Manipulator Dynamics 1

Introduction

Forward / Inverse Dynamics

Forward Dynamics

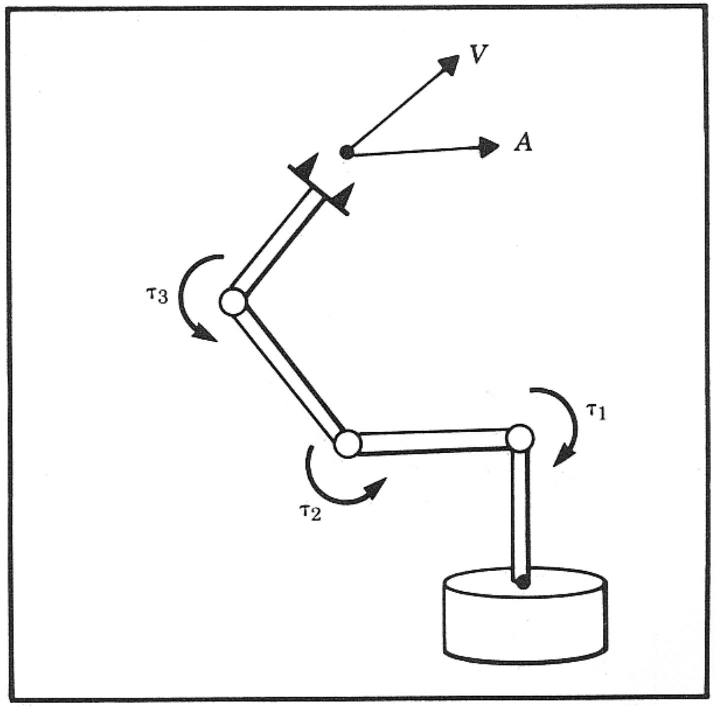
Problem

Given: *Joint torques and links geometry, mass, inertia, friction, joint torques*

Compute: *Angular acceleration of the links (solve differential equations)*

$$\begin{cases} \boldsymbol{\tau}_i \\ Link_i(x,y,z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$

$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \end{cases}$$



Solution

solve a set of differential equations

Dynamic Equations –

- Newton-Euler method
- Lagrangian Dynamics

$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

Inverse Dynamics

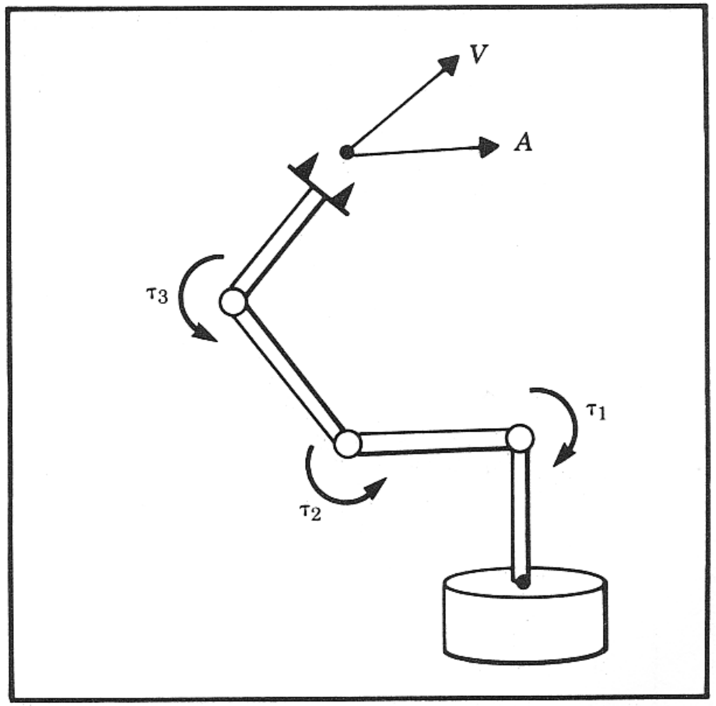
Problem

Given: Angular acceleration, velocity and angels of the links in addition to the links geometry, mass, inertia, friction

$$\begin{cases} \Theta \\ \dot{\Theta} \\ \ddot{\Theta} \\ Link_i(x, y, z) \\ m_i \\ I_i \\ P_{Ci} \\ f_i \\ n_i \end{cases}$$

Compute: Joint torques

$$\{\tau$$



Solution

Solve a set of algebraic equations

Dynamic Equations –

- Newton-Euler method
- Lagrangian Dynamics

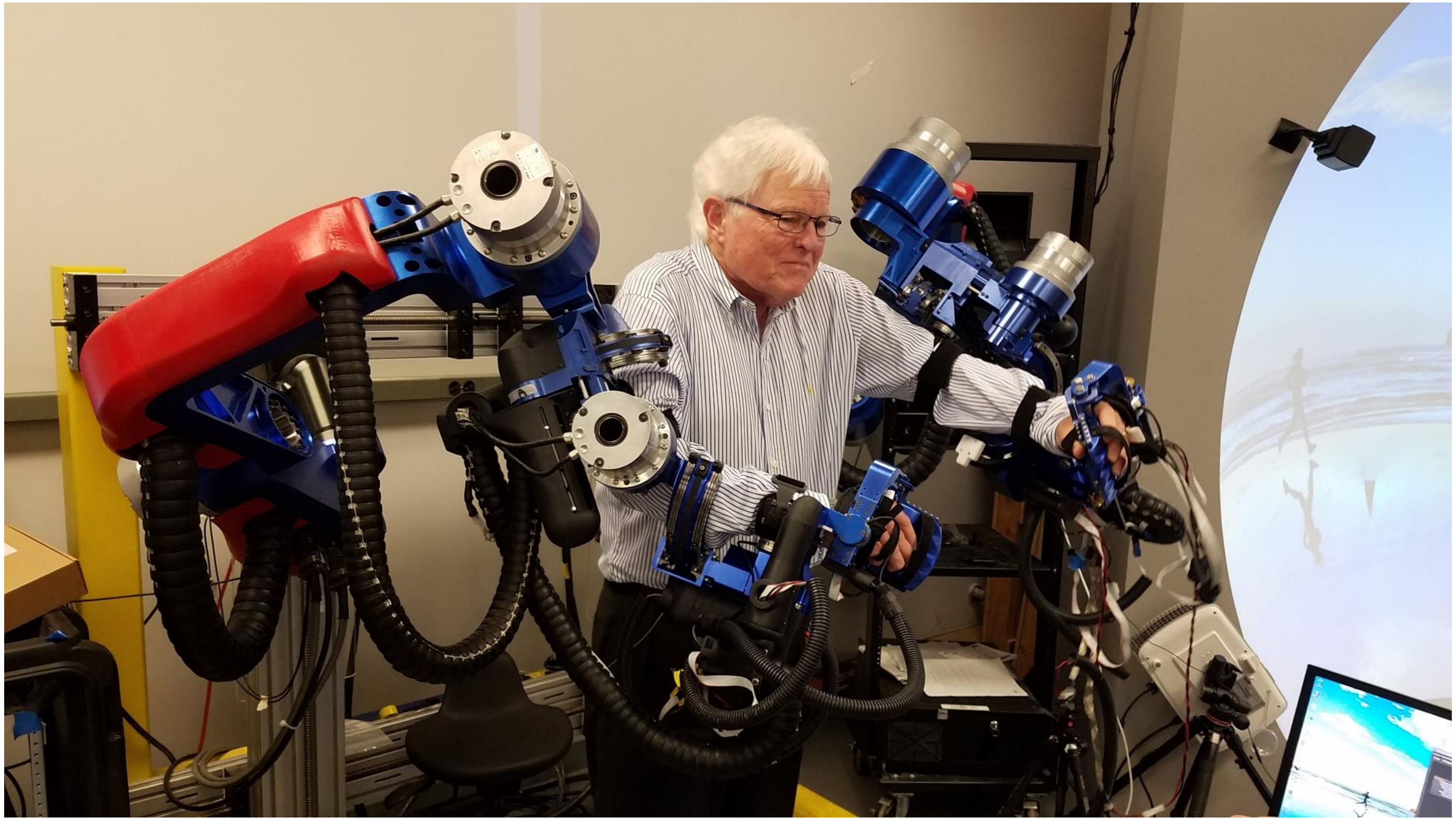
$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) + F(\Theta, \dot{\Theta})$$

Applications

Human Arm Dynamics – Exoskeleton Design

Control

Dynamics Modeling Application – Motivation
Exoskeleton Design

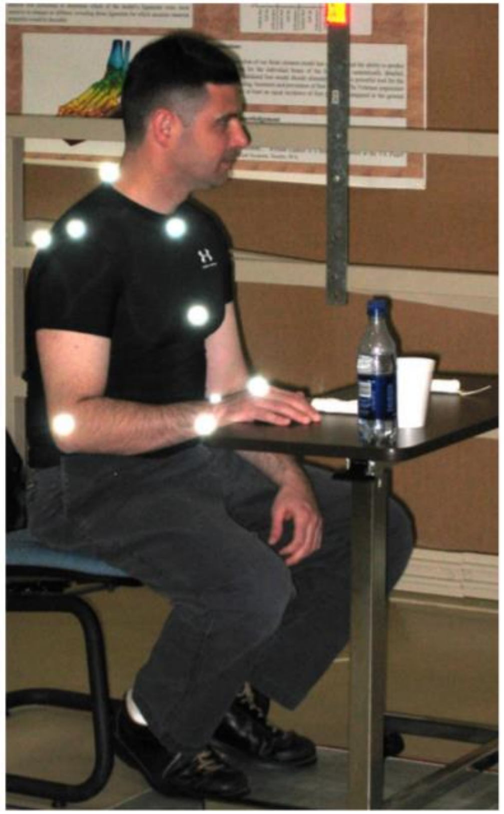


Dynamics Modeling Application – Motivation
Exoskeleton Design

- **Activities of Daily Living**
  1. Arm in lap
  2. Arm reach to head level
  3. Arm reach to right, head level
  4. Arm reach to left, head level
  5. Arm reach right, move object to left side
  6. Open door
  7. Open Drawer/Close Drawer
  8. Move object at waist level
  9. Pick up phone on table/hang up
  10. Pick up phone on wall/hang up
  11. Eat with fork
  12. Eat with spoon
  13. Eat with hands
  14. Drink with cup
  15. Eat with spoon
  16. Pour from bottle
  17. Brush teeth
  18. Comb hair
  19. Wash face
  20. Wash neck
  21. Shave
  22. Eat with fork (power – disabled grasp)
  23. Eat with spoon (power – disabled grasp)
  24. Full workspace motion

