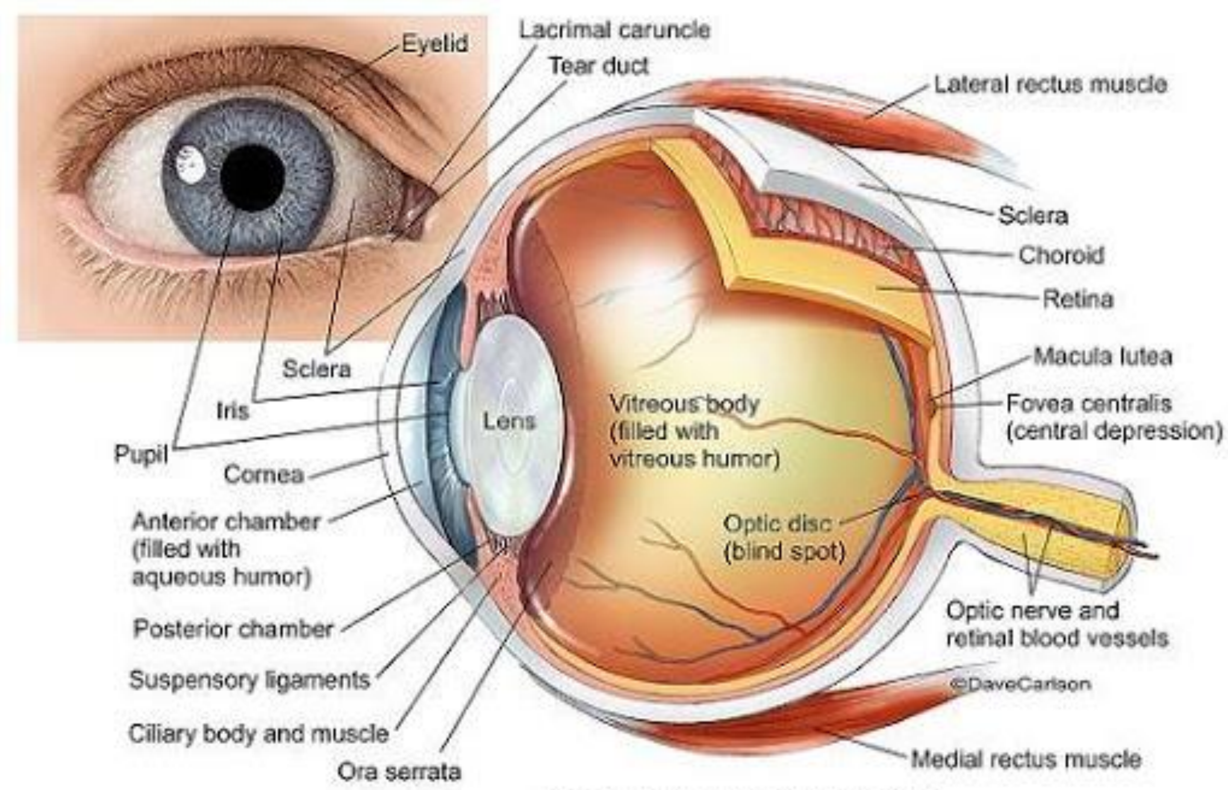
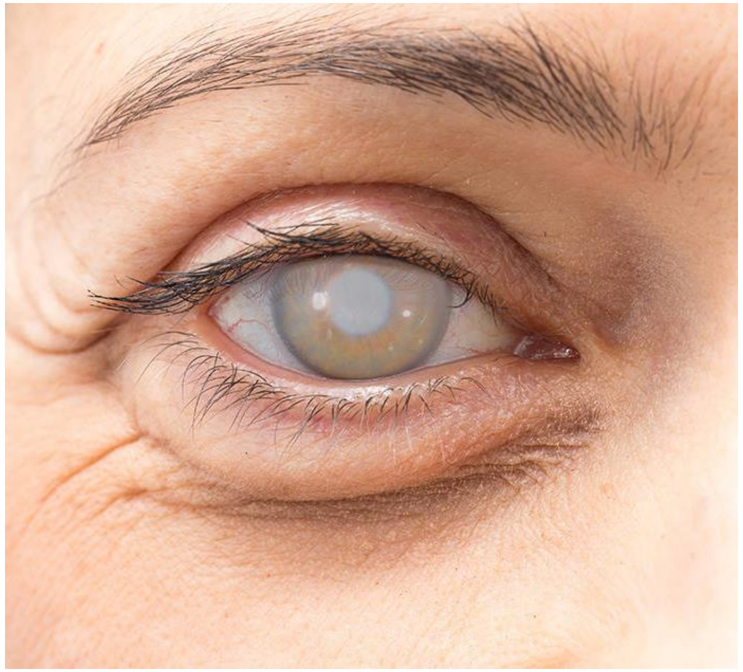
# Trajectory Generation (1/2)

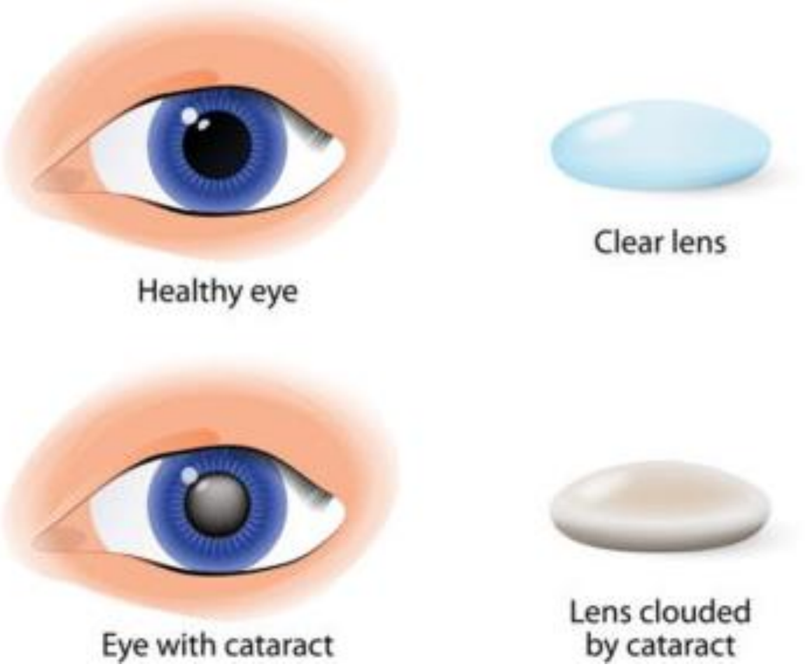
# Introduction




Right Eye (viewed from above)

# Motion Planning





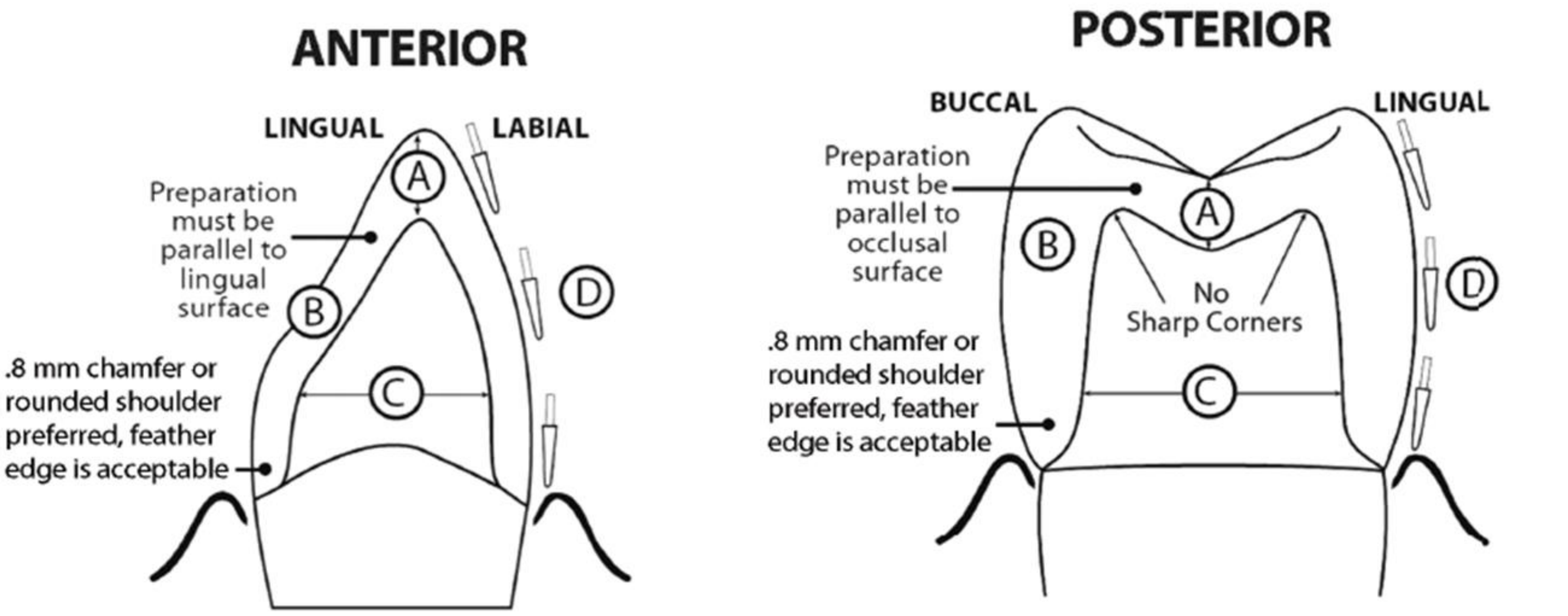


Routine Cataract Surgery

https://youtu.be/QbeI72QmFAU

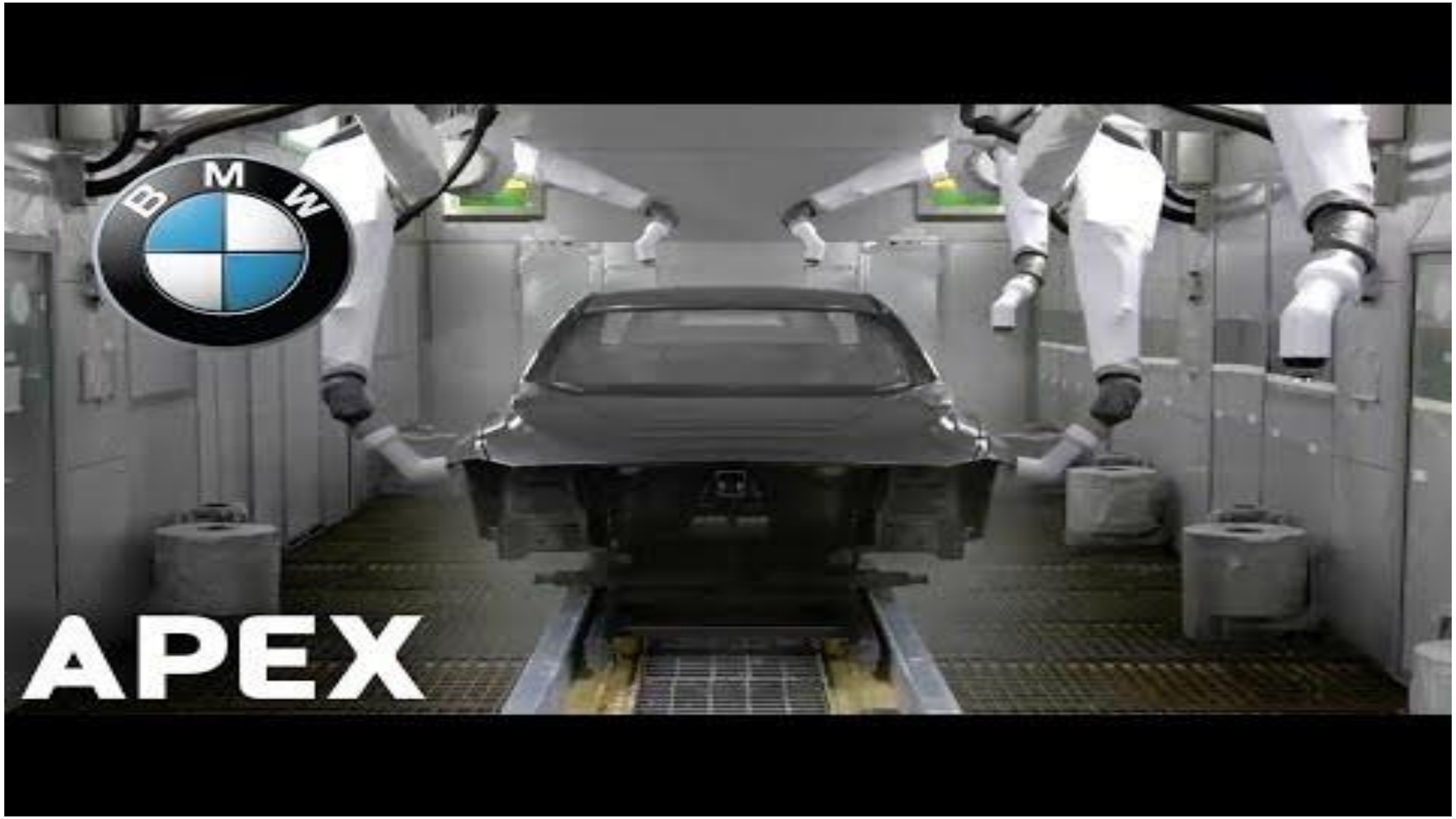
# Motion Planning

# Motion Planning




A. 1.0 mm - 1.5 mm occlusal reduction
B. 1.0 mm middle third reduction
C. Buccal and lingual walls must be convergent
D. Preparation should be cut in three planes

# Motion Planning



# Motion Planning

# Motion Planning

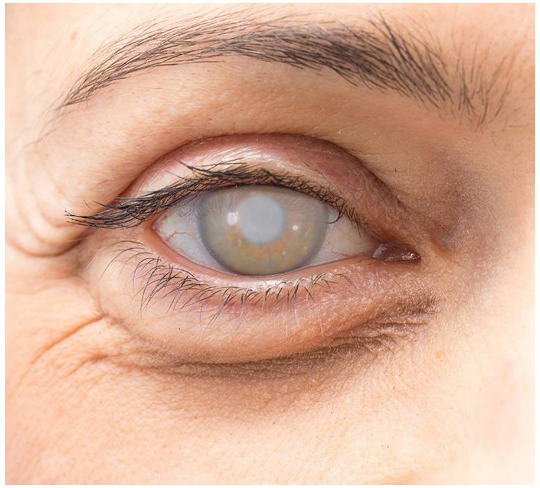
Problem Defenition

# Motion Planning – Hierarchy

- Trajectory planning is a subset of the overall problem that is ***navigation or motion planning***. The typical hierarchy of motion planning is as follows:

  - **Task planning** – Designing a set of high-level goals, such as "go pick up the object in front of you".
  - **Path planning** – Generating a feasible path from a start point to a goal point. A path usually consists of a set of connected waypoints.
  - **Trajectory planning** – Generating a time schedule for how to follow a path given constraints such as position, velocity, and acceleration.
  - **Trajectory following** – Once the entire trajectory is planned, there is a need for a control system that can execute the trajectory in a sufficiently accurate manner.

- Q: What's the difference between path planning and trajectory planning?
- A: A trajectory is a description of how to follow a path over time



# Trajectory Generation – Problem Definition

**Problem**

Given: *Manipulator geometry, End Effector Path (via point)*

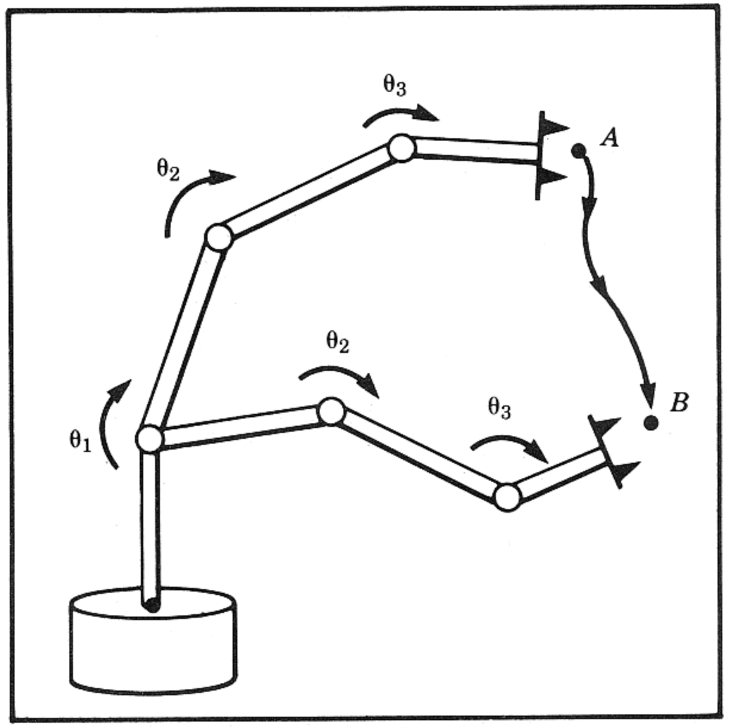
Compute: *The trajectory of each joint such that the end effector move in space from point A to Point B*

**Solution (Domains)**

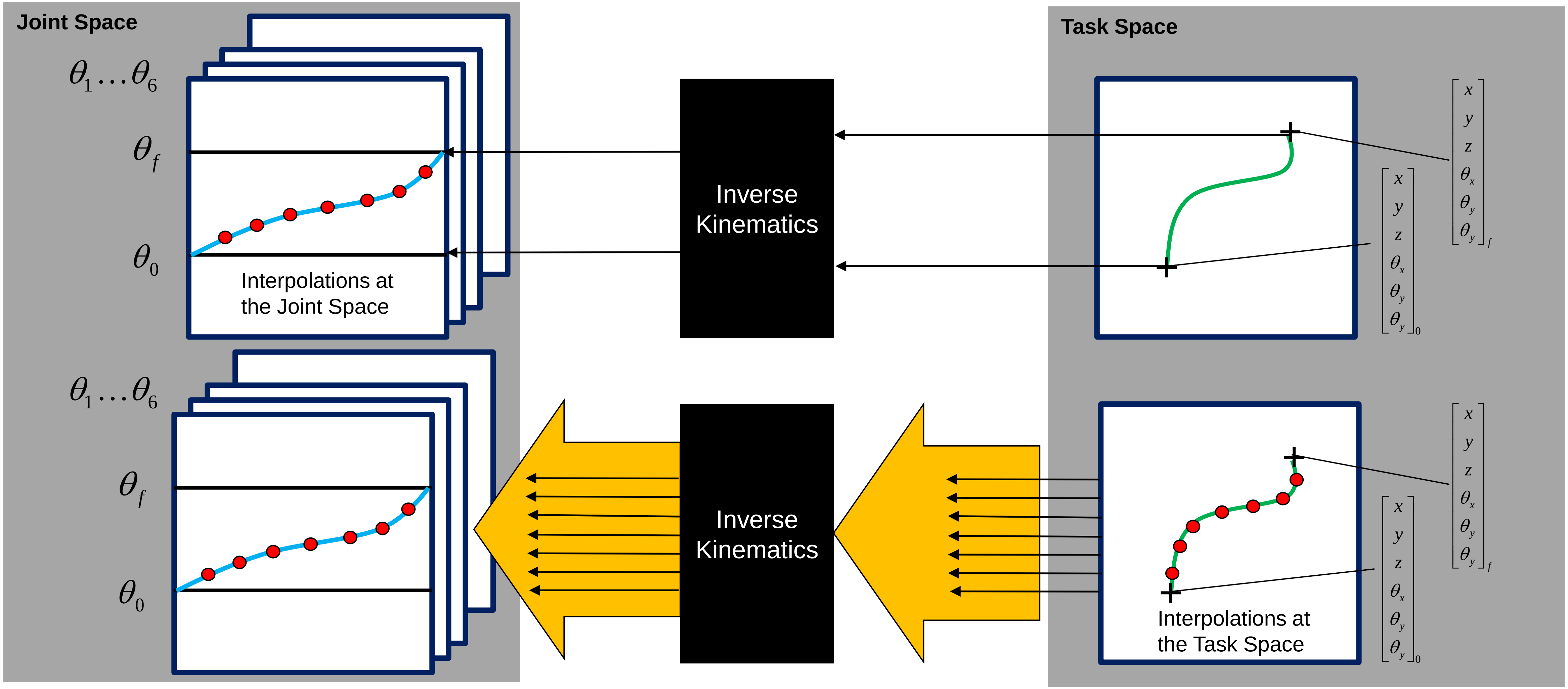
- **Joint space / Task Space**

**Definitions**

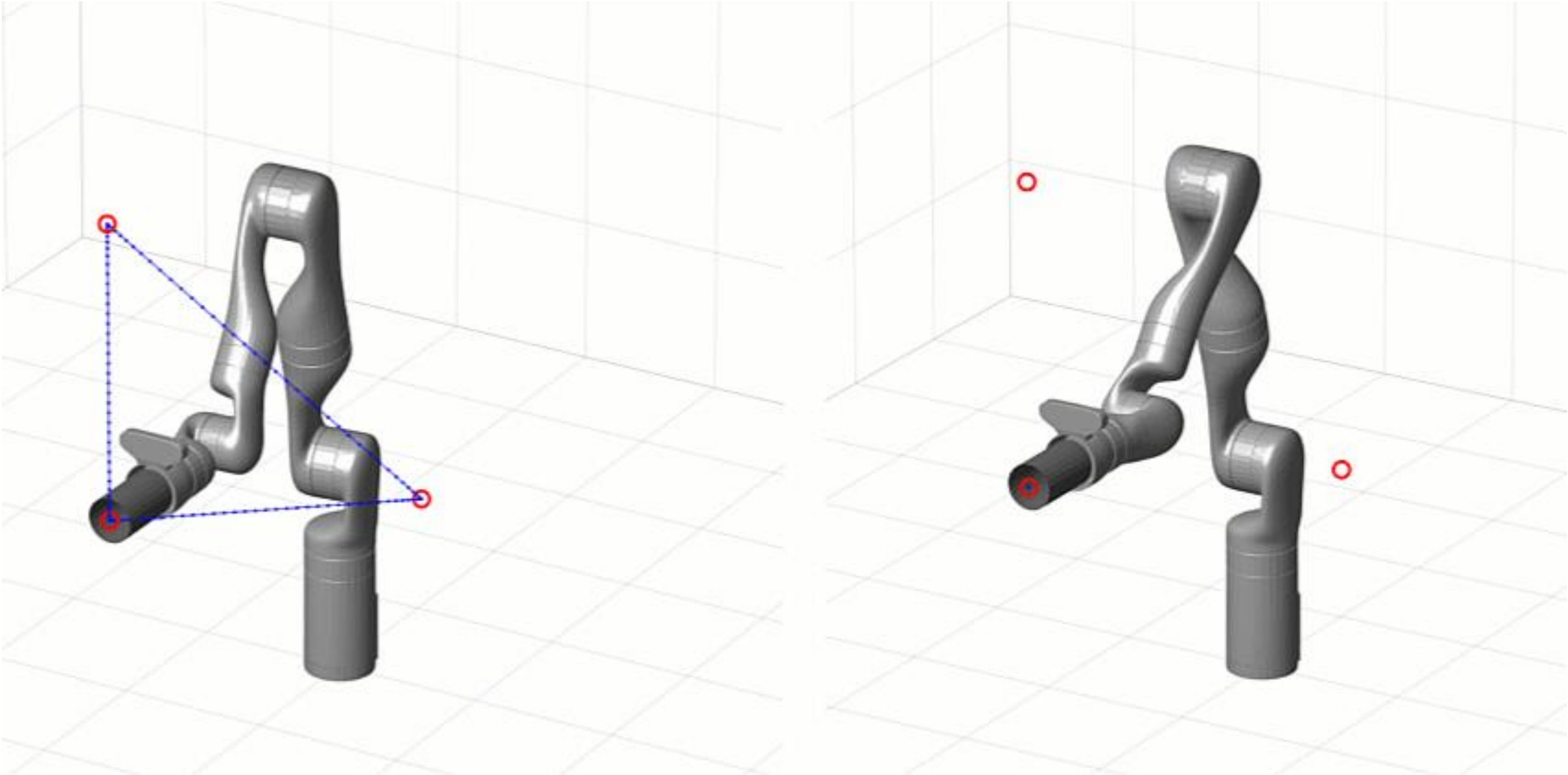
- **Trajectory (Definition)** - Time history of position, velocity, and acceleration for each DOF.
- **Trajectory Generation –** Methods of computing a trajectory that describes the desired motion of a manipulator in a multidimensional space



# Task Space Versus Joint Space - Interpolations



# Task Space Versus Joint Space



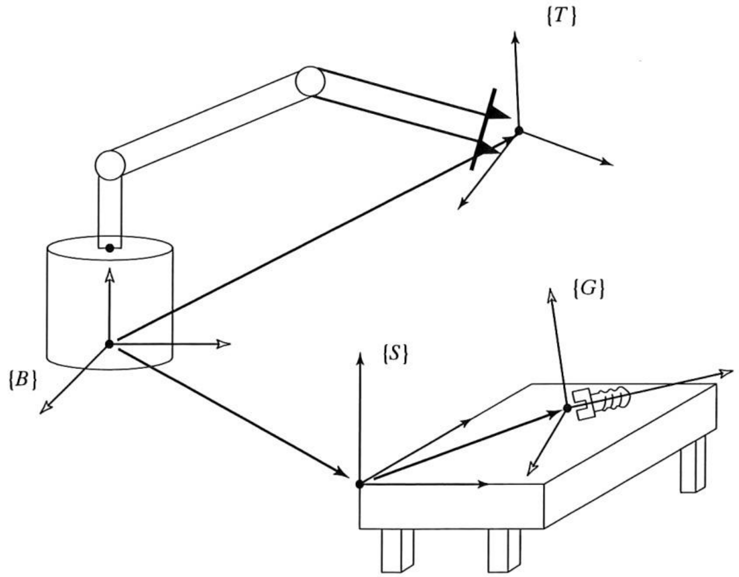
# Join Space Versus Task Space – Comparison

| Parameter | Joint Space | Task Space |
|---|---|---|
| Interpolation Space<br>intermediate points along the trajectory | Joint Space | Task Space |
| Tool Trajectory Type / Length | Curved Line / Long | Straight Lines / Short |
| Invers Kinematics (IK) Usage | Low | High |
| Computation Expense (IK) | Low<br>(IK for Start/Finish & Via Points ) | High<br>(IK for every single point / time steo on the trajectory) |
| Passing through Via Points | No<br>(Correction by establishing Pseudo Points) | Yes |
| Via Points Defined in the Task Space | No | Yes |
| Path Dependency on a Specific Manipulator | Yes | No |

# General Consideration

- **General approach for the motion of the manipulator**
  - Specify the path as a motion of the tool frame {*T*} relative to the station frame {*S*}. Frame {*G*} may change it position in time (e.g. conveyer belt)

- **Advantages**
  - Decouple the motion description from any particular robot, end effector, or workspace.
  - Modularity – Use the same path with:
    - Different robot
    - Different tool size

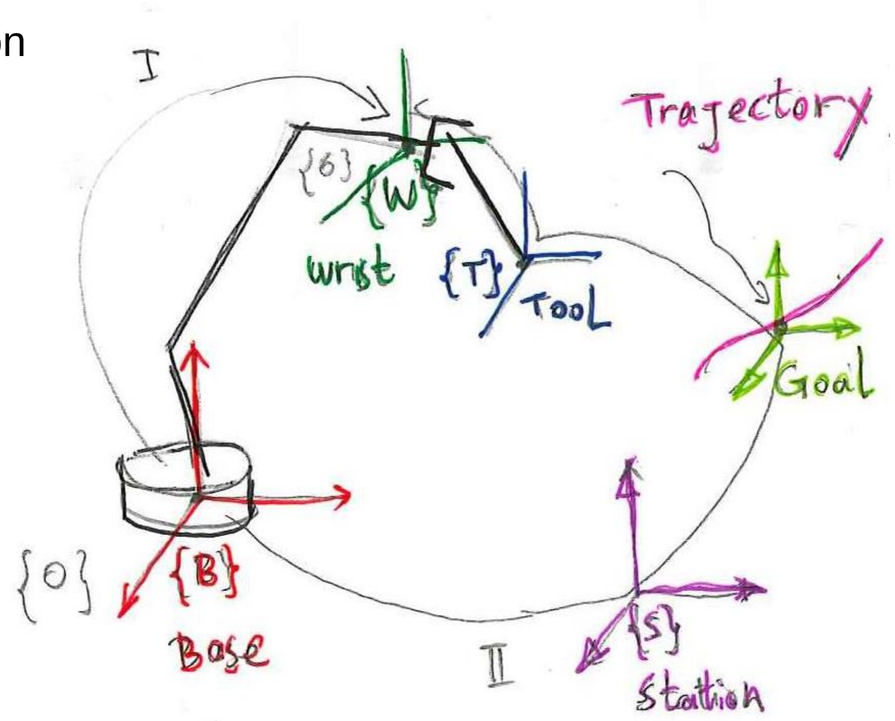


# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

- **There are two pathways from the base frame {B} to the wrist frame {W} (Path I and Path II)**

$$ {}^{0}_{6}T = \underbrace{{}^{B}_{W}T}_{\mathrm{I}} = \underbrace{\boxed{{}^{B}_{S}T\,{}^{S}_{G}T\,{}^{G}_{T}T\,{}^{T}_{W}T}}_{\mathrm{II}} $$

II → Desired position and orientation




${}^{0}_{6}T = {}^{B}_{W}T$ - Forward kinematics

${}^{B}_{S}T$ - Location of the station with respect to the base

${}^{S}_{G}T$ - Definition of the trajectory (Goal Location - Known)

$${}^{G}_{T}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ - Full alignment (position, orientation)

${}^{T}_{W}T$ - Location of the wrist with respect to the tool → usually given ${}^{W}_{T}T$

# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

User Defined

The Trajectory

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

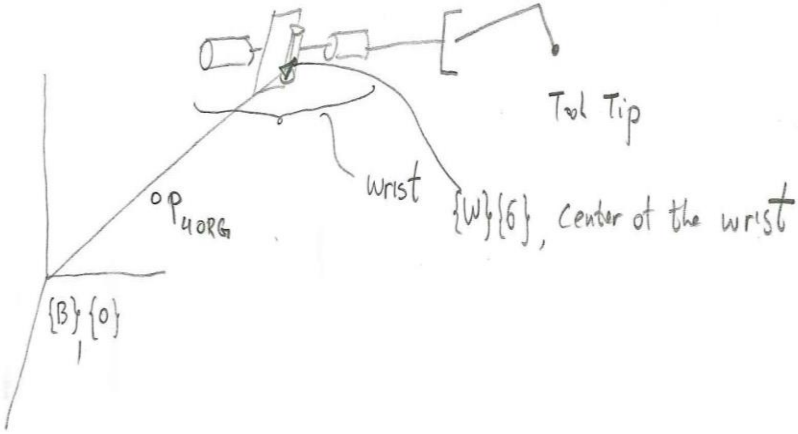
Usually given ${}^{W}_{T}T = {}^{6}_{T}T$ (Invert ${}^{W}_{T}T = \left({}^{W}_{T}T\right)^{-1}$)

Known

$${}^{B}_{W}T = {}^{0}_{6}T = \boxed{{}^{B}_{S}T\ {}^{S}_{G}T\ {}^{G}_{T}T\ {}^{T}_{W}T} = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T\ {}^{4}_{5}T\ {}^{5}_{6}T$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

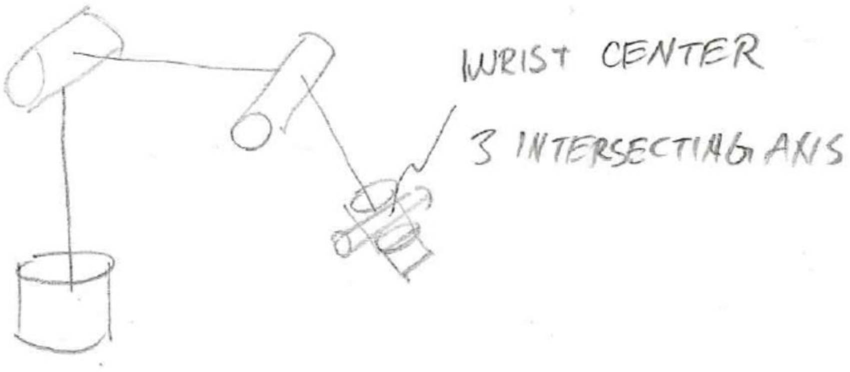
$$\theta_1 \quad \theta_2 \quad \theta_3 \quad \theta_4 \quad \theta_5 \quad \theta_6$$



# Transforming from task space to joint space - General Approach

$$
{}^{0}_{6}T = \underset{\theta_1}{\underset{\downarrow}{{}^{0}_{1}T}}\ \underset{\theta_2}{\underset{\downarrow}{{}^{1}_{2}T}}\ \underset{\theta_3}{\underset{\downarrow}{{}^{2}_{3}T}}\ \underset{\theta_4}{\underset{\downarrow}{{}^{3}_{4}T}}\ \underset{\theta_5}{\underset{\downarrow}{{}^{4}_{5}T}}\ \underset{\theta_6}{\underset{\downarrow}{{}^{5}_{6}T}} = \underset{\text{Given}}{\boxed{\left[\begin{array}{c|c} {}^{0}_{6}R & {}^{0}P_6 \\ \hline 0 & 1 \end{array}\right]}}
$$

Given 3 intersecting axis 4, 5, 6 (origins of 4, 5, 6 are at the same point)




$$
{}^{0}P_6 = {}^{0}P_4
$$



- **Solving Problem No. 1** - Solving for $\theta_1, \theta_2, \theta_3$
  - Given 3 intersecting axis 4, 5, 6 (origins of 4, 5, 6 are at the same point)

$$ {}^0P_6 = {}^0P_4 $$

  - Note that ${}^0P_6$ and ${}^0P_4$ are a function of $\theta_1, \theta_2, \theta_3$

$$ {}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T = \begin{bmatrix} {}^0_6R & {}^0P_6 \\ 0 & 1 \end{bmatrix} = \boxed{{}^B_ST\,{}^S_GT\,{}^G_TT\,{}^T_WT = {}^B_WT} $$

$$ \begin{bmatrix} {}^0_4R & \boxed{{}^0P_4} \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} {}^B_WR & \boxed{{}^BP_W} \\ 0 & 1 \end{bmatrix}} $$

# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

$$ {}^{0}_{6}T = \boxed{\underbrace{{}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T}_{}} \ | \ \boxed{{}^{3}_{4}T} \ | \ \boxed{\underbrace{{}^{4}_{5}T\ {}^{5}_{6}T}_{}} $$

Problem No. 1
**Position Problem**

Problem No. 2
**Orientation Problem**

$$ \{\text{Frame } 3\} \underset{R_{x3}(\alpha_3)}{\overset{\alpha_3}{\rightarrow}} \{\text{Frame}(R)\} \underset{D_{x3}(a_3)}{\overset{a_3}{\rightarrow}} \{\text{Frame}(Q)\} \underset{R_{z4}(\theta_4)}{\overset{\theta_4}{\rightarrow}} \{\text{Frame}(P)\} \underset{D_{z4}(d_4)}{\overset{d_4}{\rightarrow}} \{\text{Frame}(4)\} $$

# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

$${}^0_6T = \boxed{\underbrace{{}^0_1T\;{}^1_2T\;{}^2_3T\,R_{x3}(\alpha_4)\,D_{x3}(a_3)}}\;\boxed{\underbrace{R_{z4}(\theta_4)\,D_{z4}(d_4)\,{}^4_5T\;{}^5_6T}}$$

Problem No. 1
Position Problem

Problem No. 2
orientation problem

$${}^0_6T = \boxed{{}^0_1T\;{}^1_2T\;{}^2_3T\,{}^3_4T\,\Big|_{\theta_4=0}}\;\boxed{\Big|\,{}^3_4T\,\Big|_{\alpha_4=0}\;{}^4_5T\,{}^5_6T}$$

$\uparrow$ Set $\theta_4 = 0$

$\uparrow$ Set $\alpha_4 = 0$

# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

$$ {}^0_6R = {}^0_3R\;{}^3_4R\;{}^4_6R $$

$$ {}^0_6R = {}^0_3R(R(\alpha_3)I\;\;R(\theta_4)I){}^4_6R $$

$$ D_{x3}(a_3) \qquad\qquad D_{z4}(d_4) $$

$$ {}^0_6R = [{}^0_6RR_{x3}(\alpha_3)]\;[R_{z4}(\theta_4){}^4_6R] $$

Multiply both sides of the equation by $[{}^0_6RR_{x3}(\alpha_3)]^{-1}$ results in

$$ R_{z4}(\theta_4){}^4_6R = [{}^0_3RR_{x3}(\alpha_3)]^{-1}\quad{}^0_6R $$

# Trajectory Generation & Inverse Kinematics
## Transforming from task space to joint space - General Approach

- **Solving Problem No. 2** - Solving for $\theta_4, \theta_5, \theta_6$

$$\underbrace{\boxed{R_{z4}(\theta_4)\,{}^{4}_{6}R}}\; = \;\boxed{\underbrace{\left[{}^{0}_{3}R\,R_{x3}(\alpha_3)\right]^{-1}}_{\substack{\text{Solved in Problem 1}\\ \text{Known } \theta_1,\theta_2,\theta_3}}\;\underbrace{{}^{0}_{6}R}_{\substack{\text{Desired orientation given for}\\ \text{every point on the trajectory}}}}$$

$$\downarrow$$

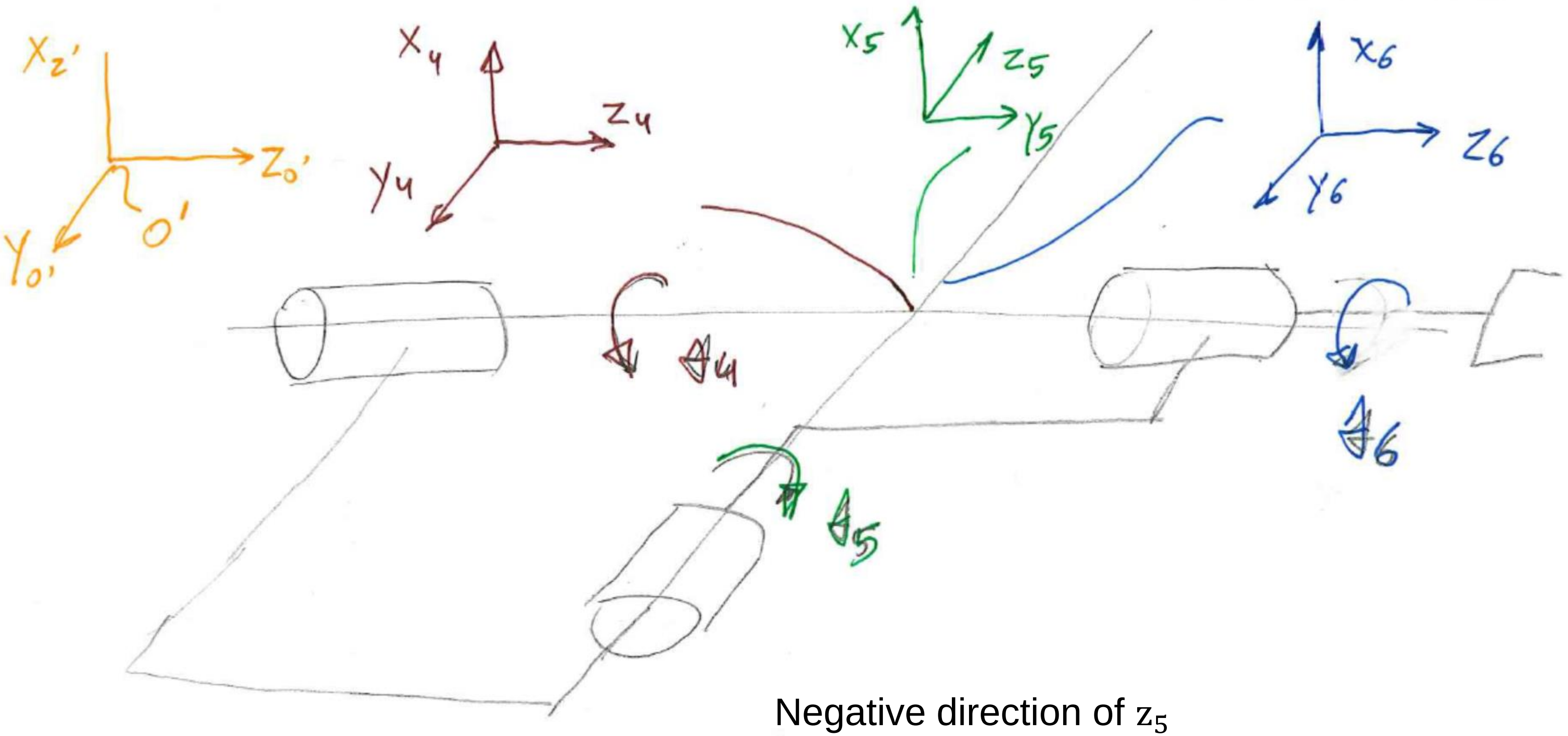
$$\boxed{R_{z4}(\theta_4)\,{}^{4}_{5}R(\theta_5)\,{}^{5}_{6}R(\theta_6)} = \boxed{\overline{R_D}} \rightarrow$$

Desired orientation of the wrist taking into account the contribution of the first 3 angles to the orientation

$$\boxed{R_{z4}(\theta_4)\,{}^{4}_{5}R(\theta_5)\,{}^{5}_{6}R(\theta_6)} = \boxed{\overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}^{\overline{R_D}}}$$

  - Solve for $\theta_4, \theta_5, \theta_6$ using the Z-Y-Z problem

# Trajectory Generation & Inverse Kinematics
# General Approach




Negative direction of $z_5$

$$R_{Euler} = R_z(\theta_4) R_{y4}(-\theta_5) R_{z4}(\theta_6)$$

# Trajectory Generation & Inverse Kinematics
## General Approach

$$R_{ZYZ}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\theta_4 \quad -\theta_5 \quad \theta_6$

$$R_{ZYZ}(\alpha,\beta,\gamma) = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\beta = \mathrm{Atan2}\,(\sqrt{r_{31}^2 + r_{32}^2}\,, r_{33})$$

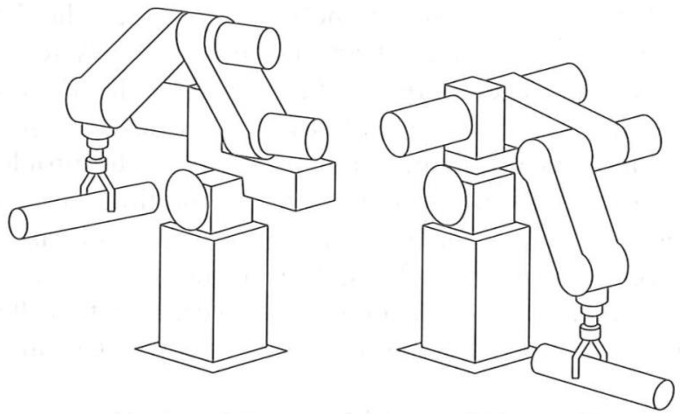
$$\alpha = \mathrm{Atan2}\,(\,r_{23}/s\beta\,,\ r_{13}/s\beta\,)$$

$$\gamma = \mathrm{Atan2}\,(r_{32}/s\beta\,)\,, -r_{31}/s\beta\,)$$

# General Consideration – Via Points

- **Basic Problem –** Move the tool frame {T} from its initial position / orientation {T_initial} to the final position / orientation {T_final}.

