Trajectory Generation (1/2)
Introduction
Motion Planning
Motion Planning

Healthy eye

Eye with cataract

Clear lens

Lens clouded by cataract
Routine Cataract Surgery

https://youtu.be/QbeI72QmFAU
Motion Planning

ANTERIOR

LINGUAL

LABIAL

Preparation must be parallel to lingual surface

.8 mm chamfer or rounded shoulder preferred, feather edge is acceptable

A. 1.0 mm - 1.5 mm occlusal reduction
B. 1.0 mm middle third reduction

POSTERIOR

BUCCAL

LINGUAL

Preparation must be parallel to occlusal surface

.8 mm chamfer or rounded shoulder preferred, feather edge is acceptable

C. Buccal and lingual walls must be convergent
D. Preparation should be cut in three planes

No Sharp Corners
Motion Planning
Motion Planning
Motion Planning

Problem Definition
Motion Planning – Hierarchy

• Trajectory planning is a subset of the overall problem that is navigation or motion planning. The typical hierarchy of motion planning is as follows:
  – Task planning – Designing a set of high-level goals, such as “go pick up the object in front of you”.
  – Path planning – Generating a feasible path from a start point to a goal point. A path usually consists of a set of connected waypoints.
  – Trajectory planning – Generating a time schedule for how to follow a path given constraints such as position, velocity, and acceleration.
  – Trajectory following – Once the entire trajectory is planned, there is a need for a control system that can execute the trajectory in a sufficiently accurate manner.

• Q: What’s the difference between path planning and trajectory planning?
  • A: A trajectory is a description of how to follow a path over time
Problem

Given: Manipulator geometry, End Effector Path (via point)

Compute: The trajectory of each joint such that the end effector move in space from point A to Point B

Solution (Domains)
• Joint space / Task Space

Definitions
• Trajectory (Definition) - Time history of position, velocity, and acceleration for each DOF.
• Trajectory Generation – Methods of computing a trajectory that describes the desired motion of a manipulator in a multidimensional space
Task Space Versus Joint Space - Interpolations

Joint Space

\[ \theta_1 \ldots \theta_6 \]

\[ \theta_f \]

\[ \theta_0 \]

Interpolations at the Joint Space

Task Space

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]

\[ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \]

\[ \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \]

\[ \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \]

Inverse Kinematics

Interpolations at the Task Space

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Instructor: Jacob Rosen
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## Join Space Versus Task Space – Comparison

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<td>Yes</td>
<td>No</td>
</tr>
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General Consideration

• **General approach for the motion of the manipulator**
  – Specify the path as a motion of the tool frame \( \{T\} \) relative to the station frame \( \{S\} \). Frame \( \{G\} \) may change its position in time (e.g. conveyor belt)

• **Advantages**
  – Decouple the motion description from any particular robot, end effector, or workspace.
  – Modularity – Use the same path with:
    • Different robot
    • Different tool size
Trajectory Generation & Inverse Kinematics

General Approach

Transforming from the task space to the joint space using inverse kinematics.

\[ B^T = \begin{bmatrix} \mathbf{t} & \mathbf{s} & \mathbf{G} & \mathbf{G} & \mathbf{T} \end{bmatrix} \]

Desired position and orientation (known)

\[ \mathbf{B}^T - \text{Forward kinematics} \]

\[ \mathbf{W} - \text{Location of the station with respect to the base} \]

\[ \mathbf{S} - \text{Definition of the Trajectory} \]

\[ \mathbf{G}^T = \begin{bmatrix} \frac{0}{0.3} & \frac{3}{0.3} & \frac{3}{0.3} \end{bmatrix} - \text{Full alignment (position, orientation)} \]

\[ \mathbf{T} - \text{Location at the wrist with respect to the tool} \]
Trajectory Generation & Inverse Kinematics
General Approach

\[ B^T = T^6 \]

\[ B^T S^T G^T S^T T^T W^T = T^1 T^2 T^3 T^4 T^5 T^6 \]

\[ \text{known} \]

\[ \text{user define} \]

\[ \begin{bmatrix}
A_{22} & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix} \]

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Trajectory Generation & Inverse Kinematics
General Approach

Given 3 intersecting axis 4, 5, 6 (origins of 4, 5, 6 are at the same point)

\[ 6P = 6P_4 \]
Trajectory Generation & Inverse Kinematics
General Approach

$^0T = ^0T_1^1T_2^2T_3^3T_4^4T_5^5T_6^6T$

Problem 1
position problem

Problem 2
orientation problem

$^3T_4$

$\{\text{Frame 3}\}$ $\xrightarrow{\alpha_3} \{\text{Frame (R)}\}$ $\xrightarrow{a_3} \{\text{Frame (Q)}\}$ $\xrightarrow{\alpha_4} \{\text{Frame (P)}\}$ $\xrightarrow{d_4} \{\text{Frame (4)}\}$

$R_{x_3}(\alpha_3)$ $D_{x_3}(a_3)$ $(R_{z_4} \theta_4)$ $D_{z_4}(d_4)$
Trajectory Generation & Inverse Kinematics
General Approach

\[ \begin{align*}
6T &= 1T_2^1 T_3^2 R_{x3}(0.0) D_{x3}(A_3) R_{zu}(4u) D_{zu}(6u) 4^T 5^T 6_T \\

&\text{Problem 1} \quad \text{position problem} \\
6T &= 1T_2^1 T_3^2 u_T | 4u = 0 | 3^T | 5^T 6_T \\
&\text{Problem 2} \quad \text{orientation problem} \\
\uparrow &\text{set } 4u = 0 \quad \text{set } A_3 = 0
\end{align*} \]
Trajectory Generation & Inverse Kinematics
General Approach

\[ \dot{^6R} = \dot{^3R} R_u \dot{^6R} \]

\[ ^6R = ^3R \left( R(\alpha_3) I \quad R(\theta_u) I \right) \dot{^6R} \]

\[ ^6R = D_{x3}(\alpha_3) \quad D_{zu}(\theta_u) \]

\[ ^6R = \left[ ^3R R_{x3}(\alpha_3) \right] \left[ R_{zu}(\theta_u) \right] ^6R \]
Solving for $\theta_4, \theta_5, \theta_6$

$$R_{z4}(\theta_4)^R_6 = [R_z(\theta_4) R_x(\alpha_3)]^{-1}_6 R$$

Solved in Problem 1

Known $\theta_1, \theta_2, \theta_3$

Desired orientation given for every point on the trajectory

$$R_{z4}(\theta_4)^9 R(\theta_5)^5 R(\theta_6) = \overrightarrow{R_D}$$

Desired orientation of the wrist taking into account the contribution of the first 3 angles to the orientation.
Trajectory Generation & Inverse Kinematics
General Approach

\[ R_{z4}(A_4) R_{s5}(A_5) R_{s6}(A_6) = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix} \]

Solve for \( A_4, A_5, A_6 \) using the Z-Y-Z problem.
Trajectory Generation & Inverse Kinematics
General Approach

\[ R_{Euler} = R_z(\theta_4) R_y(\theta_5) R_z(\theta_6) \]
Trajectory Generation & Inverse Kinematics
General Approach

\[ R_{xyz}(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha \cos \beta & \cos \alpha \cos \beta & -\sin \beta \\ \sin \alpha \sin \beta & \cos \alpha \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma \cos \beta & \cos \gamma \cos \beta & -\sin \beta \\ \sin \gamma \sin \beta & \cos \gamma \sin \beta & \cos \beta \end{bmatrix} \]

Note: \( \gamma \)
General Consideration – Via Points

- **Basic Problem** – Move the tool frame $\{T\}$ from its initial position / orientation $\{T_{\text{initial}}\}$ to the final position / orientation $\{T_{\text{final}}\}$.

