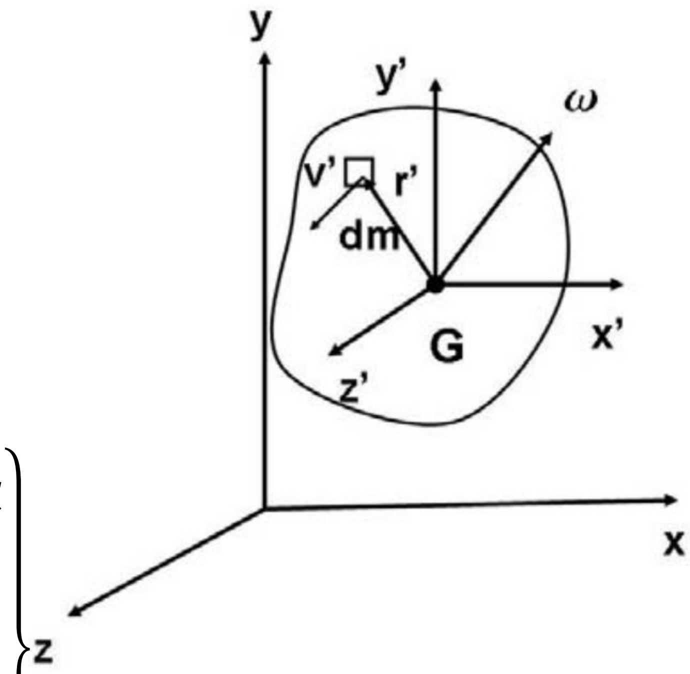
Summary

Inertia Tensor – WRT a Coordinate Frame at the CM

- The angular momentum with respect to the center of mass G can be expressed in a matrix form as

$$\boldsymbol{H}_G = [I_G]\boldsymbol{\omega}$$

$$\begin{pmatrix} H_{Gx} \\ H_{Gy} \\ H_{Gz} \end{pmatrix} = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