[CP19] Rosen Jacob, Joel C. Perry, Nathan Manning, Stephen Burns, Blake Hannaford,
The Human Arm Kinematics and Dynamics During Daily Activities – Toward a 7 DOF Upper Limb Powered Exoskeleton, -
ICAR 2005 – Seattle WA, July 2005

Dynamics Modeling Application – Motivation
Exoskeleton Design



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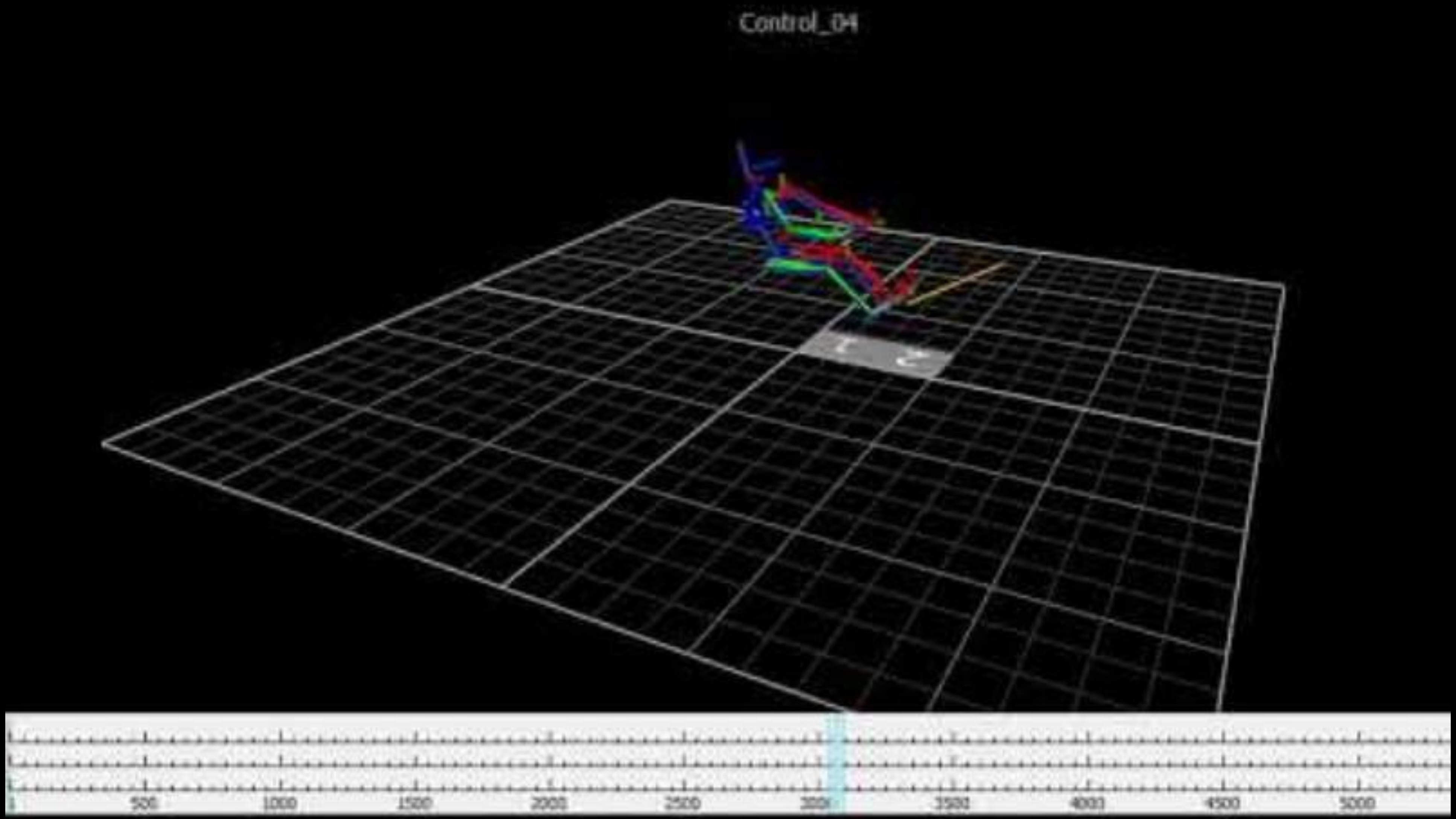
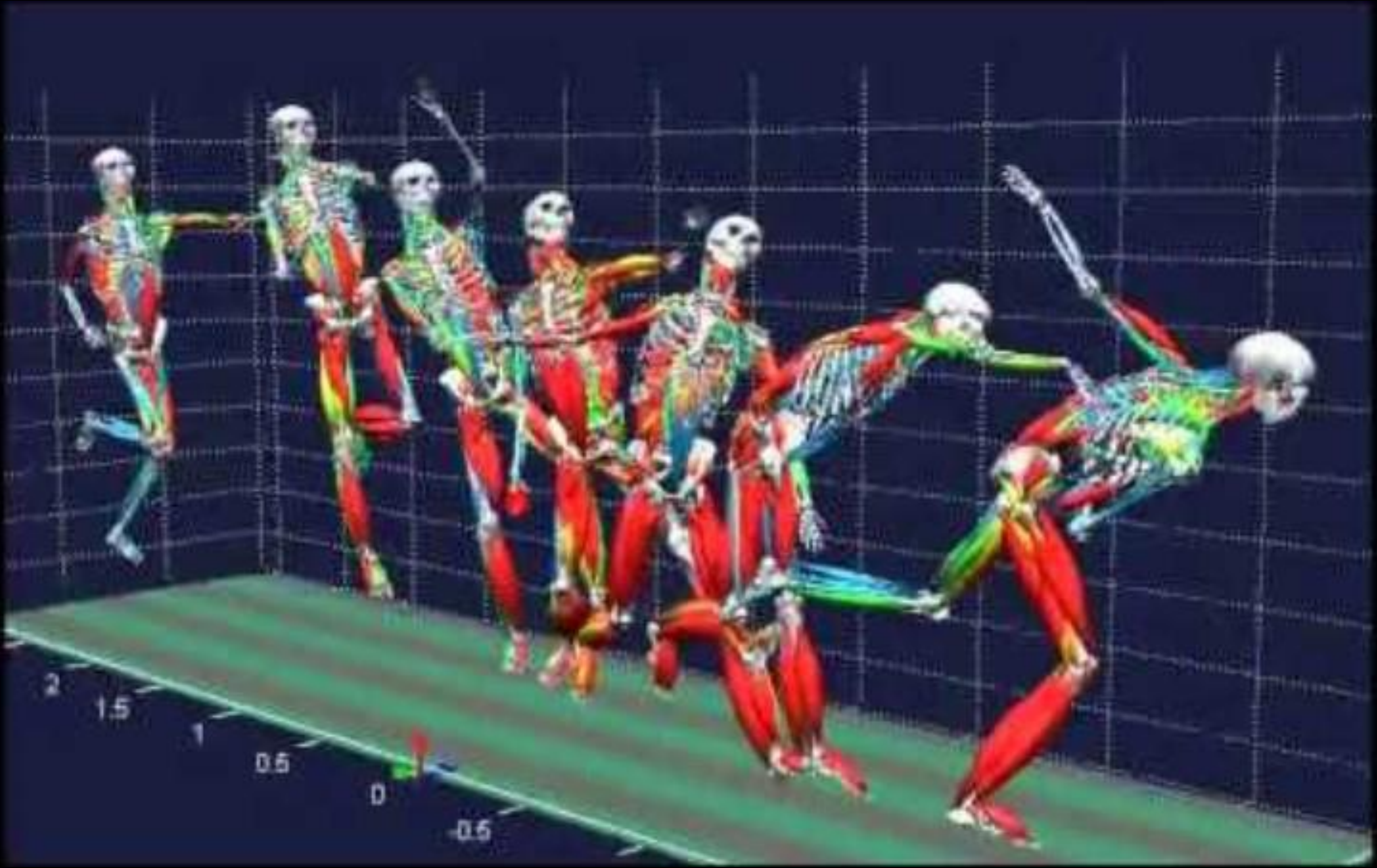
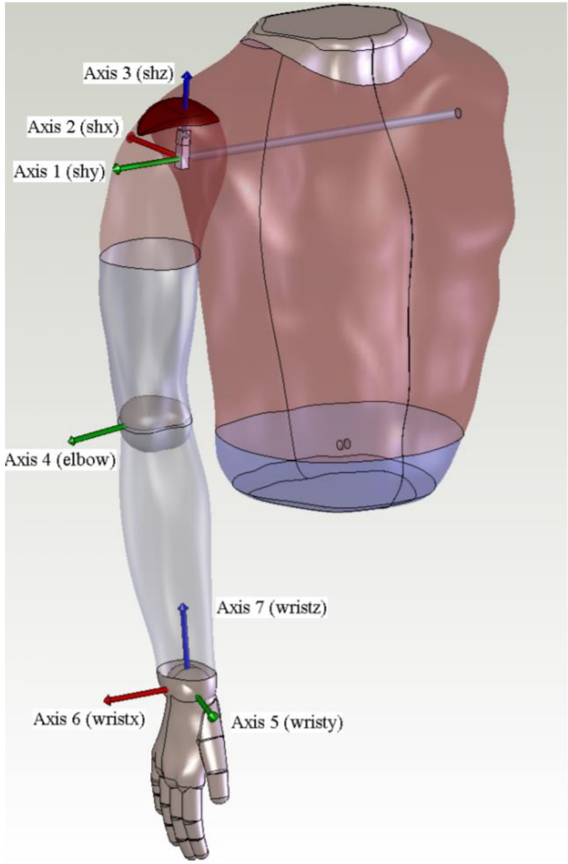