- **Specific Description**
  - **Via Point** – Intermediate points between the initial and the final end-effector locations that the end-effector must go through and match the position and orientation along the trajectory.
  - Each via point is defined by a frame defining the position/orientation of the tool with respect to the station frame.
  - **Path Points** – includes all the via points along with the initial and final points.
  - **Point (Frame)** – Every point on the trajectory is defined by a frame (spatial description).
General Consideration – Smooth Path

- **“Smooth” Path or Function**
  - Continuous path / function with first and second derivatives.
  - Add constrains on the spatial and temporal qualities of the path between the via-points.

- **Implications of non-smooth path**
  - Increase wear in the mechanism (rough jerky movement)
  - Vibration – exciting resonances.

![Example of Smooth Path](image)

- The joint angle ($\theta$), joint velocity ($\dot{\theta}$), and joint acceleration ($\ddot{\theta}$) at the entry and exit of the via point are equal.
Trajectory Generation – Joint Space Space Control

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Trajectory Generation – Task Space Control
Precision versus Accuracy

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Trajectory Generation – Roadmap Diagram

**Joint Space**
- Single Time Interval
- Polynomial

**Task Space**
- Multiple Time Intervals (Via Points)
- User Defined Function
- System Defined Function
- Linear Function with Parabolic Blend
- Heuristic
- Linear Function with Parabolic Blend

**Configuration**
- Joint Space
- Task Space

**Function Types**
- 1st Order Linear
- 3rd Order Cubic
- 5th Order Quantic
- Linear Function with Parabolic Blend

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**Course:** Advanced Robotic - Department of Mechanical & Aerospace Engineering - UCLA
Joint Space Schemes

Single Time Interval
Trajectory Generation – Roadmap Diagram

Joint Space

- Single Time Interval
  - Polynomials
    - 1st Order Linear
    - 3rd Order Cubic
    - 5th Order Quantic
    - Linear Function with Parabolic Blend

Task Space

- Multiple Time Intervals (Via Points)
  - User Defined Function
    - Heuristic
    - 3rd Order Cubic
    - Linear Function with Parabolic Blend

- System Defined Function
  - Linear Function with Parabolic Blend

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Joint Space Schemes

• **Joint space Schemes** – Path shapes (in space and in time) are described in terms of functions in the joint space.

• **General process (Steps) given initial and target P/O**
  1. Select a path point or via point (desired position and orientation of the tool frame \( \{T\} \) with respect to the base frame \( \{s\} \))
  2. Convert each of the “via point” into a set of joint angles using the invers kinematics
  3. Find a smooth function for each of the \( n \) joints that pass through the via points, and end the goal point.

**Note 1:** The time required to complete each segment is the **same for each joint** such that the all the joints will reach the via point at the same time. Thus resulting in the position and orientation of the frame \( \{T\} \) at the via point.

**Note 2:** The **joints move independently** with only one time restriction (Note 1)
Joint Space Schemes

- Define a function for each joint such that value at $t_0$ is the **initial position** of the joint and whose value at $t_f$ is the **desired goal position** of the joint.

- There are many smooth functions $\theta(t)$ that may be used to interpolate the joint value.
Joint Space Schemes

Single Time Interval
Polynomials
First Order Polynomial
Trajectory Generation – Roadmap Diagram

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3rd Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

Heuristic

3rd Order Cubic

Linear Function with Parabolic Blend

System Defined Function

Linear Function with Parabolic Blend

Task Space

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Problem - Define a function for each joint such that its value at
- $t_0$ is the initial position of the joint
- $t_f$ is the desired goal position of the joint

Given - Constraints on $\theta(t)$
\[
\begin{align*}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f
\end{align*}
\]
Solution - The two constraints can be satisfied by a first order polynomial

\[ f(t) = a_0 + a_1 t \]

Combined with the two desired constraints yields two equations in two unknowns.

\[ \begin{align*}
\theta_0(0) &= a_0 \\
\theta_f(t) &= a_0 + a_1 t_f
\end{align*} \]

\[ \begin{align*}
\theta_0 &= a_0 \\
\theta_f &= a_0 + a_1 t_f
\end{align*} \]

\[ \Delta = \theta_0 + \left( \frac{\theta_f - \theta_0}{t_f} \right) t \]
Joint Space Schemes

Single Time Interval
Polynomials
Cubic Order Polynomial
Joint Space Schemes – Order of the Polynomials
Joint Space Schemes – Cubic Polynomials - Zero Velocity

• Problem - Define a function for each joint such that it value at
  – $t_0$ is the initial position of the joint and at
  – $t_f$ is the desired goal position of the joint

• Given - Constrains on $\theta(t)$

\[
\begin{align*}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f \\
\dot{\theta}(0) &= 0 \\
\dot{\theta}(t_f) &= 0
\end{align*}
\]

• What should be the order of the polynomial function to meet these constrains?
Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial of at least third degree

\[ \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]

- The joint velocity and acceleration

\[ \dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 \]
\[ \ddot{\theta}(t) = 2a_2 + 6a_3 t \]

- Combined with the four desired constraints yields four equations in four unknowns

\[
\begin{align*}
\theta(0) &= \theta_0 \quad \theta_0 = a_0 \\
\theta(t_f) &= \theta_f \quad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
\dot{\theta}(0) &= 0 \quad 0 = a_1 \\
\dot{\theta}(t_f) &= 0 \quad 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2
\end{align*}
\]
Joint Space Schemes – Cubic Polynomials - Zero Velocity

\[
\begin{align*}
\theta_f &= \theta_0 + a_2 t_f^2 + a_3 t_f^3 \\
0 &= 2a_2 t_f + 3a_3 t_f^2 \\
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} a_2 & t_f^2 \\ a_2 & t_f^3 \end{bmatrix} + a_3 &\begin{bmatrix} t_f^2 \\ t_f^3 \end{bmatrix} = \theta_f - \theta_0 \\
\begin{bmatrix} a_2 & 2t_f \\ a_2 & 3t_f^2 \end{bmatrix} &+ a_3 \begin{bmatrix} 2t_f \\ 3t_f^2 \end{bmatrix} = 0 \\
\end{align*}
\]

\[
\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4
\]

\[
\begin{align*}
a_2 &= \frac{\begin{vmatrix} \theta_f - \theta_0 & t_f^3 \\ 0 & 3t_f^2 \end{vmatrix}}{3t_f^2 (4f - 4o)} \\
3(4f - 4o) &\quad t_f^2 \\
\end{align*}
\]

\[
\begin{align*}
a_3 &= \frac{\begin{vmatrix} t_f^2 & 4f - 4o \\ 2t_f & 0 \end{vmatrix}}{-2t_f (4f - 4o)} \\
-2(4f - 4) &\quad t_f^2 \\
\end{align*}
\]
Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solving these equations for the $a_i$, we obtain