- **Specific Description**
  - **Via Point** – Intermediate points between the initial and the final end- effector locations that the end-effector mast go through and match it position and orientation along the trajectory.
  - Each via point is defined by a **frame defining the position/orientation** of the tool with respect to the station frame
  - **Path Points** – includes all the via points along with the initial and final points
  - **Point (Frame)** – Every point on the trajectory is define by a frame (spatial description)

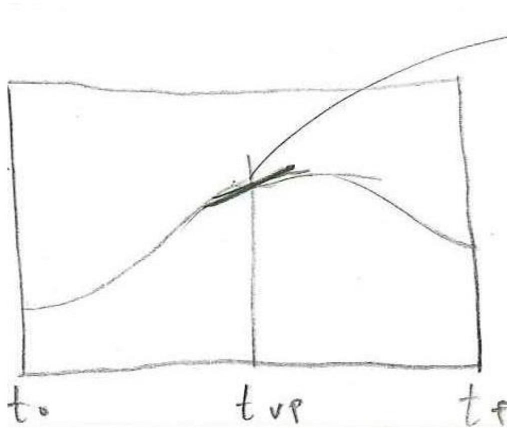


# General Consideration – Smooth Path

- **“Smooth” Path or Function**
  - Continuous path / function with first and second derivatives.
  - Add constrains on the spatial and temporal qualities of the path between the via-points

- **Implications of non-smooth path**
  - Increase wear in the mechanism (rough jerky movement)
  - Vibration – exciting resonances.

- **Example of smooth path**
  - The joint angle ($\theta$), joint velocity ($\dot{\theta}$), and joint acceleration ($\ddot{\theta}$) at the entry and exit of the via point are equal.



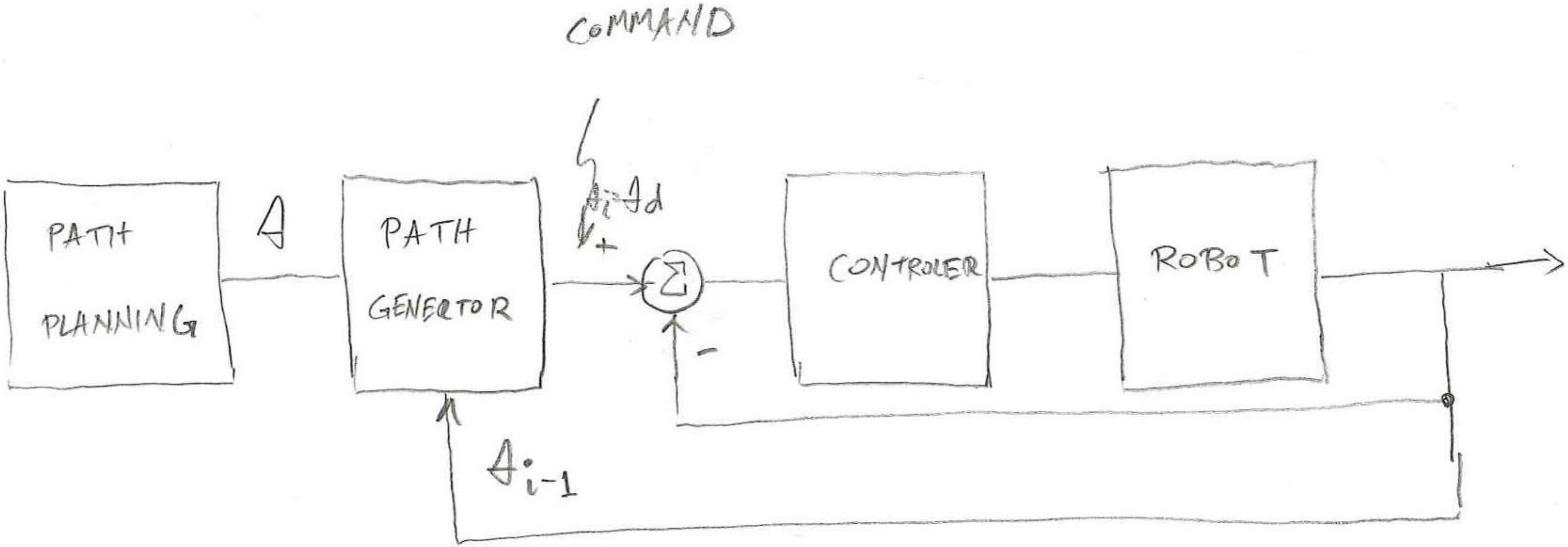

<u>Via Point</u>

$$\theta_{tvp-} = \theta_{tvp+}$$

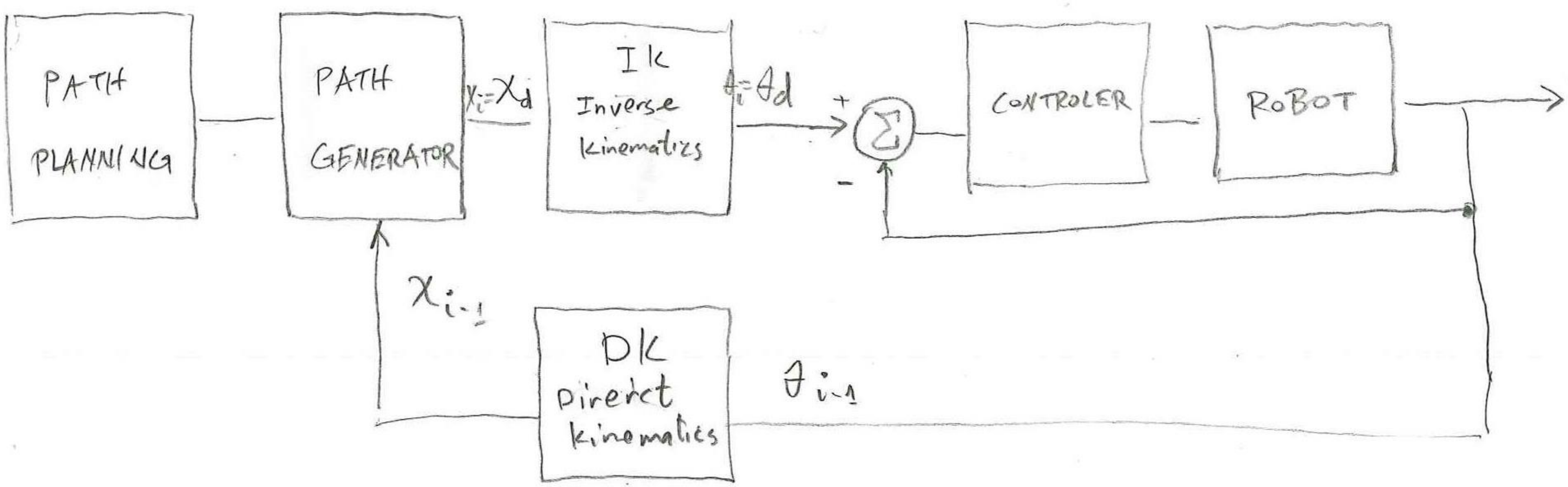
$$\dot{\theta}_{tvp-} = \dot{\theta}_{tvp+}$$

$$\ddot{\theta}_{tvp-} = \ddot{\theta}_{tvp+}$$

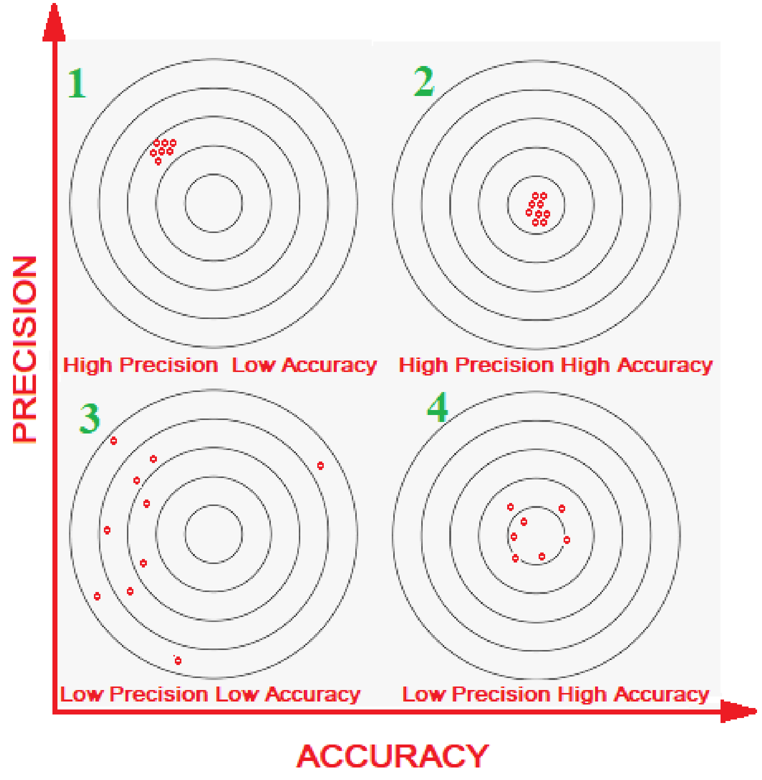
# Trajectory Generation – Joint Space Control

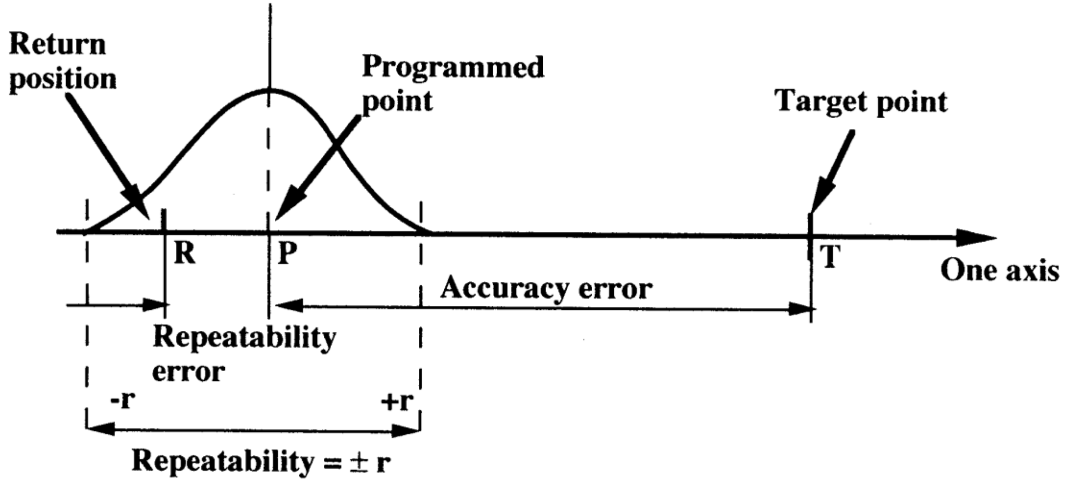


# Trajectory Generation – Task Space Control

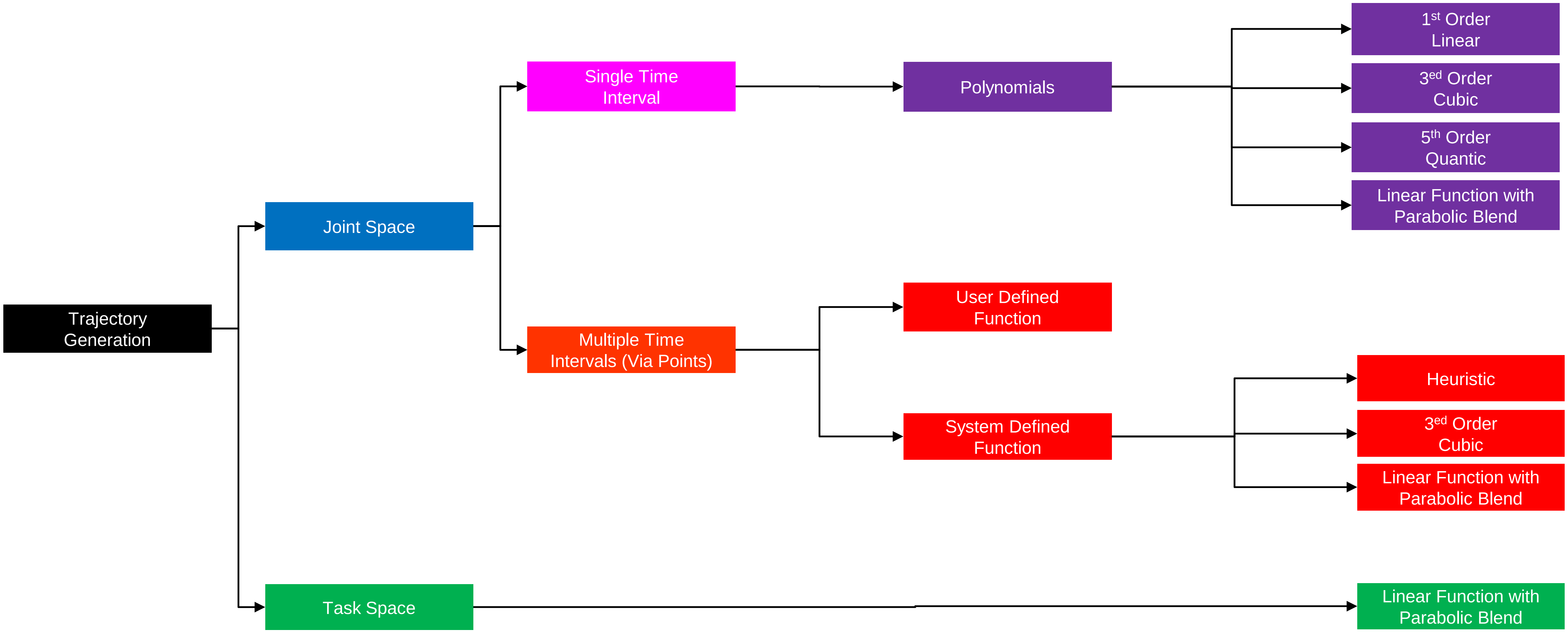


# Precision / Repeatability versus Accuracy



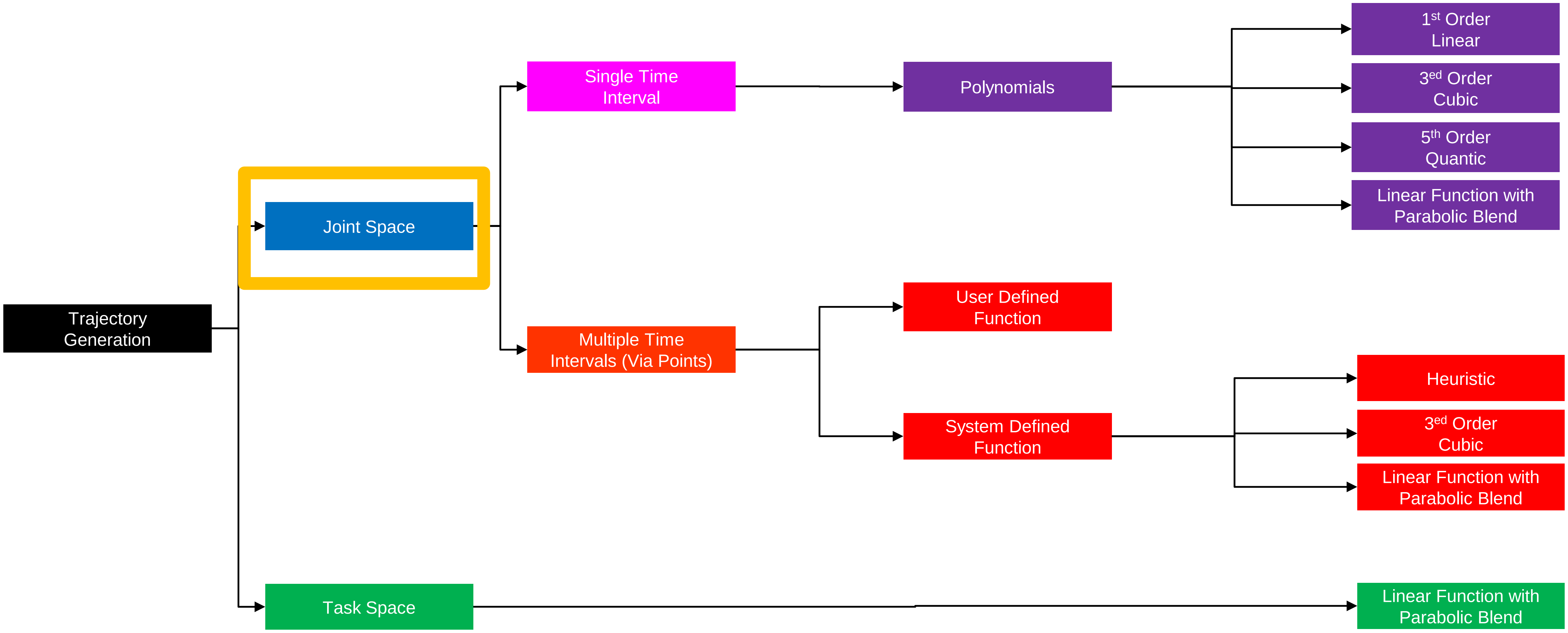
# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes

Single Time Interval

# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes

- **Joint space Schemes** – Path shapes (in space and in time) are described in terms of functions in the joint space.

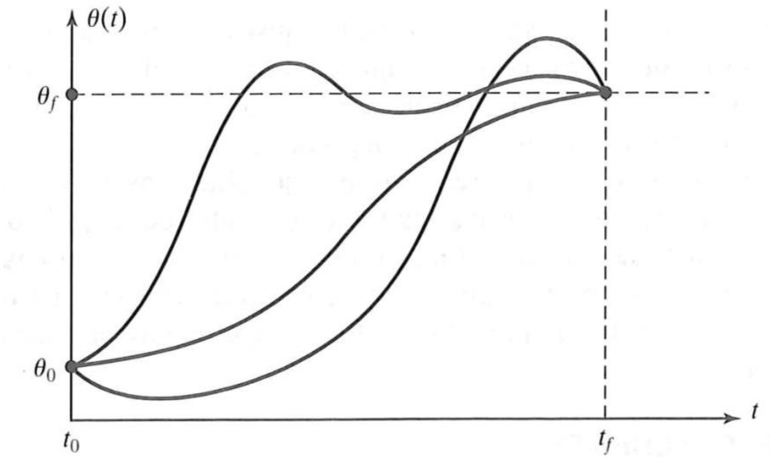
- **General process (Steps) given initial and target P/O**
  1. Select a path point or via point (desired position and orientation of the tool frame {T} with respect to the base frame {s})
  2. Convert each of the "via point" into a set of joint angles using the invers kinematics
  3. Find a smooth function for each of the *n* joints that pass trough the via points, and end the goal point.

  **Note 1:** The **time required to complete each segment is the same for each joint** such that the all the joints will reach the via point at the same time. Thus resulting in the position and orientation of the frame {T} at the via point.

  **Note 2**: The **joints move independently** with only one time restriction (Note 1)

- Define a function for each joint such that value at $t_0$ is the **initial position** of the joint and whose value at $t_f$ is the **desire goal position** of the joint

- There are many smooth functions $\theta(t)$ that may be used to interpolate the joint value.



# Joint Space Schemes

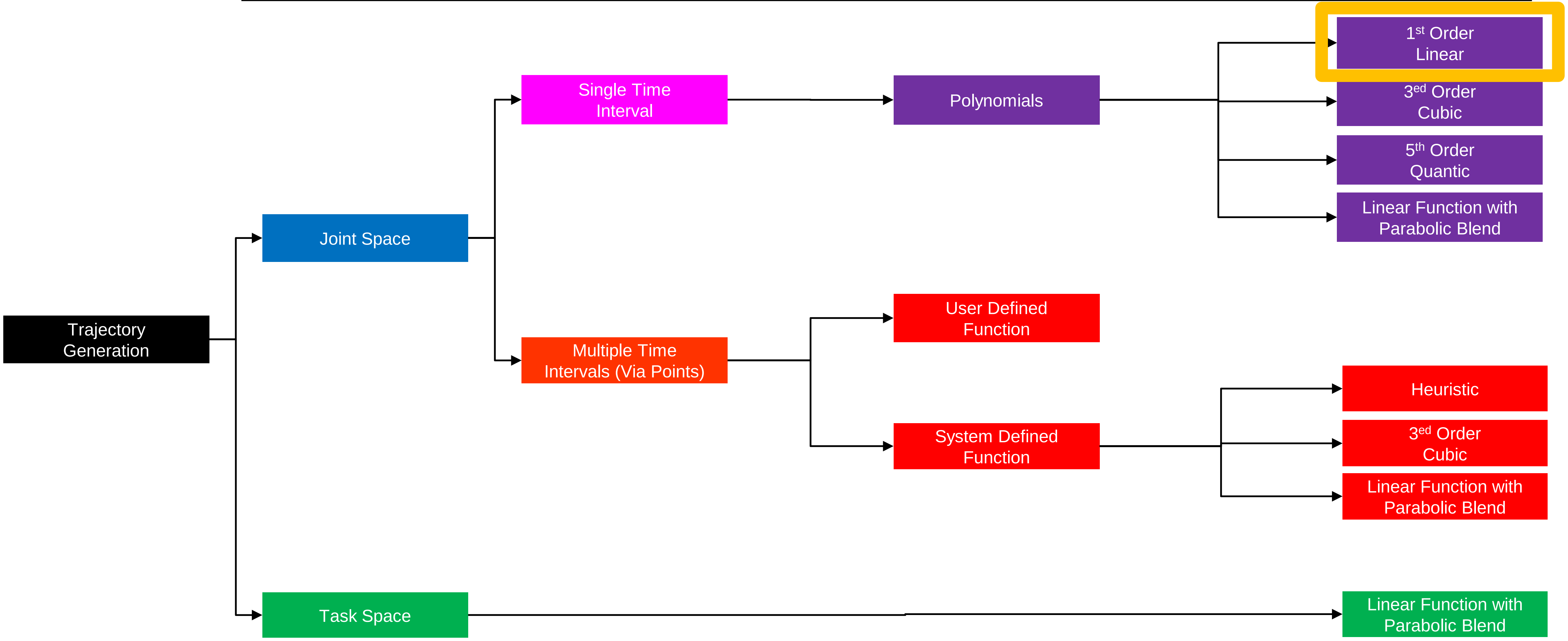
Single Time Interval

Polynomials

First Order Polynomial

# Joint Space Schemes

# Trajectory Generation – Roadmap Diagram



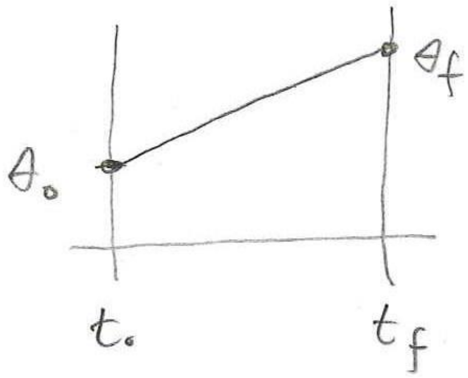
# Joint Space Schemes – Linear Polynomials

- **Problem** – Define a function for each joint such that its value at
  - $t_0$ is the initial position of the joint
  - $t_f$ is the desired goal position of the joint

- **Given** – Constrains on $\theta(t)$

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$



# Joint Space Schemes – Linear Polynomials

- Solution – the two constraints can be satisfied by a first order polynomial

$$\theta(t) = a_0 + a_1 t$$

- Combined with the two desired constrains yields two equations in two unknown

$$\theta_0(0) = a_0$$
$$\theta_f(t) = a_0 + a_1 t_f$$

$$\longrightarrow$$

$$\theta_0 = a_0$$
$$a_1 = \frac{\theta_f - \theta_0}{t_f}$$

$$\theta = \theta_0 + \left(\frac{\theta_f - \theta_0}{t_f}\right) t$$

# Joint Space Schemes

Single Time Interval

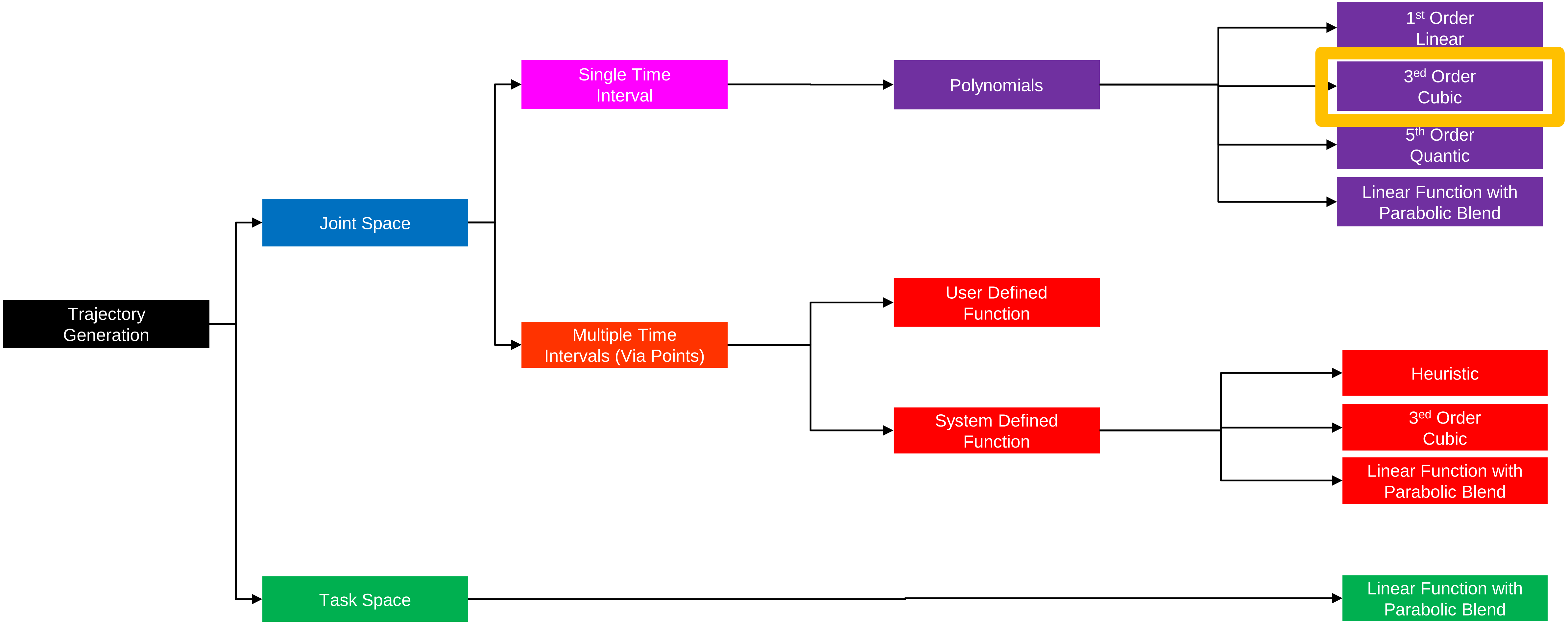
Polynomials

Cubic Order Polynomial

# Joint Space Schemes

# Joint Space Schemes

# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes – Order of the Polynomials

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the **initial position** of the joint and at
  - $t_f$ is the **desired goal position** of the joint

- Given - Constrains on

$$\theta(0) = \theta_0$$
$$\theta(t_f) = \theta_f$$
$$\dot{\theta}(0) = 0$$
$$\dot{\theta}(t_f) = 0$$

- What should be the order of the polynomial function to meet these constrains?

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial of at least third degree

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- The joint velocity and acceleration

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$$\begin{aligned} \theta(0) &= \theta_0 & \theta_0 &= a_0 \\ \theta(t_f) &= \theta_f & \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ \dot{\theta}(0) &= 0 & 0 &= a_1 \\ \dot{\theta}(t_f) &= 0 & 0 &= a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{aligned}$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$\begin{array}{c} \\ \\ \begin{bmatrix} \theta_0 \\ \theta_f \\ 0 \\ 0 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{array}$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$\theta_f = \theta_0 + a_2 t_f^2 + a_3 t_f^3$$
$$0 = 2a_2 t_f + 3a_3 t_f^2$$

$$a_2\left[t_f^2\right] + a_3\left[t_f^3\right] = \theta_f - \theta_0$$
$$a_2\left[2t_f\right] + a_3\left[3t_f^2\right] = 0$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} \theta_f - \theta_0 & t_f^3 \\ 0 & 3t_f^2 \end{vmatrix}}{\Delta} = \frac{3t_f^2\left(\theta_f - \theta_0\right)}{t_f^4} = \frac{3\left(\theta_f - \theta_0\right)}{t_f^2}$$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & \theta_f - \theta_0 \\ 2t_f & 0 \end{vmatrix}}{\Delta} = \frac{-2t_f\left(\theta_f - \theta_0\right)}{t_f^4} = \frac{-2\left(\theta_f - \theta_0\right)}{t_f^3}$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solving these equations for the $a_i$ we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0)$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

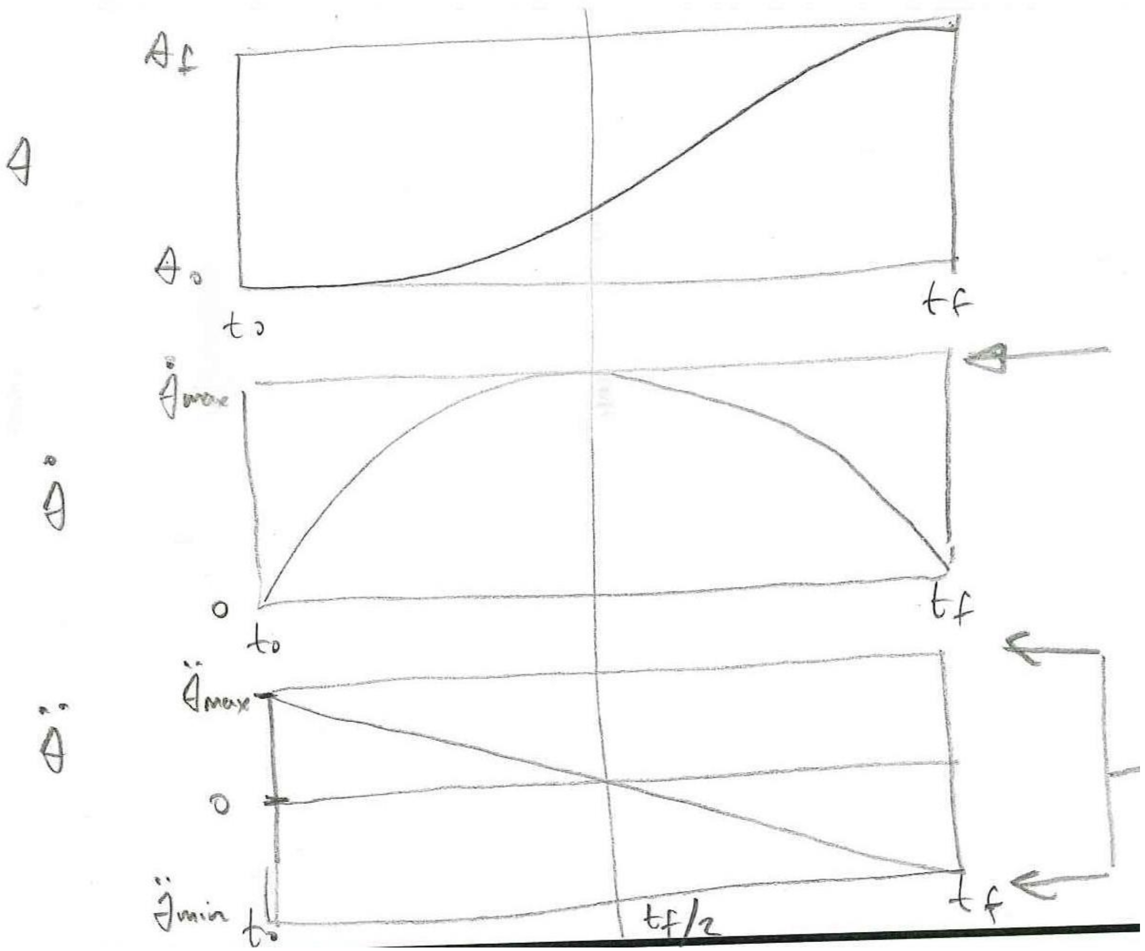
$\dot{\theta}_{max}$ - max angular velocity at $\frac{t_f}{2}$

$$\dot{\theta}_{max}\left(t=\frac{t_f}{2}\right)=\frac{6}{t_f^2}(\theta_f-\theta_0)\left[\frac{t_f}{2}\right]-\frac{6}{t_f^3}(\theta_f-\theta_0)\left[\frac{t_f}{2}\right]^2=\frac{3(\theta_f-\theta_0)}{t_f}-\frac{6}{4}\frac{(\theta_f-\theta_0)}{t_f}=\frac{3}{2}\frac{\theta_f-\theta_0}{t_f}$$

$\ddot{\theta}_{max}$ - max angular acceleration at $t=0$ and $t=t_f$

$$\ddot{\theta}_{max}=\frac{6}{t_f^2}(\theta_f-\theta_0)$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity




$$\frac{3}{2}\frac{\theta_f - \theta_0}{t_f}$$

$$\pm 6\frac{\theta_f - \theta_0}{t_f^2}$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

- **Example** – A single-link robot with a rotary joint is motionless at $\theta_0 = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta_f = 75$ degrees in 3 seconds. Find the coefficient of the cubic polynomial that accomplish this motion and brings the manipulator to rest at the goal

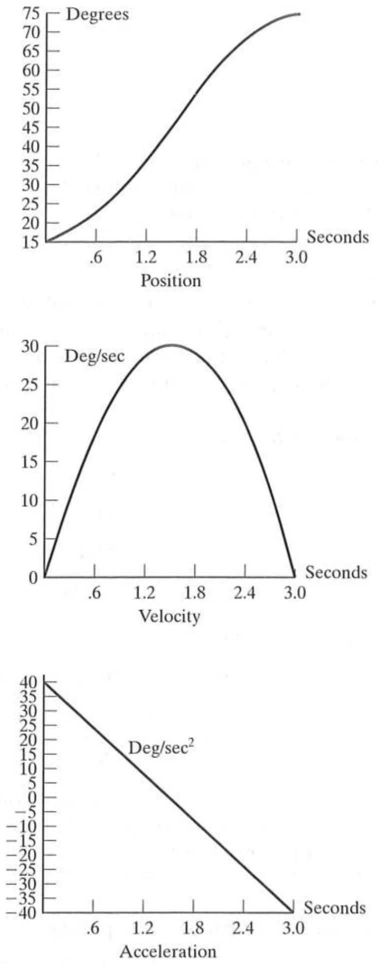
$$\theta(0) = 15$$
$$\theta(t_f) = 75$$
$$\dot{\theta}(0) = 0$$
$$\dot{\theta}(t_f) = 0$$

$$a_0 = \theta_0 = 15$$
$$a_1 = 0$$
$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) = \frac{3}{9}(75 - 15) = 20$$
$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) = -\frac{2}{27}(75 - 15) = -4.44$$

$$\theta(t) = 15 + 20t^2 - 4.44t^3$$
$$\dot{\theta}(t) = 40t - 13.33t^2$$
$$\ddot{\theta}(t) = 40 + 26.66t$$

# Joint Space Schemes – Cubic Polynomials - Zero Velocity

- The velocity profile of any cubic function is a parabola
- The acceleration profile of any cubic function is linear



## Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Previous Method - The manipulator comes to rest at each via point
- General Requirement - Pass through a point without stopping
- Problem - Define a function for each joint such that it value at
  - $t_0$ is the **initial position** of the joint and at
  - $t_f$ is the **desire goal position** of the joint
- Given - Constrains on $\theta(t)$ such that the velocities at the via points are not zero but rather some known velocities

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

$$\boxed{\begin{aligned} \dot{\theta}(0) &= \dot{\theta}_0 \\ \dot{\theta}(t_f) &= \dot{\theta}_f \end{aligned}}$$

# Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$$\begin{aligned}
\theta(0) &= \theta_0 & \qquad \theta_0 &= a_0 \\
\theta(t_f) &= \theta_f & \qquad \theta_f &= a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
\dot{\theta}(0) &= \dot{\theta}_0 & \qquad \dot{\theta}_0 &= a_1 \\
\dot{\theta}(t_f) &= \dot{\theta}_f & \qquad \dot{\theta}_f &= a_1 t_f + 2a_2 t_f + 3a_3 t_f^2
\end{aligned}$$

# Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

$$\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

# Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

$$\theta_f = \theta_0 + \dot{\theta}_0 t_f + a_2 t_f^2 + a_3 t_f^3$$
$$\dot{\theta}_f = \dot{\theta}_0 + 2a_2 t_f + 3a_3 t_f^2$$

$$a_2\left[t_f^2\right] + a_3\left[t_f^3\right] = \left(\theta_f - \theta_0\right) - \dot{\theta}_0 t_f$$
$$a_2\left[2t_f\right] + a_3\left[3t_f^2\right] = \left(\dot{\theta}_f - \dot{\theta}_0\right)$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} (\theta_f-\theta_0)-\dot{\theta}_0 t_f & t_f^3 \\ (\dot{\theta}_f-\dot{\theta}_0) & 3t_f^2 \end{vmatrix}}{\Delta}$$
$$= \frac{3t_f^2\left((\theta_f-\theta_0)-\dot{\theta}_0 t_f\right)-t_f^3(\dot{\theta}_f-\dot{\theta}_0)}{t_f^4}$$
$$= \frac{3}{t_f^2}\left(\theta_f - \theta_0\right) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & (\theta_f-\theta_0)-\dot{\theta}_0 t_f \\ 2t_f & \dot{\theta}_f-\dot{\theta}_0 \end{vmatrix}}{\Delta}$$
$$= \frac{t_f^2(\dot{\theta}_f-\dot{\theta}_0)-2t_f\left[(\theta_f-\theta_0)-\dot{\theta}_0 t_f\right]}{t_f^4}$$
$$= -\frac{2}{t_f^3}\left(\theta_f - \theta_0\right) + \frac{2}{t_f^2}\left(\theta_f + \theta_0\right)$$

# Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solving these equations for the $a_i$ we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)$$

- Given - velocities at each via point are
- Solution - Apply these equations for each segment of the trajectory.

# Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Note:
  - The Cubic polynomials ensures the continuity of velocity but not the acceleration.
  - Practically, the industrial manipulators are sufficiently rigid so this  discontinuity in acceleration is filtered by the mechanical structure
  - Therefore this trajectory is generally satisfactory for most applications

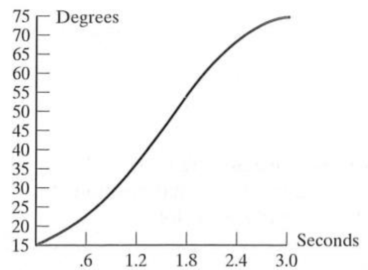
# Joint Space Schemes

Single Time Interval

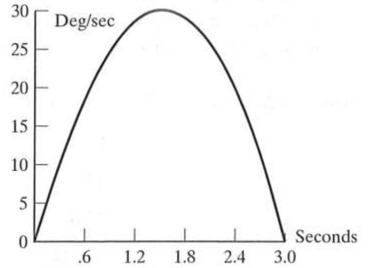
Polynomials

Quantic Order Polynomial

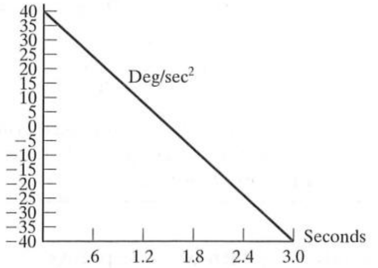
# Joint Space Schemes



Position

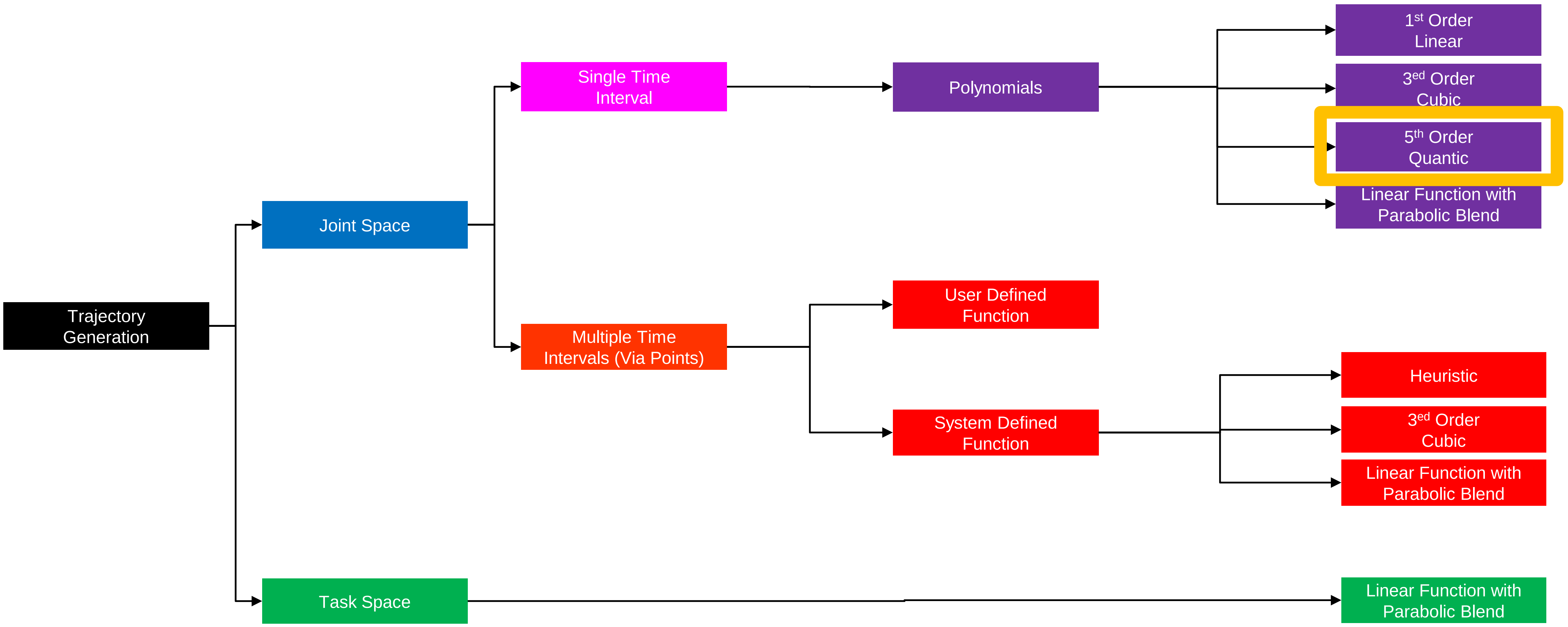


Velocity



Acceleration

# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes – Quantic Polynomials

- Rational for Quantic Polynomials (high order)
  - High Speed Robot
  - Robot Carrying heavy/delicate load
  - Non Rigid links
  - For high speed robots or when the robot is handling heavy or delicate loads. It is worth insuring the continuity of accelerations as well as avoid excitation of the resonance modes of the mechanism

# Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the time at the **initial position**
  - $t_f$ is the time at the **desired goal position**

- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

$$\dot{\theta}(0) = \dot{\theta}_0$$

$$\dot{\theta}(t_f) = \dot{\theta}_f$$

$$\ddot{\theta}(0) = \ddot{\theta}_0$$

$$\ddot{\theta}(t_f) = \ddot{\theta}_f$$

- What should be the order of the polynomial function to meet these constrains?

## Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$
$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

$$\theta(0) = \theta_0 \qquad \theta_0 = a_0$$
$$\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$
$$\dot{\theta}(0) = \dot{\theta}_0 \qquad \dot{\theta}_0 = a_1$$
$$\dot{\theta}(t_f) = \dot{\theta}_f \qquad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$$
$$\ddot{\theta}(0) = \ddot{\theta}_0 \qquad \ddot{\theta}_0 = 2a_2$$
$$\ddot{\theta}(t_f) = \ddot{\theta}_f \qquad \ddot{\theta}_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

# Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$
\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \\ \ddot{\theta}_0 \\ \ddot{\theta}_f \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3
\end{bmatrix}
\end{array}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}
$$

# Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$(2)\ \theta_f = \theta_0 + \dot{\theta}_0 t_f + \frac{\ddot{\theta}_0}{2} t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$(4)\ \dot{\theta}_f = \dot{\theta}_0 + 2\frac{\ddot{\theta}_2}{2} t_f + 3a_3 t_f^3 + 4a_4 t_f^4 + 5a_5 t_f^4$$

$$(6)\ \ddot{\theta}_f = 2\frac{\ddot{\theta}_0}{2} + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

$$\begin{bmatrix} \theta_f - \theta_0 - \dot{\theta} t_f - \frac{\ddot{\theta}_0}{2} t_f^2 \\ \dot{\theta}_f - \dot{\theta}_0 - \ddot{\theta}_0 t_f \\ \ddot{\theta}_f - \ddot{\theta}_0 \end{bmatrix} \begin{bmatrix} t_f^3 & t_f^4 & t_f^5 \\ 3t_f^3 & 4t_f^4 & 5t_f^4 \\ 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

# Joint Space Schemes – Cubic Polynomials - Non Zero Acceleration

- Solving these equations for the $a_i$ we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2}$$

$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f - (3\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f + (3\ddot{\theta}_0 - 2\ddot{\theta}_f)t_f^2}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f - (\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^5}$$

# Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

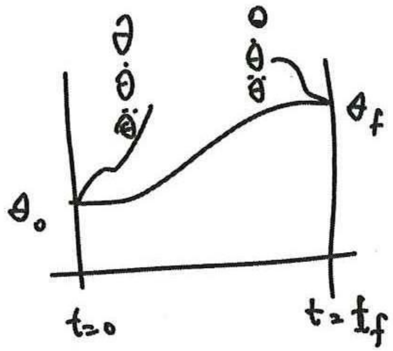
For a generalized case where $t_0 \neq 0$

$$T = t_f - t_0 \; ; \; h = \theta_f - \theta_0$$

$$\begin{aligned}
a_0 &= \theta_0 \\
a_1 &= \dot{\theta}_0 \\
a_2 &= \frac{1}{2} a_0 \\
a_3 &= \frac{1}{2T^3}\left[20h - \left(8\dot{\theta}_f + 12\dot{\theta}_0\right)T - (3a_0 - a_1)T^2\right] \\
a_4 &= \frac{1}{2T^4}\left[-30h + \left(14\dot{\theta}_f + 16\dot{\theta}_0\right)T + (3a_0 - 2a_1)T^2\right] \\
a_5 &= \frac{1}{2T^5}\left[12h - 6\left(\dot{\theta}_1 - \dot{\theta}_0\right)T + (a_1 - a_0)T^2\right]
\end{aligned}$$

# Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the time at the **initial position**
  - $t_f$ is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment



$$\theta(0)=\theta_0 \qquad \dot{\theta}(0)=\dot{\theta}_0 \qquad \boxed{\ddot{\theta}(0)=0}$$

$$\theta(t_f)=\theta_f \qquad \dot{\theta}(t_f)=\dot{\theta}_f \qquad \boxed{\ddot{\theta}(t_f)=0}$$

- What should be the order of the polynomial function to meet these constrains?

# Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\theta(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

$$\theta(0) = \theta_0 \qquad \theta_0 = a_0$$

$$\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$\dot{\theta}(0) = \dot{\theta}_0 \qquad \dot{\theta}_0 = a_1$$

$$\dot{\theta}(t_f) = \dot{\theta}_f \qquad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$$

$$\ddot{\theta}(0) = 0 \qquad 0 = 2a_2$$

$$\ddot{\theta}(t_f) = 0 \qquad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

# Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solving these equations for the $a_i$ we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2} = 0$$

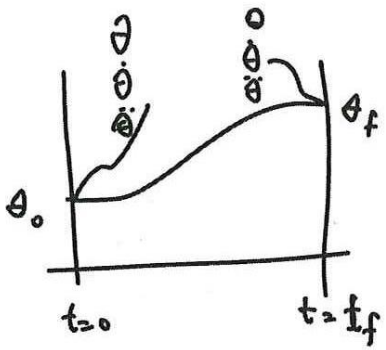
$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f}{2t_f^5}$$

# Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Problem - Define a function for each joint such that it value at
  - $t_0$ is the time at the **initial position**
  - $t_f$ is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment




$$\theta(0) = \theta_0 \qquad \boxed{\dot{\theta}(0) = 0} \qquad \boxed{\ddot{\theta}(0) = 0}$$

$$\theta(t_f) = \theta_f \qquad \boxed{\dot{\theta}(t_f) = 0} \qquad \boxed{\ddot{\theta}(t_f) = 0}$$

- What should be the order of the polynomial function to meet these constrains?

## Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\theta(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

$$\theta(0) = \theta_0 \qquad \theta_0 = a_0$$

$$\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$\dot{\theta}(0) = 0 \qquad 0 = a_1$$

$$\dot{\theta}(t_f) = 0 \qquad 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$$

$$\ddot{\theta}(0) = 0 \qquad 0 = 2a_2$$

$$\ddot{\theta}(t_f) = 0 \qquad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$$

# Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solving these equations for the $a_i$ we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = \frac{20\theta_f - 20\theta_0}{2t_f^3} = \frac{10\theta_f - 10\theta_0}{t_f^3} = 10\left[\frac{\theta_f - \theta_0}{t_f^3}\right]$$

$$a_4 = \frac{30\theta_0 - 30\theta_f}{2t_f^4} = -\frac{15\theta_f - 15\theta_0}{t_f^4} = -15\left[\frac{\theta_f - \theta_0}{t_f^4}\right]$$

$$a_5 = \frac{12\theta_f - 12\theta_0}{2t_f^5} = \frac{6\theta_f - 6\theta_0}{t_f^5} = 6\left[\frac{\theta_f - \theta_0}{t_f^5}\right]$$

# Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\theta = \overset{\overset{a_0}{\downarrow}}{\theta_0} + \overset{\overset{a_3}{\downarrow}}{\left(10\,\frac{\theta_f-\theta_0}{t_f^3}\right)} t^3 - \overset{\overset{a_4}{\downarrow}}{\left(15\,\frac{\theta_f-\theta_0}{t_f^4}\right)} t^4 + \overset{\overset{a_5}{\downarrow}}{\left(6\,\frac{\theta_f-\theta_0}{t_f^5}\right)} t^5$$

$$\theta = a_0 + a_3 t^3 + a_4 t^4 + a_5 t^5$$
$$\dot{\theta} = 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$
$$\ddot{\theta} = 6a_3 + 12a_4 t^2 + 20a_5 t^3$$

$$\theta = \theta_0 + 10\left(\frac{\theta_f-\theta_0}{t_f^3}\right) t^3 - 15\left(\frac{\theta_f-\theta_0}{t_f^4}\right) t^4 + 6\left(\frac{\theta_f-\theta_0}{t_f^5}\right) t^5$$

$$\dot{\theta} = 30\left(\frac{\theta_f-\theta_0}{t_f^3}\right) t^2 - 60\left(\frac{\theta_f-\theta_0}{t_f^4}\right) t^3 + 30\left(\frac{\theta_f-\theta_0}{t_f^5}\right) t^4$$

$$\ddot{\theta} = 60\left(\frac{\theta_f-\theta_0}{t_f^3}\right) t - 180\left(\frac{\theta_f-\theta_0}{t_f^4}\right) t^2 + 120\left(\frac{\theta_f-\theta_0}{t_f^5}\right) t^3$$

$$\dddot{\theta} = 60\left(\frac{\theta_f-\theta_0}{t_f^3}\right) - 360\left(\frac{\theta_f-\theta_0}{t_f^4}\right) t + 360\left(\frac{\theta_f-\theta_0}{t_f^5}\right) t^2$$

# Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\dot{\theta}_{max} \rightarrow at\ t = \frac{t_f}{2}$$

$$\dot{\theta}_{max} = \frac{15}{2}\left[\frac{\theta_f - \theta_0}{t_f}\right] - \frac{15}{2}\left[\frac{\theta_f - \theta_0}{t_f}\right] + \frac{15}{8}\left[\frac{\theta_f - \theta_0}{t_f}\right] = \frac{15}{8}\frac{\theta_f - \theta_0}{t_f}$$

$$\ddot{\theta}_{max} \rightarrow at\ t = \frac{t_f}{4}$$

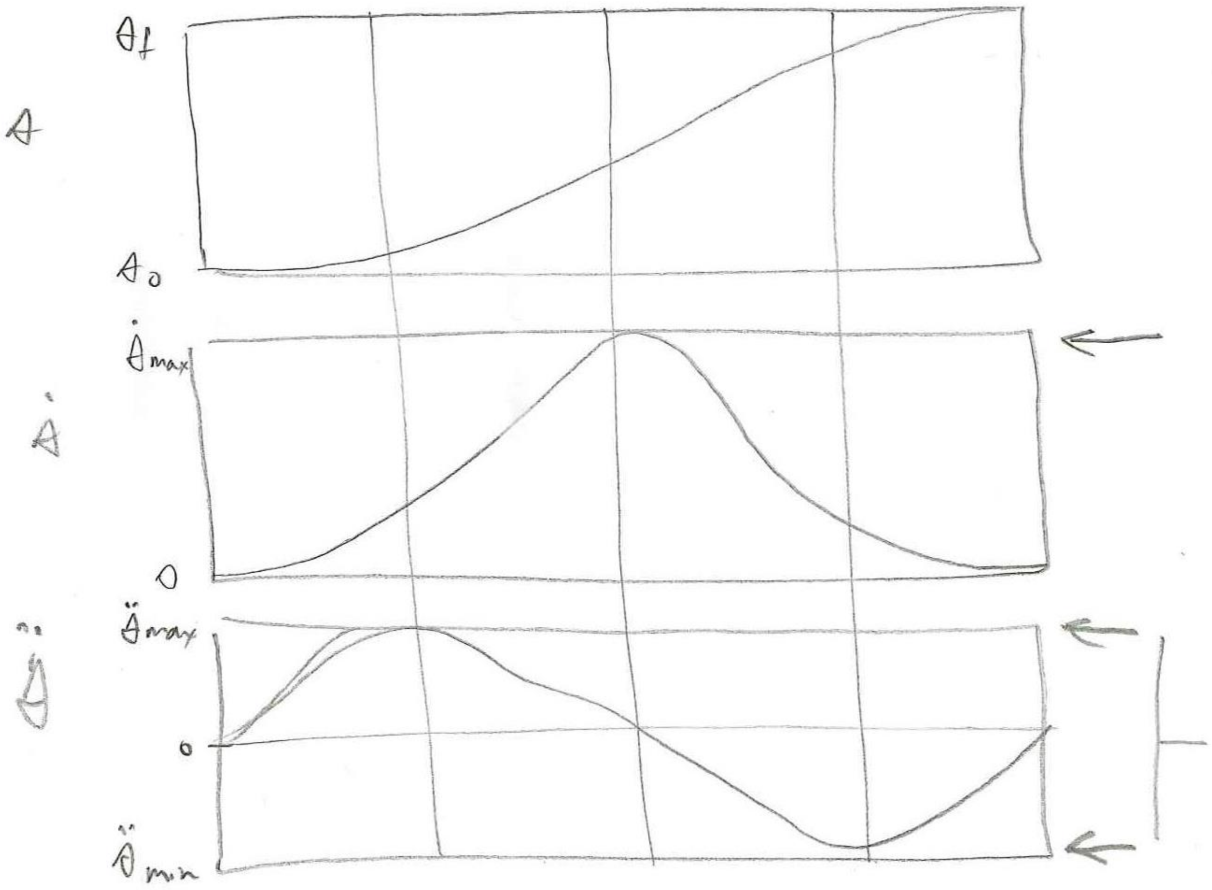
$$\ddot{\theta}_{max} = 15\left[\frac{\theta_f - \theta_0}{t_f^2}\right] - \frac{45}{4}\left[\frac{\theta_f - \theta_0}{t_f^2}\right] + \frac{15}{8}\left[\frac{\theta_f - \theta_0}{t_f^2}\right] = \left[15 - \frac{75}{8}\right]\frac{\theta_f - \theta_0}{t_f^2}$$

## Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\ddot{\theta}_{max} \rightarrow at\ t = \frac{t_f}{4}$$

$$\ddot{\theta}_{max} = 15\left[\frac{\theta_f - \theta_0}{t_f^2}\right] - \frac{45}{4}\left[\frac{\theta_f - \theta_0}{t_f^2}\right] + \frac{15}{8}\left[\frac{\theta_f - \theta_0}{t_f^2}\right] = \left[15 - \frac{75}{8}\right]\frac{\theta_f - \theta_0}{t_f^2}$$

# Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration




$$\frac{15}{8}\frac{\theta_f-\theta_0}{t_f}$$

$$\frac{10}{\sqrt{3}}\frac{\theta_f-\theta_0}{t_f^2}$$

# Joint Space Schemes

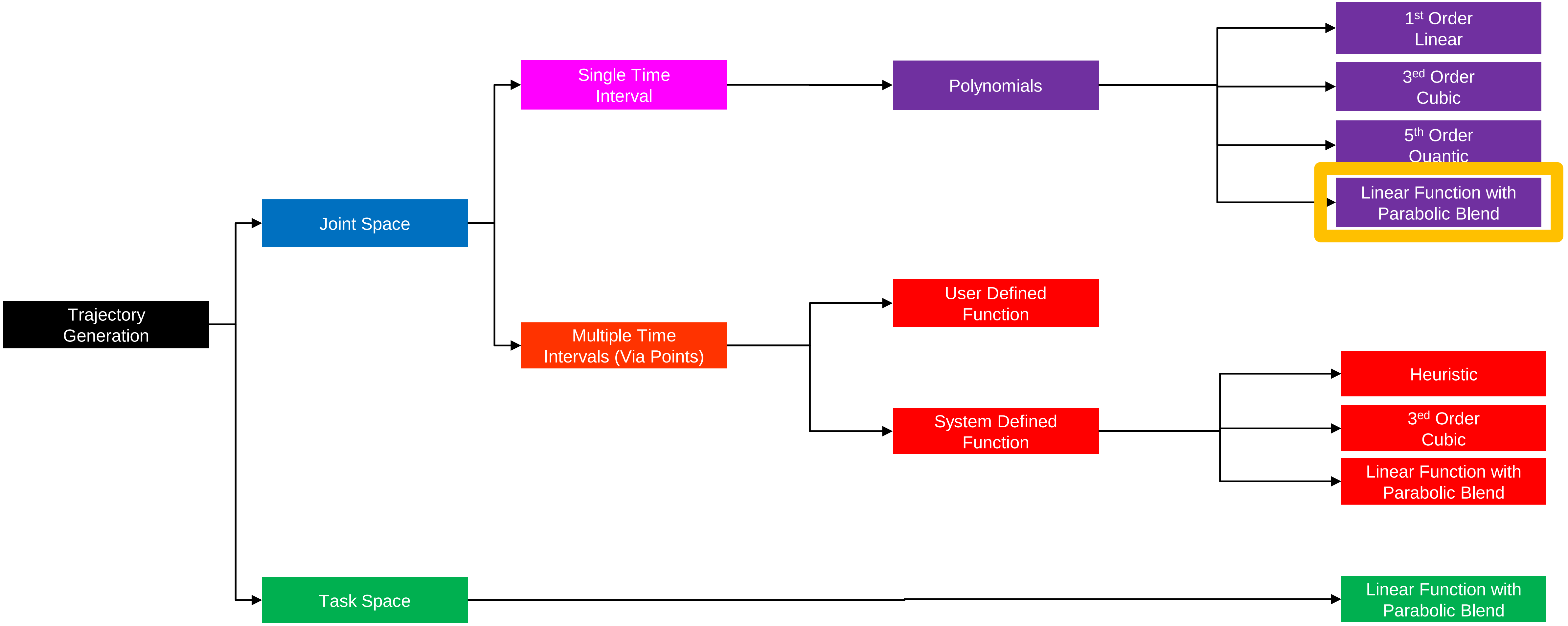
Single Time Interval

Polynomials

Linear Function with Parabolic Blend (Trapezoid Velocity Method)

# Joint Space Schemes

# Trajectory Generation – Roadmap Diagram



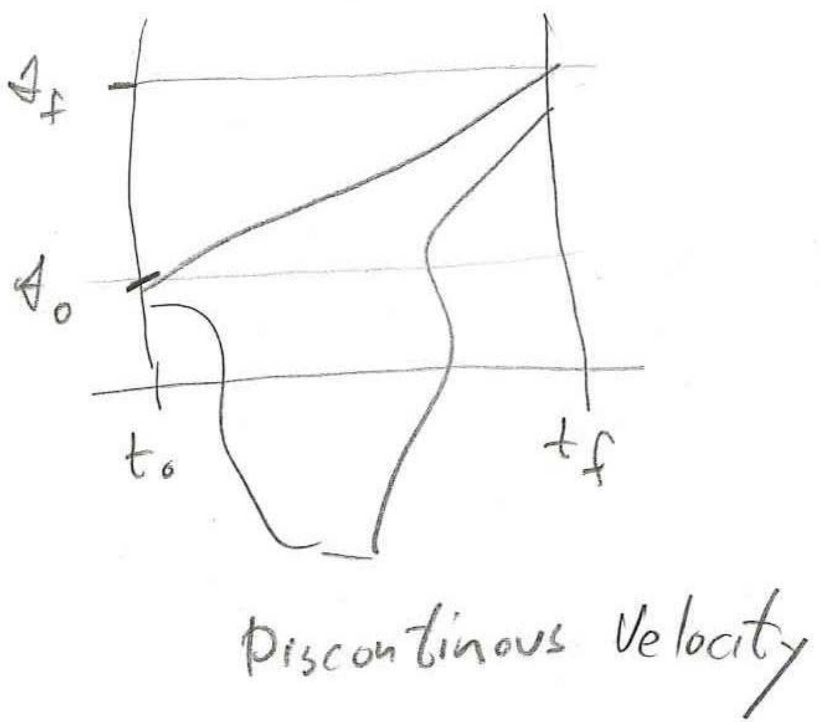
# Joint Space Schemes – Linear Function With Parabolic Blend

- Linear interpolation to move from the present joint position $\theta_0(t = t_0)$ to the final position $\theta_f(t = t_f)$

- Note: Although the motion at each joint is linear, the end-effector in general does not move in a straight line in task space

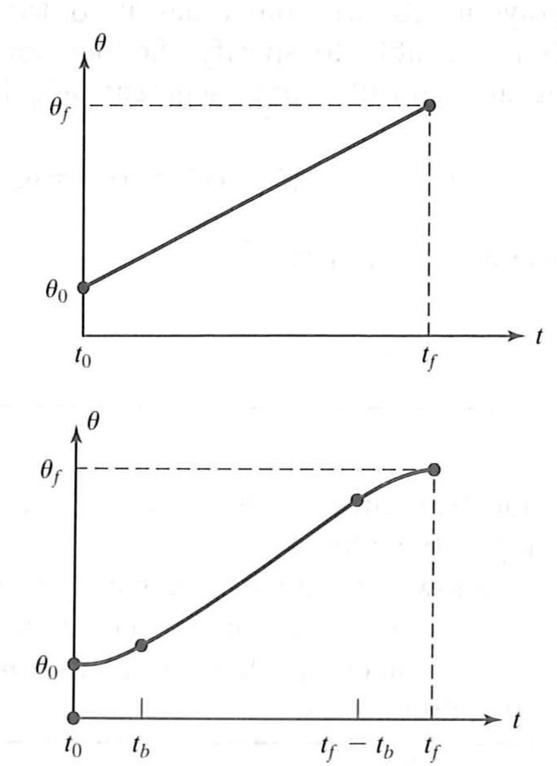
- **Problem**: Linear interpolation would cause the velocity to be discontinuous at the beginning/end

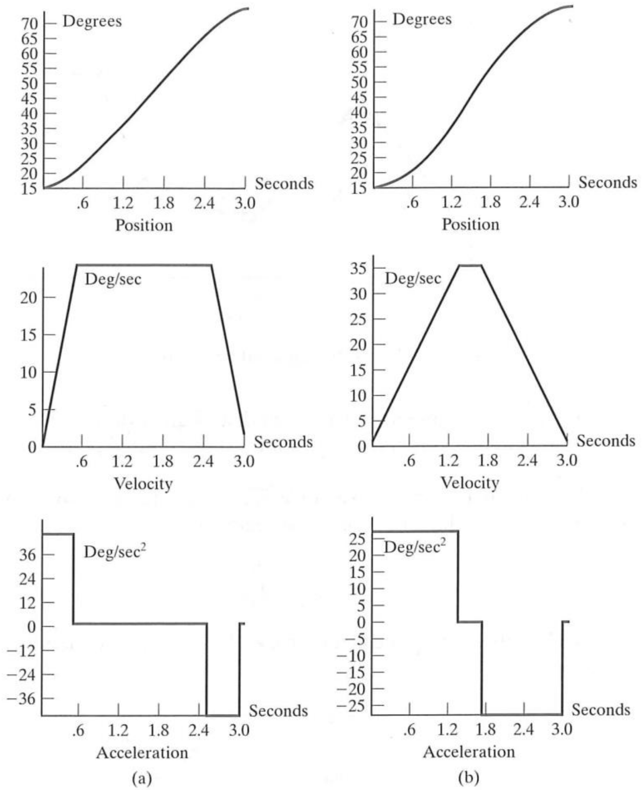
- **Solution**: Parabolic blend region



# Joint Space Schemes – Linear Function With Parabolic Blend

- **Challenge**
  - Find the relationship between the time duration of the parabolic blend and the acceleration



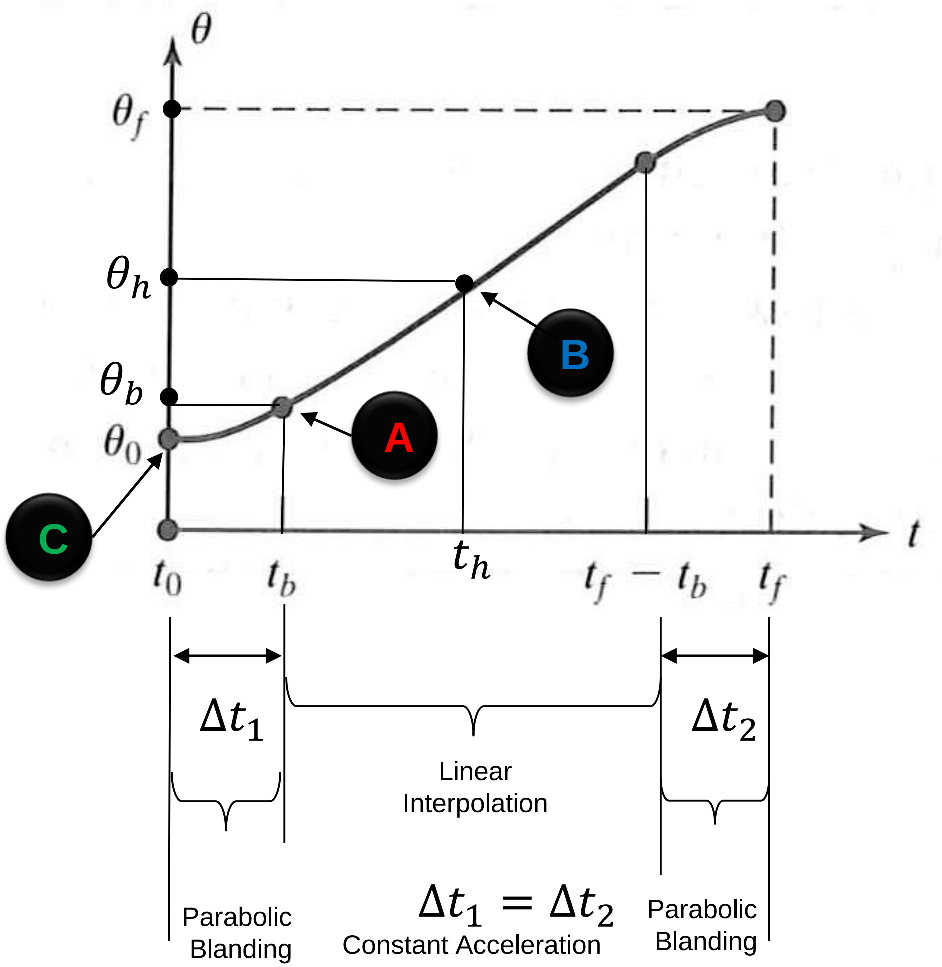



# Joint Space Schemes – Linear Function With Parabolic Blend

- **Key Points**
  - A – Transition point between the paraboloid bland (identical velocity ) segment and the linear segment
  - B – Mid point of the linear segment
  - C – Stating point of the parabolic section (zero velocity)

- **Regions**
  - C A – During the Blending section – Constant acceleration to change the velocity smoothly
  - A B – During the Linear Section – Zero Acceleration

- **Assumptions:**
  - The parabolic blend segments $(\Delta t_1, \Delta t_2)$ have the same duration $\Delta t_1 = \Delta t_2$
  - The same constant acceleration is used during both blends



# Joint Space Schemes – Linear Function With Parabolic Blend

- Conditions: Point C and Point A and the region (Point C to Point A)
  - Constant acceleration during the blend (Point C to A) (I)
  - Initial velocity is zero (Point C)

(1) $\theta_b = \theta_0 + v_0 t + \frac{1}{2}\ddot{\theta}t_b^2 \quad \rightarrow constant\ acceleration$ (I)

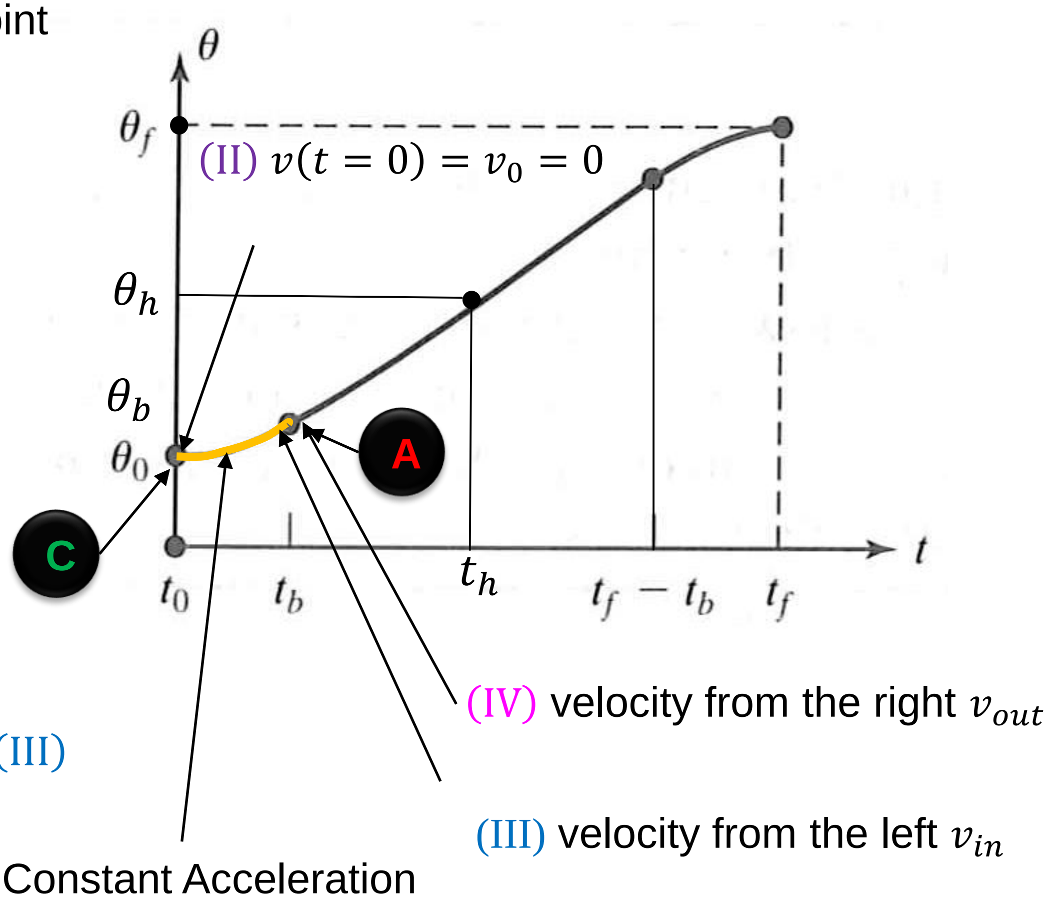
$v_0$ initial velocity = 0 (II)