Control_04
1
500
1000
1500
2000
2500
3500
4000
4500
5000


2
1.5
1
0.5
0
-0.5

Dynamics Modeling Application – Motivation
Exoskeleton Design



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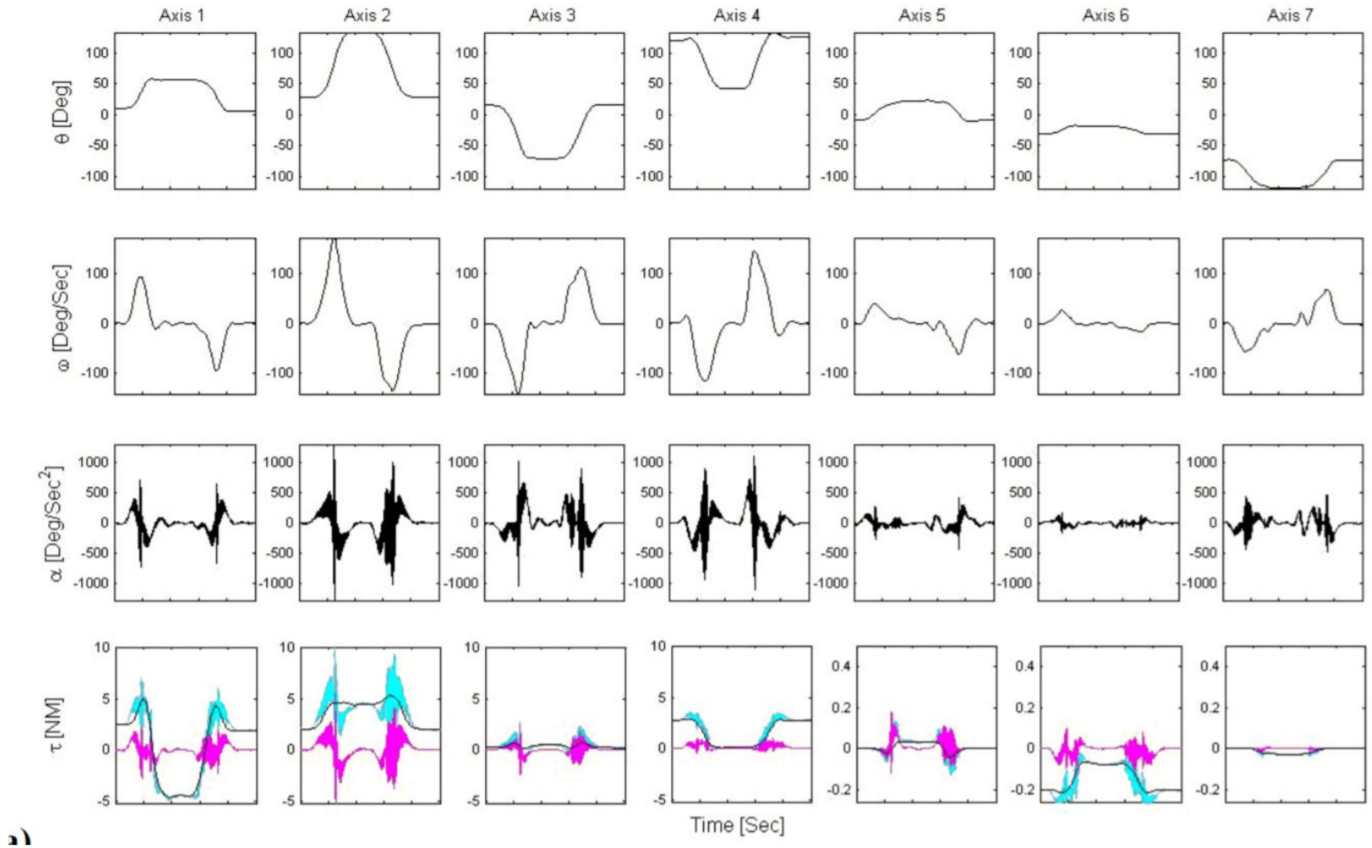
Dynamics Modeling Application – Motivation Exoskeleton Design



Time histories and statistics of the kinematics and dynamics of the human arm during an arm reach to head level (action 2): (a) Time histories of the joint kinematics and dynamics (b) Statistical distribution of the joint kinematics and dynamics. The three torque curves in (a) illustrate the total joint axis torque (cyan), in comparison to the gravitational torque (black) and the combined torque due to inertial, centrifugal, and coriolis terms (magenta). The line box plots of (b) indicate the lower quartile, median, and upper quartile values. The dashed lines extend beyond the upper and lower quartiles by one and a half times the interquartile range. Data that lies outside of this range is displayed with the symbol 'x'.

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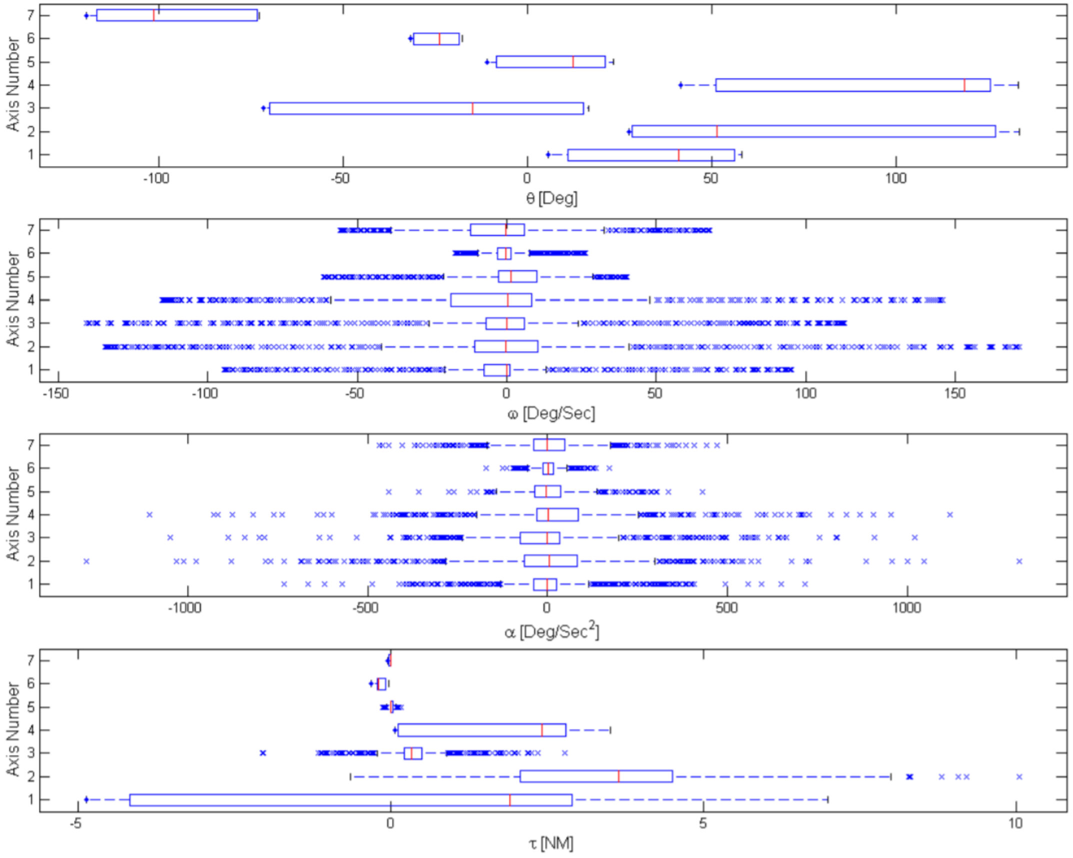
Dynamics Modeling Application – Motivation Exoskeleton Design



Time histories and statistics of the kinematics and dynamics of the human arm during an arm reach to head level (action 2): (a) Time histories of the joint kinematics and dynamics (b) Statistical distribution of the joint kinematics and dynamics. The three torque curves in (a) illustrate the total joint axis torque (cyan), in comparison to the gravitational torque (black) and the combined torque due to inertial, centrifugal, and coriolis terms (magenta). The line box plots of (b) indicate the lower quartile, median, and upper quartile values. The dashed lines extend beyond the upper and lower quartiles by one and a half times the interquartile range. Data that lies outside of this range is displayed with the symbol 'x'.

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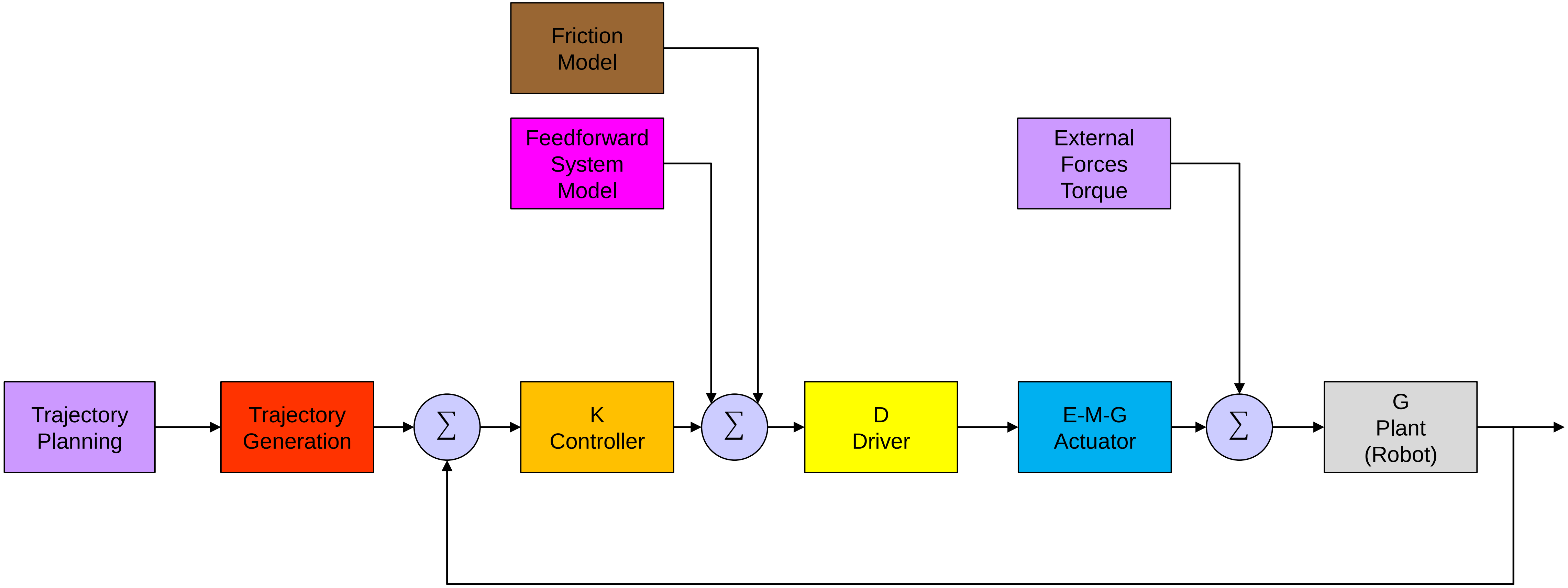
Dynamics Model Application – Motivation
Position Control