\[
\begin{align*}
  a_0 &= \theta_0 \\
  a_1 &= 0 \\
  a_2 &= \frac{3}{t_f^2} (\theta_f - \theta_0) \\
  a_3 &= -\frac{2}{t_f^3} (\theta_f - \theta_0)
\end{align*}
\]
Joint Space Schemes – Cubic Polynomials - Zero Velocity

\[ \dot{\theta}_{\text{max}} = \text{Max angular velocity at } t_f/2 \]

\[ \dot{\theta}_{\text{max}}(t = t_f/2) = \frac{6}{t_f^2} (\theta_f - \theta_0) \left[ \frac{t_f}{2} \right] - \frac{6}{t_f^3} (\theta_f - \theta_0) \left[ \frac{t_f}{2} \right]^2 \]

\[ = \frac{3(\theta_f - \theta_0)}{t_f} - \frac{6}{t_f^3} (\theta_f - \theta_0) \left[ \frac{t_f}{2} \right] \]

\[ = \frac{3}{2} \frac{\theta_f - \theta_0}{t_f} \]

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Joint Space Schemes – Cubic Polynomials - Zero Velocity

\[
\ddot{\theta}_{\text{max}} = \max \text{ angular acceleration at } t = 0 \text{ and } t = t_f
\]

\[
\dot{\theta}_{\text{max}} = \frac{6}{t_f^2} (A_f - A_o)
\]
Joint Space Schemes – Cubic Polynomials - Zero Velocity

\[ \frac{3}{2} \frac{\Delta \theta - \dot{\theta}_0}{t_f} \]

\[ + \] \[ \frac{6}{t_f^2} \frac{\Delta \theta - \dot{\theta}_0}{t_f} \]

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Joint Space Schemes – Cubic Polynomials - Zero Velocity

• Example – A single-link robot with a rotary joint is motionless at $\theta_0 = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta_f = 75$ degrees in 3 seconds. Find the coefficient of the cubic polynomial that accomplish this motion and brings the manipulator to rest at the goal

$$\theta(0) = 15 \quad a_0 = \theta_0 = 15$$
$$\theta(t_f) = 75 \quad a_1 = 0$$
$$\dot{\theta}(0) = 0 \quad a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) = \frac{3}{9}(75-15) = 20$$
$$\dot{\theta}(t_f) = 0 \quad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) = -\frac{2}{27}(75-15) = -4.44$$

$$\theta(t) = 15 + 20t^2 - 4.44t^3$$
$$\dot{\theta}(t) = 40t - 13.33t^2$$
$$\ddot{\theta}(t) = 40 + 26.66t$$
Joint Space Schemes – Cubic Polynomials - Zero Velocity

- The velocity profile of any cubic function is a parabola
- The acceleration profile of any cubic function is linear
Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

• Previous Method - The manipulator comes to rest at each via point

• General Requirement - Pass through a point without stopping

• Problem - Define a function for each joint such that it value at
  – \( t_0 \) is the initial position of the joint and at
  – \( t_f \) is the desire goal position of the joint

• Given - Constrains on \( \theta(t) \) such that the velocities at the via points are not zero
  but rather some known velocities

\[
\begin{align*}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f \\
\dot{\theta}(0) &= \dot{\theta}_0 \\
\dot{\theta}(t_f) &= \dot{\theta}_f
\end{align*}
\]
Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial

\[ \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]
\[ \dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 \]
\[ \ddot{\theta}(t) = 2a_2 + 6a_3 t \]

- Combined with the four desired constraints yields four equations in four unknowns

\[
\begin{align*}
\theta(0) &= \theta_0 & \theta_0 &= a_0 \\
\theta(t_f) &= \theta_f & \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
\dot{\theta}(0) &= \dot{\theta}_0 & \dot{\theta}_0 &= a_1 \\
\dot{\theta}(t_f) &= \dot{\theta}_f & \dot{\theta}_f &= a_1 t_f + 2a_2 t_f + 3a_3 t_f^2
\end{align*}
\]
Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

\[
\begin{bmatrix}
A_0 \\
A_f \\
\dot{A}_0 \\
\dot{A}_f
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & t_f & t_f^2 & t_f^3 \\
0 & 1 & 0 & 0 \\
0 & t_f & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
\]
Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

\[
\begin{align*}
\dot{\theta}_f &= \dot{\theta}_0 + 2a_2 t_f + 3a_3 t_f^2 \\
\ddot{\theta}_f &= \ddot{\theta}_0 + 2a_2 t_f + 3a_3 t_f^2
\end{align*}
\]

\[
a_2 &= \left( \dot{\theta}_f - \dot{\theta}_0 \right) \\
a_3 &= \left( \ddot{\theta}_f - \ddot{\theta}_0 \right)
\]

\[
\alpha_2 = \frac{\begin{vmatrix} t_f^2 & t_f^3 \\ t_f^2 & t_f^3 \end{vmatrix}}{t_f^4} = 3t_f^4 - 2t_f^4 = t_f^4
\]

\[
\alpha_3 = \frac{\begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix}}{t_f^4} = 3t_f^4 - 2t_f^4 = t_f^4
\]

\[
\Delta = \frac{\begin{vmatrix} t_f^2 & t_f^3 \\ t_f^2 & t_f^3 \end{vmatrix}}{t_f^4} = 3t_f^4 - 2t_f^4 = t_f^4
\]

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Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solving these equations for the \( a_i \) we obtain

\[
a_0 = \theta_0 \\
\dot{a}_0 = \dot{\theta}_0 \\
a_2 = \frac{3}{t_f^2} (\theta_f - \theta_0) - \frac{2}{t_f} \dot{\theta}_0 - \frac{1}{t_f^2} \ddot{\theta}_f \\
a_3 = -\frac{2}{t_f^3} (\theta_f - \theta_0) + \frac{2}{t_f^2} (\dot{\theta}_f + \dot{\theta}_0)
\]

- Given - velocities at each via point are
- Solution - Apply these equations for each segment of the trajectory.
- Note: The Cubic polynomials ensures the continuity of velocity but not the acceleration. Practically, the industrial manipulators are sufficiently rigid so this continuity in acceleration
Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

• Note:
  – The Cubic polynomials ensures the continuity of velocity but not the acceleration.
  – Practically, the industrial manipulators are sufficiently rigid so this discontinuity in acceleration is filtered by the mechanical structure
  – Therefore this trajectory is generally satisfactory for most applications
Joint Space Schemes

Single Time Interval
Polynomials
Quantic Order Polynomial
Trajectory Generation – Roadmap Diagram

- **Joint Space**
  - Single Time Interval
  - Multiple Time Intervals (Via Points)
- **Task Space**
  - User Defined Function
  - System Defined Function
- **Polynomials**
  - Linear
  - Cubic
  - Quantic
  - Linear Function with Parabolic Blend
  - Heuristic
  - Linear Function with Parabolic Blend

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Joint Space Schemes – Quantic Polynomials

- Rational for Quantic Polynomials (high order)
  - High Speed Robot
  - Robot Carrying heavy/delicate load
  - Non Rigid links
  - For high speed robots or when the robot is handling heavy or delicate loads. It is worth insuring the continuity of accelerations as well as avoid excitation of the resonance modes of the mechanism
Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the time at the initial position
  - $t_f$ is the time at the desired goal position

- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment
  \[
  \theta(0) = \theta_0, \\
  \theta(t_f) = \theta_f, \\
  \dot{\theta}(0) = \dot{\theta}_0, \\
  \dot{\theta}(t_f) = \dot{\theta}_f, \\
  \ddot{\theta}(0) = \ddot{\theta}_0, \\
  \ddot{\theta}(t_f) = \ddot{\theta}_f
  \]

- What should be the order of the polynomial function to meet these constrains?
Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

\[
\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5
\]

\[
\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4
\]

\[
\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
\]

- Combined with the six desired constraints yields six equations with six unknowns

\[
\theta(0) = \theta_0 \quad \dot{\theta}(0) = \dot{\theta}_0 \quad \ddot{\theta}(0) = \ddot{\theta}_0
\]

\[
\theta(t_f) = \theta_f \quad \dot{\theta}(t_f) = \dot{\theta}_f \quad \ddot{\theta}(t_f) = \ddot{\theta}_f
\]

\[
\theta(0) = \theta_0 = a_0
\]

\[
\theta(t_f) = \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5
\]

\[
\dot{\theta}(0) = \dot{\theta}_0 = a_1
\]

\[
\dot{\theta}(t_f) = \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4
\]

\[
\ddot{\theta}(0) = \ddot{\theta}_0 = 2a_2
\]

\[
\ddot{\theta}(t_f) = \ddot{\theta}_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
\]
Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration
Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

\[ \Theta_f = \Theta_0 + \Theta_0 t_f + \frac{\Theta_0}{2} t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5 \]

\[ \ddot{\Theta}_f = \theta_0 + 2 \frac{\dot{\Theta}_0}{\dot{t}_f} + 3 a_3 t_f^2 + 4 a_4 t_f^3 + 5 a_5 t_f^4 \]

\[ \dddot{\Theta}_f = \frac{\ddot{\Theta}_0}{\dot{t}_f^2} + 6 a_3 t_f + 12 a_4 t_f^2 + 20 a_5 t_f^3 \]
Joint Space Schemes – Cubic Polynomials - Non Zero Acceleration

Solving these equations for the \( a_i \) we obtain

\[
\begin{align*}
a_0 &= \theta_0 \\
a_1 &= \dot{\theta}_0 \\
a_2 &= \frac{\ddot{\theta}_0}{2} \\
a_3 &= \frac{20\theta_f - 20\theta_0 - (8\theta_f + 12\theta_0)t_f - (3\theta_0 - \theta_f)t_f^2}{2t_f^3} \\
a_4 &= \frac{30\theta_0 - 30\theta_0 + (14\theta_f + 16\theta_0)t_f + (3\theta_0 - 2\theta_f)t_f^2}{2t_f^4} \\
a_5 &= \frac{12\theta_f - 12\theta_0 - (6\theta_f + 6\theta_0)t_f - (\theta_0 - \theta_f)t_f^2}{2t_f^5}
\end{align*}
\]
Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

For a generalized case where \( t_0 = 0 \)

\[
T' = t_f - t_0 \quad h = \theta_f - \theta_0
\]

\[
a_0 = \theta_0
\]

\[
a_1 = \dot{\theta}_0
\]

\[
a_2 = \frac{1}{2} a_0
\]

\[
a_3 = \frac{1}{2 T^3} \left[ 20 h - 8 \left( 8 \dot{\theta}_f + 12 \dot{\theta}_0 \right) T - \left( 3 a_0 - a_1 \right) T^2 \right]
\]

\[
a_4 = \frac{1}{2 T^4} \left[ -30 h + \left( 14 \dot{\theta}_f + 16 \dot{\theta}_0 \right) T + \left( 3 a_0 - 2 a_1 \right) T^2 \right]
\]

\[
a_5 = \frac{1}{2 T^5} \left[ 12 h - 6 \left( \dot{\theta}_f - \dot{\theta}_0 \right) T + \left( a_1 - a_0 \right) T^2 \right]
\]
Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the time at the initial position
  - $t_f$ is the time at the desired goal position
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment

$$\theta(0) = \theta_0 \quad \dot{\theta}(0) = \dot{\theta}_0 \quad \ddot{\theta}(0) = 0$$
$$\theta(t_f) = \theta_f \quad \dot{\theta}(t_f) = \dot{\theta}_f \quad \ddot{\theta}(t_f) = 0$$

- What should be the order of the polynomial function to meet these constrains?
Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order
  \[
  \theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5
  \]
  \[
  \dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4
  \]
  \[
  \ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
  \]

- Combined with the six desired constraints yields six equations with six unknowns
  \[
  \theta(0) = \theta_0 \quad \theta_0 = a_0
  \]
  \[
  \theta(t_f) = \theta_f \quad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5
  \]
  \[
  \dot{\theta}(0) = \dot{\theta}_0 \quad \dot{\theta}_0 = a_1
  \]
  \[
  \dot{\theta}(t_f) = \dot{\theta}_f \quad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4
  \]
  \[
  \ddot{\theta}(0) = 0 \quad 0 = 2a_2
  \]
  \[
  \ddot{\theta}(t_f) = 0 \quad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
  \]
Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solving these equations for the \( a_i \) we obtain

\[
a_0 = \theta_0 \\
a_1 = \dot{\theta}_0 \\
a_2 = \ddot{\theta}_0 = 0 \\
a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0) t_f}{2t_f^3} \\
a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0) t_f}{2t_f^4} \\
a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0) t_f}{2t_f^5}
\]
Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Problem - Define a function for each joint such that its value at
  - $t_0$ is the time at the initial position
  - $t_f$ is the time at the desired goal position
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment

\[
\begin{align*}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f \\
\dot{\theta}(0) &= 0 \\
\dot{\theta}(t_f) &= 0 \\
\ddot{\theta}(0) &= 0 \\
\ddot{\theta}(t_f) &= 0
\end{align*}
\]

- What should be the order of the polynomial function to meet these constraints?
• Solution - The six constraints can be satisfied by a polynomial of at least fifth order

\[
\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5
\]

\[
\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4
\]

\[
\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
\]

• Combined with the six desired constraints yields six equations with six unknowns

\[
\theta(0) = \theta_0 \quad \Rightarrow \quad \theta_0 = a_0
\]

\[
\theta(t_f) = \theta_f \quad \Rightarrow \quad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5
\]

\[
\dot{\theta}(0) = 0 \quad \Rightarrow \quad 0 = a_1
\]

\[
\dot{\theta}(t_f) = 0 \quad \Rightarrow \quad 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4
\]

\[
\ddot{\theta}(0) = 0 \quad \Rightarrow \quad 0 = 2a_2
\]

\[
\ddot{\theta}(t_f) = 0 \quad \Rightarrow \quad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3
\]
Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solving these equations for the $a_i$ we obtain

\[
\begin{align*}
  a_0 &= \theta_0 \\
  a_1 &= 0 \\
  a_2 &= 0 \\
  a_3 &= \frac{20\theta_f - 20\theta_0}{2t_f^3} = \frac{10\theta_f - 10\theta_0}{t_f^3} = 10 \left[ \frac{\theta_f - \theta_0}{t_f^3} \right] \\
  a_4 &= \frac{30\theta_0 - 30\theta_f}{2t_f^4} = \frac{15\theta_f - 15\theta_0}{t_f^4} = -15 \left[ \frac{\theta_f - \theta_0}{t_f^4} \right] \\
  a_5 &= \frac{12\theta_f - 12\theta_0}{2t_f^5} = \frac{6\theta_f - 6\theta_0}{t_f^5} = 6 \left[ \frac{\theta_f - \theta_0}{t_f^5} \right]
\end{align*}
\]
Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