  - The slope at point A must be equal on both sides

(2) $\ddot{\theta}t_b = \frac{\theta_h - \theta_b}{t_h - t_b}$

$\ddot{\theta}t_b$ : velocity from the left $v_{in}$ (time derivative of eq. 1) (III)

$\frac{\theta_h - \theta_b}{t_h - t_b}$ : velocity from the right $v_{out}$ (IV)



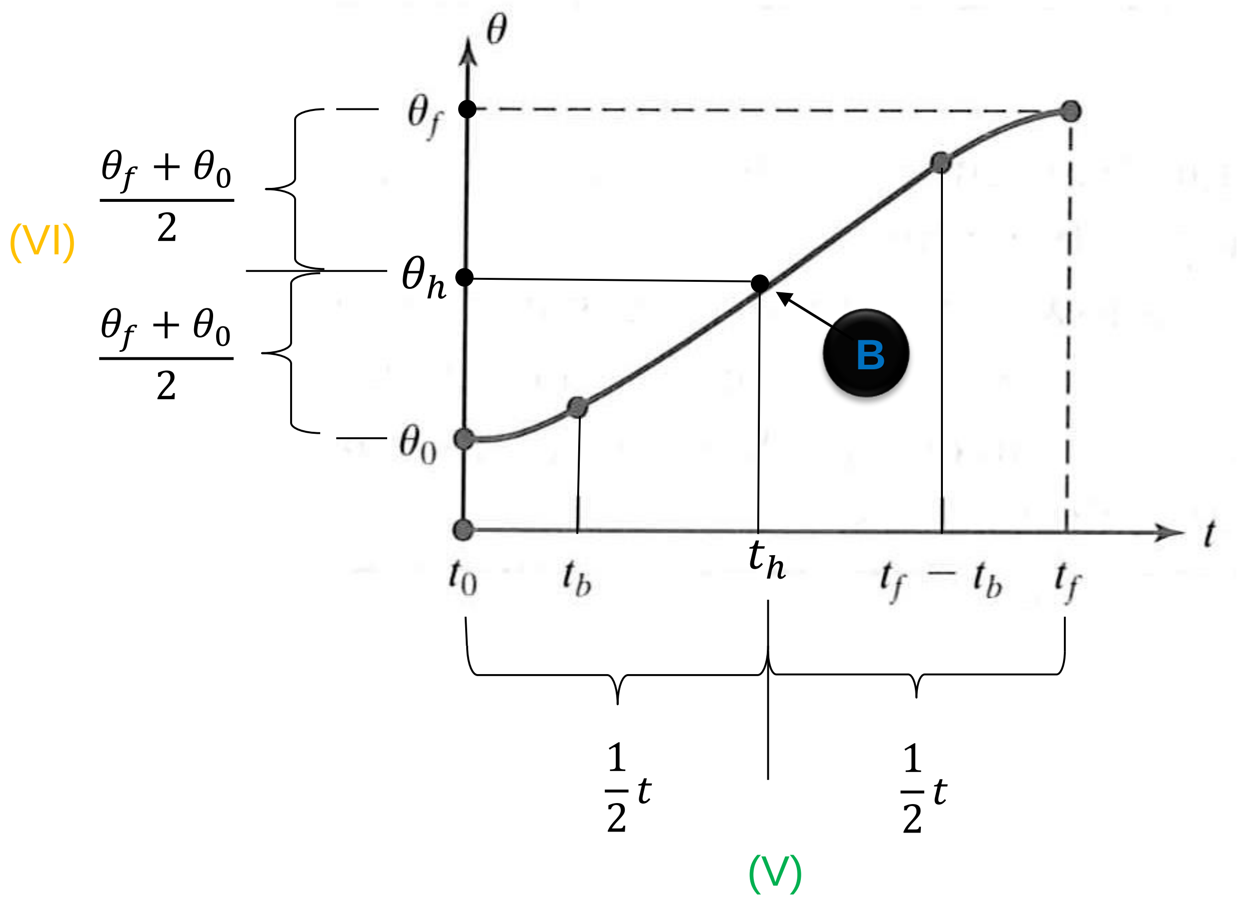
# Joint Space Schemes –
# Linear Function With Parabolic Blend

- Conditions at Point (B)
  - Point B is at the middle of the segment
    - In the time domain (V)

(3) $t_h = \frac{t}{2}$

    - In the angle domain (VI)

(4) $\theta_h = \frac{\theta_f - \theta_0}{2} + \theta_0 = \frac{\theta_f - \theta_0 + 2\theta_0}{2} = \frac{\theta_f + \theta_0}{2}$



# Joint Space Schemes – Linear Function With Parabolic Blend

- Equation Summary

$$(1)\ \theta_b = \theta_0 + \frac{1}{2}\ddot{\theta} t_b^2$$

$$(2)\ \ddot{\theta} t_b = \frac{\theta_h - \theta_b}{t_h - t_b}$$

$$(3)\ t_h = \frac{t}{2}$$

$$(4)\ \theta_h = \frac{\theta_f + \theta_0}{2}$$

- Plug
  - $\mathrm{Eqn}(4) \rightarrow \mathrm{Eqn}(2)$
  - $\mathrm{Eqn}(3) \rightarrow \mathrm{Eqn}(2)$
  - Resulting in a relationship between the acceleration ($\ddot{\theta}$) at the parabolic bland, the joint angles ($\theta_f$ , $\theta_0$) and the time durations ($t$ , $t_b$)

$$\ddot{\theta} t_b = \frac{\theta_h - \theta_b}{t_h - t_b} = \frac{\frac{\theta_f + \theta_0}{2} - \theta_b}{\frac{t}{2} - t_b}$$

# Joint Space Schemes – Linear Function With Parabolic Blend

Simplifying the expression

$$\ddot{\theta} t_b \left(\frac{t}{2} - t_b\right) = \frac{\theta_f + \theta_0}{2} - \theta_b$$

$$\ddot{\theta} t_b \left(\frac{t - 2t_b}{2}\right) = \frac{\theta_f + \theta_0 - 2\theta_b}{2}$$

$$\ddot{\theta} t_b t - 2\ddot{\theta} t_b^2 = \theta_f + \theta_0 - 2\theta_b$$

$$\ddot{\theta}(t_b t) - 2\ddot{\theta} t_b^2 - \theta_f - \theta_0 + 2\theta_0 + \ddot{\theta} t_b^2 = 0$$

Plug Eq. 1 $\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta} t_b^2$

$$\ddot{\theta}(t_b t) - \ddot{\theta} t_b^2 + \theta_f + \theta_0 = 0$$

# Joint Space Schemes – Linear Function With Parabolic Blend

(5) $(\ddot{\theta})t_b^2 + (\ddot{\theta}t)t_b + (\theta_f - \theta_0) = 0$

a b c

- **Option 1**
  - Given: $\theta_f, \theta_0, t, t_b$ (desired duration of motion)
  - Calculate: $\ddot{\theta}$ (Eq 5)

- **Option 2**
  - Given: $\ddot{\theta}(\text{chosen}), \text{t}, \theta_f, \theta_0$
  - Calculate: $t_b$

$$t_b = \frac{\ddot{\theta}t \pm \sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}} = \frac{t}{2} \pm \frac{\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

# Joint Space Schemes – Linear Function With Parabolic Blend

- Constraint on the acceleration used in the blend

$$\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}\left(\theta_f - \theta_0\right)} \geq 0$$

$$\ddot{\theta} \geq \frac{4\left(\theta_f - \theta_0\right)}{t^2}$$

If $$\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}\left(\theta_f - \theta_0\right)} = 0$$

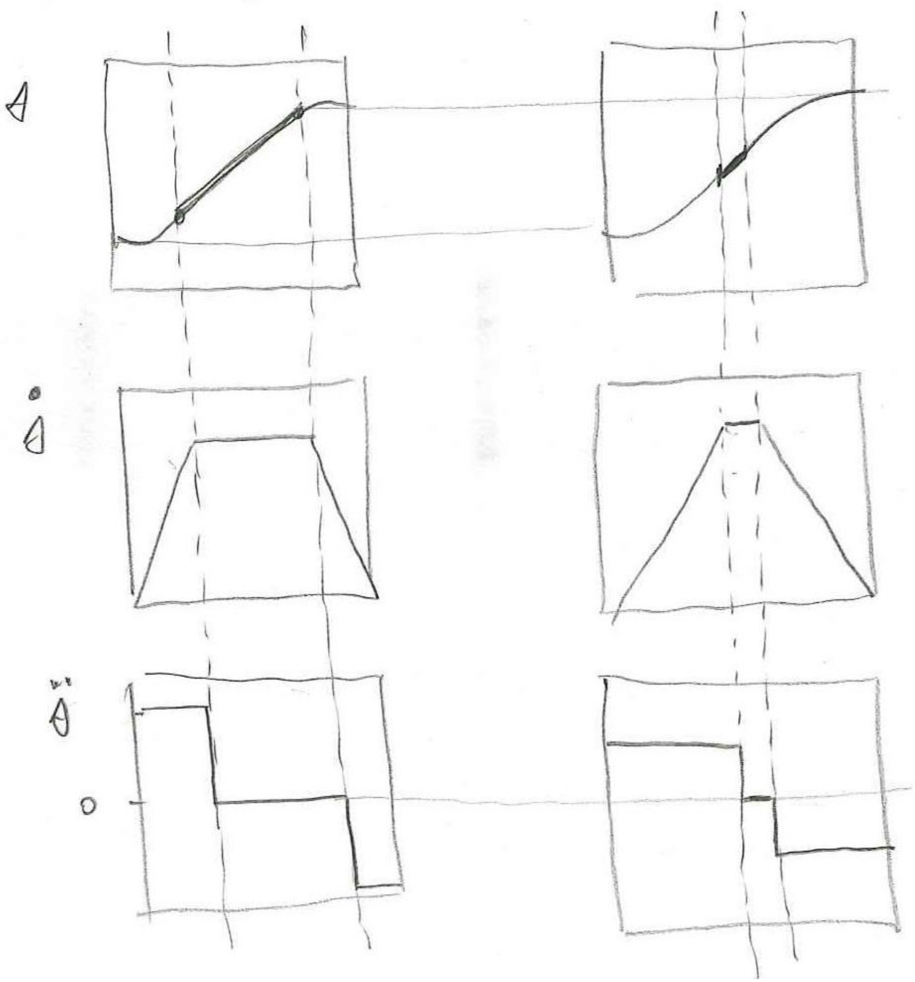
The minimal acceleration is defined as $\ddot{\theta} = \frac{4\left(\theta_f - \theta_0\right)}{t^2}$

$$t_b = \frac{t}{2} \pm \frac{\sqrt{0}}{2\ddot{\theta}} \rightarrow t_b = \frac{t}{2}$$

Meaning – The linear segment is eliminated and we are left with two parabolic blends

# Joint Space Schemes –
# Linear Function With Parabolic Blend

- The length of the linear portion and the parabolic portion may vary
  - High acceleration $\ddot{\theta}$ → short blend
  - Low acceleration $\ddot{\theta}$ → long blend

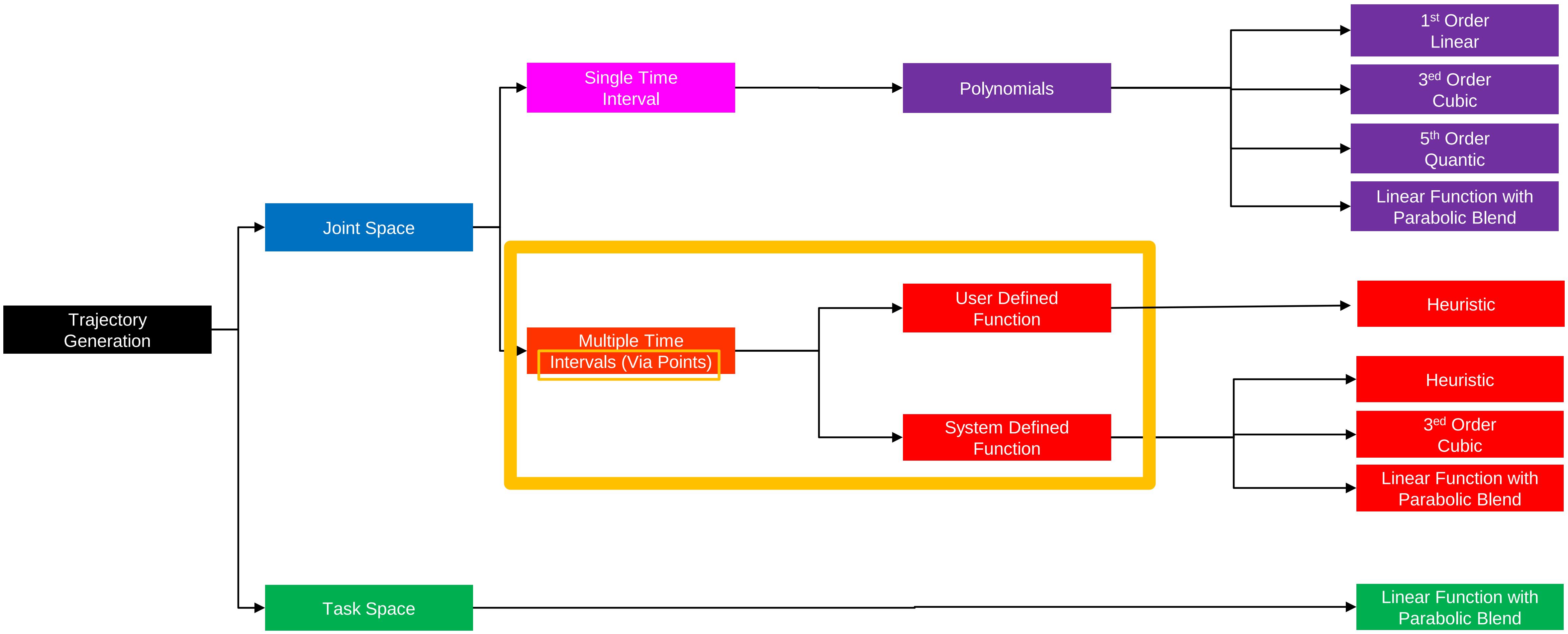


# Joint Space Schemes

Multiple Time Interval

Via Point

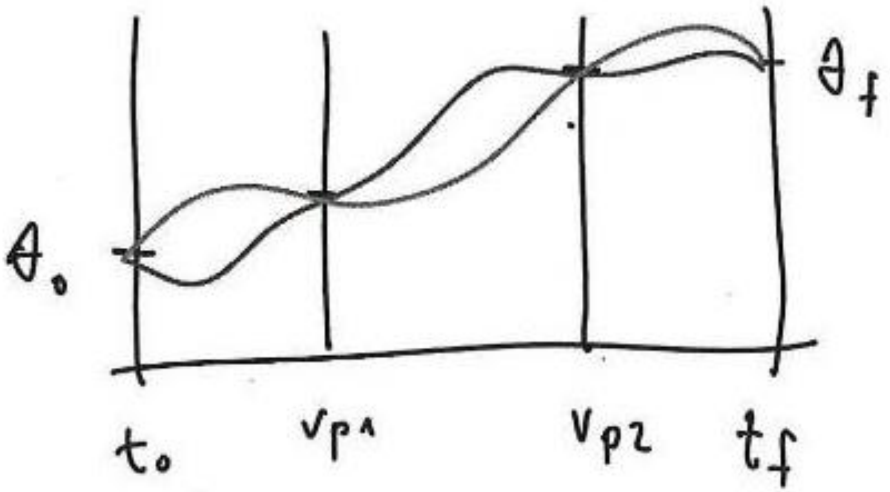
# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes – Multiple Time – Via Points

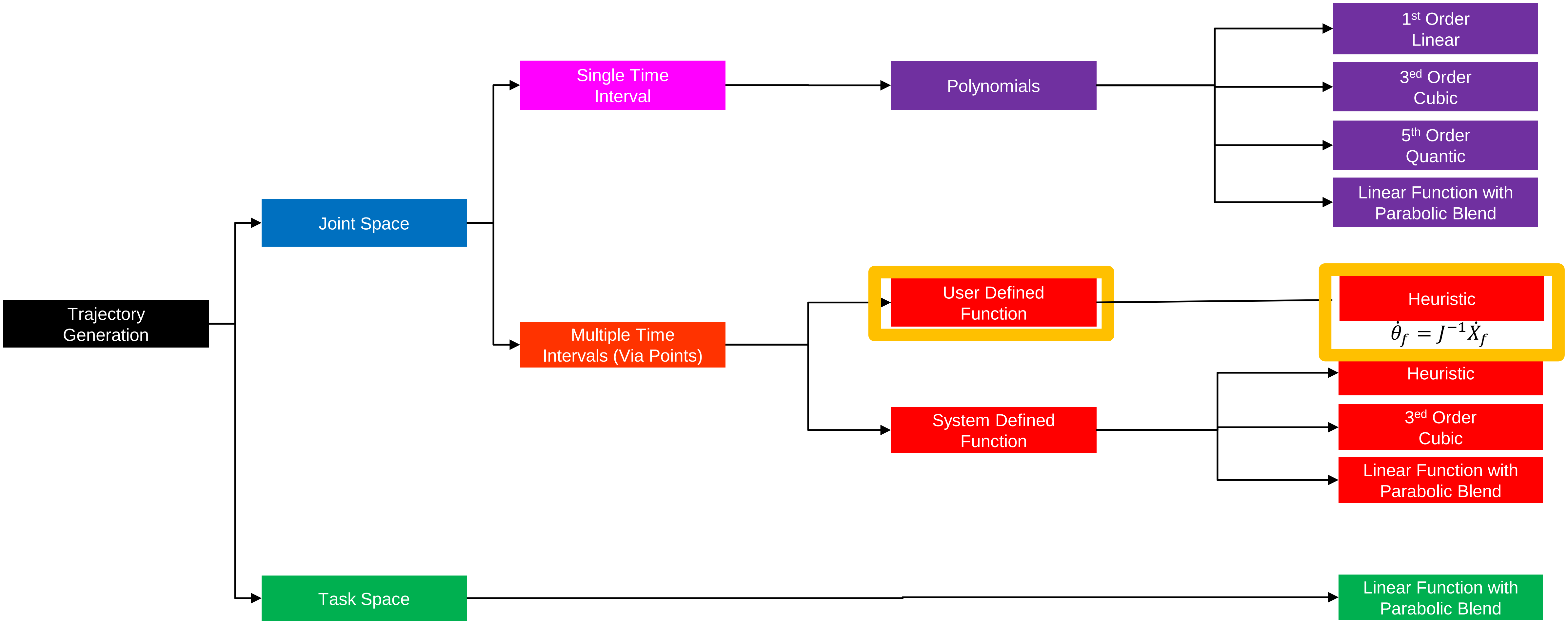
- Define function for each joints such that:
  1. **Initial and Final Points** – The joint angle value at $t_0$ is the **initial joint angle** $\theta_0$ and the joint angle at $t_f$ is the final joint angle $\theta_0$ .

  2. **Via Points** – in between the beginning/ending points the **user** defines via points that each joint must pass through




- **Notes:**
  - **Time Sync –** All the joints reach the via point at the same time to guarantee specific pose (position and orientation) of the end effector.
  - Multiple Solutions – There are multiple smooth functions that may be used to interpolate the joint value

# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

**User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.

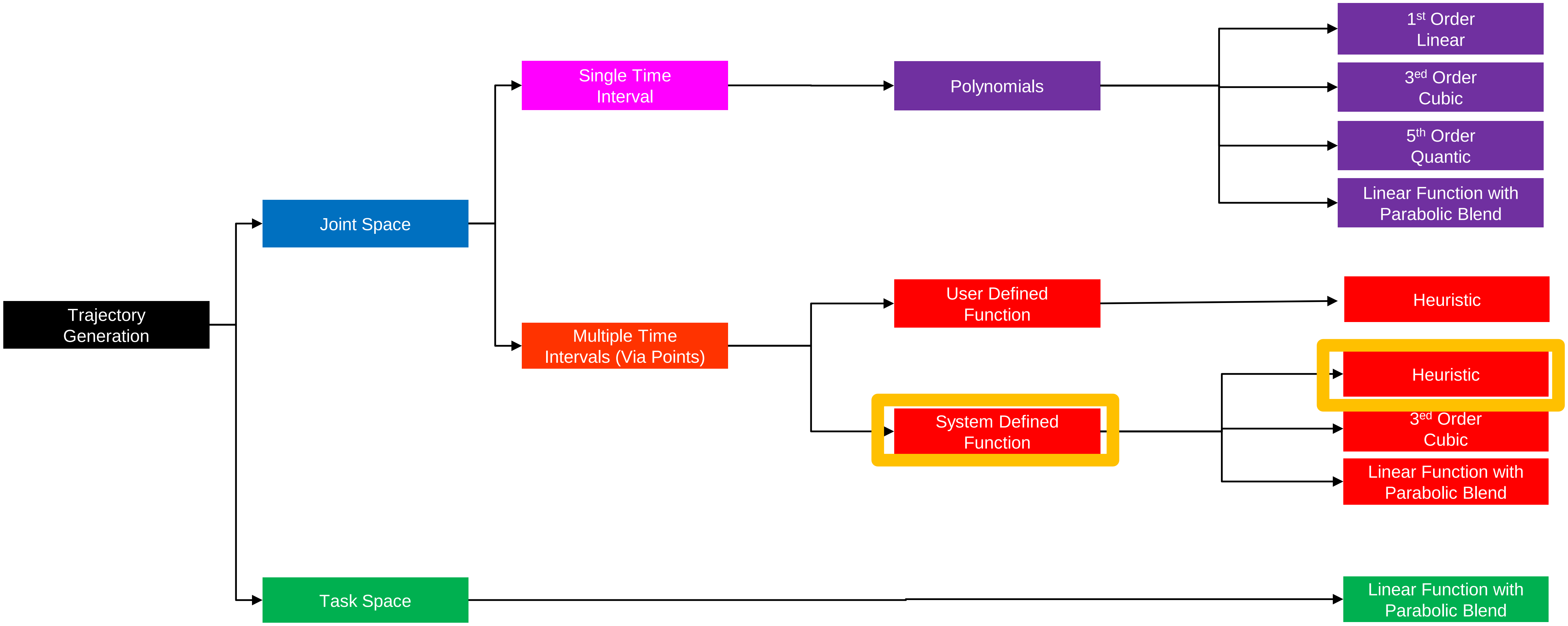
- **Mapping** (Cartesian Space to Joint Space) - Cartesian velocities at the via point are “mapped” to desired joint rates by using the inverse Jacobian

$$\dot{\theta}_f = J^{-1}\dot{X}_f$$

- **Singularity** - If the manipulator is at a singular point at a particular via point then the user is not free to choose an arbitrarily velocity at this point.

- **Difficult**

# Trajectory Generation – Roadmap Diagram

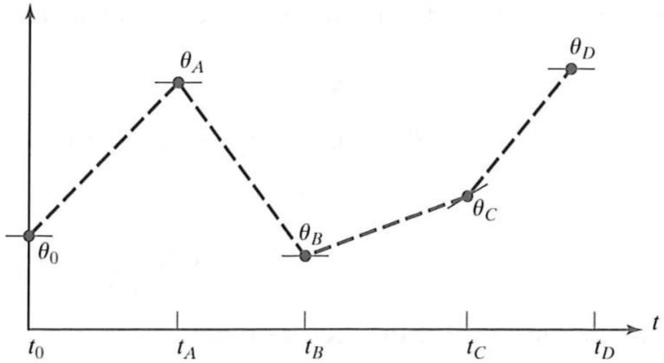


# Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

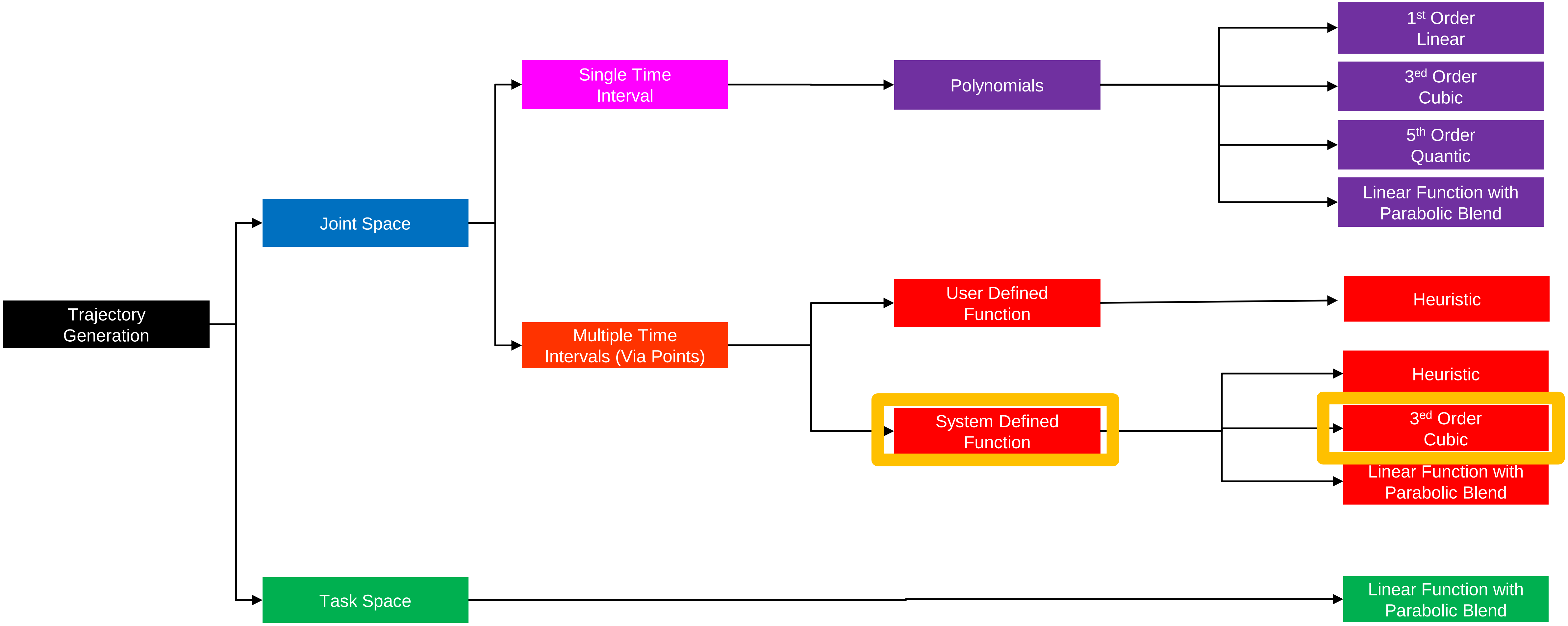
**System Definition (Heuristic Approach)** – The system automatically chooses the velocities (Cartesian or angular) using a suitable heuristic method given a trajectory .

**Heuristic method**

- Consider a path defined by via points
- Connect the via points with straight lines
- If the slope change sign
  - Set the velocity at the via point to be zero
- If the slope have the same sign
  - Calculate the average between the to velocities at the via point.



# Trajectory Generation – Roadmap Diagram



# Joint Space Schemes – Multiple Time – Via Points – Velocity Definition

**System Definition** (Continues Accelerations) – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.

– **Spline** - Enforcing the velocity and the acceleration to be continuous at the via point

# Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Solve for the coefficients of two cubic functions that are connected in a two segment spline with a continuous acceleration at the intermediate via point.

$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$
$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t \ \ + 3a_{13}t^2$$
$$\dot{\theta}(t) = a_{21} + 2a_{22}t \ \ + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$
$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

## Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- The joint angle velocity and acceleration for each segment (8 unknowns )

$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$
$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t \ + 3a_{13}t^2$$
$$\dot{\theta}(t) = a_{21} + 2a_{22}t \ + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$
$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

# Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Position at the beginning and end of each segment

- Segment 1

$$\theta_0(t=0) = a_{10}$$
$$\theta_{via}(t=t_{f_1}) = a_{10} + a_{11}t_{f_1} + a_{12}t_{f_1}{}^2 + a_{13}t_{f_1}{}^3$$

- Segment 2

$$\theta_{via}(t=0) = a_{20}$$
$$\theta_g(t=t_{f_2}) = a_{20} + a_{21}t_{f_2} + a_{22}t_{f_2}{}^2 + a_{23}t_{f_2}{}^3$$

# Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Velocity at the beginning of the interval

$$\dot{\theta}(t=0) = a_{11}$$

- Velocity at the end of the interval

$$\dot{\theta}(t=t_{f_2}) = a_{21} + 2a_{22}t_{f_2} + 3a_{23}t_{f_2}{}^2$$

- Velocity at the mid point between the intervals

$$\dot{\theta}\left[Function1(t=t_{f_1})\right] = \dot{\theta}\left[Function2(t=0)\right]$$

$$a_{11} + 2a_{12}t_{f_1} + 3a_{13}t_{f_1}{}^2 = a_{21}$$

## Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Acceleration at the mid point between the intervals

$$\ddot{\theta}\left[Function1(t=t_{f_1})\right]=\ddot{\theta}\left[Function2(t=0)\right]$$

$$2a_{12}+6a_{13}t_{f_1}=2a_{22}$$

- Solve 8 equations with 8 unknown

# Joint Space Schemes – Multiple Time – Via Points – Cubic Polynomials

- Solution for the 8 equations

$$a_{10} = \theta_0$$

$$a_{11} = 0$$

$$a_{12} = \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2}$$

$$a_{13} = \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3}$$

$$a_{10} = \theta_v$$

$$a_{21} = \frac{3\theta_g - 3\theta_0}{4t_f}$$

$$a_{22} = \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2}$$

$$a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}$$

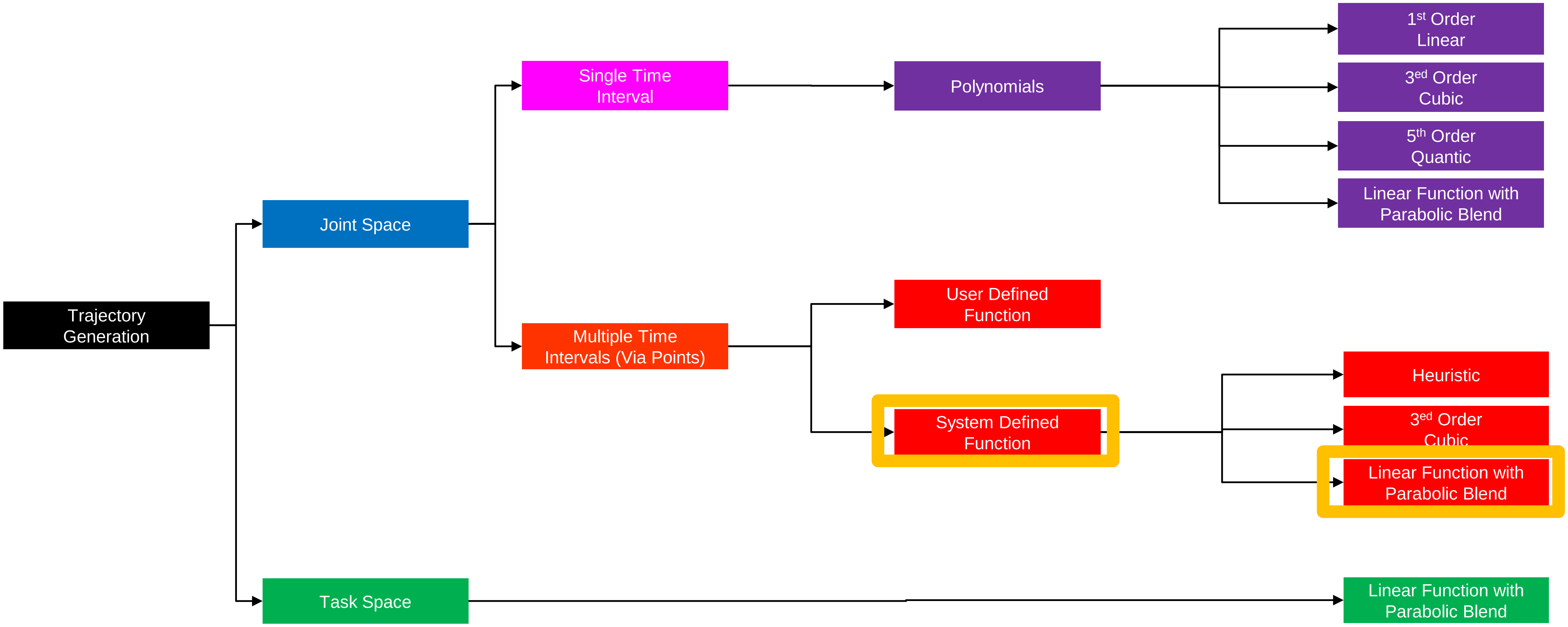
# Joint Space Schemes

Multiple Time Intervals

Via Point

**System Defined Function – Linear Function With Parabolic Blend**

# Trajectory Generation – Roadmap Diagram

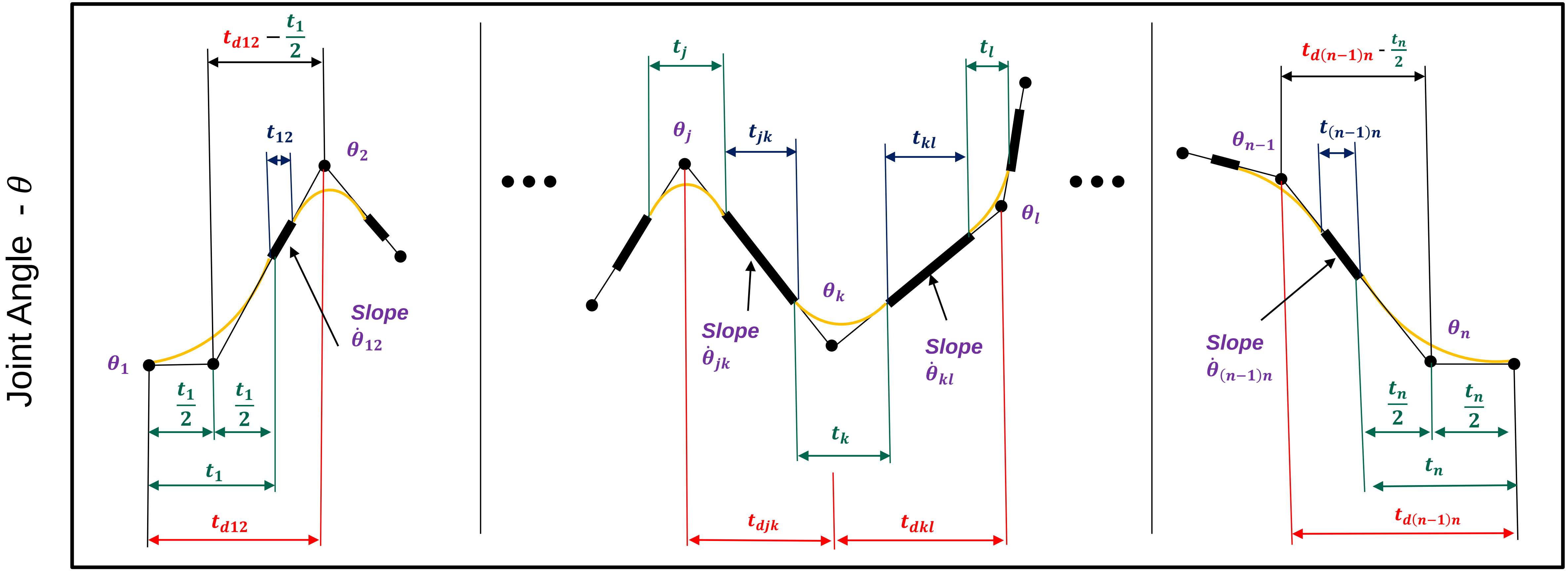


# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- The Need
  - Linear path with parabolic blends is used in cases where there are **arbitrary number of via points specified**

- Method Anatomy
  - **Linear Functions** – Connecting the via points
  - **Parabolic Blend** – Connecting the linear functions around the via points

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend



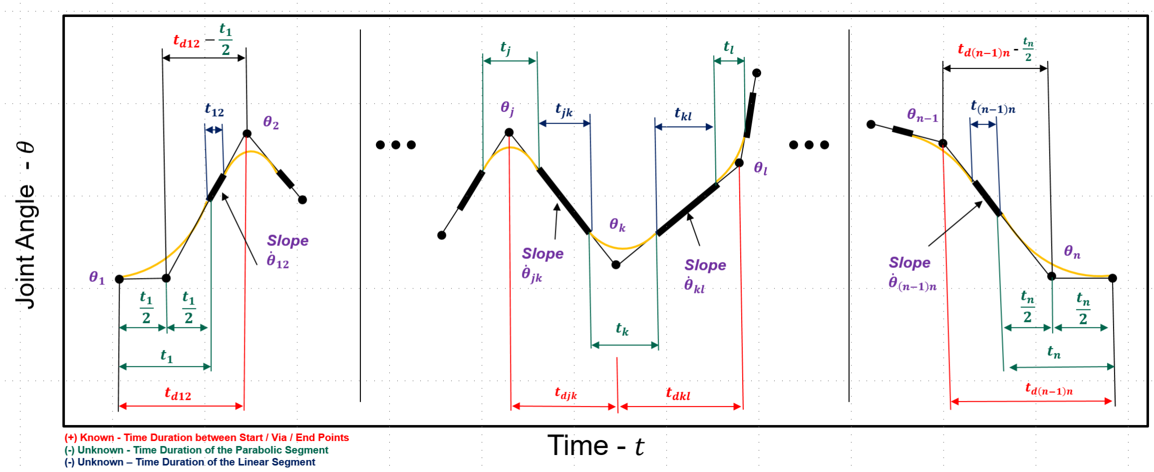

(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

Time - $t$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1 – Time Intervals**
  - Calculate the **time intervals** of the parabolic blending (marked in green)
  - Calculate the **time intervals** of the linear functions (marked in blue)

- **Task No. 2 – Functions**
  - Define the Linear Functions (marked in black)
  - Defined the parabolic blend functions (marked in gray)



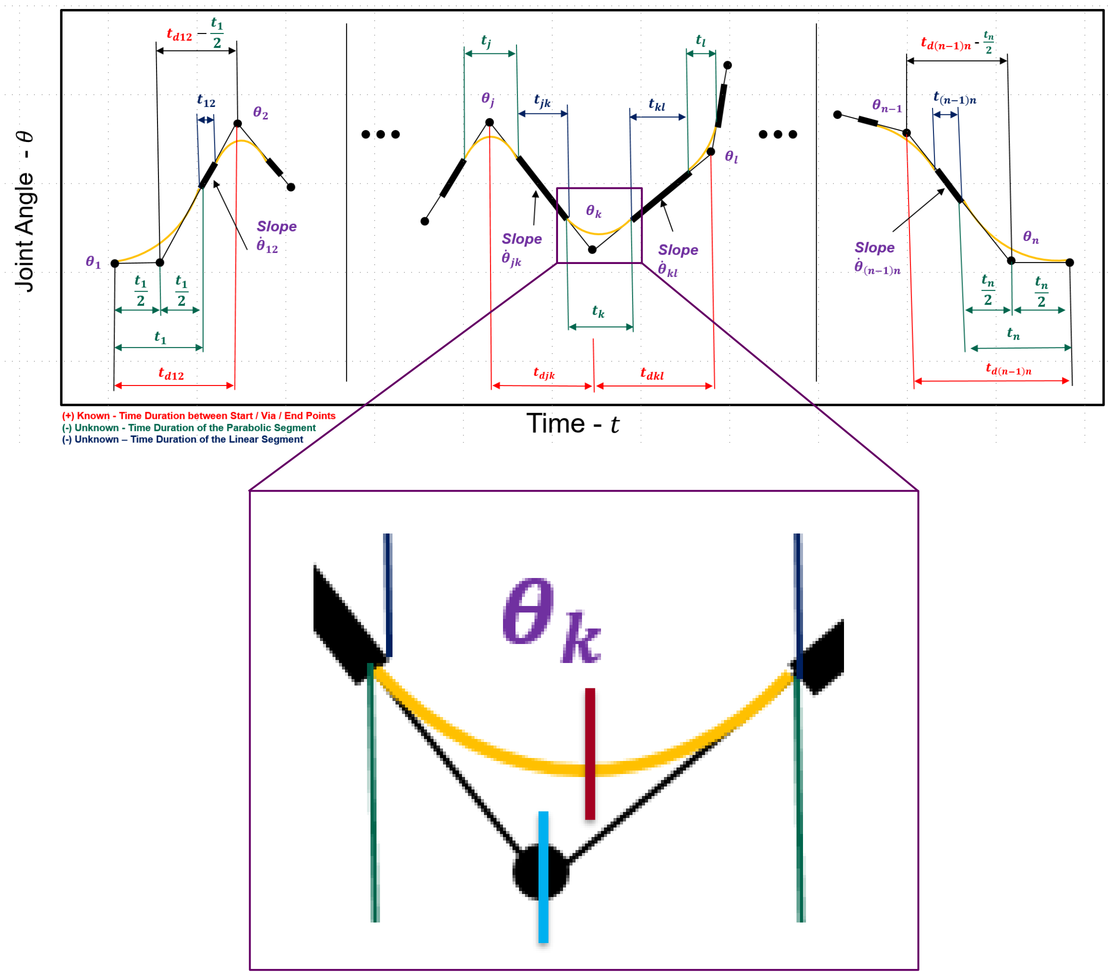
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Assumptions**
  - The velocity at the first and last through point must be zero as well as the transition from one segment to the other.
  - The velocity and acceleration of the trajectory must be:

$$\left|\dot{\theta}_{jk}\right| < \dot{\theta}_{max}$$

$$\left|\ddot{\theta}_{jk}\right| < \ddot{\theta}_{max}$$

- **Limitations**
  - Accelerations - Sufficiently large acceleration is required so as to obtain linear portion in the segment.
  - The manipulator's velocity must be zero as it passes into waypoints.
  - The system should generate two pseudo points so as to make the manipulator passes exactly through a path point without stopping.
  - The parabolic portion is assumed to be centered equally in time about through point. This later assumption makes the apex of parabolic part to be shifted away from through point, i.e. For interior path point, the apex of the parabolic portion is not equally centered in time about waypoint.