Manipulator Dynamics – Newton Euler Equations

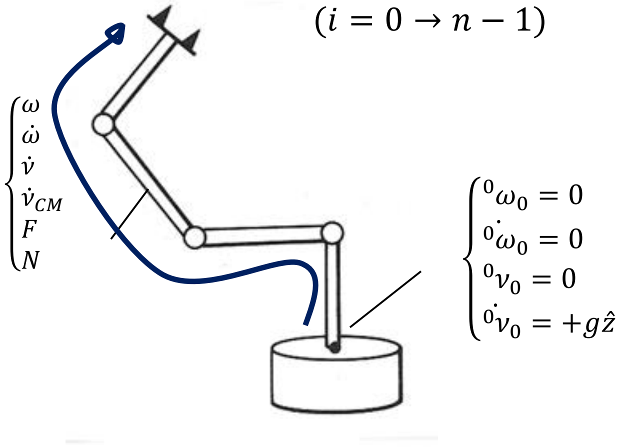
Iterative Newton Euler Equations
Steps of the Algorithm

- (1) Outward Iterations $(i = 0 \rightarrow n-1)$
  - Starting With velocities and accelerations of the base

    $${}^{0}\omega_0 = 0,\ {}^{0}\dot{\omega}_0 = 0,\ {}^{0}v_0 = 0,\ {}^{0}\dot{v}_0 = +g\hat{z}$$

  - Calculate velocities accelerations, along with forces and torques (at the CM)

    $$\omega, \dot{\omega}, \dot{v}, \dot{v}_{CM}, F, N$$

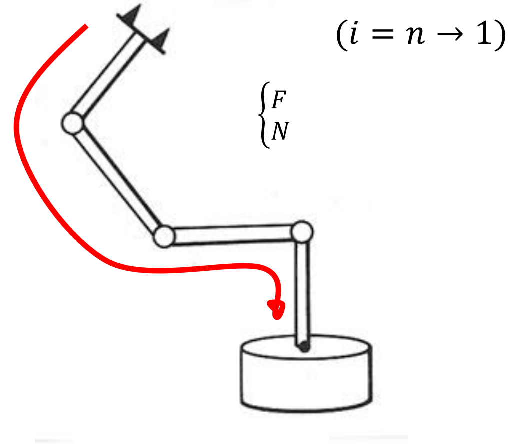


- (2) Inward Iteration $(i = n \rightarrow 1)$
  - Starting with forces and torques (at the CM)

    $$F, N$$

  - Calculate forces and torques at the joints

    $$f, n$$



Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i : 0 \rightarrow 5$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i) $$

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1} $$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C\,i+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\,i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i\,:\; 6 \rightarrow 1$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector all the way the robot's base.

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i+1}n^{T}{}_{i+1}\;{}^{i}\hat{Z}_i$$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

  $\dot{v}_c$ - The linear acceleration of the center of mass
  $\dot{\omega}$ - The angular acceleration
  ${}^{c}I$ - The Inertia tensor (moment of inertia)
  F - The sum of all the forces applied on the center of mass
  N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

Newton Euler Equations
Derivation Based on Momentum

Dynamics - Newton-Euler Equations

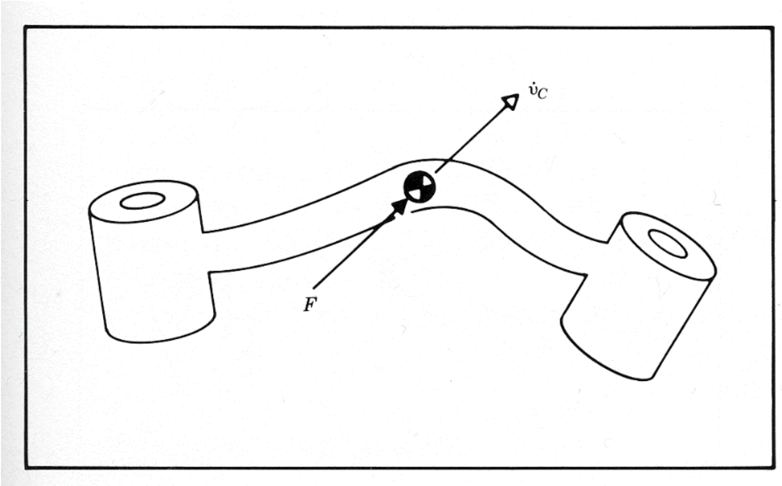
Newton Equation

- For a rigid body (like a robot link) whose center of mass is experiencing an acceleration, the force acting at the center of mass that caused the acceleration is given by Newton's equation

$$F = \frac{d(mv_c)}{dt}$$

- For our robot manipulators, whose link masses are constant, this equation simplifies to

$$F = m\dot{v}_C = ma_C$$



Dynamics - Newton-Euler Equations

Eular Equation

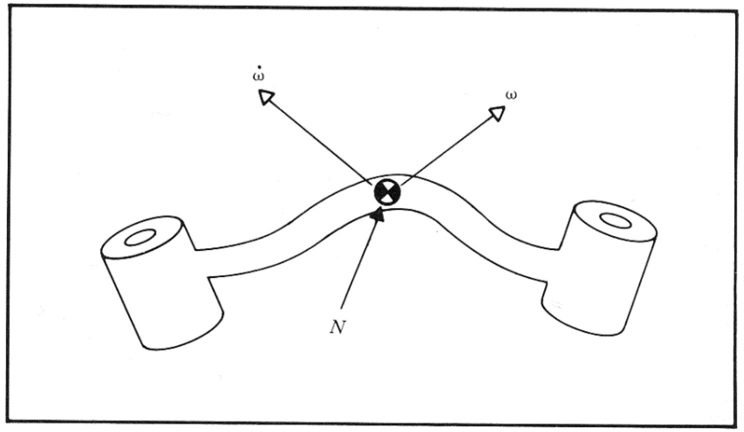
- For a rotating rigid body, the moment that causes an angular acceleration is given by Euler's equation

$$N = \frac{d(^{c}I\omega)}{dt}$$

- For our robot manipulators, whose **link moment of inertia is constant**, this equation simplifies to

$$N = {^{c}I}\dot{\omega} + \omega \times {^{c}I}\omega$$

- The second term on the right will be non-zero when the link's angular velocity vector is not coincident with the link's principle axis of inertia.



Linear Momentum – Particle

- The liner momentum of a particle is defined as the mass of the particle times its linear velocity or the time derivative of its position

$$L = mv = m\frac{dP}{dt}$$

- The time derivative of the linear momentum is equal to the sum of all the external forces applied on the particle

$$\frac{dL}{dt} = \frac{d}{dt}(mv) = m\frac{d^2P}{dt^2} = \sum f_{i\ \ exterenal}$$

- It is than van be summarized as Newton’s second law

$$\sum f_{i\ \ exterenal} = ma$$

Linear Momentum – Rigid Body

- In a similar fashion a rigid body can be view as an assembly of particles

$$\sum f_{i \quad exterenal} = dm_i \frac{d^2 P}{dt^2}$$

- Integrate over the entire volume of the rigid body is resulted in

$$F = M \frac{d^2}{dt^2} \underbrace{\int_V \frac{P dm}{M}}_{P_{CM}}$$

$$F = M \frac{d^2}{dt^2} P_{CM} = m a_{CM}$$

Angular Momentum – Particle

- The rigid body is a composition of infinitesimal particles i with a mass m_i in a location defined by the vector P_{iG} with respect to a coordinate system with the origin at the CM . Each particle i has a linear velocity v_{iG} as a result of the rotation of the entire rigid body with an angular velocity defined by the vector ω . The angular momentum of the particle H_{iG} is defined by

$$\vec{H}_{i_G} = \vec{P}_{i_G} \times m_i \vec{v}_{i_G}$$

- The particle's i linear velocity v_{iG} is defined by

$$\vec{v}_{i_G} = \vec{\omega} \times \vec{P}_{i_G}$$

- Substituting the expression of linear velocity v_{iG} in the equation defined the angular momentum of the particle H_{iG} is resulted by

$$\vec{H}_{i_G} = \vec{P}_{i_G} \times m_i(\vec{\omega} \times \vec{P}_{i_G}) = m_i \vec{P}_{i_G} \times (\vec{\omega} \times \vec{P}_{i_G})$$

Angular Momentum – Rigid Body

$$H_G = \sum H_{i_G} = \sum_i m_i \vec{P}_{i_G} \times (\vec{\omega} \times \vec{P}_{i_G})$$

$$m_i = \rho dv$$

$$H_G = \int_v P \times \vec{\omega} \times P \, \rho dv = w \int_v P(-P) \, \rho dv = \vec{\omega} I$$