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Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

\[ \dot{a}_{max} \rightarrow at \quad t = \frac{t_f}{2} \]

\[ \dot{a}_{max} = \begin{array}{c}
\frac{a_3}{t_f^2} \left[ \frac{t_f}{4} \right] \\
\frac{15}{4} \left[ \frac{t_f}{t_f^2} \right] \\
\frac{26}{16} \left[ \frac{t_f}{t_f^2} \right] \\
\frac{15}{8} \left[ \frac{t_f}{t_f^2} \right]
\end{array} + 3 \left[ \frac{\theta_f - \theta_o}{t_f^2} \right] \begin{array}{c}
\frac{a_4}{t_f^3} \\
\frac{15}{8} \left[ \frac{t_f}{t_f^2} \right] \\
\frac{26}{16} \left[ \frac{t_f}{t_f^2} \right]
\end{array} - 4 \left[ \frac{\theta_f - \theta_o}{t_f^4} \right] \begin{array}{c}
\frac{a_5}{t_f^5} \\
\frac{15}{8} \left[ \frac{t_f}{t_f^2} \right] \\
\frac{26}{16} \left[ \frac{t_f}{t_f^2} \right]
\end{array} + 5 \left[ \frac{\theta_f - \theta_o}{t_f^4} \right] \begin{array}{c}
\frac{a_s}{t_f^5} \\
\frac{15}{8} \left[ \frac{t_f}{t_f^2} \right] \\
\frac{26}{16} \left[ \frac{t_f}{t_f^2} \right]
\end{array} \]

\[ = \frac{15}{8} \left[ \frac{\theta_f - \theta_o}{t_f} \right] \]
Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

\[ \ddot{A}_{\text{max}} \rightarrow \alpha t \quad t = t_f/4 \]

\[ \ddot{A}_{\text{max}} = 6 \left[ \frac{10 \left( \theta_f - \theta_o \right)}{t_f^3} \right] \frac{t_f^2}{4} - \frac{3}{12} \left[ 15 \left( \frac{\theta_f - \theta_o}{t_f^4} \right) \right] \frac{t_f^2}{16} + \frac{5}{26} \left[ 6 \left( \frac{\theta_f - \theta_o}{t_f^5} \right) \right] \frac{t_f^3}{32} \]

\[ = 15 \left[ \frac{\theta_f - \theta_o}{t_f^2} \right] - \frac{45}{8} \left[ \frac{\theta_f - \theta_o}{t_f^2} \right] + \frac{15}{8} \left[ \frac{4\theta_f - 5\theta_o}{t_f^2} \right] \]

\[ = \left[ 15 - \frac{75}{8} \right] \frac{\theta_f - \theta_o}{t_f^2} \]

5.625
Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

\[ \frac{15}{8} \frac{\theta_f - \theta_0}{t_f} \]

\[ \frac{10}{13} \frac{\theta_f - \theta_0}{t_r^2} \]
Joint Space Schemes

Single Time Interval
Polynomials
Linear Function with Parabolic Blend (Trapezoid Velocity Method)
Trajectory Generation – Roadmap Diagram

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3rd Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

Heuristic

3rd Order Cubic

Linear Function with Parabolic Blend

System Defined Function

Linear Function with Parabolic Blend

Task Space

Trajectory Generation
Joint Space Schemes –
Linear Function With Parabolic Blend

- Linear interpolation to move from the present joint position \( \theta_0 \) to the final position \( \theta_f \) at \( t = t_f \).

- Note: Although motion of each joint is linear, the EE in general does not move in a straight line in space.

- Problem: Linear interpolation would cause the velocity to be discontinuous at the beginning/end.

- Solution: Parabolic blend region.
Joint Space Schemes –
Linear Function With Parabolic Blend

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Joint Space Schemes –
Linear Function With Parabolic Blend

• During the blend - Constant Acceleration to change the velocity smoothly.
• Assumptions:
  1. The parabolic blend segments (Δt_1, Δt_2) have the same duration.
     \[ Δt_1 = Δt_2 \]
  2. The same constant acceleration is used during both blends.
Joint Space Schemes –
Linear Function With Parabolic Blend

1. \[ \theta_b = \theta_0 + \theta_f t + \frac{1}{2} \dot{\theta}_b t^2 \]
   - \( \theta_b \): \( \theta \) at \( t_b \)
   - Initial \( \theta \) is zero (Point C)
   - Constant acceleration during the blend (Point C to A)
   - \( \dot{\theta}_b \): Initial velocity (initial \( \theta \) is zero)
   - The slope at point A must be equal on both sides

2. \[ \frac{\theta_f - \theta_0}{t_f - t_b} = \frac{\theta_h - \theta_b}{th - t_b} \]
   - \( \theta_h \): \( \theta \) at \( t_h \)
   - Velocity from the left (\( \dot{\theta}_{in} \))
   - Velocity from the right (\( \dot{\theta}_{out} \))
Joint Space Schemes –
Linear Function With Parabolic Blend

Point \( B \)
- Point \( B \) is at the middle of the segment

(3) \[ t_b = \frac{t}{2} \]

(4) \[ \theta_b = \frac{\theta_f - \theta_0}{2} + \theta_0 = \frac{\theta_f - \theta_0 + 2\theta_0}{2} = \frac{\theta_f + \theta_0}{2} \]

plug Eq (4) into Eq (2) and Eq (3) into Eq (2)

\[ t_b = \frac{4b - \theta_f}{\theta_f - t_b} = \frac{4b + \theta_0}{2} - t_b \]
Joint Space Schemes –
Linear Function With Parabolic Blend

\[ \dot{t}_b \left( \frac{t}{2} - t_b \right) = \frac{\theta_f + \theta_o}{2} - \dot{\theta}_b \]

\[ \ddot{t}_b \left( \frac{t}{2} - t_b \right) = \frac{\dot{\theta_f} + \dot{\theta_o} - 2\ddot{\theta}_b}{2} \]

\[ \dot{t}_b \cdot t - 2 \dot{\theta}_b \dot{t}_b = \theta_f + \theta_o - 2 \ddot{\theta}_b \]

\[ \ddot{t}_b (t_b t) - 2 \dot{\theta}_b \dot{t}_b^2 - \theta_f - \theta_o + 2 \ddot{\theta}_b = 0 \]

\[ \text{plug Eq. (1)} \]

\[ \dot{t}_b (t_b t) - 2 \dot{\theta}_b \dot{t}_b^2 - \theta_f - \theta_o + 2 \ddot{\theta}_b + \dot{\theta}_b \dot{t}_b^2 = 0 \]

\[ \ddot{t}_b (t_b t) - \dot{\theta}_b + \theta_f + \theta_o = 0 \]
Joint Space Schemes –
Linear Function With Parabolic Blend

\[
(5) \quad (\ddot{\theta}) t_b^2 + (\dot{\theta}) t_b + (\theta_f - \theta_o) = 0
\]