# Joint Space Schemes

Multiple Time Intervals

Via Point

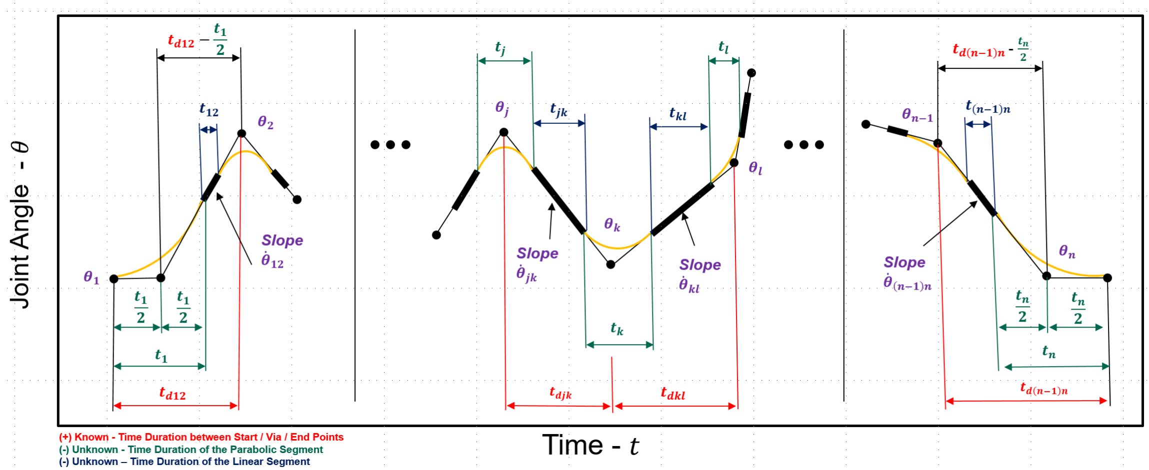
**System Defined Function – Linear Function With Parabolic Blend**

**Time Interval Analysis**

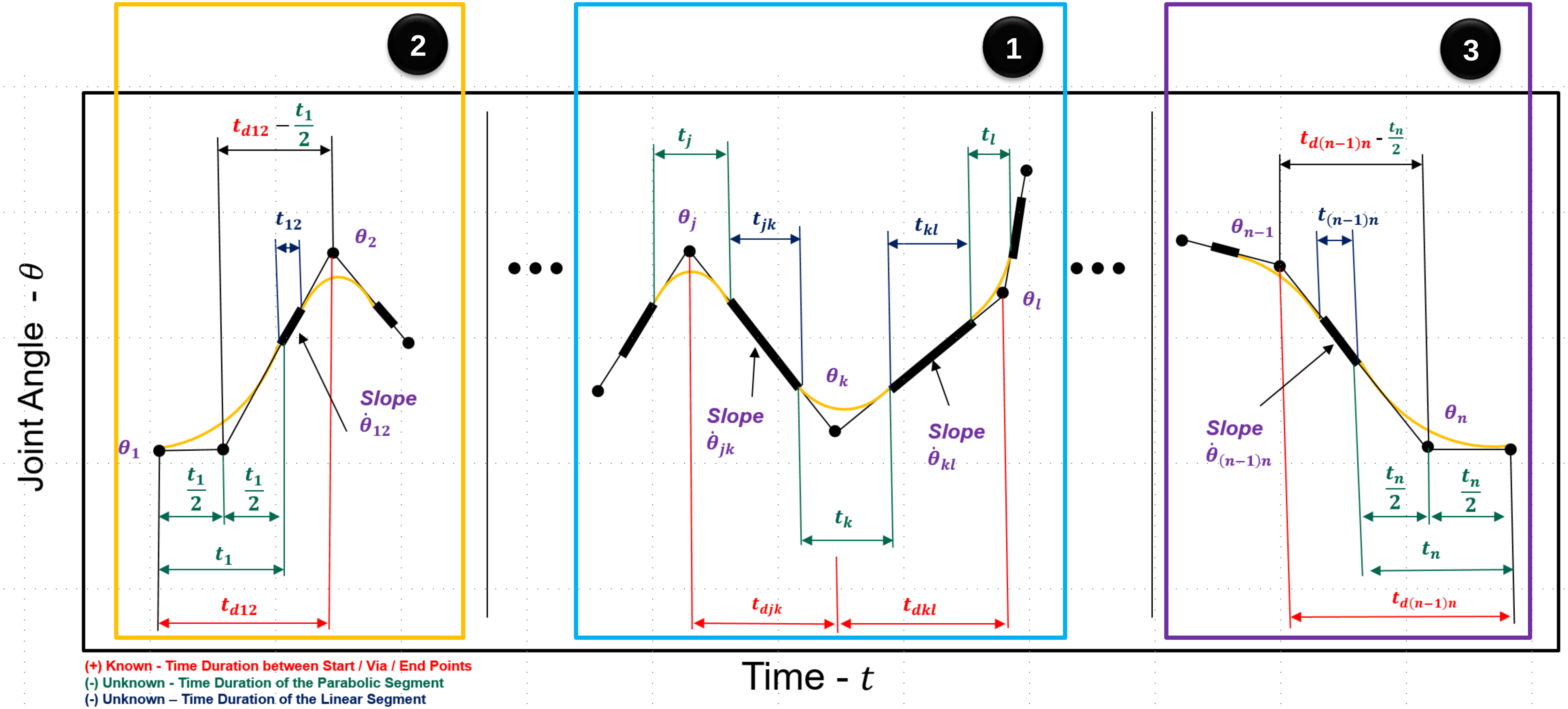
**Task No. 1 – Time Intervals**

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1 – Time Intervals**
  - Calculate the **time intervals** of the parabolic blending (marked in green)
  - Calculate the **time intervals** of the linear functions (marked in blue)
- **Task No. 2 – Functions**
  - Define the Linear Functions (marked in black)
  - Defined the parabolic blend functions (marked in yellow)

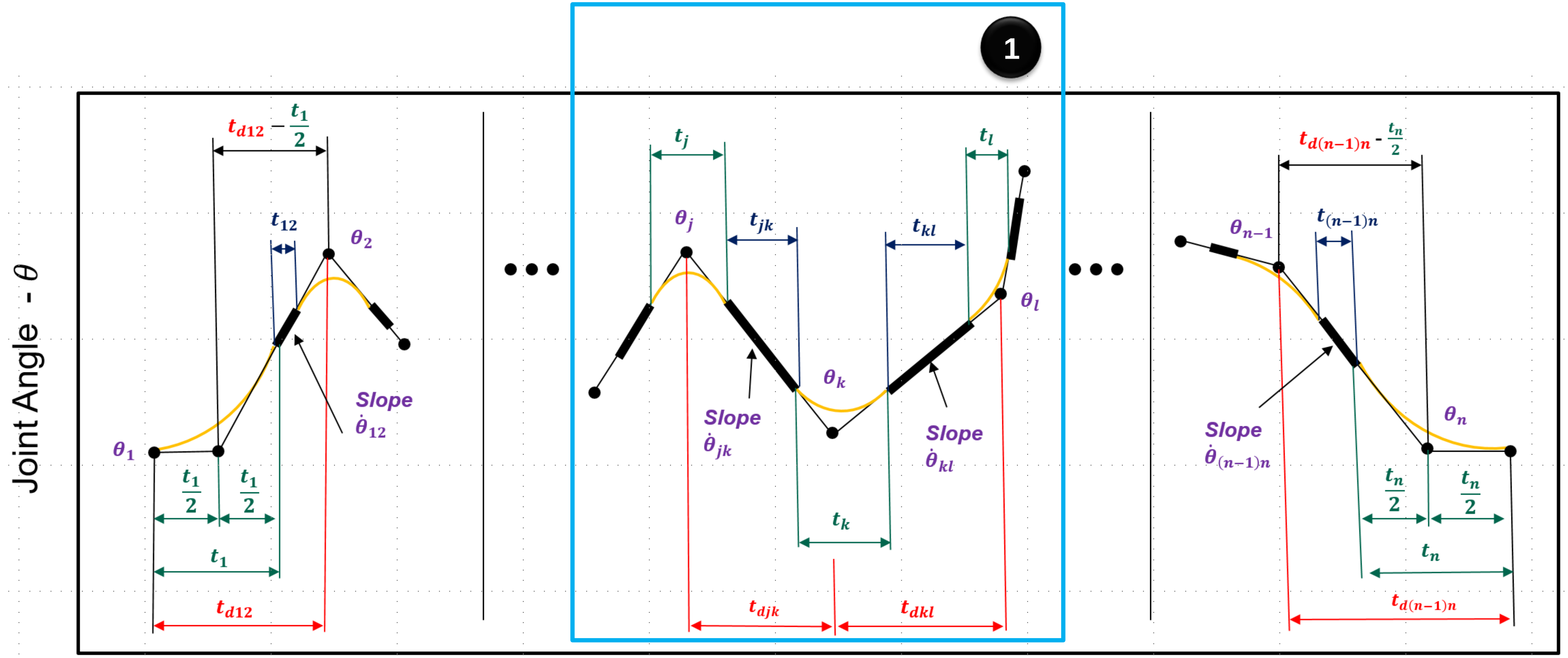


# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis




(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

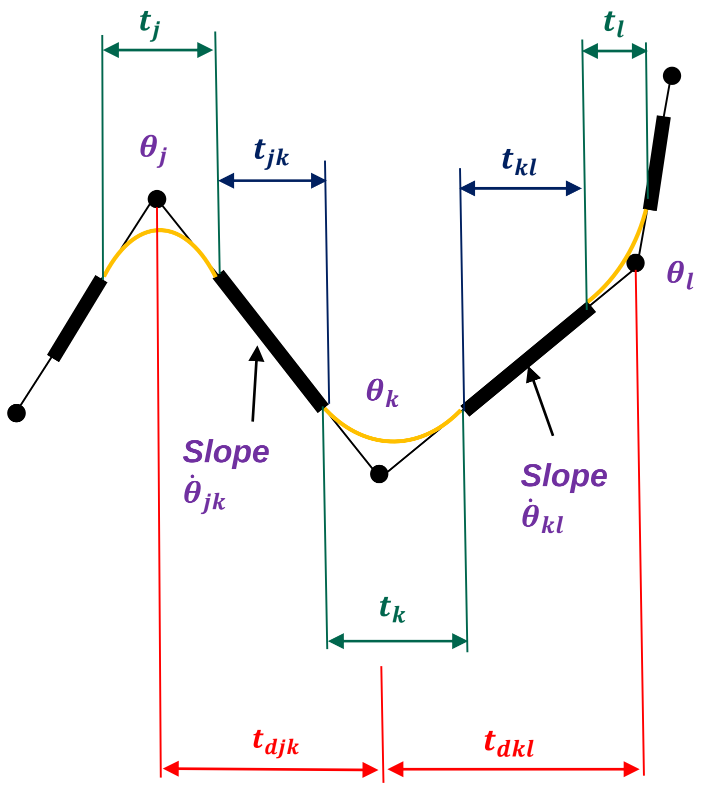



(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

**1** **Mid Points $(\boldsymbol{j}, \boldsymbol{k}, \boldsymbol{l})$**

- **Calculate**
  - $t_k$ - The time duration during the parabolic bland
  - $t_{jk}$ - The time duration during the linear interpolation
- **Given**
  - $\theta_j, \theta_k, \theta_l$ - The angles at points j,k,l - $\theta_j, \theta_k, \theta_l$
  - $t_{djk}, t_{dkl}$ - The time durations from point J to K and from point K to L
  - $|\ddot{\theta}_k|$ The desired acceleration during the parabolic blend through point K




- **Velocity**

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}$$

$$\dot{\theta}_{kl} = \frac{\theta_l - \theta_k}{t_{dkl}}$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

**1** Acceleration

$$ax^2 + bx + c \qquad \begin{matrix} a > 0 & \cup \\ a < 0 & \cap \end{matrix}$$

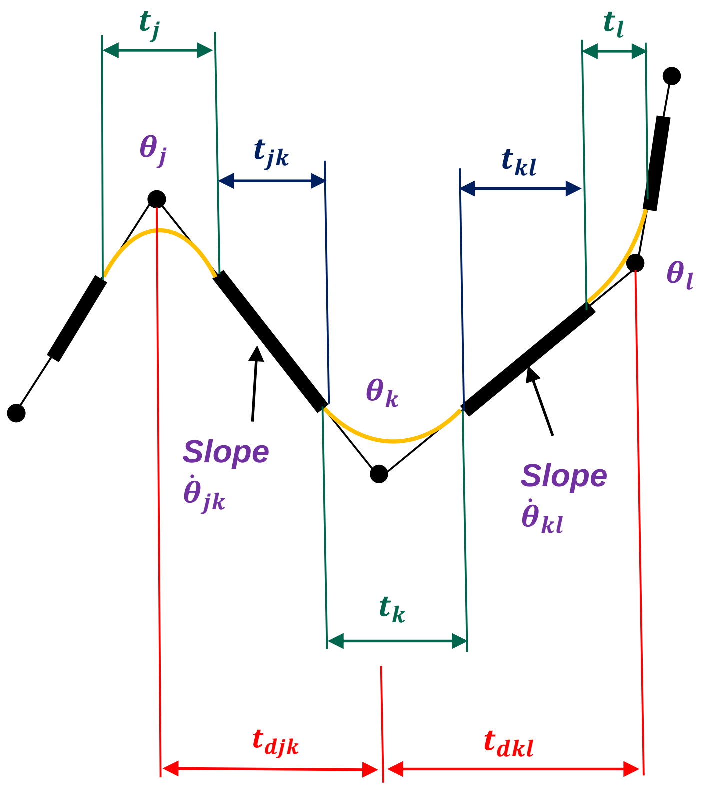
$$v = at$$

$$t = \frac{v}{a}$$

$$\ddot{\theta}_k = [\overbrace{SGN(\dot{\theta}_{KL} - \dot{\theta}_{JK})}^{+\ \cup}_{\underbrace{}_{-\ \cap}}]\ddot{\theta}_k$$

$$t_k = \frac{\dot{\theta}_{KL} - \dot{\theta}_{JK}}{\ddot{\theta}_k}$$

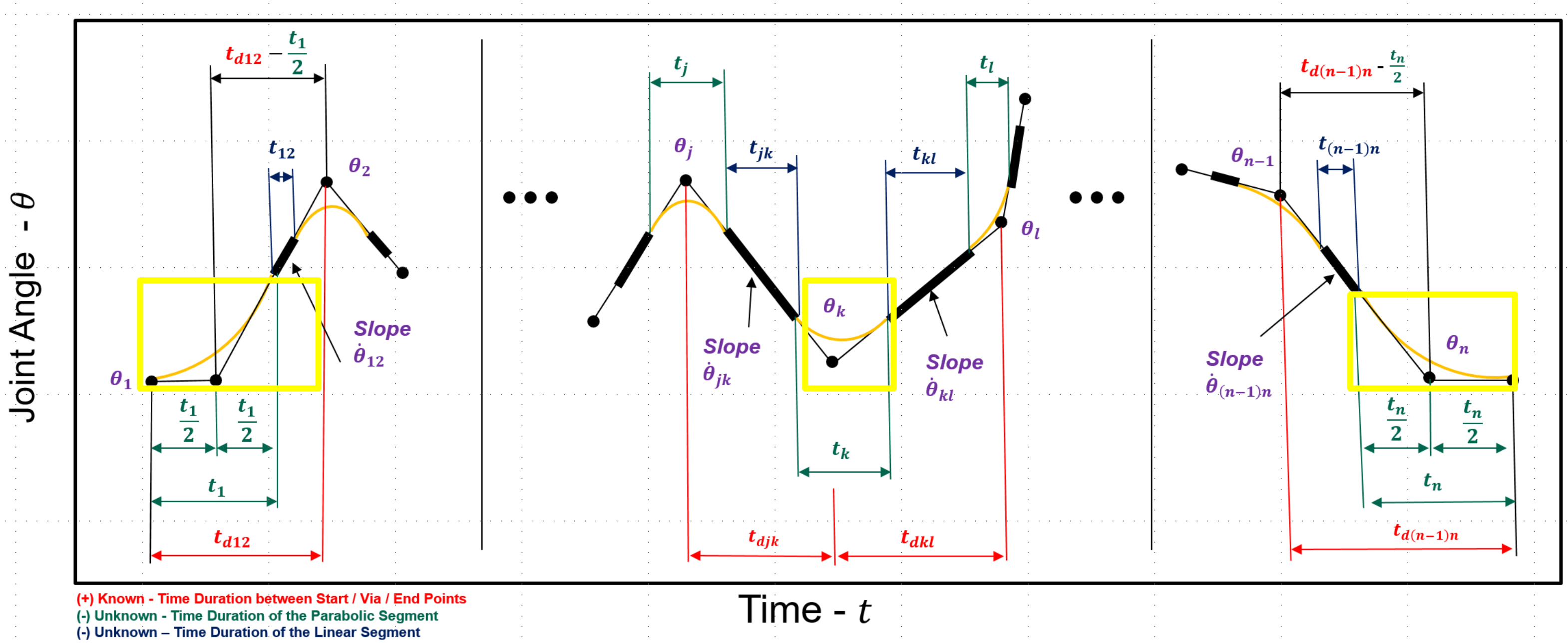
$$t_{JK} = t_{dJK} - \frac{1}{2}t_J - \frac{1}{2}t_K$$



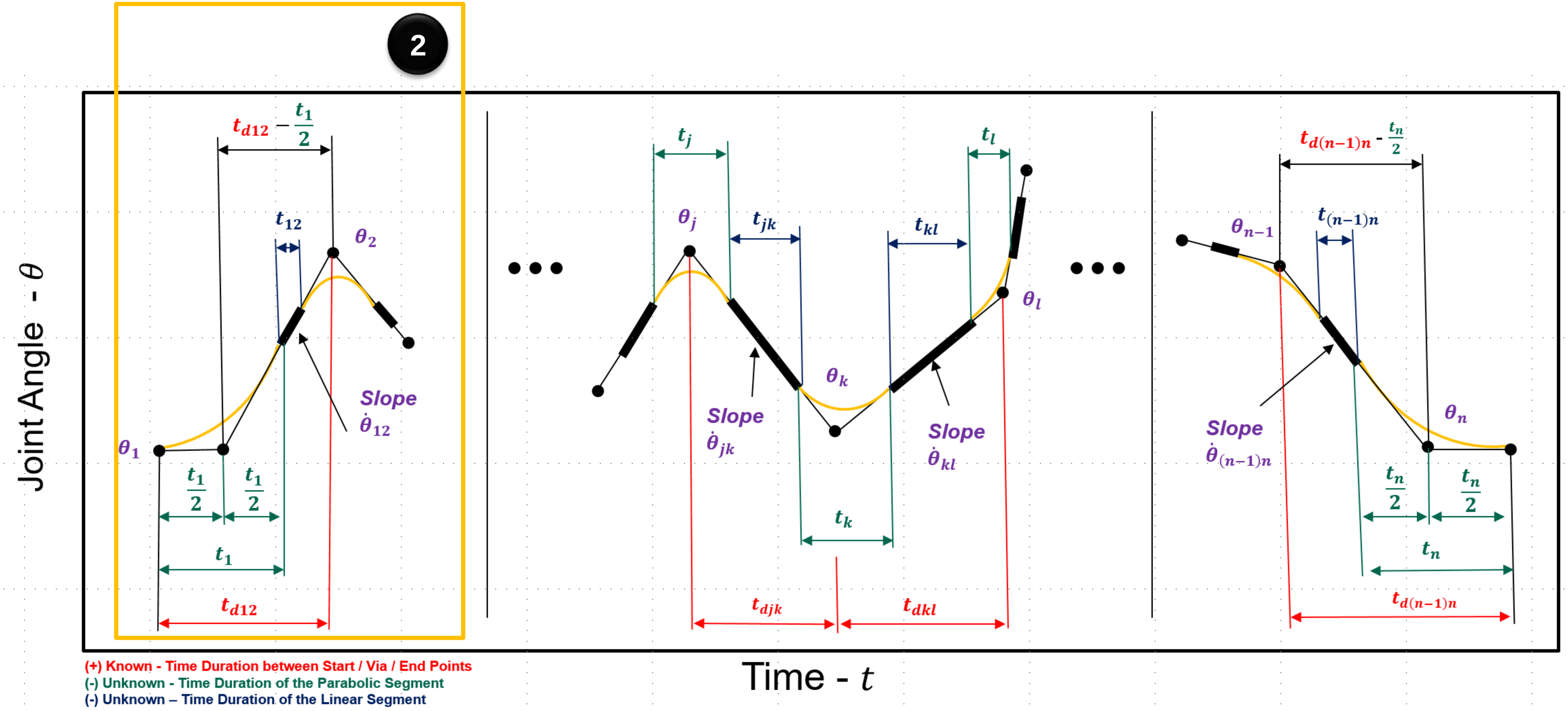
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

- **Note: First and Last Segments** – The first and the last segments must be handled slightly differently because the entire bland regeion at one end of the segment must be counted in the total segment's time duration.

- Note the difference between
  - Mid Points $t_{djk}$ $t_{dkl}$
  - First Point $t_{d12}$
  - Last Point $t_{d(n-1)n}$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

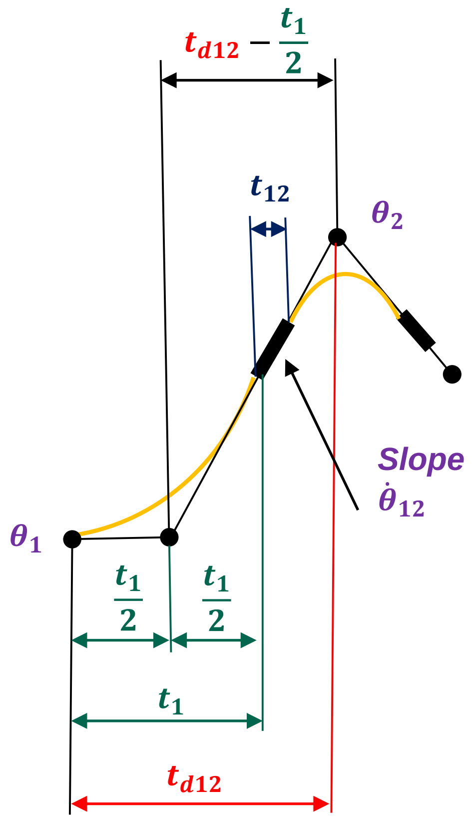
**2**

## First Segment $(1, 2)$

- **Calculate**
  - $t_1$ - The time duration during the parabolic bland
  - $t_{12}$ - The time duration during the linear interpolation
- **Given**
  - $\theta_1, \theta_2$ - The angles at points 1,2
  - $t_{d12}$ - The time durations from point 1 to 2
  - $|\ddot{\theta}_1|$ The desired acceleration during the parabolic blend through point 1
- **Velocity**

$$v = \frac{x}{t} = at$$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2}t_1} = \ddot{\theta}_1 t_1$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

**2** Acceleration

$$ax^2 + bx + c \qquad \begin{matrix} a > 0 \\ a < 0 \end{matrix}$$

$$\ddot{\theta}_1 = \overbrace{\underbrace{SGN(\theta_2 - \theta_1)}_{-}}^{+}|\ddot{\theta}_1|$$

Using the previous velocity equation

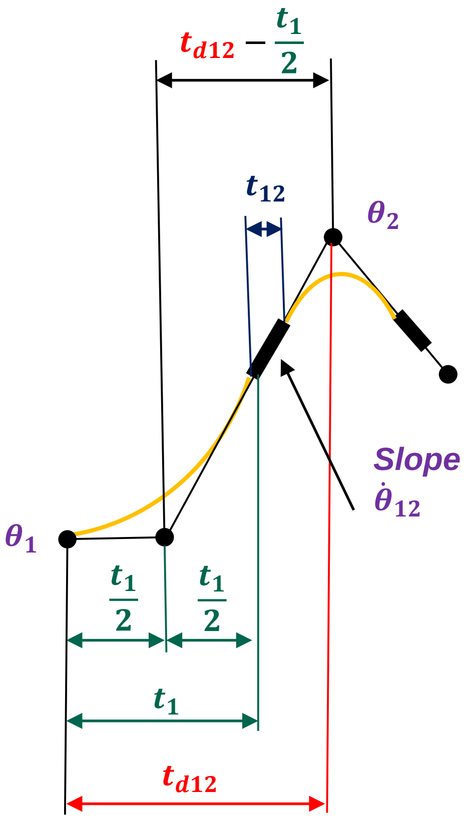
$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2}t_1} = \ddot{\theta}_1 t_1$$

resulted in

$$\theta_2 - \theta_1 = \ddot{\theta} t_1 \left( t_{d12} - \frac{1}{2} t_1 \right)$$

rearranging into a quadratic equation

$$\left(\frac{\ddot{\theta}_1}{2}\right) t_1^2 - (\ddot{\theta}_1 t_{d12})t + (\theta_2 - \theta_1) = 0$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

**2** Solving the quadratic equation

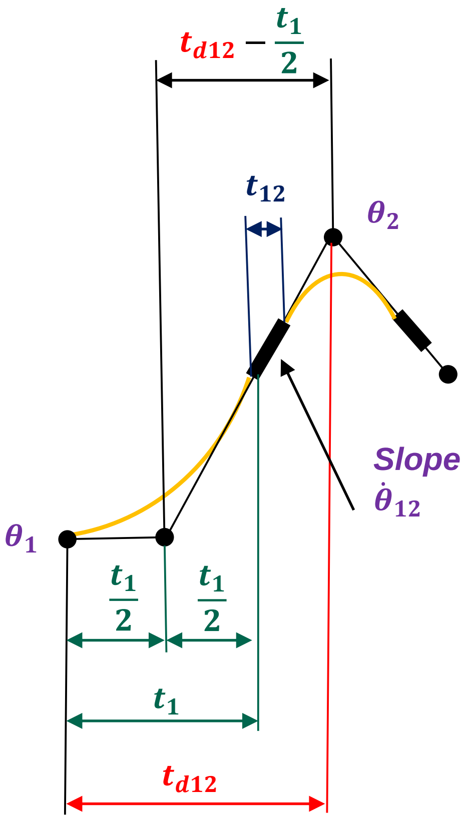
$$\left(\frac{\ddot{\theta}_1}{2}\right)t_1^2-(\ddot{\theta}_1 t_{d12})t+(\theta_2-\theta_1)=0$$

Solving the quadratic equation resulted in

$$t_1=\frac{\ddot{\theta}_1 t_{d12}\pm\sqrt{\ddot{\theta}_1^2 t_{d12}^2-\frac{4\ddot{\theta}_1}{2}(\theta_2-\theta_1)}}{\ddot{\theta}_1}$$

Simplifying the result by eliminating $\ddot{\theta}_1$ resulted in the time for the parabolic blending

$$t_1=t_{d12}\pm\sqrt{t_{d12}^2+\frac{2(\theta_1-\theta_2)}{\ddot{\theta}_1}}$$



2

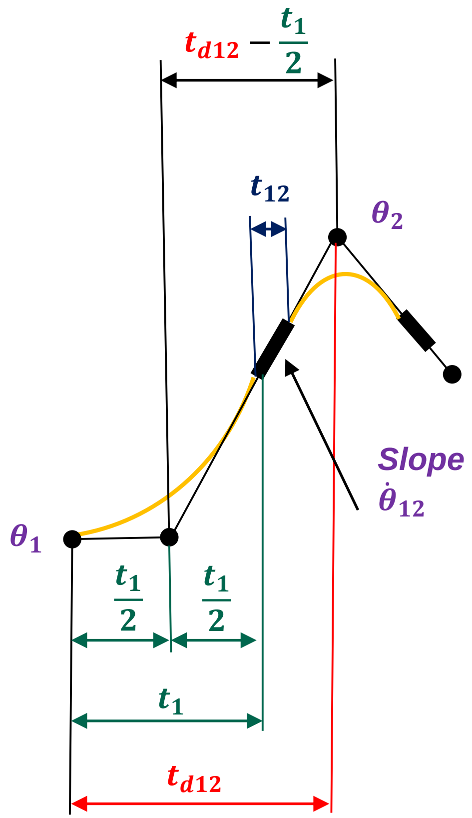
$$t_1 = t_{d12} \pm \sqrt{t_{d12}^2 + \frac{2(\theta_1 - \theta_2)}{\ddot{\theta}_1}}$$

Out of the two solution use the solution for $t_1$ including the minus sign (-)
Because

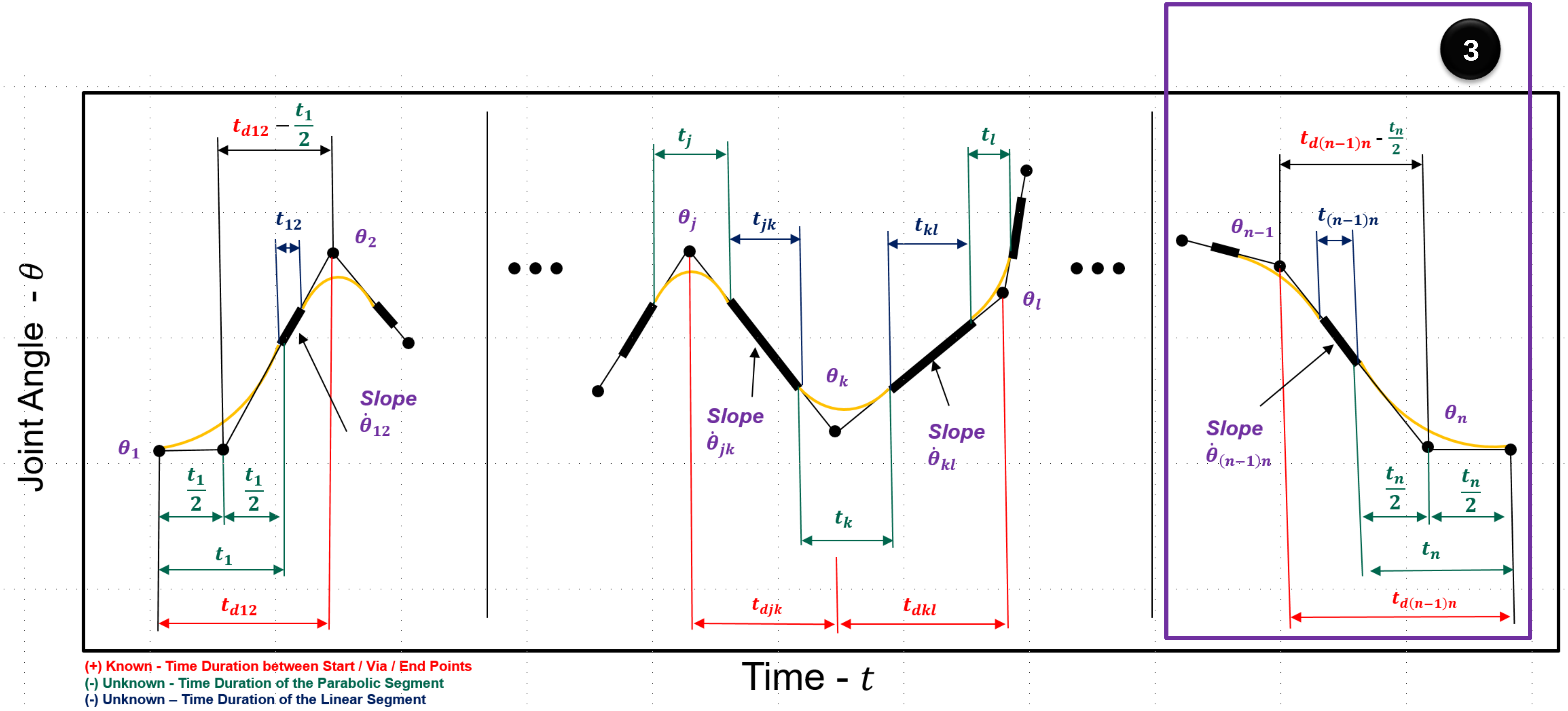
$$t_1 < t_{d12}$$

The time duration of the liner interpolation is than calculated as

$$t_{12} = t_{d12} - t_1 - \frac{1}{2} t_2$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis






UCLA



3 **Last Segment** $\left((n-1),n\right)$

- **Calculate**
  - $t_n$ - The time duration during the parabolic bland
  - $t_{(n-1)n}$ - The time duration during the linear interpolation
- **Given**
  - $\theta_{n-1}, \theta_n$ - The angles at points $n-1$ , and $n$
  - $t_{d(n-1)n}$ - The time durations from $n-1$ to $n$
  - $\left|\ddot{\theta}_n\right|$ The desired acceleration during the parabolic blend through point 1
- **Velocity**

$$v = \frac{x}{t} = at$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_{n-1}-\theta_n}{t_{d(n-1)n}-\frac{1}{2}t_n} = \ddot{\theta}_n t_n$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

**3** Acceleration

$$ax^2 + bx + c \qquad a > 0 \qquad a < 0$$

$$\ddot{\theta}_n = \underbrace{\overbrace{SGN(\theta_{n-1} - \theta_n)}^{+}}_{-}|\ddot{\theta}_n|$$