Angular Momentum – Rigid Body

$$\sum \tau = \frac{d}{dt} H_G = \frac{d}{dt} I\omega$$

$$\sum \tau = I\dot{\omega} + \omega \times I\omega$$

Transport Theorem –
Differentiation of a Vector in a Rotating Frame of Reference

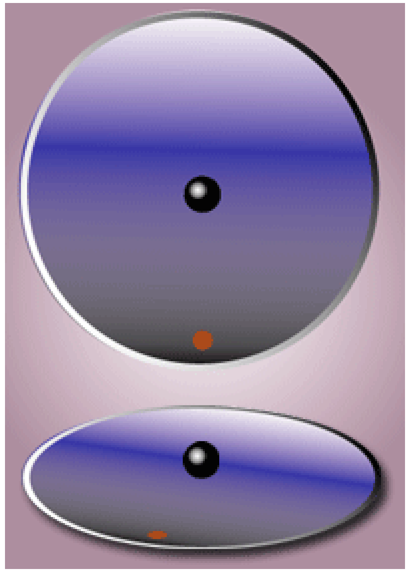
- The transport theorem is a vector equation that relates the time derivative of a Euclidean vector as evaluated in a non-rotating (inertial) coordinate system to its time derivative in a rotating frame.

$$\left(\frac{dA}{dt}\right)_{Inertial} = \left(\frac{dA}{dt}\right)_{Rotating} + \Omega \times A = \left[\left(\frac{d}{dt}\right)_{Rotating} + \Omega \times\right] A$$

- Additional names
  - transport equation,
  - rate of change transport theorem
  - basic kinematic equation
  - Bour's formula, (Edmond Bour)

Transport Theorem –
Differentiation of a Vector in a Rotating Frame of Reference

- **A Euclidean vector** represents a certain magnitude and direction in space that is independent of the coordinate system in which it is measured.
- When taking a time derivative of such a vector one actually takes the difference between two vectors measured at two different times t and t+dt.
- In a rotating coordinate system, the coordinate axes can have different directions at these two times, such that even a constant vector can have a non-zero time derivative.
- As a consequence, the time derivative of a vector measured in a rotating coordinate system can be different from the time derivative of the same vector in a non-rotating reference system.



Transport Theorem – Differentiation of a Vector in a Rotating Frame of Reference

- The velocity vector of an airplane as evaluated using a coordinate system that is fixed to the earth (a rotating reference system) is different from its velocity as evaluated using a coordinate system that is fixed in space.

Transport Theorem – Derivation

- We need to develop a relationship between the total derivative of a vector in an inertial reference frame and the corresponding derivative in a rotating system

- Let A be an arbitrary vector with Cartesian components

$$A = A_x\hat{\imath} + A_x\hat{\jmath} + A_x\hat{k}$$

in an inertial frame of reference

- The same vector A can be expressed in a rotating frame of reference as

$$A = A'_x\,\hat{\imath}' + A'_y\,\hat{\jmath}' + A'_z\,\hat{k}'$$

Transport Theorem – Derivation

- If the vector A is an inertial frame of reference, then the time derivative of this vector is

$$\frac{dA}{dt} = \frac{d}{dt}(A_x\hat{\imath} + A_x\hat{\jmath} + A_x\hat{k})$$

$$\frac{dA}{dt} = \left(\frac{dA_x}{dt}\hat{\imath} + \frac{d\hat{\imath}}{dt}A_x\right) + \left(\frac{dA_y}{dt}\hat{\jmath} + \frac{d\hat{\jmath}}{dt}A_y\right) + \left(\frac{dA_z}{dt}\hat{k} + \frac{d\hat{k}}{dt}A_z\right)$$

- Since the coordinate axes are in an inertial frame of reference the time derivative of the unite vectors are equal to zero,

$$\frac{d\hat{\imath}}{dt} = \frac{d\hat{\jmath}}{dt} = \frac{d\hat{k}}{dt} = 0$$

Transport Theorem – Derivation

- Therefore the time derivative of this vector can be rewritten as

$$\frac{dA}{dt} = \frac{dA_x}{dt}\hat{\imath} + \frac{dA_y}{dt}\hat{\jmath} + \frac{dA_z}{dt}\hat{k} \qquad \text{Eq. 1}$$

- If the same vector is described in the rotated

$$A = A'_x\,\hat{\imath}' + A'_y\,\hat{\jmath}' + A'_z\,\hat{k}'$$

- Then the time derivative of this vector is

$$\frac{dA}{dt} = \left(\frac{dA'_x}{dt}\hat{\imath}' + \frac{d\hat{\imath}'}{dt}A'_x\right) + \left(\frac{dA'_y}{dt}\hat{\jmath}' + \frac{d\hat{\jmath}'}{dt}A'_y\right) + \left(\frac{dA'_z}{dt}\hat{k}' + \frac{d\hat{k}'}{dt}A'_z\right) \qquad \text{Eq. 2}$$

Transport Theorem – Derivation

- Since the left hand sides of equation 1 and 2 are identical then we can equate the right and side of these to equations to form

$$\frac{dA_x}{dt}\hat{\imath}+\frac{dA_y}{dt}\hat{\jmath}+\frac{dA_z}{dt}\hat{k}=\left(\frac{dA'_x}{dt}\hat{\imath}'+\frac{d\hat{\imath}'}{dt}A'_x\right)+\left(\frac{dA'_y}{dt}\hat{\jmath}'+\frac{d\hat{\jmath}'}{dt}A'_y\right)+\left(\frac{dA'_z}{dt}\hat{k}'+\frac{d\hat{k}'}{dt}A'_z\right)$$

- Regrouping the terms

$$\frac{dA_x}{dt}\hat{\imath}+\frac{dA_y}{dt}\hat{\jmath}+\frac{dA_z}{dt}\hat{k}=\left(\frac{dA'_x}{dt}\hat{\imath}'+\frac{dA'_y}{dt}\hat{\jmath}'\frac{dA'_z}{dt}\hat{k}'\right)+\left(A'_x\frac{d\hat{\imath}'}{dt}+A'_y\frac{d\hat{\jmath}'}{dt}+A'_z\frac{d\hat{k}'}{dt}\right)$$

$\left(\frac{dA}{dt}\right)_{Inertial}$ $\left(\frac{dA}{dt}\right)_{Rotataing}$ Effect of Rotation

Transport Theorem – Derivation

- In order to provide an interpretation to the terms

$$A'_x \frac{d\hat{\imath}'}{dt} + A'_y \frac{d\hat{\jmath}'}{dt} + A'_z \frac{d\hat{k}'}{dt}$$

- Assume that each unite vector $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ is a position vector r rotating with an angular velocity Ω

$$\mathrm{v} = \frac{dr}{dt} = \Omega \times \mathrm{r}$$

- Thus

$$\frac{d\hat{\imath}'}{dt} = \Omega \times \hat{\imath}' \qquad \frac{d\hat{\jmath}'}{dt} = \Omega \times \hat{\jmath}' \qquad \frac{d\hat{k}'}{dt} = \Omega \times \hat{k}'$$

Transport Theorem – Derivation

- The combined equation can be rewritten as

$$\frac{dA_x}{dt}\hat{\imath} + \frac{dA_y}{dt}\hat{\jmath} + \frac{dA_z}{dt}\hat{k} = \left(\frac{dA'_x}{dt}\hat{\imath}' + \frac{dA'_y}{dt}\hat{\jmath}'\,\frac{dA'_z}{dt}\hat{k}'\right) + \left(A'_x(\Omega \times \hat{\imath}') + A'_y\,(\Omega \times \hat{\jmath}') + +A'_z\,(\Omega \times \hat{k}')\right)$$

$$\left(\frac{dA}{dt}\right)_{Inertial} \qquad \left(\frac{dA}{dt}\right)_{Rotating} \qquad \Omega \times A$$

Effect of Rotation

$$\left(\frac{dA}{dt}\right)_{Inertial} = \left(\frac{dA}{dt}\right)_{Rotating} + \Omega \times A = \left[\left(\frac{d}{dt}\right)_{Rotating} + \Omega \times\right] A$$

Transport Theorem – Derivation

- Applying the Transport Theorem to the Euler Equation in case $A = I\omega$ and $\Omega = \omega$

$$\left(\frac{dA}{dt}\right)_{Inertial} = \left(\frac{dA}{dt}\right)_{Rotating} + \Omega \times A = \left[\left(\frac{d}{dt}\right)_{Rotating} + \Omega \times\right] A$$

$$\sum \tau = \frac{d}{dt} H_G = \frac{d}{dt} I\omega$$

$$\left(\frac{dI\omega}{dt}\right)_{Inertial} = \left(\frac{dI\omega}{dt}\right)_{Rotating} + \omega \times I\omega = \left[\left(\frac{d}{dt}\right)_{Rotating} + \omega \times\right] I\omega$$

$$\sum \tau = I\dot{\omega} + \omega \times I\omega$$

Manipulator Dynamics – Newton Euler Equations

The Angular Acceleration

$$\dot{\omega}$$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

$\dot{v}_c$ - The linear acceleration of the center of mass

$\dot{\omega}$ - The angular acceleration

${}^{c}I$ - The Inertia tensor (moment of inertia)

F - The sum of all the forces applied on the center of mass

N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

Deriving Angular Acceleration – Matrix Approach

$$\dot{\omega}$$

(*) Review at home

Propagation of Acceleration – Angular Matrix Approach

- To derive a general formula for the angular acceleration, we will differentiate the angular velocity

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + [0 \quad 0 \quad \dot{\theta}_{i+1}]^T$$

- Applying the chain rule, we find:

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + \boxed{{}^{i+1}_{i}\dot{R}}\,{}^{i}\omega_i + [0 \quad 0 \quad \ddot{\theta}_{i+1}]^T$$

- Recall that

$$\boxed{{}^{A}_{B}\dot{R}} = \frac{d}{dt}[{}^{A}_{B}R] = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} {}^{A}_{B}R = \boxed{{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R}$$

- Substitution of this result yields

Propagation of Acceleration - Angular

Matrix form (Revolute Joint)

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{\ i}\dot{R}_{\Omega}\,{}^{i+1}_{\ i}R\,{}^{i}\omega_i + [0 \quad 0 \quad \ddot{\theta}_{i+1}]^T $$

- Converting from matrix to vector form gives the angular acceleration vector

Vector form (Revolute Joint)

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{\ i}R\,{}^{i}\omega_i \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix} $$

- If joint ***i+1*** is prismatic, the angular terms are zero ($\dot{\theta}_{i+1} = \ddot{\theta}_{i+1} = 0$) and the above equation simplifies to:

Matrix form (Prismatic Joint)

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ i}R\,{}^{i}\dot{\omega}_i $$