- Given: \( \theta_f, \theta_o, t, t_b \) (desired duration of motion)
- Calculate: \( \ddot{\theta} \) (Eq. 5)

\[
t_b = \frac{\pm \sqrt{\ddot{\theta}^2 - 4 \ddot{\theta} (\theta_f - \theta_o)}}{2 \ddot{\theta}} = \frac{t}{2} \pm \frac{\sqrt{\ddot{\theta}^2 - 4 \ddot{\theta} (\theta_f - \theta_o)}}{2 \ddot{\theta}}
\]

- Given: \( \ddot{\theta} \) (chosen), \( t, \theta_f, \theta_o \)
- Calculate: \( t_b \)
Joint Space Schemes –
Linear Function With Parabolic Blend

Constraint on the acceleration used in the blend

\[ \sqrt{\dddot{q}^2 - 4\ddot{q}(\dot{q} - \dot{q}_0)} > 0 \]

\[ \dddot{q}^2 > 4\ddot{q}(\dot{q} - \dot{q}_0) \]

\[ \dot{q} \geq \frac{4(\dot{q} - \dot{q}_0)}{1^2} \]

If equal, \( t_b = \frac{t}{2} \pm \frac{1}{2\ddot{q}} \) → \( t_b = \frac{t}{2} \)
Joint Space Schemes –
Linear Function With Parabolic Blend

- The length of the linear portion and the parabolic portion may vary.
  - High Acceleration ($\ddot{a}$) → Short Blend
  - Low Acceleration ($\ddot{a}$) → Long Blend
Joint Space Schemes –
Linear Function With Parabolic Blend
Joint Space Schemes

Multiple Time Interval
Via Point
Trajectory Generation – Roadmap Diagram

- **Joint Space**
  - Single Time Interval
  - Polynomials
    - 1st Order Linear
    - 3rd Order Cubic
    - 5th Order Quantic
    - Linear Function with Parabolic Blend

- **Multiple Time Intervals (Via Points)**
  - User Defined Function
  - System Defined Function
    - 3rd Order Cubic
    - Linear Function with Parabolic Blend
    - Linear Function with Parabolic Blend

- **Task Space**

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Joint Space Schemes – Multiple Time – Via Points

- Define multiple functions for each joint such that
  - The value at $t_0$ is the initial position of the joint and whose value at $t_f$ is the desire goal position of the joint $\theta(t)$ at the end of the time interval
  - In between the beginning/ending points the user define via points that each joint must pass trough.
  - Note that all the joints reach the via point at the same time to guarantee a specific position and orientation of the end effector.
- There are many smooth functions that may be used to interpolate the joint value.
Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

• Specify the desired velocities at each segment:
  1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.

  2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) automatically using a suitable heuristic method.

  3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.
Trajectory Generation – Roadmap Diagram

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3rd Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

Heuristic

3rd Order Cubic

Linear Function with Parabolic Blend

Task Space

System Defined Function

Linear Function with Parabolic Blend

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Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.

   - **Mapping** (Cartesian Space to Joint Space) - Cartesian velocities at the via point are “mapped” to desired joint rates by using the inverse Jacobian

     \[ \dot{\theta}_f = J^{-1} \dot{X}_f \]

   - **Singularity** - If the manipulator is at a singular point at a particular via point then the user is not free to choose an arbitrarily velocity at this point.

   - **Difficult**
2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) using a suitable heuristic method given a trajectory.

**Heuristic method**
- Consider a path defined by via points
- Connect the via points with straight lines
- If the slope change sign
  - Set the velocity at the via point to be zero
- If the slope have the same sign
  - Calculate the average between the velocities at the via point.
3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.

   - **Spline** - Enforcing the velocity and the acceleration to be continuous at the via point
Trajectory Generation – Roadmap Diagram

- **Joint Space**
  - Single Time Interval
  - Polynomials
    - 1\(^{st}\) Order Linear
    - 3\(^{rd}\) Order Cubic
    - 5\(^{th}\) Order Quantic
    - Linear Function with Parabolic Blend
  - Multiple Time Intervals (Via Points)
    - User Defined Function
      - 3\(^{rd}\) Order Cubic
      - Linear Function with Parabolic Blend
    - System Defined Function
      - Heuristic
        - 3\(^{rd}\) Order Cubic
        - Linear Function with Parabolic Blend
  - Task Space

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Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

• Solve for the coefficients of two cubic functions that are connected in a two segment spline with a continuous acceleration at the intermediate via point.

\[
\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3
\]
\[
\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3
\]
\[
\dot{\theta}(t) = a_{11} + 2a_{12}t + 3a_{13}t^2
\]
\[
\dot{\theta}(t) = a_{21} + 2a_{22}t + 3a_{23}t^2
\]
\[
\ddot{\theta}(t) = 2a_{12} + 6a_{13}t
\]
\[
\ddot{\theta}(t) = 2a_{22} + 6a_{23}t
\]
Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- The joint angle velocity and acceleration for each segment (8 unknowns)

\[
\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3
\]
\[
\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3
\]
\[
\dot{\theta}(t) = a_{11} + 2a_{12}t + 3a_{13}t^2
\]
\[
\ddot{\theta}(t) = a_{21} + 2a_{22}t + 3a_{23}t^2
\]
\[
\dddot{\theta}(t) = 2a_{12} + 6a_{13}t
\]
\[
\dddot{\theta}(t) = 2a_{22} + 6a_{23}t
\]
Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Position at the beginning and end of each segment

- Segment 1

\[ \theta_0(t = 0) = a_{10} \]
\[ \theta_{via}(t = t_{f_1}) = a_{10} + a_{11}t_{f_1} + a_{12}t_{f_1}^2 + a_{13}t_{f_1}^3 \]

- Segment 2

\[ \theta_{via}(t = 0) = a_{20} \]
\[ \theta_g(t = t_{f_2}) = a_{20} + a_{21}t_{f_2} + a_{22}t_{f_2}^2 + a_{23}t_{f_2}^3 \]
Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- **Velocity at the beginning** of the interval
  \[ \dot{\theta}(t = 0) = a_{i1} \]

- **Velocity at the end** of the interval
  \[ \dot{\theta}(t = t_{f_2}) = a_{21} + 2a_{22}t_{f_2} + 3a_{23}t_{f_2}^2 \]

- **Velocity at the mid point** between the intervals
  \[ \dot{\theta}[\text{Function1}(t = t_{f_1})] = \dot{\theta}[\text{Function2}(t = 0)] \]
  \[ a_{11} + 2a_{12}t_{f_1} + 3a_{13}t_{f_1}^2 = a_{21} \]
Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

• Acceleration at the mid point between the intervals

\[ \ddot{\theta}[\text{Function}1(t = t_f)] = \ddot{\theta}[\text{Function}2(t = 0)] \]

\[ 2a_{12} + 6a_{13}t_f = 2a_{22} \]