Using the previous velocity equation

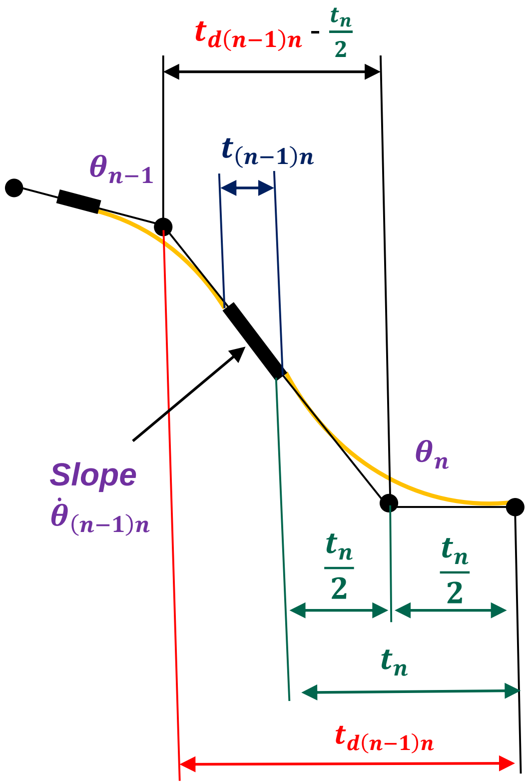
$$\dot{\theta}_{(n-1)n} = \frac{\theta_{n-1} - \theta_n}{t_{d(n-1)n} - \frac{1}{2}t_n} = \ddot{\theta}_n t_n$$

resulted in

$$\theta_{n-1} - \theta_n = \ddot{\theta}_n t_n \left( t_{d(n-1)n} - \frac{1}{2} t_n \right)$$

rearranging into a quadratic equation

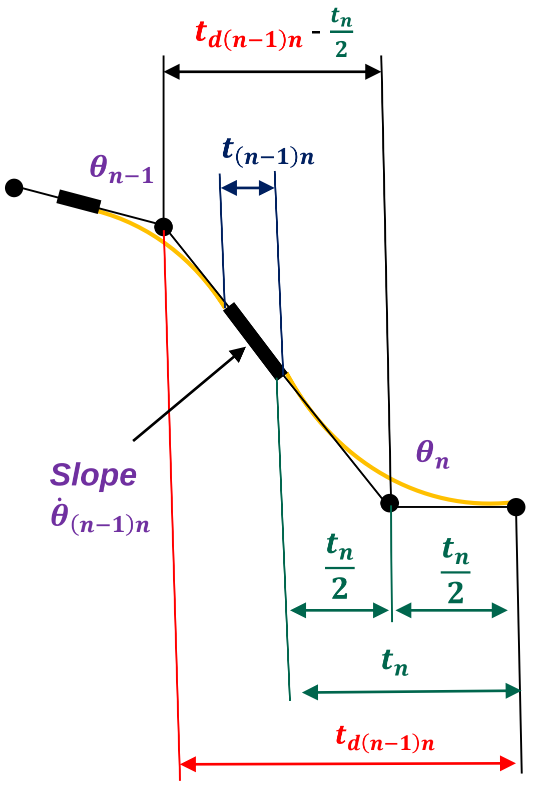
$$\left(\frac{\ddot{\theta}_n}{2}\right) t_n^2 - \left(\ddot{\theta}_n t_{d(n-1)n}\right) t_n + (\theta_{n-1} - \theta_n) = 0$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

3 Solving the quadratic equation

$$\left(\frac{\ddot{\theta}_n}{2}\right)t_n^2 - \left(\ddot{\theta}_n t_{d(n-1)n}\right)t_n + \left(\theta_{n-1} - \theta_n\right) = 0$$




Solving the quadratic equation resulted in

$$t_n = \frac{\ddot{\theta}_n t_{d(n-1)n} \pm \sqrt{\ddot{\theta}_n^2 t_{d(n-1)n}^2 - \frac{4\ddot{\theta}_n}{2}\left(\theta_{n-1} - \theta_n\right)}}{\ddot{\theta}_n}$$

Simplifying the result by eliminating $\ddot{\theta}_n$ resulted in the time for the parabolic blending

$$t_n = t_{d(n-1)n} \pm \sqrt{t_{d(n-1)n}^2 - \frac{2\left(\theta_{n-1} - \theta_n\right)}{\ddot{\theta}_n}}$$

3

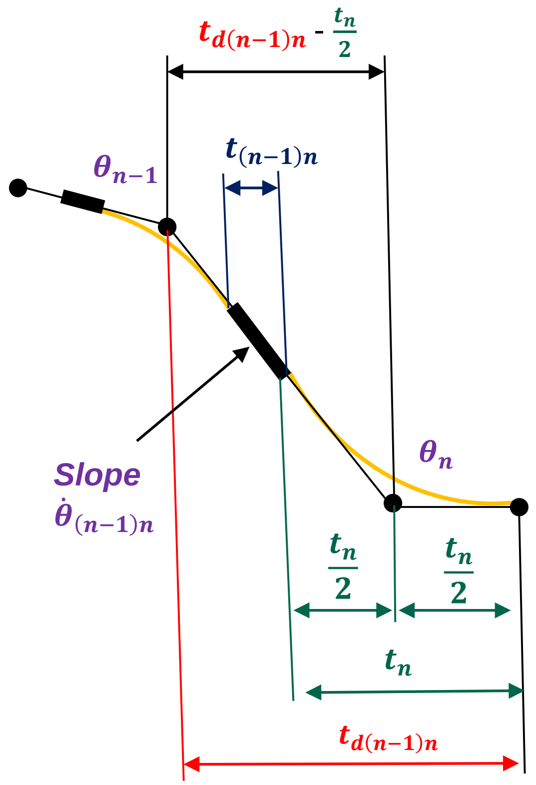
$$t_n = t_{d(n-1)n} \boxed{\pm} \sqrt{t_{d(n-1)n}^2 - \frac{2(\theta_{n-1} - \theta_n)}{\ddot{\theta}_n}}$$

Out of the two solution use the solution for $t_1$ including the minus sign (-) Because

$$t_n < t_{d(n-1)n}$$

The time duration of the liner interpolation is than calculated as

$$t_{(n-1)n} = t_{d(n-1)} - t_n - \frac{1}{2} t_{n-1}$$



# Joint Space Schemes

Multiple Time Intervals

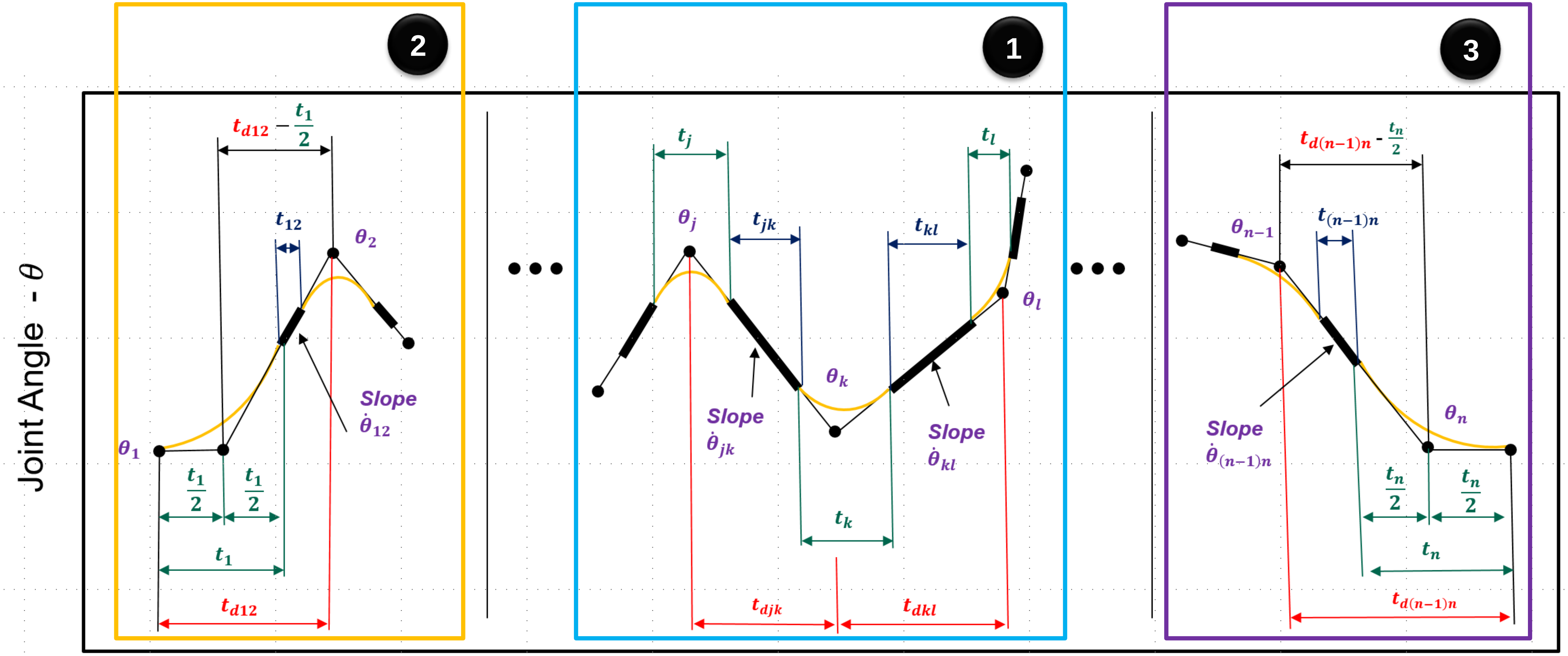
Via Point

**System Defined Function – Linear Function With Parabolic Blend**

**Time Interval Analysis**

**Task No. 2 – Function (Linear & Parabolic)**

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Interpolation Function Analysis



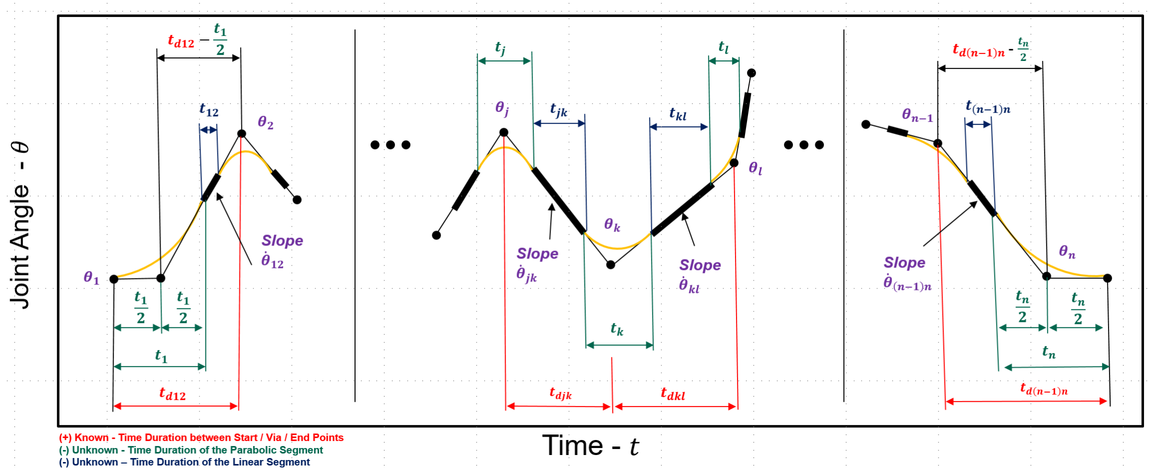

(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

Time - $t$

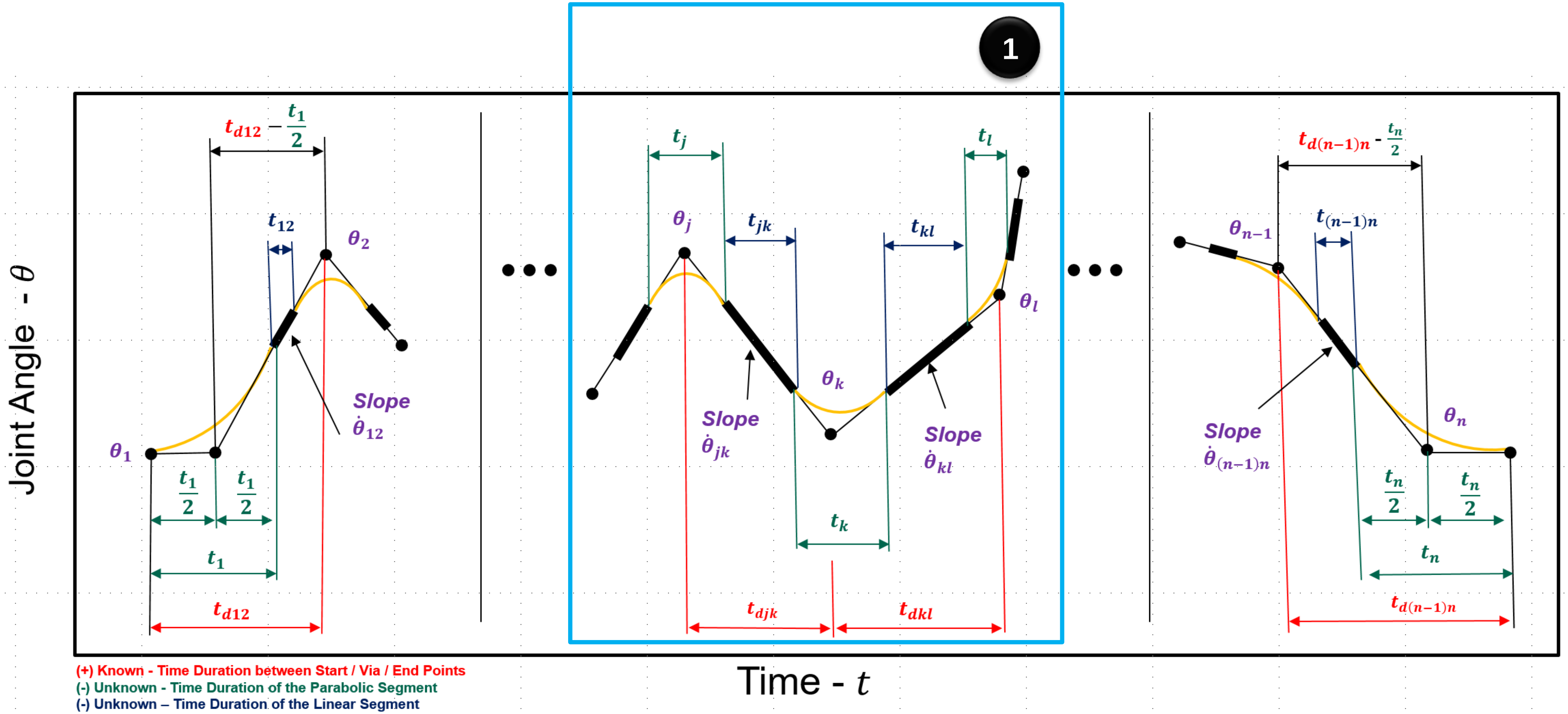
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1 – Time Intervals**
  - Calculate the **time intervals** of the parabolic blending (marked in green)
  - Calculate the **time intervals** of the linear functions (marked in blue)

- **Task No. 2 – Functions**
  - Define the Linear Functions (marked in black)
  - Defined the parabolic blend functions (marked in yellow)



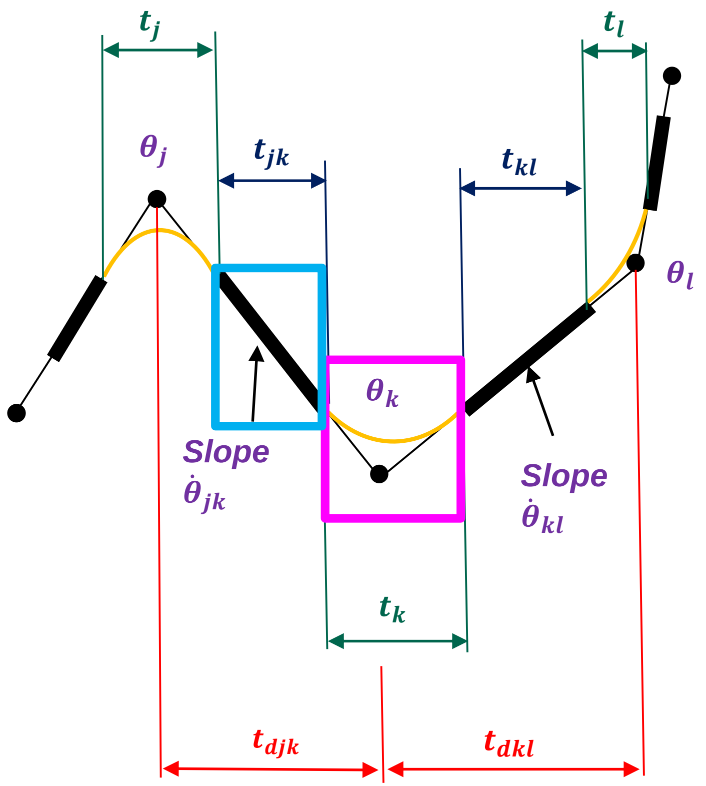
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Interpolation Function Analysis




(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

1 Middle Segment Points $(\boldsymbol{j}, \boldsymbol{k})$

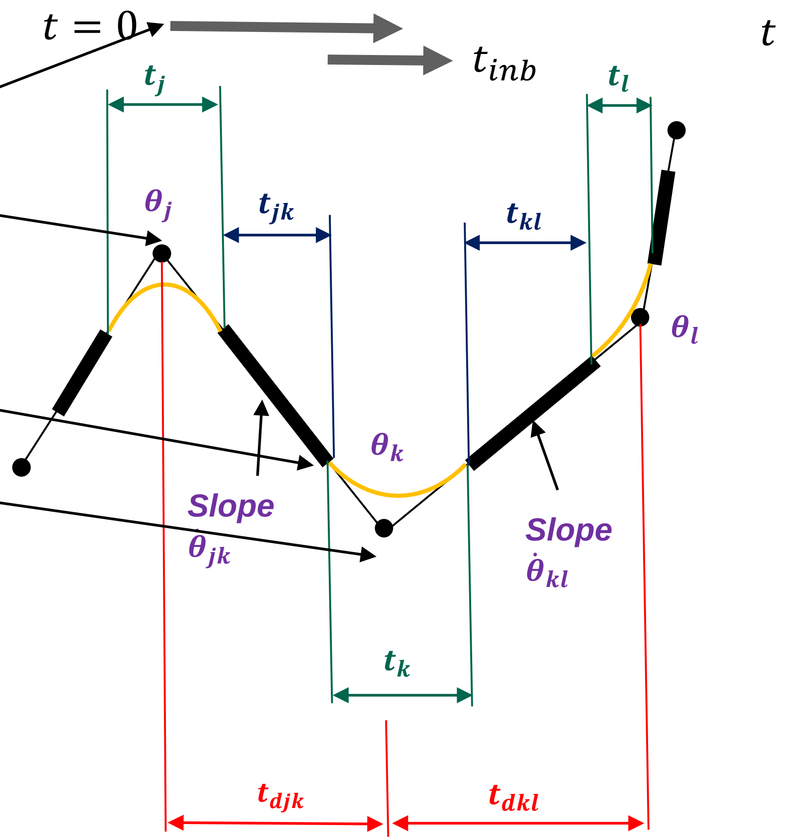
- Calculate
  - Liner Spline
  - Parabolic Spine
- Given
  - $t_k$ - The time duration during the parabolic bland
  - $t_{jk}$ - The time duration during the linear interpolation
  - $\theta_j, \theta_k, \theta_l$ - The angles at points j,k,l - $\theta_j, \theta_k, \theta_l$
  - $t_{djk}, t_{dkl}$ - The time durations from point J to K and from point K to L
  - $|\ddot{\theta}_k|$ The desired acceleration during the parabolic blend through point K



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

**1** **Middle Segment $(\boldsymbol{j}, \boldsymbol{k})$**

- Start measuring time from the via point $\theta_j$ for the segment $(\boldsymbol{j}, \boldsymbol{k})$
- Reset the time to 0 at the parabolic blend
- Reset the time to 0 at $\theta_K$ for the following segment



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

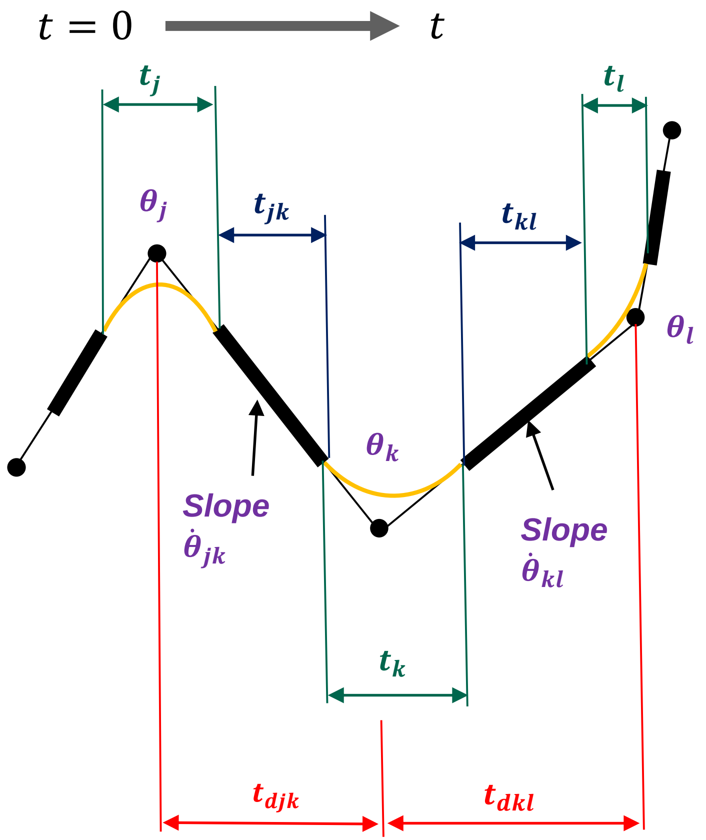
**1** **Middle Segment $(\boldsymbol{j}, \boldsymbol{k})$**

– **LINEAR SPLINE** $t \in t_{jk}$

$$\theta = \theta_j + \dot{\theta}_{jk} t$$

$$\dot{\theta} = \dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k}$$

$$\ddot{\theta} = 0$$



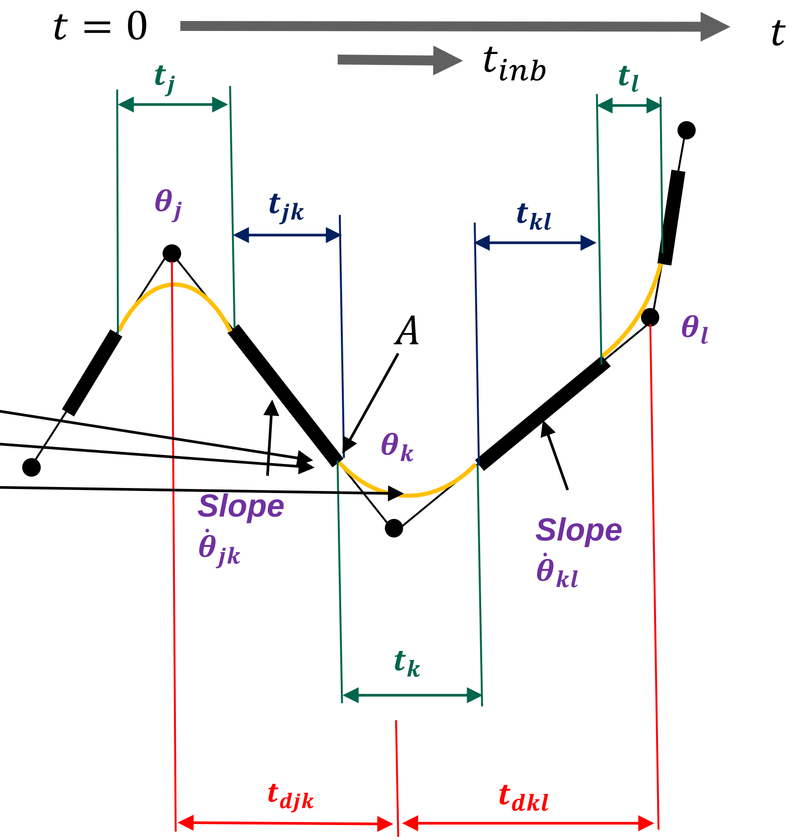
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

1 **Middle Segment $(\boldsymbol{j},\boldsymbol{k})$**

– PARABOLIC SPLINE $\quad t \in t_k$

$$\theta = a_2 t_{inb}^2 + a_1 t_{inb} + a_0$$

$$conditions \begin{cases} \theta(t_{inb}=0) = \theta_j + \dot{\theta}_{jk} t \\ \dot{\theta}(t_{inb}=0) = \dot{\theta}_{jk} \\ \ddot{\theta}\,(t_{inb}=0) = \ddot{\theta}_k \end{cases}$$




Condition 1: The angle at the beginning of the parabolic spline is equal to the angle of at the end of the linear spline

Condition 2: The velocity at the beginning of the parabolic spline is equal to the velocity of the linear spline

Condition 3: The acceleration at the beginning of the parabolic spline as well as throughout this segment is constant



1 **Middle Segment $(\boldsymbol{j},\boldsymbol{k})$**

– PARABOLIC SPLINE $t \in t_k$

$$\theta = a_2 t_{inb}^2 + a_1 t_{inb} + a_0$$

$$conditions \begin{cases} \theta(t_{inb}=0) = \theta_j + \dot{\theta}_{jk} t_A \\ \dot{\theta}(t_{inb}=0) = \dot{\theta}_{jk} \\ \ddot{\theta}(t_{inb}=0) = \ddot{\theta}_k \end{cases}$$

Condition 1

$$\theta(t_{inb}=0) = a_2 0 + a_1 0 + a_0 = \theta_j + \dot{\theta}_{jk} t_A$$
$$a_0 = \theta_j + \dot{\theta}_{jk} t_A$$

Condition 2

$$\dot{\theta}(t_{inb}=0) = \dot{\theta}_{jk} = 2a_2 0 + a_1 + \dot{\theta}_{jk} = \dot{\theta}_{jk}$$
$$a_1 = 0$$



**1** **Middle Segment $(\boldsymbol{j}, \boldsymbol{k})$**

– PARABOLIC SPLINE $\quad t \in t_k$

$$\theta = a_2 t_{inb}^2 + a_1 t_{inb} + a_0$$

$$conditions \begin{cases} \theta(t_{inb} = 0) = \theta_j + \dot{\theta}_{jk} t_A \\ \dot{\theta}(t_{inb} = 0) = \dot{\theta}_{jk} \\ \ddot{\theta}(t_{inb} = 0) = \ddot{\theta}_k \end{cases}$$

Condition 3

$$\theta = a_2 t_{inb}^2 + \theta_j + \dot{\theta}_{jk} t_A$$

$$\dot{\theta} = 2a_2 t_{inb} + \theta_j$$

$$\ddot{\theta}(t_{inb} = 0) = 2a_2 = \ddot{\theta}_k$$

$$a_2 = \frac{1}{2}\ddot{\theta}_k$$

Summary

$$\theta = \frac{1}{2}\ddot{\theta}_k t_{inb}^2 + \dot{\theta}_{jk} t_A + \theta_j$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

1 **Middle Segment $(\boldsymbol{j}, \boldsymbol{k})$**

– PARABOLIC SPLINE $\quad t \in t_k$

At point A the time can be expressed in two different ways

$$t_A \rightarrow t_{inb} = 0$$

$$t_A = \frac{1}{2} t_J + t_{jk}$$

The time during the blending can be expressed as

$$t_{inb} = t - \left(\frac{1}{2} t_j + t_{jk}\right)$$

Therefore, the following equation describing the joint angle during the parabolic blend

$$\theta = \frac{1}{2} \ddot{\theta}_k t_{inb}^2 + \dot{\theta}_{jk} t_A + \theta_j$$

Can be rewritten as

$$\theta = \frac{1}{2} \ddot{\theta}_k \left(t - \left(\frac{1}{2} t_j + t_{jk}\right)\right)^2 + \dot{\theta}_{jk} \left(\frac{1}{2} t_J + t_{jk}\right) + \theta_j$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

**1** **Middle Segment $(\boldsymbol{j}, \boldsymbol{k})$**

– PARABOLIC SPLINE $\quad t \in t_k$

Therefore, the following equation describing the joint angle during the parabolic blend

$$\theta = \frac{1}{2}\ddot{\theta}_k t_{inb}^2 + \dot{\theta}_{jk} t_A + \theta_j$$

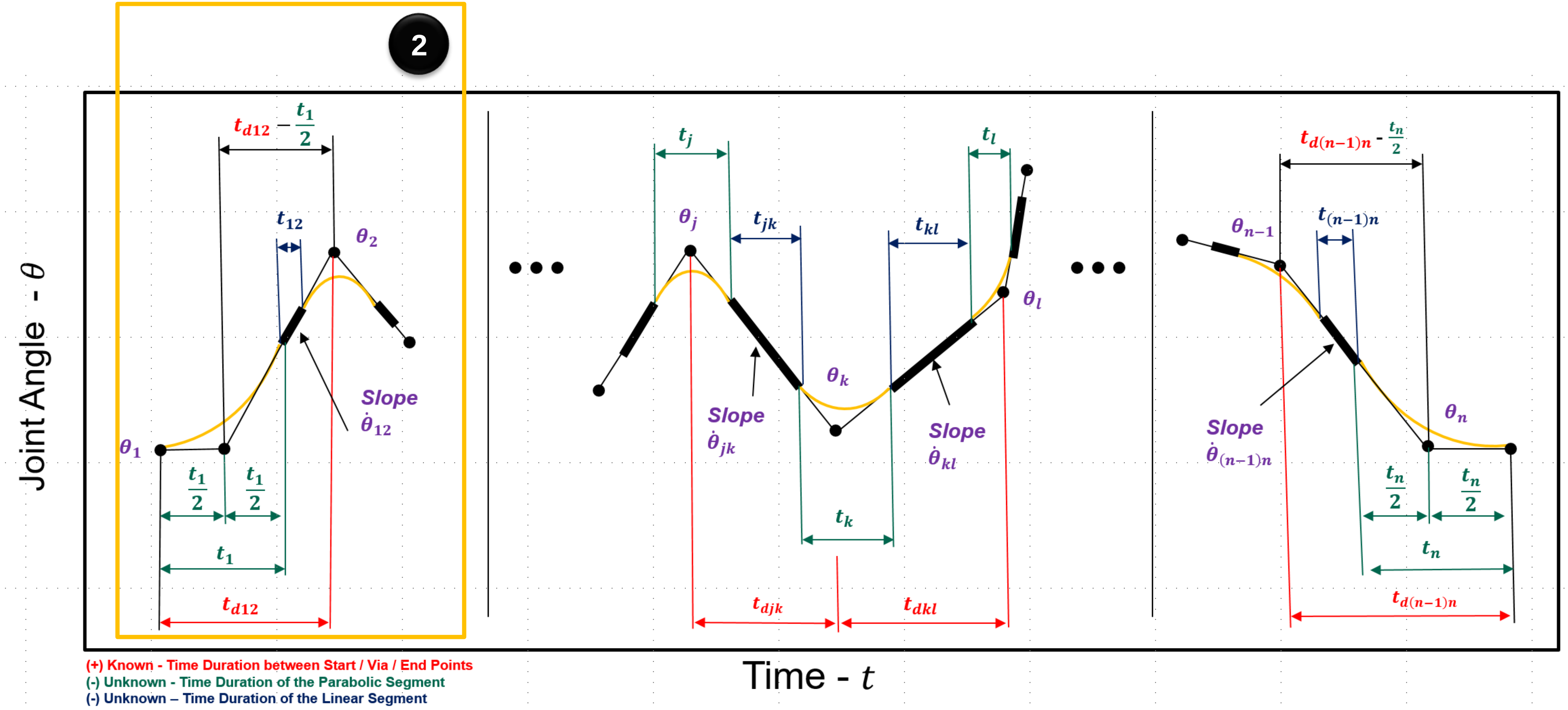
can be rewritten in two different way the first is as a function of time $t$ starting at point $\boldsymbol{j}$

$$\theta = \frac{1}{2}\ddot{\theta}_k \left(t - \left(\frac{1}{2}t_j + t_{jk}\right)\right)^2 + \dot{\theta}_{jk}\left(\frac{1}{2}t_J + t_{jk}\right) + \theta_j$$

and the second of time $t_{inb}$ Starting at the beginning of the parabolic blend

$$\theta = \frac{1}{2}\ddot{\theta}_k t_{inb}^2 + \dot{\theta}_{jk}\left(\frac{1}{2}t_J + t_{jk}\right) + \theta_j$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Interpolation Function Analysis




Instructor: Jacob Rosen
Advanced Robotic - Department of Mechanical & Aerospace Engineering - UCLA

## 2 First Segment (1,2)

- **Linear Spline** $t \in t_{d12}$

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

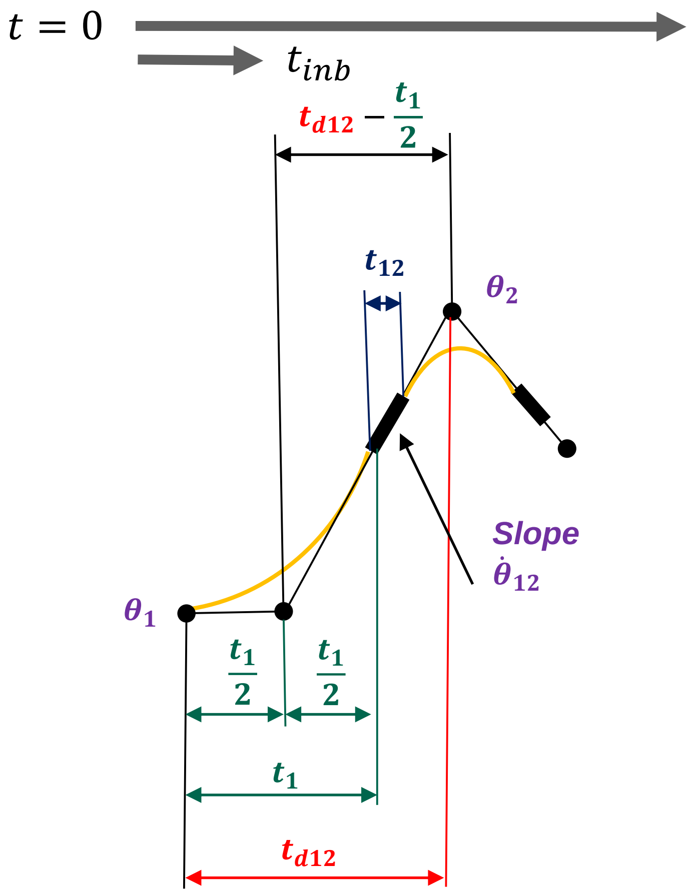
$$\dot{\theta} = \dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1}$$

$$\ddot{\theta} = 0$$

- **Parabolic Spline** $t \in t_1$

$$\theta = a_0 + a_1 t + a_2 t^2$$

$$conditions \begin{cases} \theta(t=0) = \theta_0 \\ \dot{\theta}(t=0) = 0 \\ \dot{\theta}(t=t_1) = \dot{\theta}_{12} \end{cases}$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

2

$$\begin{cases} \theta = a_0 + a_1 t + a_2 t^2 \\ \dot{\theta} = a_1 + 2a_2 t \\ \ddot{\theta} = 2a_2 \end{cases}$$

Plugging the conditions into the various equations and solve for $a_0, a_1, a_2$

$$\theta(t=0) = \theta_0 = a_0 + a_1 0 + a_2 0 \rightarrow a_0 = \theta_0$$

$$\dot{\theta}(t=0) = 0 = a_1 + 2a_2 0 \rightarrow a_1 = 0$$

$$\dot{\theta}(t=t_1) = \dot{\theta}_{12} = 0 + 2a_2 t_1 \rightarrow a_2 = \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1}$$

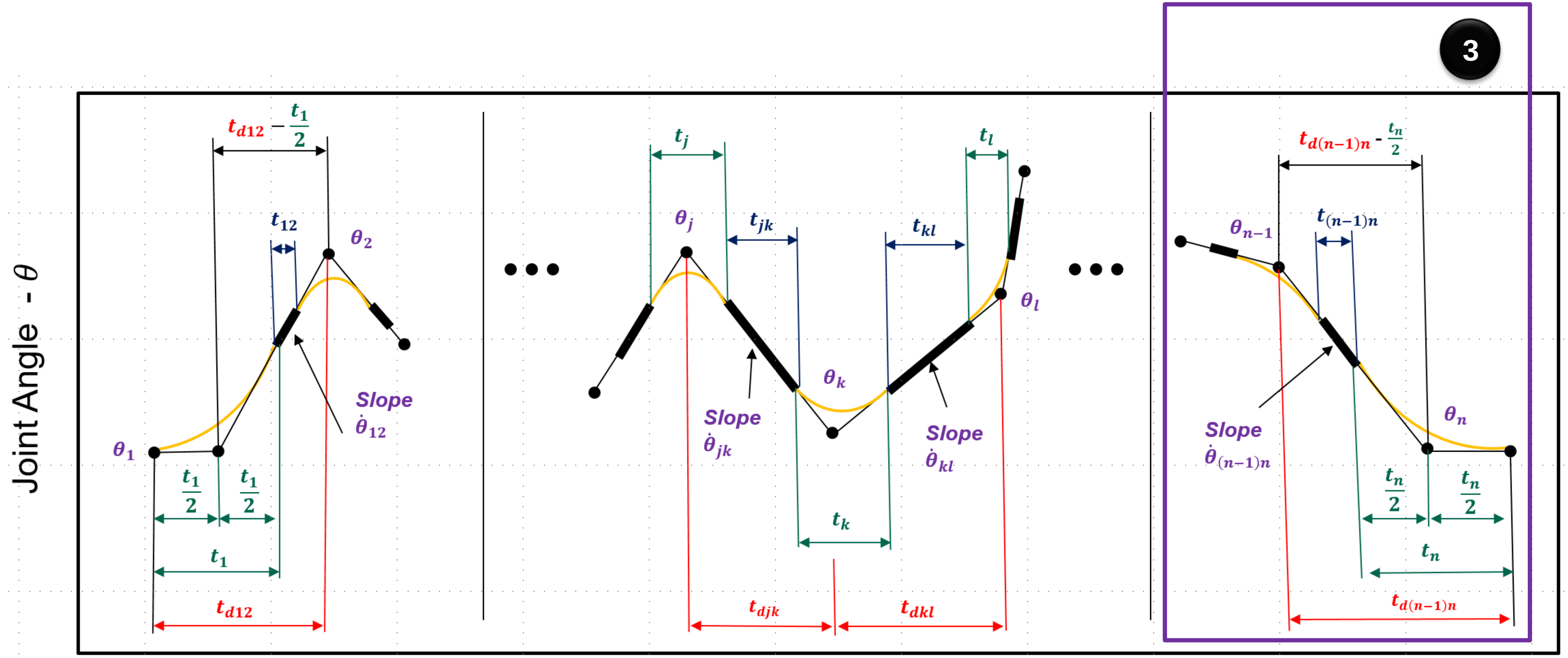
Resulting in the joint angle, angular velocity, and angular acceleration