Deriving Angular Acceleration – Vector Approach (*)

$$\dot{\omega}$$

(*) Review at home

Propagation of Acceleration – Angular Vector Approach

<u>3 Frames</u>

{B} rotates relative to {A} with ${}^{A}\Omega_{B}$
{C} rotates relative to {B} with ${}^{B}\Omega_{C}$

<u>Calculate</u>

${}^{A}\dot{\Omega}_{C}$

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} \qquad (*) $$

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \boxed{\frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}\Omega_{C}\right)} $$

$$ \left(\frac{dA}{dt}\right)_{Inertial} = \left(\frac{dA}{dt}\right)_{Rotating} + \Omega \times A = \left[\left(\frac{d}{dt}\right)_{Rotating} + \Omega \times\right] A $$

$$ \boxed{\frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}\Omega_{C}\right) = {}^{A}_{B}R\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}\Omega_{C}} $$

$$ {}^{A}\dot{\Omega}_{C} = {}^{A}\dot{\Omega}_{B} + \boxed{{}^{A}_{B}R\,{}^{B}\dot{\Omega}_{C} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}\Omega_{C}} $$

$A \rightarrow 0$
$B \rightarrow i$
$C \rightarrow i+1$

Propagation of Acceleration – Angular Vector Approach

$$ {}^{0}\dot{\Omega}_{i+1} = {}^{0}\dot{\Omega}_{i} + {}^{0}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + {}^{0}\Omega_{i} \times {}^{0}_{i}R\,{}^{i}\Omega_{i+1} $$

Multiply both side of the equation from the left by ${}^{i+1}_{0}R$

$$ {}^{i+1}_{0}R\,{}^{0}\dot{\Omega}_{i+1} = {}^{i+1}_{0}R\,{}^{0}\dot{\Omega}_{i} + {}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + {}^{i+1}_{0}R\left({}^{0}\Omega_{i} \times {}^{0}_{i}R\,{}^{i}\Omega_{i+1}\right) $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}\dot{\omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + {}^{i+1}_{0}R\,{}^{0}\Omega_{i} \times {}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}\Omega_{i+1} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}\dot{\omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\dot{\Omega}_{i+1} + {}^{i+1}\omega_{i} \times {}^{i+1}_{i}R\,{}^{i}\Omega_{i+1} $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix} + {}^{i+1}_{i}R\,{}^{i}\omega_{i} \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

Manipulator Dynamics – Newton Euler Equations

The Liner Acceleration of the Center of Mass

$$\dot{v}_C$$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

$\dot{v}_c$ - The linear acceleration of the center of mass
$\dot{\omega}$ - The angular acceleration
${}^{c}I$ - The Inertia tensor (moment of inertia)
F - The sum of all the forces applied on the center of mass
N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

Propagation of Acceleration - Linear
Simultaneous Linear and Rotational Velocity

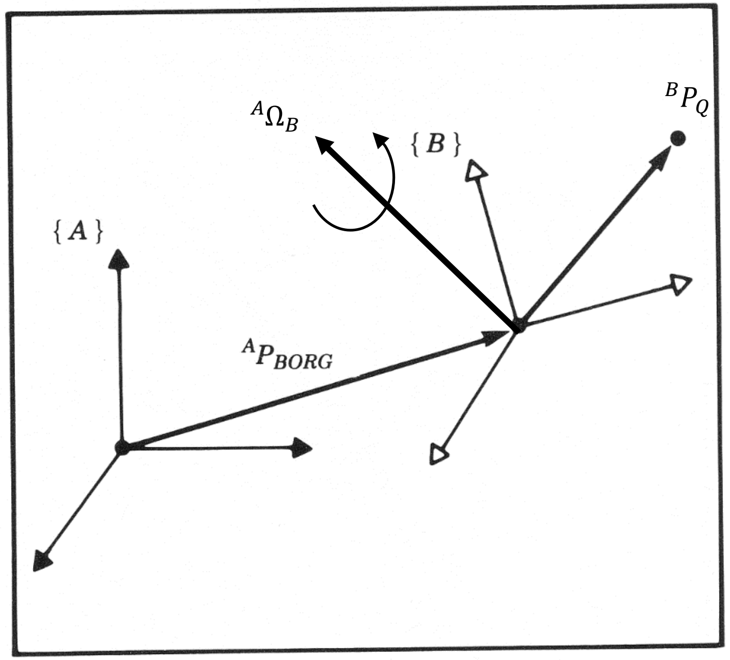
$$ {}^{A}V_{Q} = f\left({}^{B}P_{Q},\ {}^{B}V_{Q},\ {}^{A}V_{BORG},\ {}^{A}\Omega_{B},\ {}^{A}_{B}R\right) $$

- Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



Deriving Linear Acceleration – Matrix Approach

$$\dot{v}_c$$

Propagation of Acceleration – Linear Matrix Approach

- To derive a general formula for the linear acceleration, we will differentiate the linear velocity. However, instead of differentiating the recursive equation like we did for the angular acceleration derivation, we'll begin at a slightly earlier step. Recall the three-part expression:

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q\ org}\right) + {}^{A}V_{B\ org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

- Re-assigning the link frames ($A \rightarrow 0 \quad B \rightarrow i \quad Q \rightarrow i+1$), we find

$$ {}^{0}V_{i+1} = {}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{0}V_{i} + {}^{0}_{i}R\,{}^{i}V_{i+1} $$

- Differentiating using the chain rule gives:

$$ {}^{0}\dot{V}_{i+1} = {}^{0}_{i}\ddot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}\dot{P}_{i+1} + {}^{0}\dot{V}_{i} + {}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}V_{i+1} + {}^{0}_{i}R\,{}^{i}\dot{V}_{i+1} $$

Propagation of Acceleration – Linear Matrix Approach

$$ {}^0\dot{V}_{i+1} = {}^0_i\ddot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + {}^0_i\dot{R}_\Omega\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + \boxed{{}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^i\dot{P}_{i+1}} + {}^0\dot{V}_i + \boxed{{}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iV_{i+1}} + {}^0_iR\, {}^i\dot{V}_{i+1} $$

- Combining the two like terms, we find:

$$ {}^0\dot{V}_{i+1} = {}^0_i\ddot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + {}^0_i\dot{R}_\Omega\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + {}^0\dot{V}_i + \boxed{2\,{}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iV_{i+1}} + {}^0_iR\, {}^i\dot{V}_{i+1} $$

- Pre-multiplying both sides of the equation by $\boxed{{}^{i+1}_0R}$ gives:

$$ \boxed{{}^{i+1}_0R}\, {}^0\dot{V}_{i+1} = \boxed{{}^{i+1}_0R}\, {}^0_i\ddot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + \boxed{{}^{i+1}_0R}\, {}^0_i\dot{R}_\Omega\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + \boxed{{}^{i+1}_0R}\, {}^0\dot{V}_i + 2\boxed{{}^{i+1}_0R}\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iV_{i+1} + \boxed{{}^{i+1}_0R}\, {}^0_iR\, {}^i\dot{V}_{i+1} $$

- Expanding term ${}^{i+1}_0R$ into ${}^{i+1}_iR\, {}^i_0R$ gives

$$ {}^{i+1}_0R\, {}^0\dot{V}_{i+1} = \boxed{{}^{i+1}_iR\, {}^i_0R}\, {}^0_i\ddot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + \boxed{{}^{i+1}_iR\, {}^i_0R}\, {}^0_i\dot{R}_\Omega\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iP_{i+1} + \boxed{{}^{i+1}_0R}\, {}^0\dot{V}_i + 2\boxed{{}^{i+1}_iR\, {}^i_0R}\, {}^0_i\dot{R}_\Omega\, {}^0_iR\, {}^iV_{i+1} + \boxed{{}^{i+1}_0R\, {}^0_iR}\, {}^i\dot{V}_{i+1} $$

Propagation of Acceleration – Linear Matrix Approach

$$ {}^{i+1}_{0}R\,{}^{0}\dot{V}_{i+1} = {}^{i+1}_{i}R\,{}^{i}_{0}R\,{}^{0}_{i}\ddot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i+1}_{i}R\,{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i+1}_{0}R\,{}^{0}\dot{V}_{i} + 2\,{}^{i+1}_{i}R\,{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}V_{i+1} + {}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}\dot{V}_{i+1} $$

- Simplifying the previous equation using (Note: ${}^{s}_{t}R\,{}^{A}_{B}\dot{R}_{\Omega}\,{}^{s}_{t}R^{T} = {}^{s}_{t}R\,{}^{A}\Omega_{B}$)

$$ {}^{i}_{0}R\,{}^{0}_{i}\ddot{R}_{\Omega}\,{}^{0}_{i}R = {}^{i}_{0}R\,{}^{0}_{i}\ddot{R}_{\Omega}\,{}^{i}_{0}R^{T} = {}^{i}_{0}R\,{}^{0}\dot{\Omega}_{i} = {}^{i}_{0}R\,\dot{\omega}_{i} = {}^{i}\dot{\omega}_{i} $$

$$ {}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R = {}^{i}\omega_{i}\times{}^{i}\omega_{i} $$

$$ {}^{i+1}_{0}R\,{}^{0}\dot{V}_{i} = {}^{i+1}_{i}R\,{}^{i}_{0}R\,{}^{0}\dot{V}_{i} = {}^{i+1}_{i}R\,{}^{i}\dot{v}_{i} $$

$$ {}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R = {}^{i}_{0}R\,{}^{0}\omega_{i} = {}^{i}\omega_{i} $$

- we have

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_{i}\times{}^{i}P_{i+1} + {}^{i}\omega_{i}\times{}^{i}\omega_{i}\times{}^{i}P_{i+1} + {}^{i}\dot{v}_{i}\right] + 2\,{}^{i}\omega_{i}\times{}^{i+1}v_{i+1} + {}^{i+1}\dot{v}_{i+1} $$

Propagation of Acceleration – Linear Matrix Approach

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}\dot{v}_i\right] + 2\,{}^{i+1}\omega_{i+1} \times {}^{i+1}v_{i+1} + {}^{i+1}\dot{v}_{i+1}$$

- This equation can be written equivalently as:

General form

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}\dot{v}_i\right] + 2\,{}^{i+1}\omega_{i+1} \times \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix}$$

- If joint ***i+1*** is revolute joint, the linear velocity terms are zero and the above equation simplifies to:

Revolute Joint

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}\dot{v}_i\right] + 0 + 0$$

Propagation of Acceleration – Linear Matrix Approach

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}\dot{v}_i\right]$$

- Note that frame $\{\boldsymbol{i}\}$ and the frame at the center of mass $\{\boldsymbol{C}\}$ are parallel to each other. Therefore the rotation matrix of frame $\{\boldsymbol{i}\}$ with respect to frame at the center of mass $\{\boldsymbol{C}\}$ is equal to identity ${}^{C_i}_{i}R = [I]$

- Substituting in the above equation $i + 1 \rightarrow i_C$ and multiplying both side by ${}^{i}_{C_i}R$ resulting in

$${}^{i}_{C_i}R\,{}^{C_i}\dot{v}_{C_i} = {}^{i}_{C_i}R\,{}^{C_i}_{i}R\left[{}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{C_i} + {}^{i}\dot{v}_i\right]$$

- Since

$${}^{i}_{C_i}R\,{}^{C_i}\dot{v}_{C_i} = {}^{i}\dot{v}_{C_i}$$

Propagation of Acceleration - Linear

- From the general equation, we can also get the solution for the acceleration of the center of mass for link i. Appropriate frame substitution and elimination of prismatic terms gives we find:

$$ {}^{i}\dot{v}_{C_i} = {}^{i}\dot{\omega}_i \times {}^{i}P_{C_i} + {}^{i}\omega_i \times {}^{i}\omega_i \times {}^{i}P_{C_i} + {}^{i}\dot{v}_i $$

- Frame $\{C_i\}$ attached to each link with its origin located at the Center of mass of the link, and with the same orientation as the link frame $\{i\}$
- Increasing the index i by 1 to i+1 resulted in the equation used in the algorithm

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1} $$

Deriving Linear Acceleration – Vector Approach (*)

$$\dot{v}_c$$

(*) Review at home

Propagation of Acceleration – Linear Vector Approach

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad \text{(Eq. 1)}$$

If the origins are coincident $V_{BORG} = 0$

$$^{A}V_{Q} = \frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad \text{(Eq. 2a)}$$

Differentiating the term ${}^{A}_{B}R\,{}^{B}V_{Q}$

$$^{B}\left({}^{A}a_{Q}\right) = {}^{B}\left({}^{A}\dot{V}_{Q}\right) = \frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} \qquad \text{(Eq. 2a)}$$

Differentiating Eq. 1 (Assuming $V_{BORG} = 0$) and applying Eq. 2 for the first and the third term

$$^{A}\dot{V}_{Q} = \frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}V_{Q}\right) + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} + {}^{A}\Omega_{B} \times \frac{d}{dt}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

$$^{A}\dot{V}_{Q} = {}^{A}_{B}R\,{}^{B}\dot{V}_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\dot{\Omega}_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} + {}^{A}\Omega_{B} \times \left[{}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}\right]$$

Propagation of Acceleration – Linear Vector Approach

$$^A\dot{V}_Q = {}^A_BR\,{}^B\dot{V}_Q + 2\,{}^A\Omega_B \times {}^A_BR\,{}^BV_Q + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BP_Q + {}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_BR\,{}^BP_Q\right)$$

$$^Aa_Q = {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR\,{}^B\dot{V}_Q + 2\,{}^A\Omega_B \times {}^A_BR\,{}^BV_Q + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BP_Q + {}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_BR\,{}^BP_Q\right)$$

$^Aa_Q = {}^A\dot{V}_Q$ = Absolute acc. of point Q expressed in frame A

$^A\dot{V}_{BORG}$ - Absolute acc. of the origin of frame B expressed in frame A

${}^A_BR\,{}^B\dot{V}_Q$ - Acc. of point Q with respect to frame B and expressed in frame A

$2\,{}^A\Omega_B \times {}^A_BR\,{}^BV_Q$(Coriolis Acc.) – Combined effect of point Q moving with a velocity ^BV_Q relative to frame B and the rotation of frame B wrt frame A

${}^A\dot{\Omega}_B \times {}^A_BR\,{}^BP_Q$ - Angular acc. effect caused by the rotation of frame B wrt frame A

${}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_BR\,{}^BP_Q\right)$ - Centrifugal Acc. – Angular Acc. caused by the rotation of frame B wrt frame A

Propagation of Acceleration – Linear Vector Approach

Special Case ${}^BP_Q$ is constant

$$ {}^B\dot{P}_Q = {}^BV_Q = 0 $$

$$ {}^B\ddot{P}_Q = {}^B\dot{V}_Q = {}^Ba_Q = 0 $$

$$ {}^Aa_Q = {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^A_BR\,\cancelto{0}{{}^B\dot{V}_Q} + 2\,{}^A\Omega_B \times {}^A_BR\,\cancelto{0}{{}^BV_Q} + {}^A\dot{\Omega}_B \times {}^A_BR\,{}^BP_Q + {}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_BR\,{}^BP_Q\right) $$

$$ {}^Aa_Q = {}^A\dot{V}_Q = {}^A\dot{V}_{BORG} + {}^B\dot{\Omega}_Q \times {}^A_BR\,{}^BP_Q + {}^A\Omega_B \times \left({}^A\Omega_B \times {}^A_BR\,{}^BP_Q\right) $$

$$ {}^0\dot{V}_{i+1} = {}^0\dot{V}_i + {}^0\dot{\Omega}_i \times {}^0_iR\,{}^iP_{i+1} + {}^0\Omega_i \times \left({}^0\Omega_i \times {}^0_iR\,{}^iP_{i+1}\right) $$

Multiply by ${}^{i+1}_{0}R$

$$ {}^{i+1}_0R\,{}^0\dot{V}_{i+1} = {}^{i+1}_0R\,{}^0\dot{V}_i + {}^{i+1}_0R\left({}^0\dot{\Omega}_i \times {}^0_iR\,{}^iP_{i+1}\right) + {}^{i+1}_0R\,{}^0\Omega_i \times \left({}^0\Omega_i \times {}^0_iR\,{}^iP_{i+1}\right) $$

$$ {}^{i+1}\dot{V}_{i+1} = {}^{i+1}\dot{V}_i + {}^{i+1}_0R\,{}^0_iR\left({}^0_iR\,{}^0\dot{R}_i \times {}^iP_{i+1}\right) + {}^{i+1}_0R\,{}^0_iR\left[{}^i_0R\,{}^0\Omega_i \times {}^i_0R\,{}^0\Omega_i \times {}^iP_{i+1}\right] $$

$$ {}^{i+1}\dot{V}_{i+1} = {}^{i+1}R_i\left[{}^i\dot{v}_i + {}^i\omega_i \times {}^iP_{i+1}\right] + {}^i\omega_i \times \left({}^i\omega_i \times {}^iP_{i+1}\right) $$

Manipulator Dynamics – Newton Euler Equations

Forces & Torques

Applied at the Center of Mass $F \quad N$

Applied at the Joints ${}^{i}f_i \quad {}^{i}n_i$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

  $\dot{v}_c$ - The linear acceleration of the center of mass
  $\dot{\omega}$ - The angular acceleration
  ${}^cI$ - The Inertia tensor (moment of inertia)
  F - The sum of all the forces applied on the center of mass
  N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^cI\dot{\omega} + \omega \times {}^cI\omega$$

Expression of Force & Torque With Respect to the CM

Iterative Newton-Euler Equations

- The Newton and Euler equations are re-written for the forces and moments at each link:

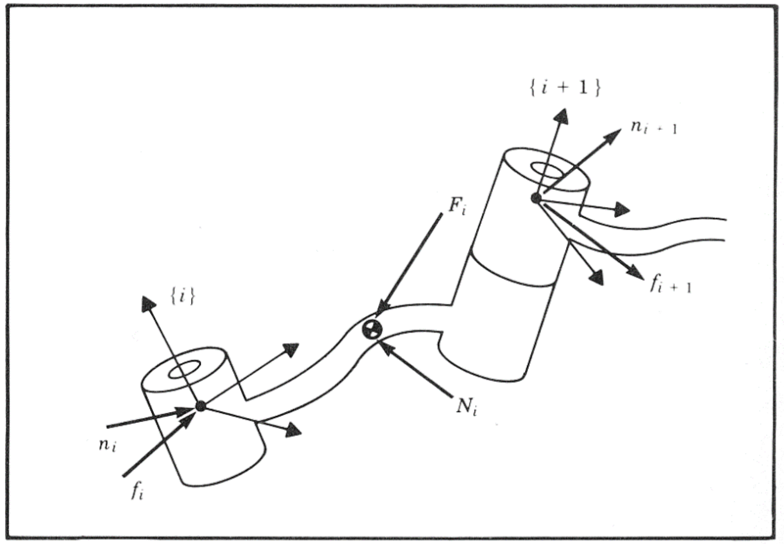
$$ {}^{i}F_i = m_i\,{}^{i}\dot{v}_{c\,i} $$

$$ {}^{i}N_i = {}^{c\,i}I_i\,{}^{i}\dot{\omega}_i + {}^{i}\omega_i \times {}^{c\,i}I_i\,{}^{i}\omega_i $$

- Where $\{C_i\}$ is a frame who has its origin at the link's center of mass and has the same orientation as the link frame $\{i\}$.

Sum of Forces and Moment on a Link

- In addition to calculating the forces and torques arising from link accelerations, we also need to account for how they affect the neighboring links as well as the end effectors interactions with the environment.