• Solve 8 equations with 8 unknown
Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Solution for the 8 equations

\[
\begin{align*}
a_{10} &= \theta_0 \\
a_{11} &= 0 \\
a_{12} &= \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2} \\
a_{13} &= \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3} \\

a_{10} &= \theta_v \\
a_{21} &= \frac{3\theta_g - 3\theta_0}{4t_f} \\
a_{22} &= \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2} \\
a_{23} &= \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}
\end{align*}
\]
Joint Space Schemes

Multiple Time Intervals
Via Point

System Defined Function – Linear Function With Parabolic Blend
Trajectory Generation – Roadmap Diagram

Joint Space

Single Time Interval

Polynomials

Multiple Time Intervals (Via Points)

User Defined Function

System Defined Function

1st Order Linear

3rd Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Heuristic

3rd Order Cubic

Linear Function with Parabolic Blend

Linear Function with Parabolic Blend

Task Space
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

• The Need
  – Linear path with parabolic blends is used in cases where there are **arbitrary number of via points specified**

• Method Anatomy
  – **Linear Functions** – Connecting the via points
  – **Parabolic Blend** – Connecting the linear functions around the via points
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend
Joint Space Schemes

Multiple Time Intervals
Via Point
System Defined Function – Linear Function With Parabolic Blend
Time Interval Analysis
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

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Mid Points \((j, k, l)\)

Given: \(\theta_j, \theta_k, \theta_l\)
\(t_{djk}, t_{dkl}\)

\(\ddot{\theta}_k\) Desired acceleration at each path point

\(- Velocity\)

\(\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}\)

\(\dot{\theta}_{kl} = \frac{\theta_l - \theta_k}{t_{dkl}}\)
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

- Acceleration

\[
\dddot{\theta}_k = \begin{cases} 0 & \text{if } \dot{\theta}_{kL} - \dot{\theta}_{jk} > 0 \\ \text{sgn} (\dot{\theta}_{kL} - \dot{\theta}_{jk}) \dddot{\theta}_k & \text{otherwise} \end{cases},
\]

\[
\dot{t}_k = \frac{\dddot{\theta}_k (\dot{\theta}_{kL} - \dot{\theta}_{jk})}{\dddot{\theta}_k}
\]

\[
t_{jk} = t_{dk} - \frac{1}{2} t_j - \frac{1}{2} t_k
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

- **Note: First and Last Segments** – The first and the last segments must be handled slightly differently because the entire bland region at one end of the segment must be counted in the total segment’s time duration.

- **Note the difference between**
  - Mid Points $t_{dj} \quad t_{dj}$
  - First Point $t_{d12}$
  - Last Point $t_{d(n-1)n}$

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Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

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Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

\[ t_{12} = \frac{A_2 - A_1}{td_{12} - \frac{1}{2} t_1} \]

\[ t_{1} = \frac{td_{12} + \sqrt{\left( \frac{2}{A_2} \right)^2 + 2 \left( \frac{2}{A_1} \right)} - \frac{2}{A_1}}{\frac{2}{A_2} - \frac{2}{A_1}} \]

Taking only the (-) sign since \( t_1 < td_{12} \)

\[ t_{12} = td_{12} - t_1 - \frac{1}{2} t_2 \]

\[ t_1 \rightarrow \text{Blending time} \]

\[ t_{12} \rightarrow \text{Straight line time} \]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

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Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

\[ t_n = t_{dn-1}n \pm \sqrt{t_{dn-1}^2 - \frac{2(\theta_{n-1} - \theta_n)}{\dot{\theta}_n}} \]

- Taking only the (-) sign since \( t_n < t_{dn-1} \)

\[ \theta_{n-1} = \frac{\theta_n - \theta_{n-1}}{t_{dn-1}n - \frac{1}{2}t_n} \]

\[ t_{n-1} = td(n-1) - t_n - \frac{1}{2}t_{n-1} \]
Joint Space Schemes

Multiple Time Intervals
Via Point
System Defined Function – Linear Function With Parabolic Blend
Linear & Parabolic Spline Analysis
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- Middle Segment

\[
\text{LINEAR SPLINE} \quad t = t_{jk}
\]

\[
\dot{\theta} = \dot{\theta}_{jk} \\
\ddot{\theta} = 0
\]

\[
\text{PARABOLIC SPLINE} \quad t \leq t_k
\]

\[
t_{\text{inb}} = t - (\frac{1}{2} t_{j} + t_{jk})
\]

\[
\dot{\theta} = \dot{\theta}_{j} + \dot{\theta}_{jk} (t - t_{\text{inb}}) + \frac{1}{2} \ddot{\theta}_{k} t_{\text{inb}}^2
\]

\[
\ddot{\theta} = \ddot{\theta}_{jk} + \dddot{\theta}_{k} t_{\text{inb}}
\]

\[
\dddot{\theta} = \dddot{\theta}_{k}
\]

Note:
\[
\dddot{\theta}_k = \frac{\dddot{\theta}_k - \dddot{\theta}_j}{t_{dj_k}}
\]

\[ \dot{\theta}_k - \text{max allowed acceleration(used to define } t_k) \]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- First Segment

\[
\text{LINEAR SPLINE } t \leq t_{12} \\
\theta = \dot{\theta}_{1} + \ddot{\theta}_{12} t \\
\dot{\theta} = \ddot{\theta}_{12} \\
\ddot{\theta} = 0
\]

\[
\text{PARABOLIC SPLINE } t = t_1 \\
\theta = a_0 + a_1 t + a_2 t^2
\]

Conditions:
\[
\begin{cases} 
\theta(t=0) = \theta_0 \\
\dot{\theta}(t=0) = \dot{\theta}_0 \\
\ddot{\theta}(t=t_1) = \ddot{\theta}_{12}
\end{cases}
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

\[
\begin{align*}
\theta &= a_o + a_1 t + a_2 t^2 \\
\dot{\theta} &= a_1 + 2a_2 t \\
\ddot{\theta} &= 2a_2
\end{align*}
\]

\[
\begin{align*}
\theta(t = t_0) = \theta_0 &= a_o + a_1 t_0 + a_2 t_0^2 \\
\dot{\theta}(t = t_0) = 0 &= a_1 + 2a_2 t_0 \\
\ddot{\theta}(t = t_0) = \ddot{\theta}_{t_0} &= 2a_2 t_0
\end{align*}
\]

\[
\begin{align*}
\dot{\theta} = \frac{\ddot{\theta}_{t_0}}{t_1} \frac{t_1}{t_1} t^2 \\
\ddot{\theta} = \frac{\dddot{\theta}_{t_0}}{t_1} \frac{t_1}{t_1} \\
\theta = \frac{\ddot{\theta}_{t_0}}{t_1} \frac{t_1}{t_1} t
\end{align*}
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- Last Segment

**Linear Spline** \( t < t_{n-1} \)

\[
\theta = \theta_{n-1} + \dot{\theta}_{n-1} t \\
\ddot{\theta} = \ddot{\theta}_{n-1} \\
\theta' = 0
\]

**Parabolic Spline**

\[
t_{mb} = t - \left( \frac{1}{2} t_{n-1} + t_{n-1} \right)
\]

\[
\theta = \theta_0 + \alpha_1 t + \alpha_2 t^2
\]