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

$$\dot{\theta} = \frac{\dot{\theta}_{12}}{t_1} t$$

$$\ddot{\theta} = \frac{\dot{\theta}_{12}}{t_1}$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Interpolation Function Analysis




(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

**3** **Last Segment**

– **Linear Spline** $t \in t(n-1)n$

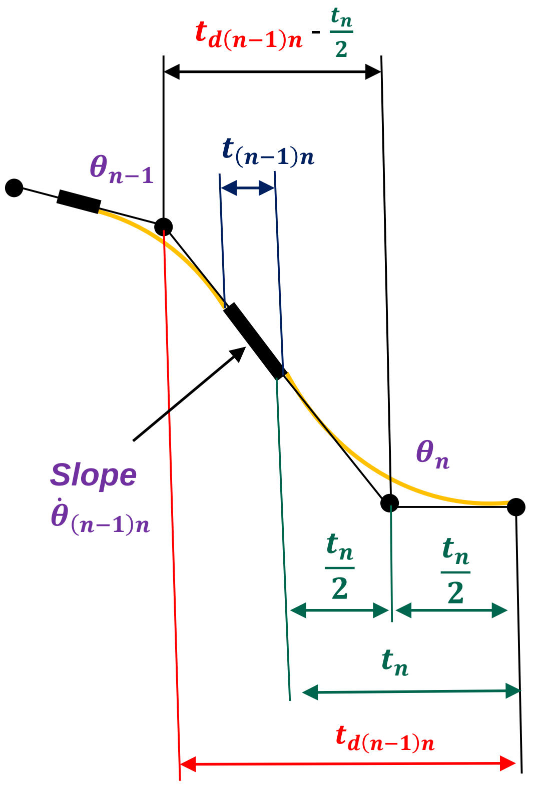
$$\begin{cases} \theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t \\ \dot{\theta} = \dot{\theta}_{(n-1)n} \\ \ddot{\theta} = 0 \end{cases}$$

– **Parabolic Bland** $t \in t_n$

$$t_{inb} = t - \left(\frac{1}{2} t_{n-1} + t_{(n-1)n}\right)$$

$$\theta = a_0 + a_1 t_{inb} + a_2 t_{inb}^2$$

$$\text{CONDITIONS} \begin{cases} \theta(t_{inb} = 0) = \theta_{n-1} + \dot{\theta}_{(n-1)n} t_{(n-1)n} = \theta_{inb} \\ \dot{\theta}(t_{inb} = 0) = \dot{\theta}_{(n-1)n} \\ \dot{\theta}(t = t_{inb}) = 0 \end{cases}$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

$$
\begin{cases}
\theta = a_0 + a_1 t + a_2 t^2 \\
\dot{\theta} = a_1 + 2a_2 t \\
\ddot{\theta} = 2a_2
\end{cases}
$$

Plugging the conditions into the various equations and solve for $a_0, a_1, a_2$

$$\theta(t_{inb} = 0) = \theta_{inb} = a_0 + a_1 0 + a0 \;\rightarrow a_0 = \theta_{inb}$$

$$\dot{\theta}(t_{inb} = 0) = \dot{\theta}_{(n-1)n} = a_1 + 2a_2 0 \;\rightarrow a_1 = \dot{\theta}_{(n-1)n}$$

$$\dot{\theta}(t = t_{inb}) = 0 = \dot{\theta}_{(n-1)n} + 2a_2 t_{inb} \rightarrow a_2 = -\frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}}$$

Resulting in the joint angle

$$\theta = \theta_{inb} + \theta_{(n-1)n} t - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

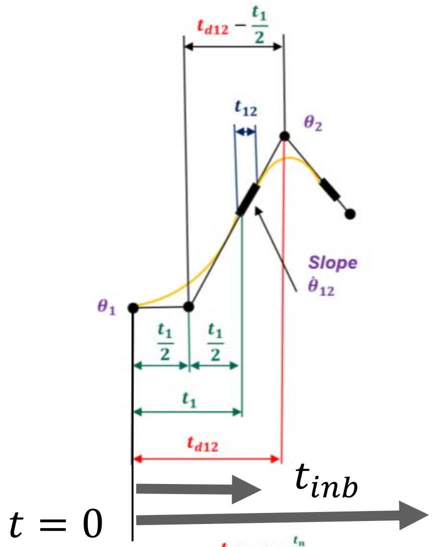
Linear / Parabolic Splines – Summary Reminder

First Segment

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

$$t_{inb} = t$$



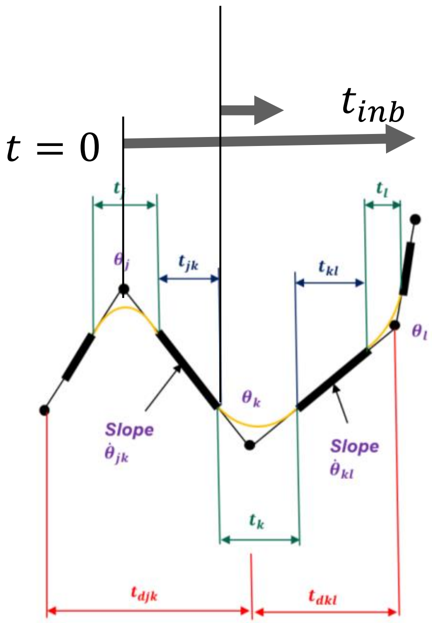

Mid Segment

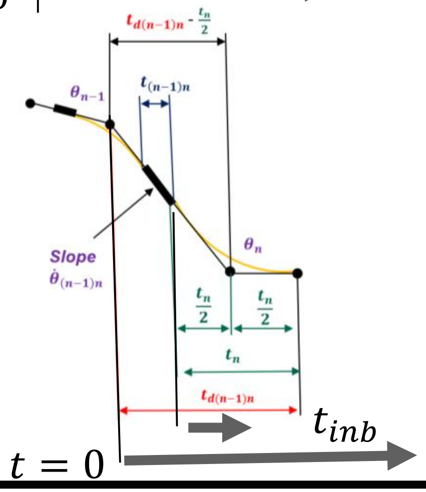
$$\theta = \theta_j + \dot{\theta}_{jk} t$$

$$\theta = \theta_j + \dot{\theta}_{jk}(t - t_{inb}) + \frac{1}{2}\ \ddot{\theta}_k^2 t_{inb} =$$

$$\theta_j + \dot{\theta}_{jk}\left(\frac{1}{2}t_j + t_{jk}\right) + \frac{1}{2}\ \ddot{\theta}_k^2 \left( t - \left(\frac{1}{2}t_j + t_{jk}\right)\right)^2$$

$$t_{inb} = t - \left(\frac{1}{2}t_j + t_{jk}\right)$$




Last Segment

$$\theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t$$

$$\theta = \theta_{inb} + \theta_{(n-1)n} t - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

$$t_{inb} = t - \left(\frac{1}{2}t_{n-1} + t_{(n-1)n}\right)$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Calculation Approaches

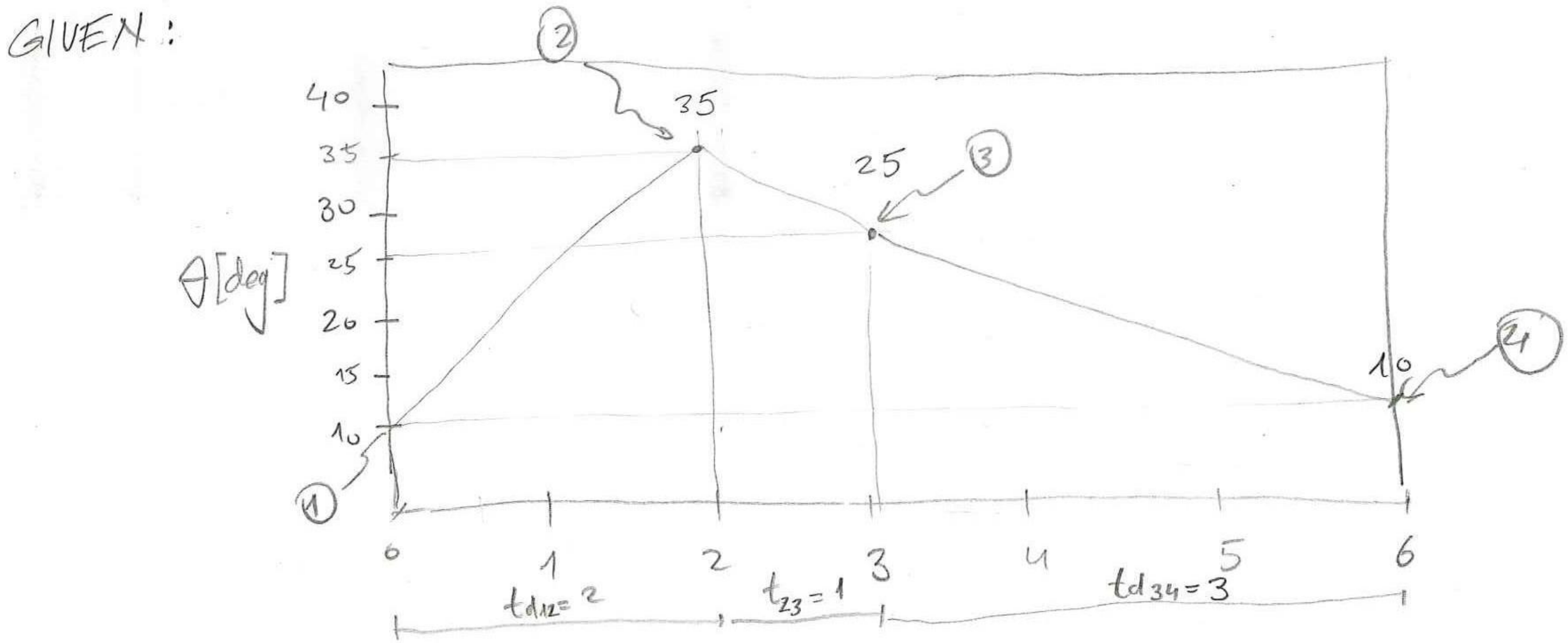
- **Calculation Approach No. 1**
  - User Defines
    - Via Points
    - Desired time duration of segments
  - System Defines
    - Use default value of acceleration for each joint

- **Calculation Approach No. 2**
  - System calculate time durations based on default velocities

- **Note for both Approaches –** At all the blends, sufficiently large acceleration must be used so that the system has  sufficient time to get into the linear portion of the segment before the next blend region starts

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

The trajectory of a particular joint is specified as follows:



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Acceleration – magnitude of the default acceleration to use at all blend points is

$$|\ddot{\theta}|_{max} = 50\ \frac{\circ}{sec^2}$$

CALCULATE:

- Segment velocities $\dot{\theta}$
- Blend times $t_i$ $(t_1, t_2, t_3, t_4)$
- Linear segments times

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

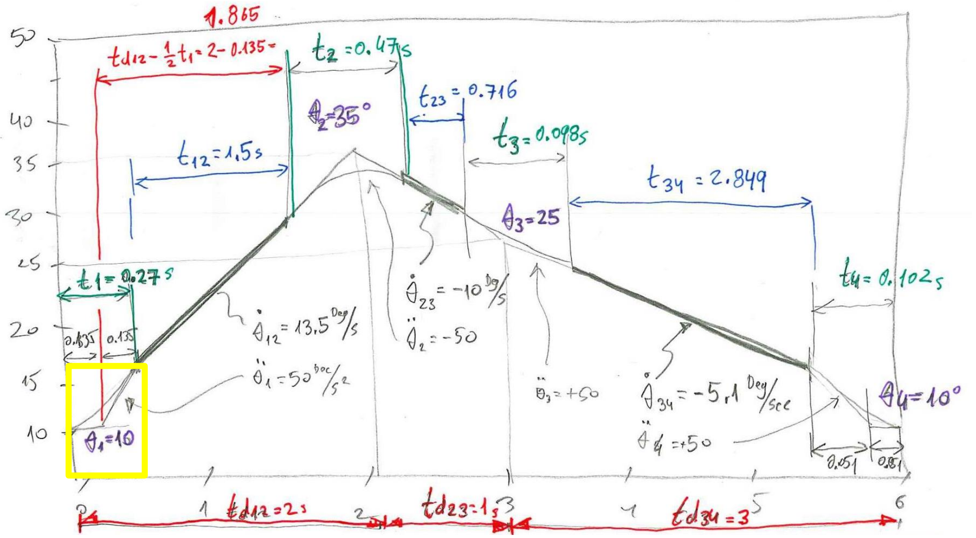
- **Solution Step 1 – Time Interval Analysis**

**First Segment (t: 0→2)**

$$\ddot{\theta}_1 = 50 = SGN(\theta_2 - \theta_1)\left|\ddot{\theta}_1\right| = SGN(35-10)|50| = +50\,\frac{deg}{s}$$

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(\theta_2-\theta_1)}{\ddot{\theta}_1}} = 2 - \sqrt{4 - \frac{2(35-10)}{50}} = \boxed{0.27\ sec}$$

$$\dot{\theta}_{12} = \frac{\theta_2-\theta_1}{t_{d12}-\frac{1}{2}t_1} = \frac{35-10}{2-\frac{1}{2}(0.27)} = 13.5\,\frac{deg}{s}$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

## Second Segment (t: 2→3)

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}$$

$$\begin{cases} j = 2 \\ k = 3 \end{cases} \rightarrow \dot{\theta}_{23} = \frac{\theta_3 - \theta_1}{t_{d23}} = \frac{25-35}{1} = -10\,\frac{deg}{s}$$

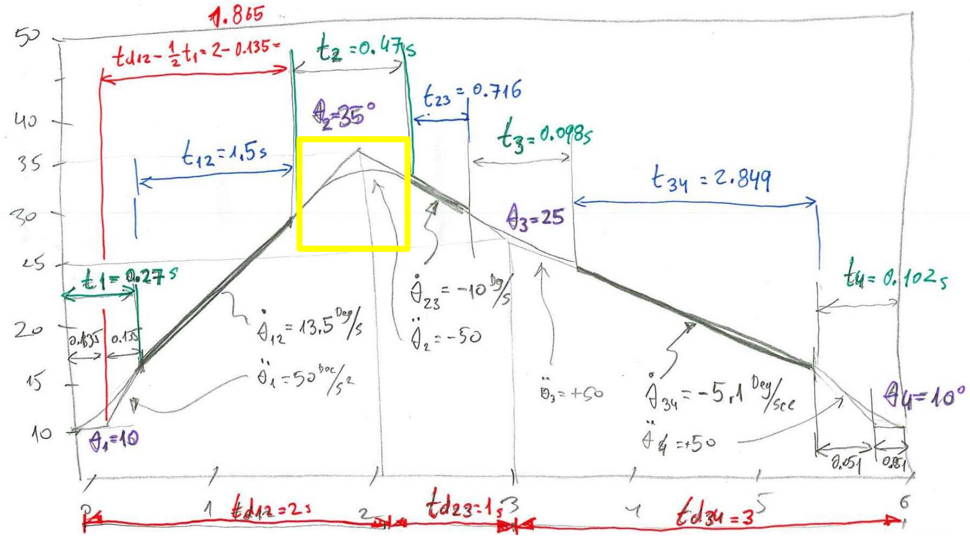
$$\ddot{\theta}_k = SIG\left(\dot{\theta}_{kl} - \dot{\theta}_{jk}\right)\left|\ddot{\theta}_k\right|$$

$$\begin{cases} j = 1 \\ k = 2 \\ l = 3 \end{cases} \rightarrow \ddot{\theta}_2 = SIG\left(\dot{\theta}_{23} - \dot{\theta}_{12}\right)\left|\ddot{\theta}_2\right| = SIG(-10-13)\left|\ddot{\theta}_2\right| = -50\,\frac{deg}{s}$$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$\begin{cases} j = 1 \\ k = 2 \\ l = 3 \end{cases} \rightarrow t_2 = \frac{\dot{\theta}_{23} - \dot{\theta}_{12}}{\ddot{\theta}_2} = \frac{(-10)-(13.5)}{-50} = \boxed{0.47 sec}$$

$$t_{12} = t_{d12} - t_1 - \frac{1}{2}t_2 = 2 - 0.27 - \frac{1}{2}(0.47) = 1.5$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

**Third Segment (t: 3→6)**

$$\ddot{\theta}_n = SGN(\theta_{n-1} - \theta_n)|\ddot{\theta}_n|$$

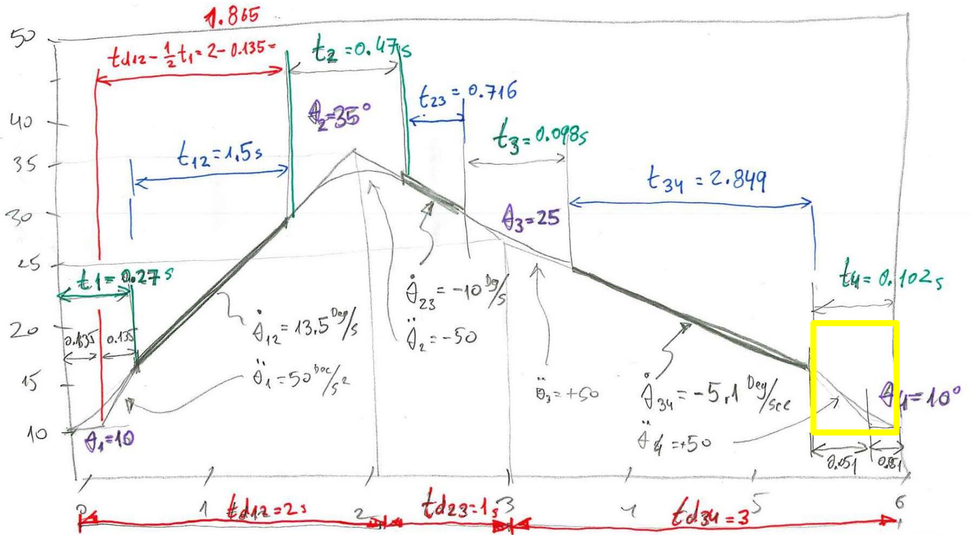
$$\text{n} = 4 \rightarrow \ddot{\theta}_4 = SGN(\theta_3 - \theta_4)|\ddot{\theta}_4| = (25 - 10)|50| = +50\frac{deg}{s}$$

$$t_n = t_{d(n-1)n} - \sqrt{t_{d(n-1)n}^2 - \frac{2(\theta_{n-1} - \theta_n)}{\ddot{\theta}_n}}$$

$$\text{n} = 4 \rightarrow t_4 = t_{d34} - \sqrt{t_{d34}^2 - \frac{2(\theta_3 - \theta_4)}{\ddot{\theta}_4}} = 3 - \sqrt{9 - \frac{2(25-10)}{50}} = \boxed{0.102\ s}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n}$$

$$n = 4 \rightarrow \dot{\theta}_{34} = \frac{\theta_4 - \theta_3}{t_{d34} - \frac{1}{2}t_4} = \frac{10-25}{3 - \frac{1}{2}0.102} = -5.10\frac{deg}{s}$$

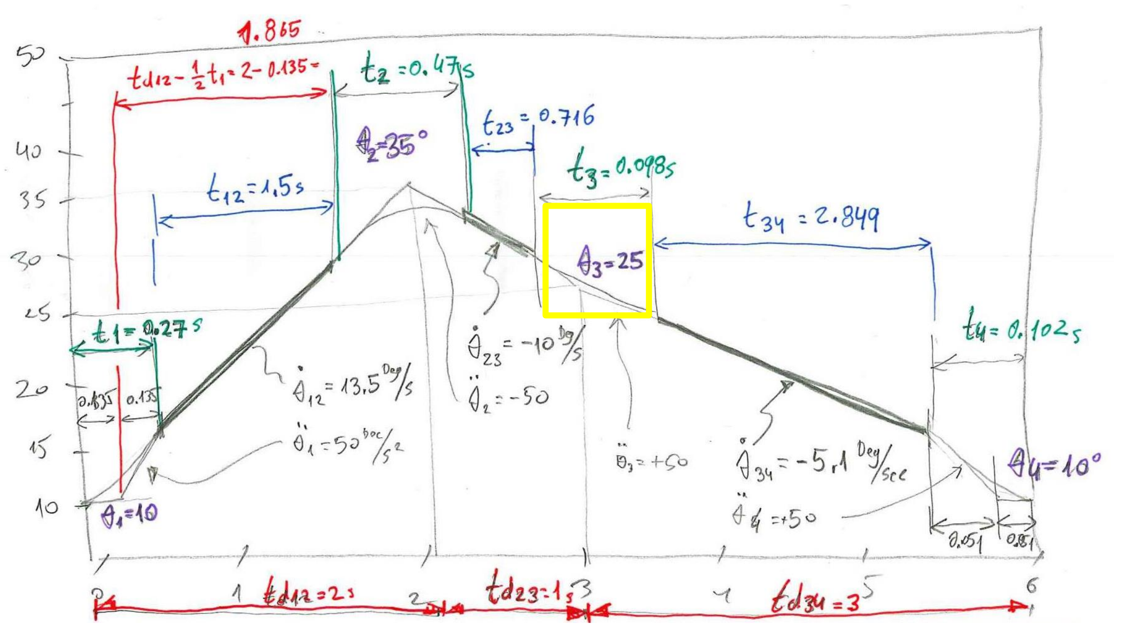


# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

## Third Segment (t: 3→6) Cont.

$$\ddot{\theta}_k = SGN(\dot{\theta}_{kl} - \dot{\theta}_{jk})|\ddot{\theta}_k|$$

$$\begin{cases} j = 2 \\ k = 3 \\ l = 4 \end{cases} \rightarrow \begin{array}{c} \ddot{\theta}_3 = SGN(\dot{\theta}_{34} - \dot{\theta}_{23})|\ddot{\theta}_3| = SGN((-5.1)-(-10))|\ddot{\theta}_3| \\ \ddot{\theta}_3 = +50\frac{deg}{s^2} \end{array}$$




$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$\begin{cases} j = 2 \\ k = 3 \\ l = 4 \end{cases} \rightarrow t_3 = \frac{\dot{\theta}_{34} - \dot{\theta}_{23}}{\ddot{\theta}_3} = \frac{(-5.1)-(-10)}{50} = \boxed{0.098\ s}$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

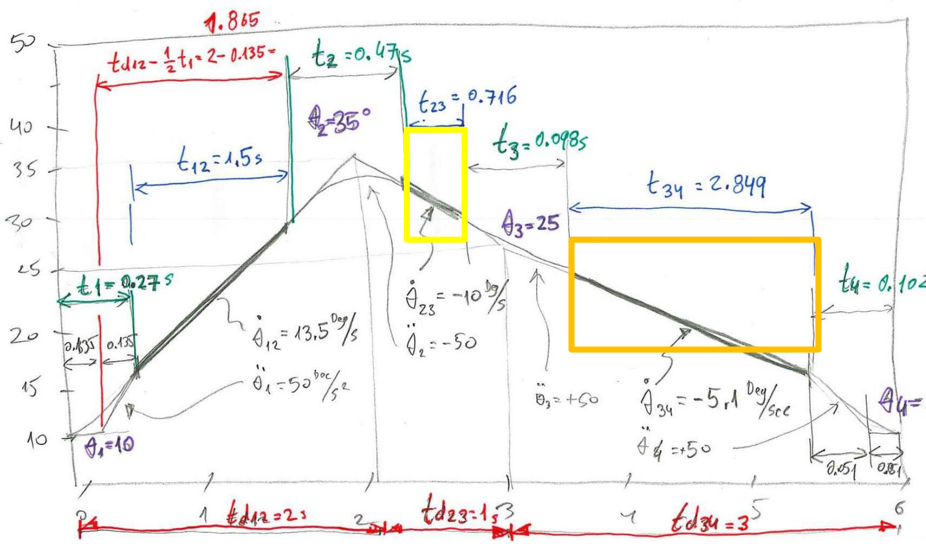
**Third Segment (t: 3→6) Cont.**

$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k$$

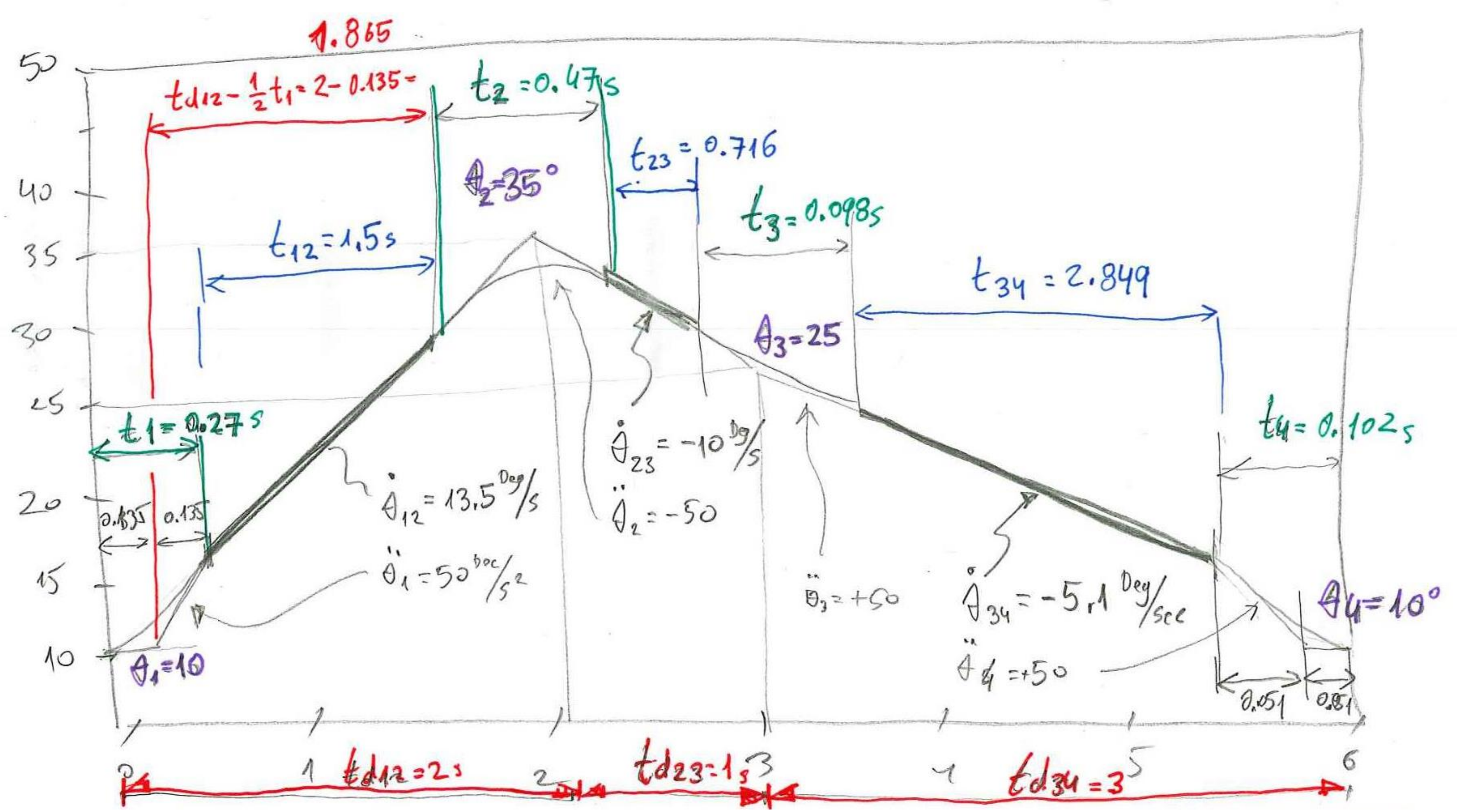
$$\begin{cases} j = 2 \\ k = 3 \end{cases} \rightarrow t_{23} = t_{d23} - \frac{1}{2}t_2 - \frac{1}{2}t_3 = \boxed{0.716\, s}$$

$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2}t_{n-1}$$

$$n = 4 \rightarrow t_{34} = t_{d34} - t_4 - \frac{1}{2}t_3 = 1 - 0.102 - \frac{1}{2}0.098 = \boxed{2.849\, s}$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example



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# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear Polynomial

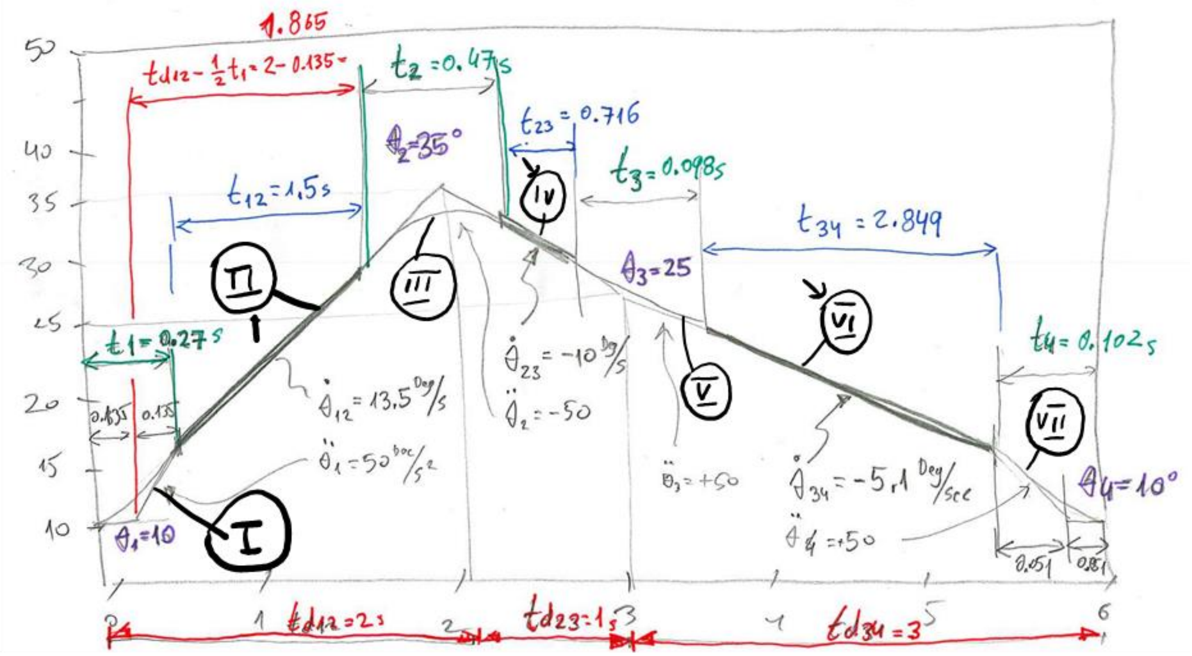
<u>LINEAR SEGMENTS</u>

$\text{II}: \theta = 10 + 13.5t$

$\text{IV}: \theta = 35 - 10t$

$\text{VI}: \theta = 25 - 5.1t$

$$\theta_{inb} = 25 - 5.1\left(\frac{0.098}{2} + 2.849\right)$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Parabolic Blend

$$I: \quad \theta = 10 + \frac{1}{2}\frac{13.5}{0.27}t^2 \text{ (FIRST)}$$

$$III: \theta = 10 + 13.5(t - t_{inb}) + \frac{1}{2}(50)^2 t_{inb}$$

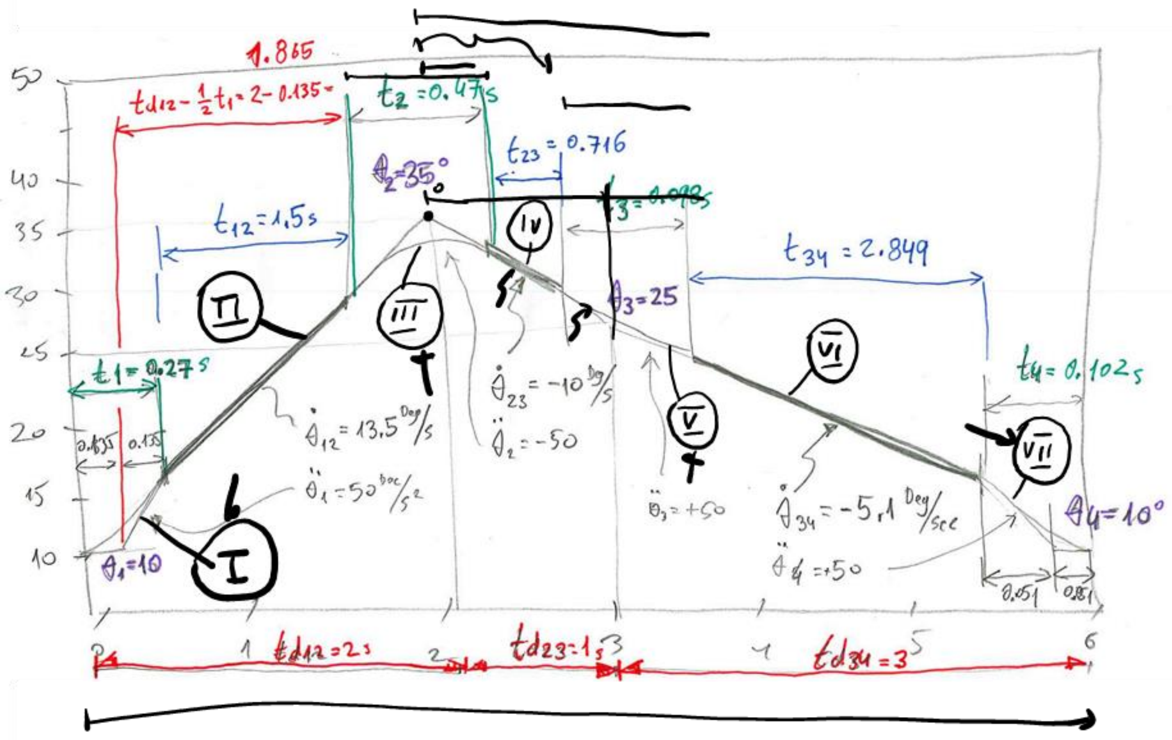
$$t_{inb} = t - \left(\frac{1}{2}0.27 + 1.5\right)$$

$$V: \quad \theta = 35 - 10(t - t_{inb}) + \frac{1}{2}(50)^2 t_{inb}$$

$$t_{inb} = t - \left(\frac{1}{2}0.47 + 0.716\right)$$

$$VII: \theta = 10 + \frac{1}{2}\frac{13.5}{0.27}t^2$$

$$t_{inb} = t - \left(\frac{1}{2}0.098 + 2.849\right)$$



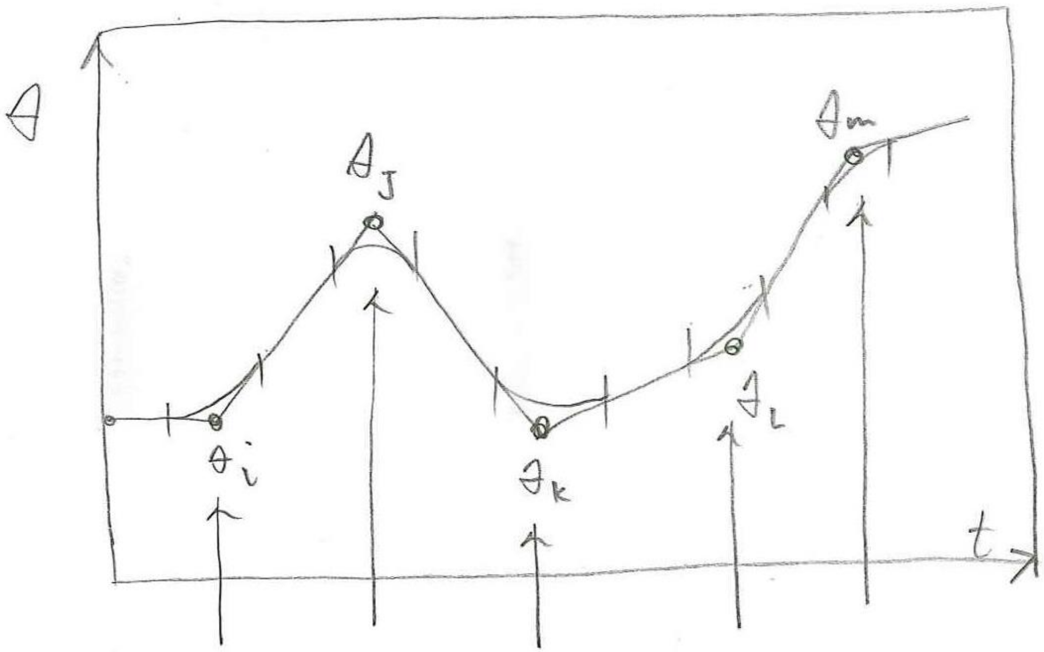
# Joint Space Schemes

Multiple Time Intervals

**Pseudo Via Point**

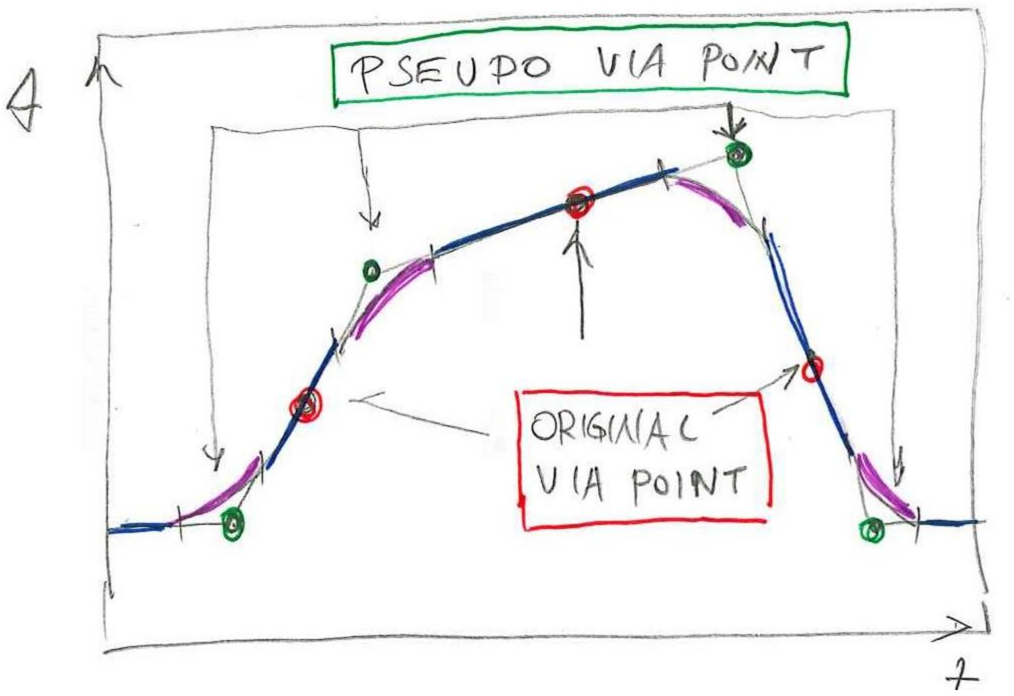
# Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