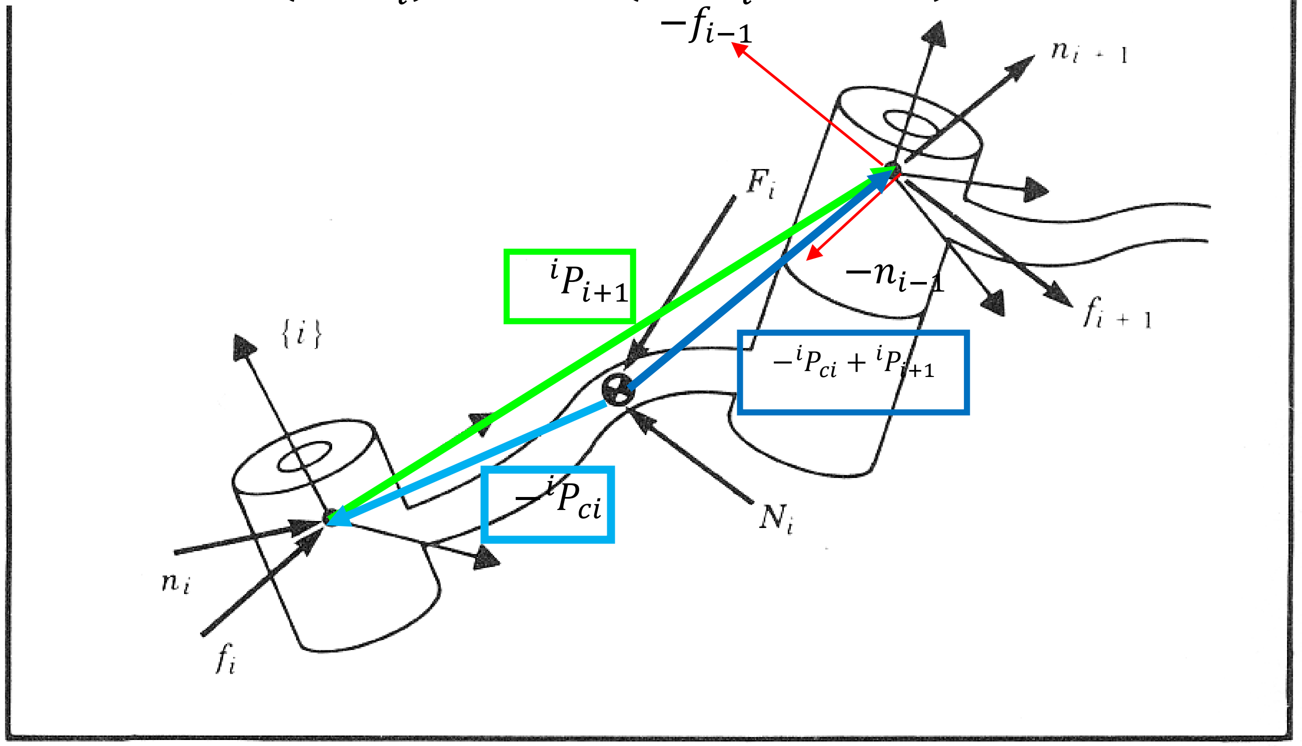


- Balancing the forces shown in the above figure, we can find the total force and torque on each link.

Sum of Forces and Moment on a Link

$$\sum F_{cm} = {}^iF_i = {}^if_i - {}^if_{i-1} = {}^if_i - {}_{i+1}^{\ \ i}R\,{}^if_{i-1}$$

$$\sum N_{cm} = {}^iN_i = {}^in_i - {}^in_{i-1} + (-{}^iP_{c_i}) \times {}^if_i - (-{}^iP_{c_i} + {}^iP_{i+1}) \times {}^if_{i+1}$$



Sum of Forces and Moment on a Link

$${}^{i}F_i = m\dot{v}_{ci} \qquad {}^{i}N_i = {}^{ci}I\dot{\omega}_i + \omega_i \times {}^{c}I\omega_i$$

$${}^{i}F_i = {}^{i}f_i - {}^{i}f_{i+1} = {}^{i}f_i - {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1}$$

$${}^{i}N_i = {}^{i}n_i - {}^{i}n_{i+1} + (-{}^{i}P_{ci}) \times {}^{i}f_i - (-{}^{i}P_{ci} + {}^{i}P_{i+1}) \times {}^{i}f_{i+1}$$

$${}^{i}N_i = {}^{i}n_i - {}_{i+1}^{\ i}R\,{}^{i+1}n_{i+1} - {}^{i}P_{ci} \times {}^{i}f_i + {}^{i}P_{ci} \times {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1} - {}^{i}P_{i+1} \times {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1}$$

$$-{}^{i}P_{ci} \times ({}^{i}f_i - {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1})$$

$$-{}^{i}P_{ci} \times {}^{i}F_i$$

$${}^{i}N_i = {}^{i}n_i - {}_{i+1}^{\ i}R\,{}^{i+1}n_{i+1} - {}^{i}P_{ci} \times {}^{i}F_i - {}^{i}P_{i+1} \times {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1}$$

Sum of Forces and Moment on a Link

- Rearranging the force / torque equations so that they appear as iterative relationship from higher number neighbor to lower number neighbor. The total force and torque on each link.

$$ {}^{i}f_i = {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1} + \boxed{{}^{i}F_i} $$

$$ {}^{i}n_i = \boxed{{}^{i}N_i} + {}_{i+1}^{\ i}R\,{}^{i+1}n_{i+1} + \boxed{{}^{i}P_{c\,i} \times {}^{i}F_i} + {}^{i}P_{i+1} \times {}_{i+1}^{\ i}R\,{}^{i+1}f_{i+1} $$

- Compare with the same equation for the static conditions

$$ {}^{i}f_i = {}_{i+1}^{\ i}R\ {}^{i+1}f_{i+1} $$

$$ {}^{i}n_i = {}_{i+1}^{\ i}R\ {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} $$

Sum of Forces and Moment on a Link

- The joint torque is simply the component of torque that projects onto the joint axis (Z axis by definition)

$$\tau_i = {}^{i}n_i \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Sum of Forces and Moment on a Link

- For a robot moving in free space

$$^{N+1}f_{N+1} = 0$$

$$^{N+1}n_{N+1} = 0$$

- If the robot is contacting the environment, the forces/ torques due to this contact may be included in the equations

$$^{N+1}f_{N+1} \neq 0$$

$$^{N+1}n_{N+1} \neq 0$$

Manipulator Dynamics – Newton Euler Equations

Equation Formulation Procedure

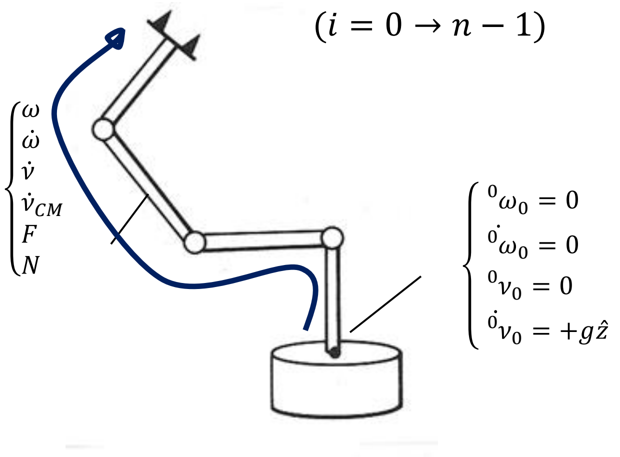
Iterative Newton Euler Equations
Steps of the Algorithm

- (1) Outward Iterations $(i = 0 \rightarrow n-1)$
  - Starting With velocities and accelerations of the base

$$^{0}\omega_0 = 0,\ ^{0}\dot{\omega}_0 = 0,\ ^{0}v_0 = 0,\ ^{0}\dot{v}_0 = +g\hat{z}$$

  - Calculate velocities accelerations, along with forces and torques (at the CM)

$$\omega, \dot{\omega}, \dot{v}, \dot{v}_{CM}, F, N$$

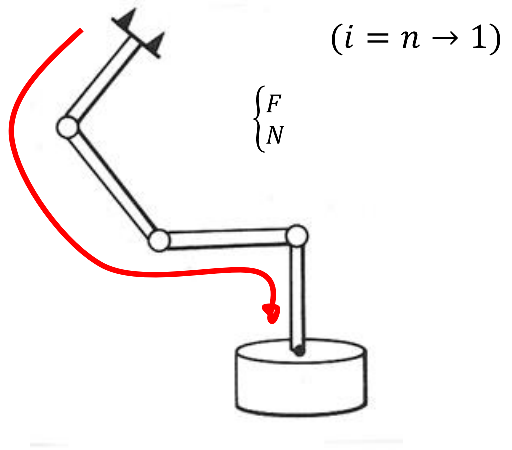


- (2) Inward Iteration $(i = n \rightarrow 1)$
  - Starting with forces and torques (at the CM)

$$F, N$$

  - Calculate forces and torques at the joints

$$f, n$$



Iterative Newton-Euler Equations - Solution Procedure
Phase 1: Outward Iteration

Outward Iteration: $i: 0 \rightarrow 5$

- Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^{i}\omega_i + \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} $$

$$ {}^{i+1}\dot\omega_{i+1} = {}^{i+1}_{\;\;i}R\,{}^{i}\dot\omega_i + {}^{i+1}_{\;\;i}R\,{}^{i}\omega_i \times \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} + \ddot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} $$

$$ {}^{i+1}\dot v_{i+1} = {}^{i+1}_{\;\;i}R\left({}^{i}\dot\omega_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot v_i\right) $$

$$ {}^{i+1}\dot v_{C_{i+1}} = {}^{i+1}\dot\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot v_{i+1} $$

- Calculate the force and torques applied on the CM of each link using the Newton and Euler equations

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot v_{C_{i+1}} $$

$$ {}^{i+1}N_{i+1} = {}^{C\;i+1}I_{i+1}\,{}^{i+1}\dot\omega_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C\;i+1}I_{i+1}\,{}^{i+1}\omega_{i+1} $$

Iterative Newton-Euler Equations - Solution Procedure
Phase 2: Inward Iteration

Inward Iteration: $i : 6 \rightarrow 1$

- Use the forces and torques generated at the joints starting with forces and torques generating by interacting with the environment (that is, tools, work stations, parts etc.) at the end effector al the way the robot's base.

$$^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i$$

$$^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\tau_i = {}^{i+1}n^{T}{}_{i+1}\;{}^{i}\hat{Z}_i$$

Iterative Newton-Euler Equations - Solution Procedure

- **Error Checking -** Check the units of each term in the resulting equations

- **Gravity Effect** - The effect of gravity can be included by setting ${}^{0}\dot{v}_0 = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.