Conditions:

\[
\left\{ \begin{array}{l}
\theta (t_{mb} = 0) = \theta_{n-1} + \dot{\theta}_{n-1} t_{mb} = \dot{\theta}_{mb} \\
\dot{\theta} (t_{mb} = 0) = \ddot{\theta}_{n-1} \\
\ddot{\theta}(t_{mb}) = 0
\end{array} \right.
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

\[
\begin{align*}
\theta &= a_0 + a_1 t + a_2 t^2 \\
\dot{\theta} &= a_1 + 2a_2 t \\
\ddot{\theta} &= 2a_2
\end{align*}
\]

\[
\begin{align*}
\theta(t_{inb} = 0) &= \dot{\theta}_{inb} = a_0 + a_1 t + a_2 t^2 \\
\dot{\theta}(t_{inb} = 0) &= \ddot{\theta}_{inb} = a_1 + 2a_2 t \\
\dot{\theta}(t = t_{inb}) &= 0 = \ddot{\theta}(n-1) + 2a_2 t_{inb} \\
\theta &= \theta_{inb} + \dot{\theta}_{inb} t_{inb} - \frac{1}{2} \ddot{\theta}_{inb} t_{inb}^2
\end{align*}
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Calculation Approaches

- **Calculation Approach No. 1**
  - User Defines
    - Via Points
    - Desired time duration of segments
  - System Defines
    - Use default value of acceleration for each joint

- **Calculation Approach No. 2**
  - System calculate time durations based on default velocities

- **Note for both Approaches** – At all the blends, sufficiently large acceleration must be used so that the system has sufficient time to get into the linear portion of the segment before the next blend region starts
The trajectory of a particular joint is specified as follows:

**Given:**

\[ \theta \text{[deg]} \]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
10 & 20 & 30 & 35 & 40 & 50 \\
15 & 25 & 35 & & & \\
10 & 20 & 30 & & & \\
\end{array}
\]

\[ t_{dev} = 2 \quad t_{t1} = 1 \quad t_{d3} = 3 \quad t_{d3} = 3 \]

Example
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Acceleration - Magnitude of the default acceleration to use at all blend points is

\[ \| \dot{\theta} \|_{\text{max}} = 50 \text{ deg/sec}^2 \]

CALCULATE: - Segment velocities \( \dot{\theta} \)
- Blend times \( t_i (t_1, t_2, t_3, t_4) \)
- Linear segments times

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- Solution Step 1 – Time Interval Analysis

First Segment (t: 0 → 2)

\[ \dot{q}_1 = 50 \]

\[ t_1 = t_{d12} - \frac{v^2}{\ddot{q}_1} = 2 - \frac{2(35 - 10)}{50} = 0.27 \text{ sec} \]

\[ \dot{q}_{12} = \frac{q_2 - q_1}{t_{d12} - \frac{t_1}{2}} = \frac{35 - 10}{2 - 1(0.27)} = 13.50 \text{ deg/s} \]
Second Segment (t: 2→3)

$t_{12} = t_{d12} - t_1 - \frac{1}{2} t_2 = 2 - 0.27 - \frac{1}{2}(0.47) = 1.5$
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Third Segment (t: 3 → 6)
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Third Segment (t: 3→6) Cont.
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Third Segment (t: 3→6) Cont.

\[ t_{jk} = t_{dj}k - \frac{1}{2} t_j - \frac{1}{2} t_k = t_{23} = t_{d2} - \frac{1}{2} t_e - \frac{1}{2} t_5 = 1 - \frac{1}{2} (0.47) - \frac{1}{2} (0.098) = 0.716 \text{ sec} \]

\[ t_{(n-1)n} = t_{dn} - \frac{1}{2} t_n = t_{34} = t_{d3} - \frac{1}{2} t_5 = \]

\[ n = 5 \]

\[ = 1 - 0.192 - \frac{1}{2} (0.098) = 2.849 \text{ sec} \]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear / parabolic Splines – Summary Reminder

\[ \theta = \dot{\theta}_1 + \ddot{\theta}_1 t \]
\[ \theta = \theta_0 + \frac{1}{2} \ddot{\theta}_1 t^2 \]
\[ \theta = \dot{\theta}_J + \ddot{\theta}_J t \]
\[ \theta = \dot{\theta}_J + \ddot{\theta}_J (t - t_{inb}) + \frac{1}{2} \dddot{\theta}_J t_{inb} \]
\[ t_{inb} = t \left( \frac{1}{2} t_J + t_Jk \right) \]
\[ \theta = \dot{\theta}_{n-1} + \ddot{\theta}_{n-1} t \]
\[ \theta = \dot{\theta}_{inb} + \ddot{\theta}_{inb} t - \frac{1}{2} \dddot{\theta}_{inb} t_{inb}^2 \]
\[ t_{inb} = t - \left( \frac{1}{2} t_{(n-1)} + t_{(n-1)m} \right) \]
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear Polynomial

\[
\text{Linear Segments}
\]

\[
\begin{align*}
\text{II} & : \theta = 10 + 13.5t \\
\text{IV} & : \theta = 35 - 10t \\
\text{VI} & : \theta = 25 - 5.1t \\
\end{align*}
\]

\[
\theta_{in} = 25 - 5.1t \left( \frac{0.098}{2} + 2.849 \right)
\]
Joint Space Schemes – Multiple Time Intervals – Via Points – 
Linear Function With Parabolic Blend – Example

Parabolic Blend

\[
\begin{align*}
\text{I} & \quad \theta = 10 + \frac{1}{2} \left( \frac{12.5}{0.27} \right) t^2 \quad (t \leq 1.85) \\
\text{II} & \quad \theta = 10 + 13.5 (t - t_{ib}) + \frac{1}{2} (50)^2 t_{ib} \\
& \quad t_{ib} = t \left( \frac{1}{2} \cdot 0.27 + 1.5 \right) \\
\text{III} & \quad \theta = 35 - 10 (t - t_{ib}) + \frac{1}{2} (50)^2 t_{ib} \\
& \quad t_{ib} = t \left( \frac{1}{2} \cdot 0.47 + 0.716 \right) \\
\text{IV} & \quad \theta = t_{ib} + 35 t - \frac{1}{2} \left( \frac{12.5}{t_{ib}} \right) t^2 \quad (t \geq 1.85) \\
& \quad t_{ib} = t - \left( \frac{1}{2} \cdot 0.98 + 2.849 \right)
\end{align*}
\]
Joint Space Schemes

Multiple Time Intervals

Pseudo Via Point
Joint Space Schemes – Multiple Time Intervals –
Pseudo Via Points

• Problem:
  – In the linear parabolic blend spline, note that the via points are not actually reached unless the manipulator come to a stop
  – Often when the acceleration is sufficiently high the path will come quite close to the desire via point

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Joint Space Schemes – Multiple Time Intervals –
Pseudo Via Points

• Solution
  – Define Pseudo via Points - The system automatically replace the via point through which we wish the manipulator to pass through with two pseudo via points one on each side of the original.
  – The original via point will now lie in the linear region of the path connecting the two pseudo via points
  – Define Velocity at the original Via Points - In addition to requesting that the manipulator pass exactly through the via point, the user can also request that it pass through with a certain velocity. If the user does not specify this velocity the system chooses it by means of suitable heuristic
  – Define Through Point - Through Point rather than via point is used to specify a path through which we force the manipulator to pass exactly through