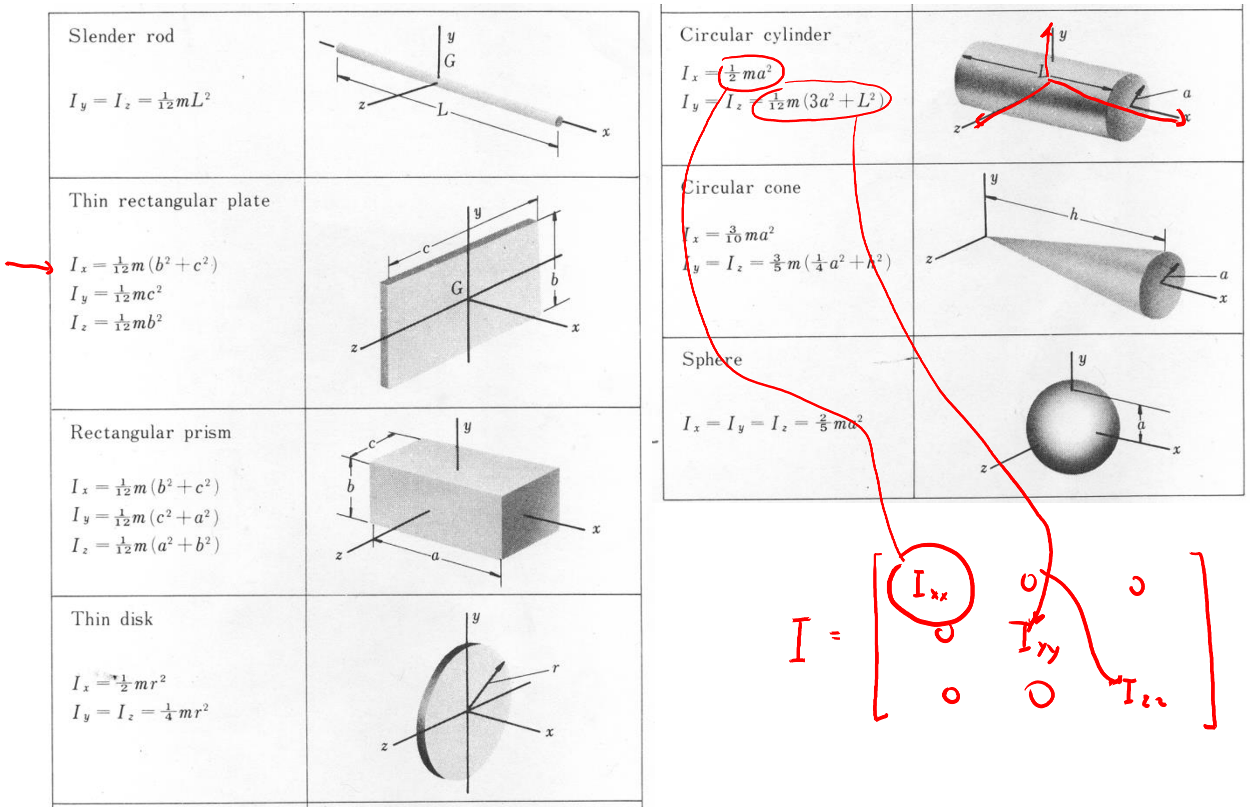


$$\left.\begin{aligned} I_{xx} &= \int_m (y'^2 + z'^2) dm = \iiint_V (y'^2 + z'^2)\, \rho dv \\ I_{yy} &= \int_m (x'^2 + z'^2) dm = \iiint_V (x^2 + z'^2)\, \rho d \\ I_{zz} &= \int_m (x'^2 + y'^2) dm = \iiint_V (x'^2 + y'^2)\, \rho d \end{aligned}\right\}$$

$$\left.\begin{aligned} I_{xy} &= I_{yx} = \int_m (x'y') dm = \iiint_V x'y'\, \rho d \\ I_{xz} &= I_{zx} = \int_m (x'z') dm = \iiint_V x'z'\, \rho d \\ I_{yz} &= I_{zy} = \int_m (y'z') dm = \iiint_V y'z'\, \rho d \end{aligned}\right\}$$

Inertia Tensor

- The elements for relatively simple shapes can be solved from the equations describing the shape of the links and their density. However, most robot arms are far from simple shapes and as a result, these terms are simply measured in practice.

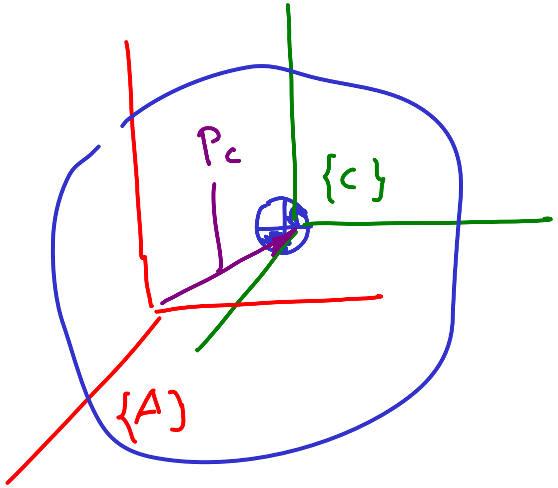


Parallel Axis Theorem – 3D

$$^{A}I_{xx} = {}^{C}I_{xx} + m(z_c^2 + y_c^2)$$

$$^{A}I_{yy} = {}^{C}I_{yy} + m(x_c^2 + z_c^2)$$

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_c^2 + y_c^2)$$



$$^{A}I_{xy} = {}^{C}I_{xy} - m x_c y_c$$

$$^{A}I_{xz} = {}^{C}I_{xz} - m x_c z_c$$

$$^{A}I_{yz} = {}^{C}I_{yz} - m y_c z_c$$

$P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$ - Location of the CM (origin of C) relative to frame [A]

Rotation of the Inertia Tensor

$$I_A = {}^A_B R \, {}^B I \, {}^A_B R^T$$