- **Problem**:
  - In the linear parabolic blend spline, note that the via points are not actually reached unless the manipulator come to a stop
  - Often when the acceleration is sufficiently high the path will come quite close to the desire via point

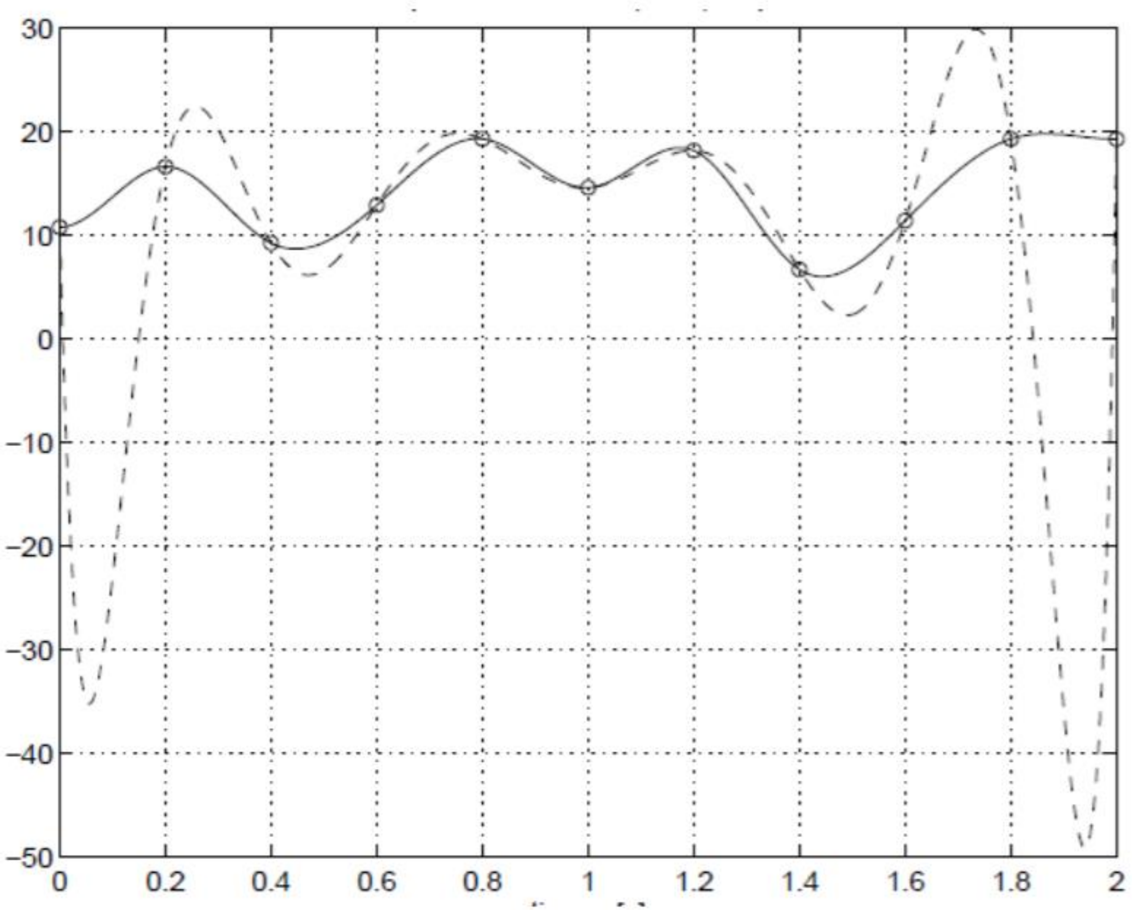


# Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

- **Solution**
  - **Define Pseudo via Points** - The system automatically replace the via point through which we wish the manipulator to pass through with two pseudo via points one on each side of the original.
  - The original via point will now lie in the linear region of the path connecting the two pseudo via points
  - **Define Velocity at the original Via Points** - In addition to requesting that the manipulator pass exactly through the via point, the user can also request that it pass through with a certain velocity. If the user does not specify this velocity the system chooses it by means of suitable heuristic
  - **Define Through Point** - Through Point rather than via point is used to specify a path through which we force the manipulator to pass exactly through



# Joint Space Schemes – High Order Polynomials




In general, given:

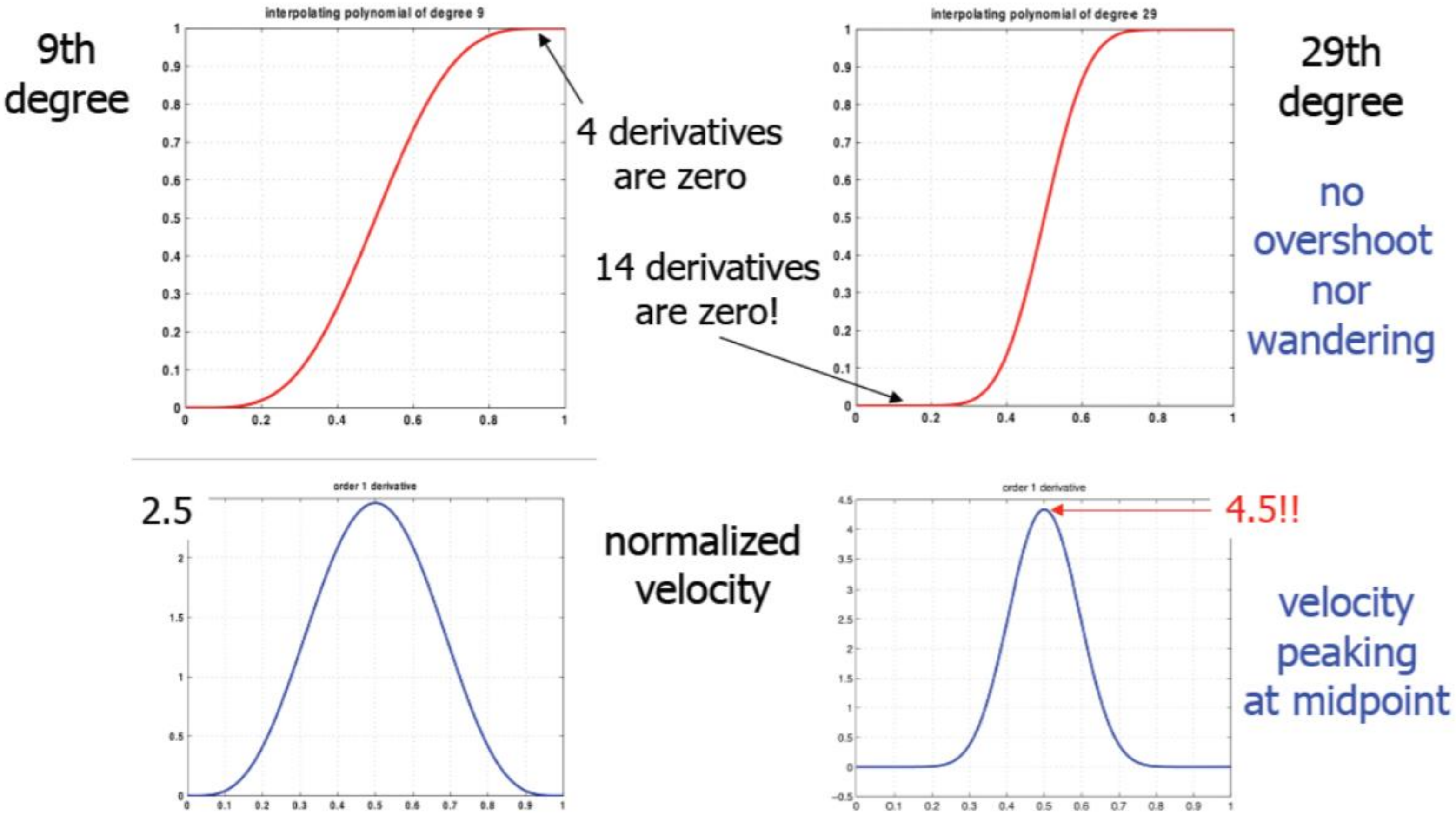
2 points $\Longrightarrow$ unique line

3 points $\Longrightarrow$ unique quadric

...

$n$ points $\Longrightarrow$ unique polynomial with degree $n-1$

# Joint Space Schemes – High Order Polynomials

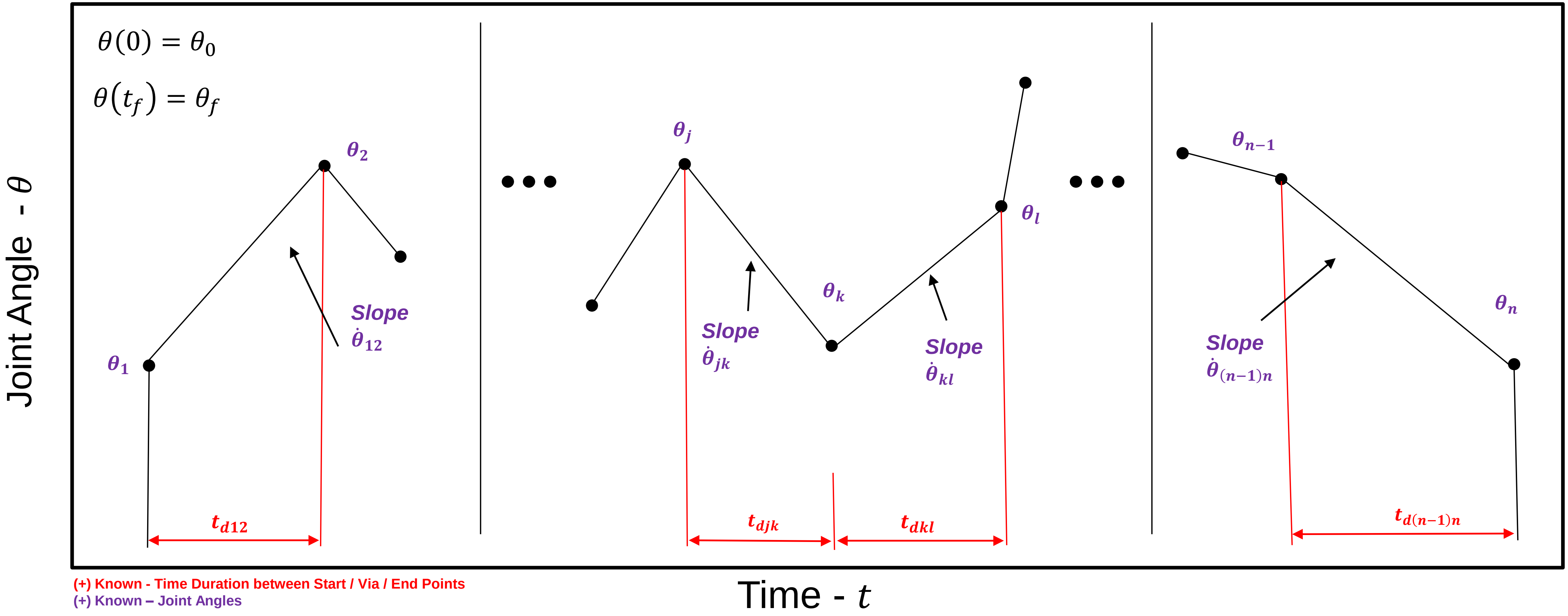


# Joint Space Schemes

Multiple Time Intervals

**Summary**

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Polynomials




(+) Known - Time Duration between Start / Via / End Points
(+) Known – Joint Angles

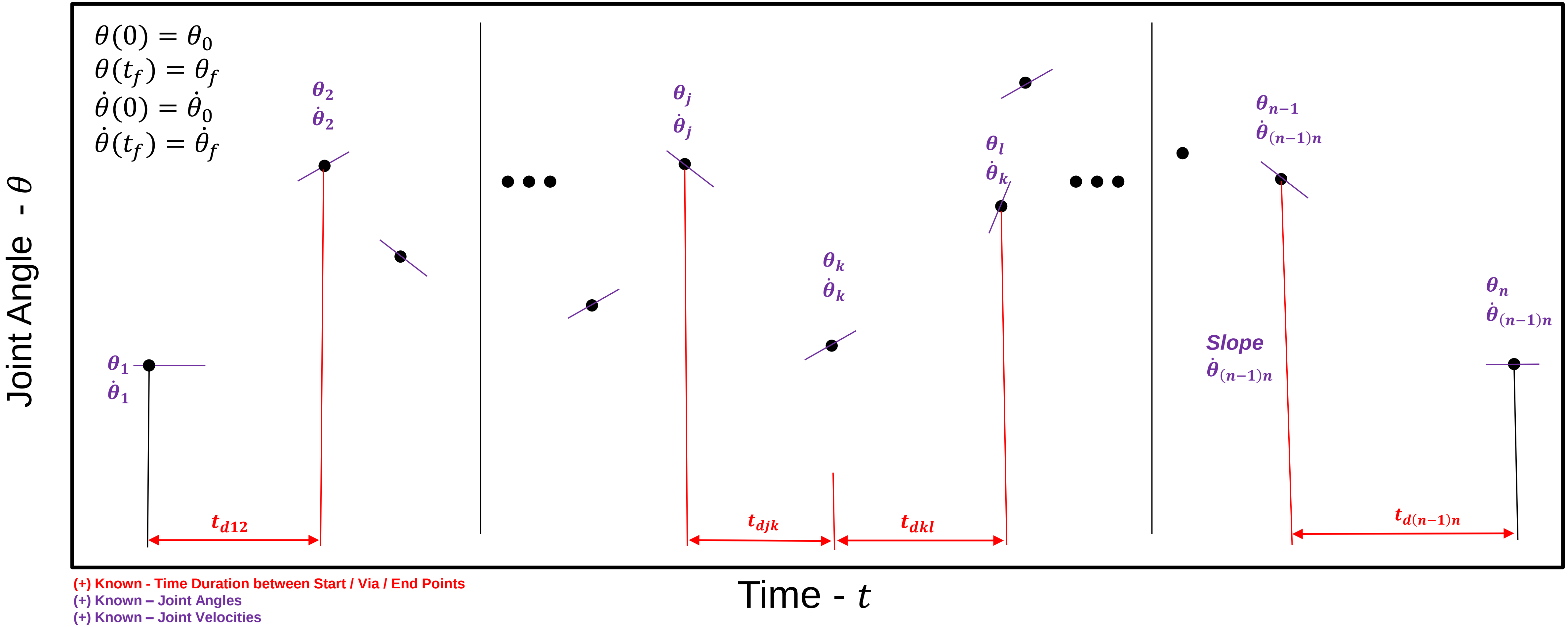
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Polynomials

$$\theta = \theta_0 + \left(\frac{\theta_f - \theta_0}{t_f}\right) t$$

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials – Non Zero Velocity




(+) Known - Time Duration between Start / Via / End Points
(+) Known – Joint Angles
(+) Known – Joint Velocities

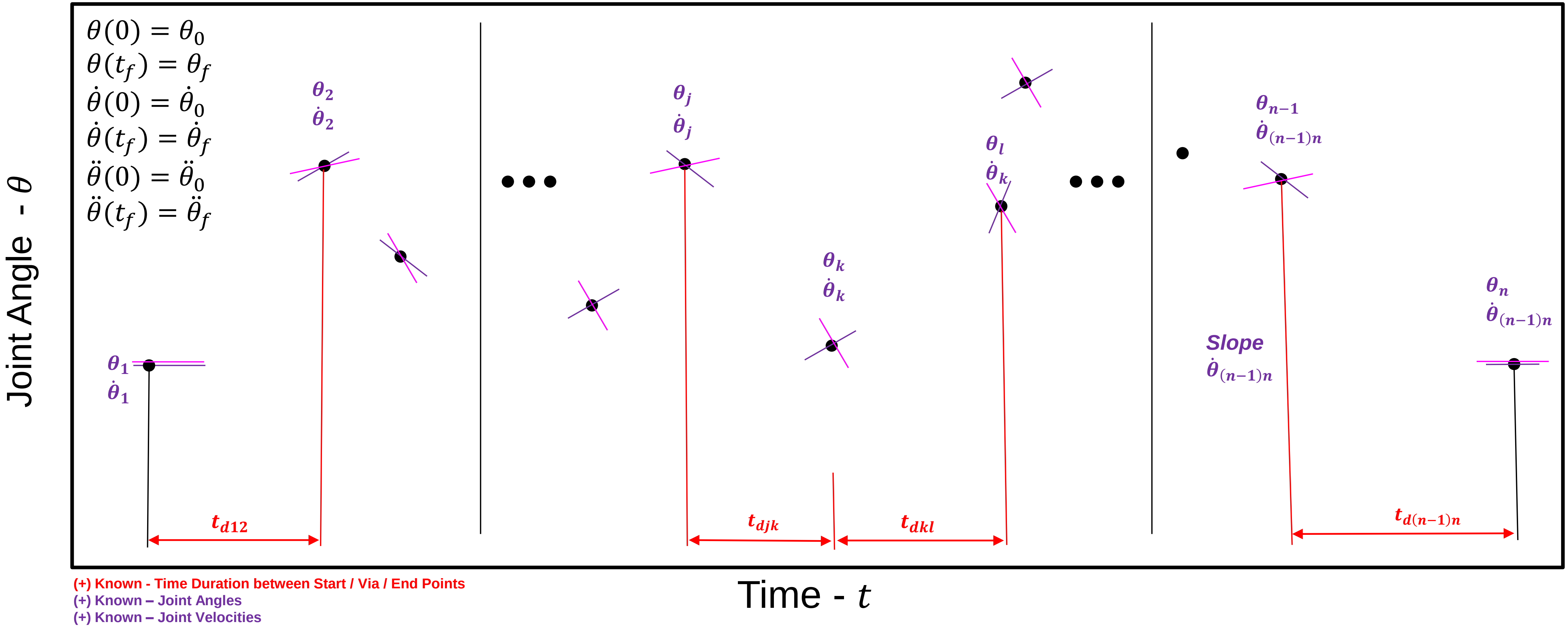
# Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials – Non Zero Velocity

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$
$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

$$\theta(0) = \theta_0$$
$$\theta(t_f) = \theta_f$$
$$\dot{\theta}(0) = \dot{\theta}_0$$
$$\dot{\theta}(t_f) = \dot{\theta}_f$$

$$a_0 = \theta_0$$
$$a_1 = \dot{\theta}_0$$
$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$
$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)$$

# Joint Space Schemes – Multiple Time Intervals – Via Points – Quantic Polynomials - Non Zero Acceleration




(+) Known - Time Duration between Start / Via / End Points
(+) Known – Joint Angles
(+) Known – Joint Velocities
(+) Known – Joint Acceleration

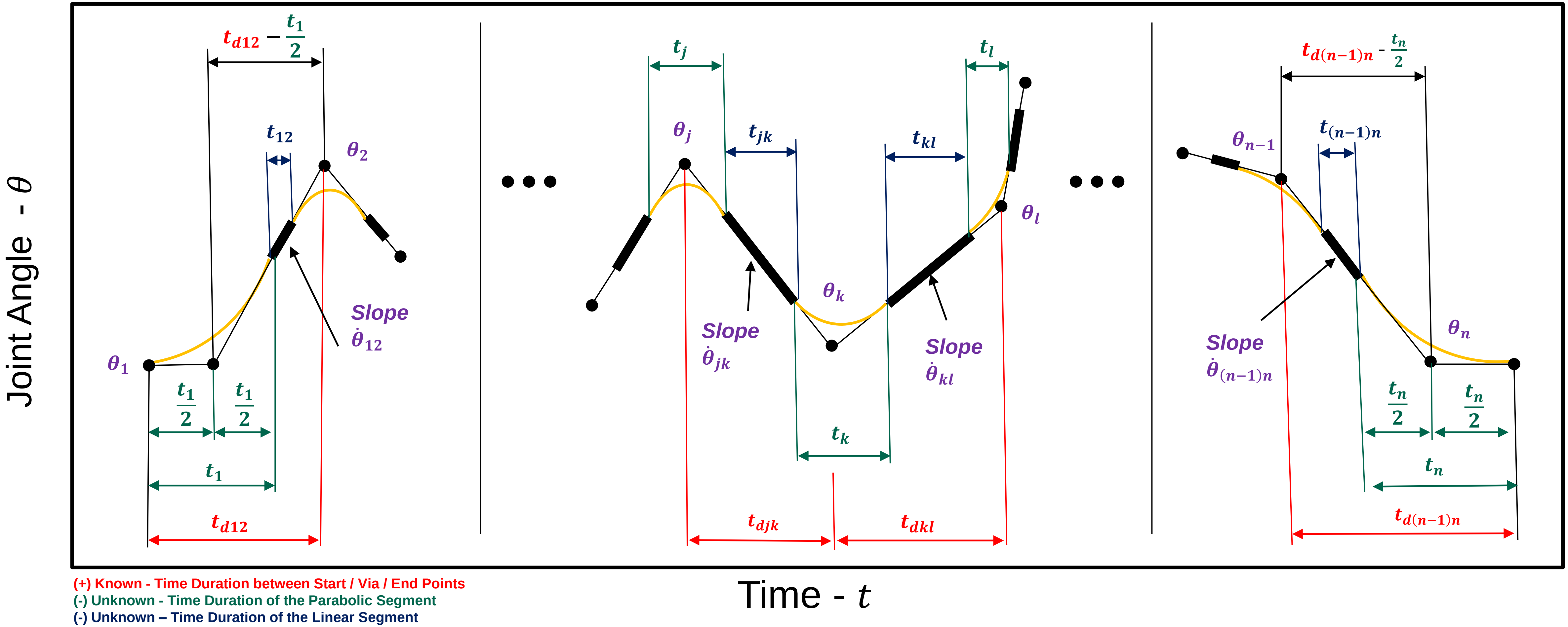
# Joint Space Schemes – Multiple Time Intervals – Via Points – Quantic Polynomials - Non Zero Acceleration

$$
\begin{aligned}
\theta(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 \\
\dot{\theta}(t) &= a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 \\
\ddot{\theta}(t) &= 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3
\end{aligned}
$$

$$
\begin{aligned}
\theta(0) &= \theta_0 \\
\theta(t_f) &= \theta_f \\
\dot{\theta}(0) &= \dot{\theta}_0 \\
\dot{\theta}(t_f) &= \dot{\theta}_f \\
\ddot{\theta}(0) &= \ddot{\theta}_0 \\
\ddot{\theta}(t_f) &= \ddot{\theta}_f
\end{aligned}
$$

$$
\begin{aligned}
a_0 &= \theta_0 \\
a_1 &= \dot{\theta}_0 \\
a_2 &= \frac{\ddot{\theta}_0}{2} \\
a_3 &= \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f - (3\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^3} \\
a_4 &= \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f + (3\ddot{\theta}_0 - 2\ddot{\theta}_f)t_f^2}{2t_f^4} \\
a_5 &= \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f - (\ddot{\theta}_0 - \ddot{\theta}_f)t_f^2}{2t_f^5}
\end{aligned}
$$

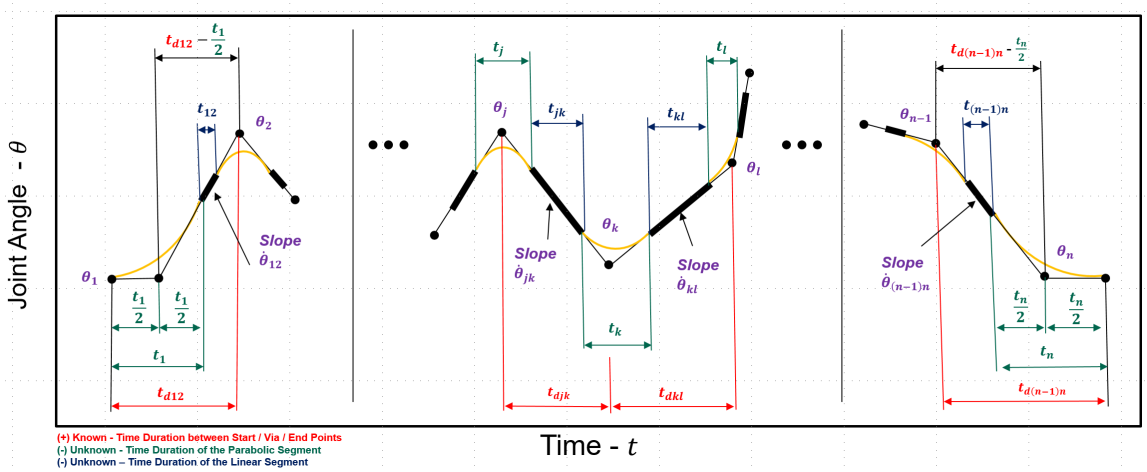
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend




(+) Known - Time Duration between Start / Via / End Points
(-) Unknown - Time Duration of the Parabolic Segment
(-) Unknown – Time Duration of the Linear Segment

# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1 – Time Intervals / Velocity / Acceleration**
  - Calculate the **time intervals** of the parabolic blending (marked in green)
  - Calculate the **time intervals** of the linear functions (marked in blue)
  - Calculate the **direction of the acceleration** during the linear bland
  - Calculate the velocity during the linear spline




- **Task No. 2 – Functions**
  - Define the Linear Functions (marked in black)
  - Defined the parabolic blend functions (marked in gray)

Assume 4 intervals
- First Interval: 1→2
- Intermediate Interval: 2 → 3
- Last Interval: 2 → 3

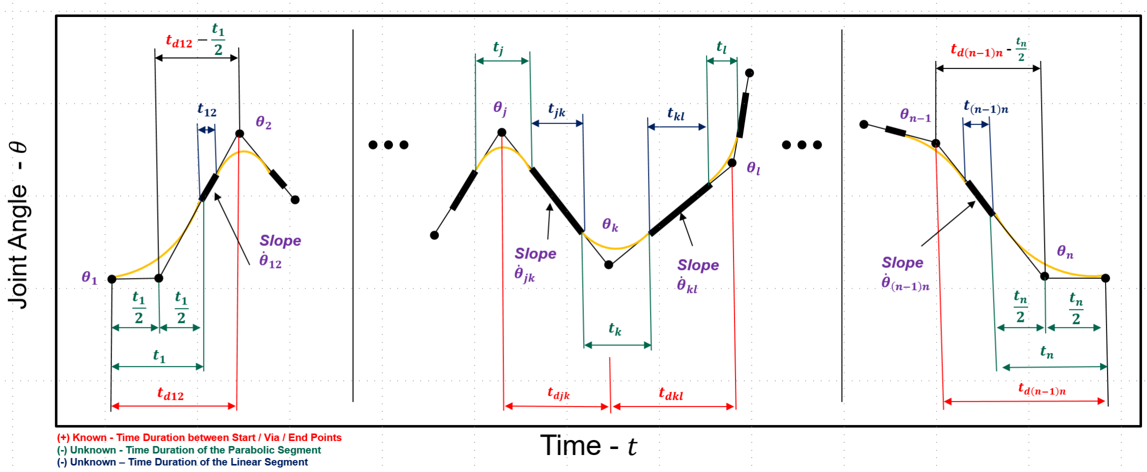
# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1**
  - **First Interval -** 1→2

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}}$$

$$\ddot{\theta}_1 = SGN(\theta_2 - \theta_1)\left|\ddot{\theta}_1\right|$$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2}t_1}$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1**
  - **Intermediate Interval** (Repeat for any intermediate interval ) - 2→3

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}$$

$$\begin{cases} j = 2 \\ k = 3 \end{cases} \rightarrow \dot{\theta}_{23} = \frac{\theta_3 - \theta_1}{t_{d23}}$$

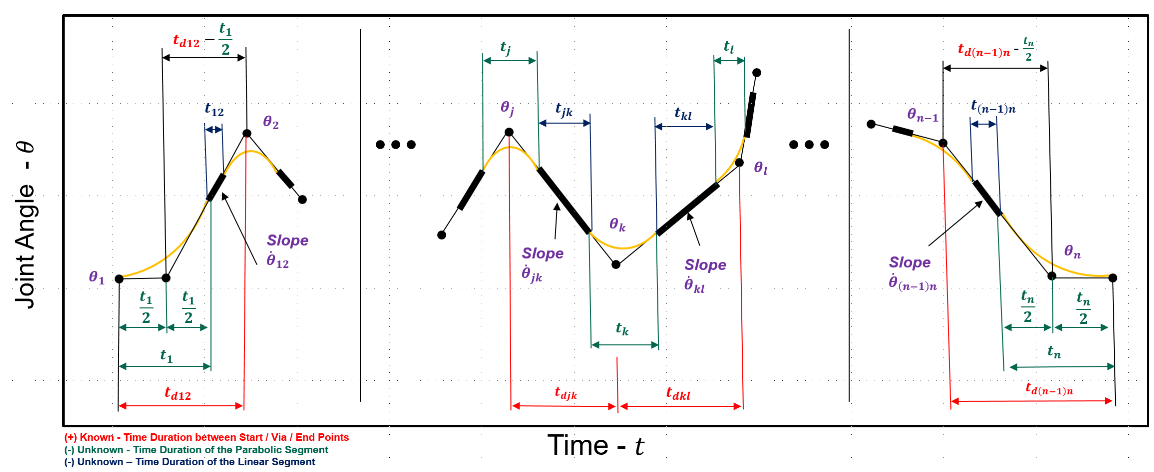
$$\ddot{\theta}_k = SIG(\dot{\theta}_{kl} - \dot{\theta}_{jk})|\ddot{\theta}_k|$$

$$\begin{cases} j = 1 \\ k = 2 \\ l = 3 \end{cases} \rightarrow \ddot{\theta}_2 = SIG(\dot{\theta}_{23} - \dot{\theta}_{12})|\ddot{\theta}_2|$$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$\begin{cases} j = 1 \\ k = 2 \\ l = 3 \end{cases} \rightarrow t_2 = \frac{\dot{\theta}_{23} - \dot{\theta}_{12}}{\ddot{\theta}_2}$$

$$t_{d12} - t_1 - \frac{1}{2} t_2$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- **Tasks No. 1**
  - **Final Interval -** 3→4

$$\ddot{\theta}_n = SGN(\theta_{n-1} - \theta_n)\left|\ddot{\theta}_n\right|$$

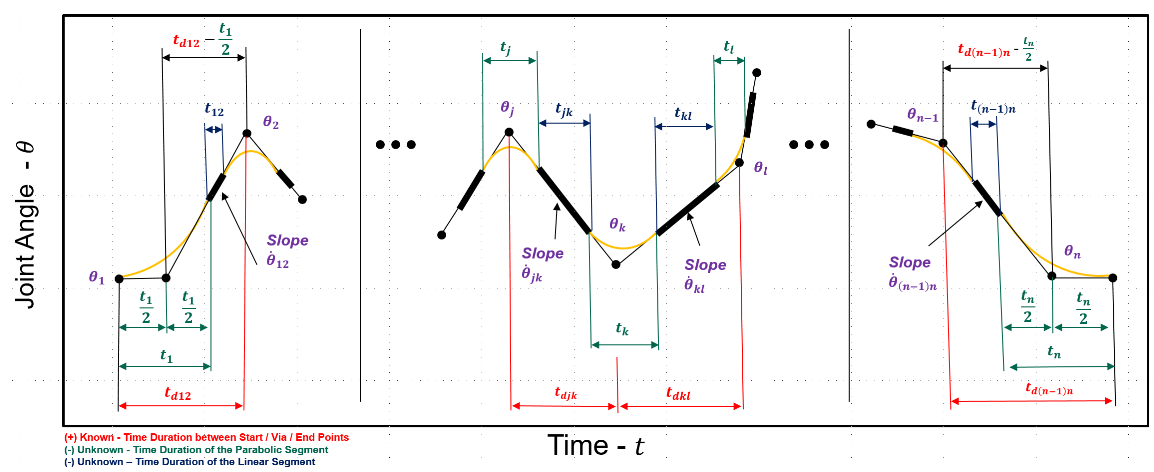
$$\mathrm{n} = 4 \rightarrow \ddot{\theta}_4 = SGN(\theta_3 - \theta_4)\left|\ddot{\theta}_4\right|$$

$$t_n = t_{d(n-1)n} - \sqrt{t_{d(n-1)n}^2 - \frac{2(\theta_{n-1} - \theta_n)}{\ddot{\theta}_n}}$$

$$\mathrm{n} = 4 \rightarrow t_4 = t_{d34} - \sqrt{t_{d34}^2 - \frac{2(\theta_3 - \theta_4)}{\ddot{\theta}_4}}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n}$$

$$n = 4 \rightarrow \dot{\theta}_{34} = \frac{\theta_4 - \theta_3}{t_{d34} - \frac{1}{2}t_4}$$



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# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

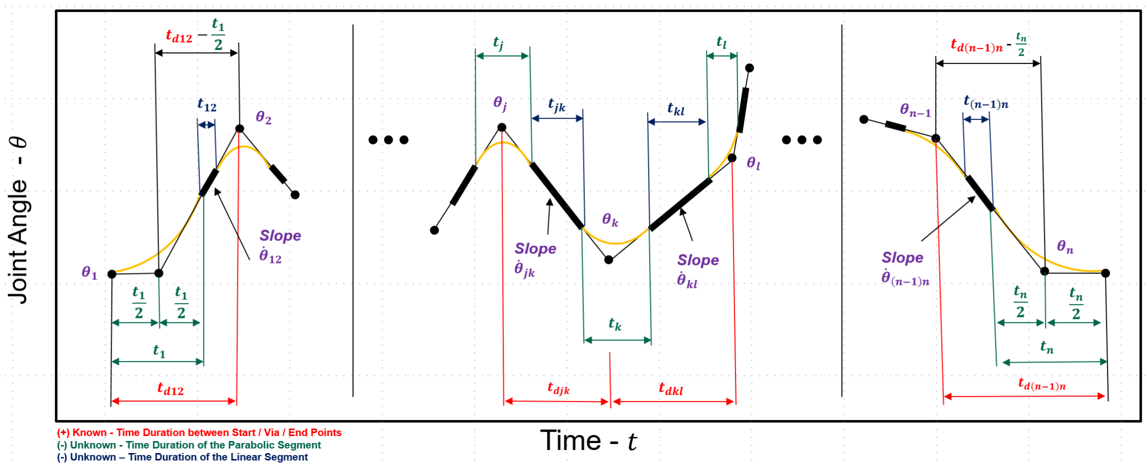
- **Tasks No. 1**
  - **Final Interval -** 3→4 (Continue)
  - (Interval Prior to the final interval)

$$\ddot{\theta}_k = SGN(\dot{\theta}_{kl} - \dot{\theta}_{jk})|\ddot{\theta}_k|$$

$$\begin{cases} j = 2 \\ k = 3 \\ l = 4 \end{cases} \rightarrow \ddot{\theta}_3 = SGN(\dot{\theta}_{34} - \dot{\theta}_{23})|\ddot{\theta}_3|$$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$\begin{cases} j = 2 \\ k = 3 \\ l = 4 \end{cases} \rightarrow t_3 = \frac{\dot{\theta}_{34} - \dot{\theta}_{23}}{\ddot{\theta}_3}$$



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# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

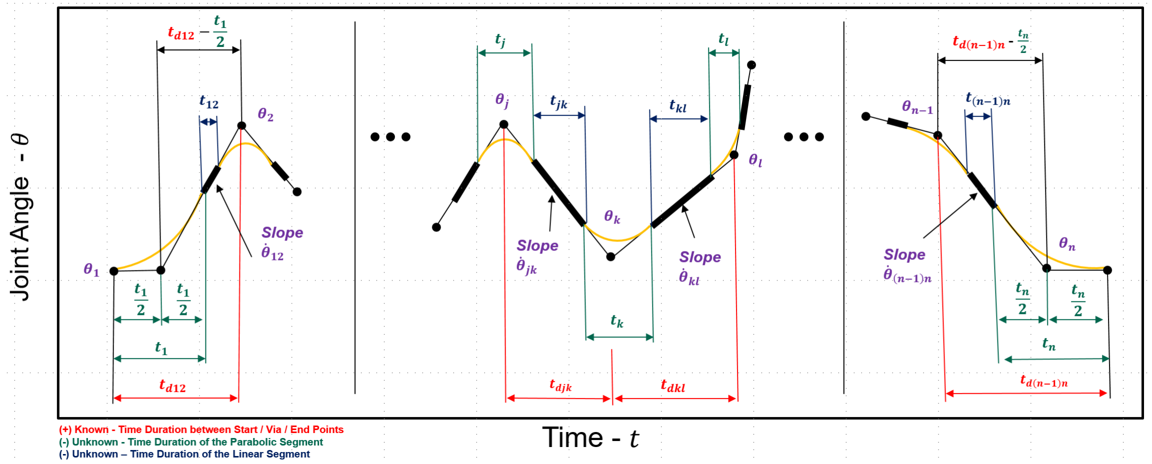
- **Tasks No. 1**
  - **Final Interval -** 3→4 (Continue)
  - (Interval Prior to final interval & Final Inetrval)

$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k$$

$$\begin{cases} j = 2 \\ k = 3 \end{cases} \rightarrow t_{23} = t_{d23} - \frac{1}{2}t_2 - \frac{1}{2}t_3$$

$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2}t_{n-1}$$

$$n = 4 \rightarrow t_{34} = t_{d34} - t_4 - \frac{1}{2}t_3$$



# Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- **Tasks No. 2 – Linear & parabolic functions**
  - First Segment

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

$$t_{inb} = t$$

  - Mid Segment

$$\theta = \theta_j + \dot{\theta}_{jk} t$$

$$\theta = \theta_j + \dot{\theta}_{jk}(t - t_{inb}) + \frac{1}{2}\ \ddot{\theta}_k^2 t_{inb}$$

$$t_{inb} = t - \left(\frac{1}{2} t_j + t_{jk}\right)$$

  - Last Segment

$$\theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t$$

$$\theta = \theta_{inb} + \theta_{(n-1)n} t_{inb} - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t_{inb}^2$$

$$t_{inb} = t - \left(\frac{1}{2} t_{n-1} + t_{(n-1)n}\right)$$