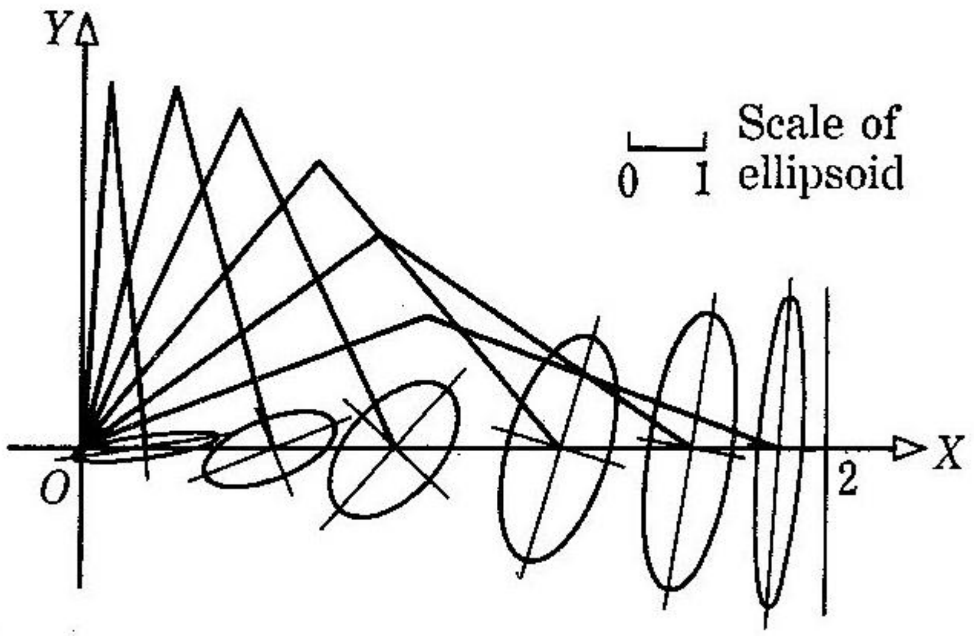
Jacobian – Implications & Applications

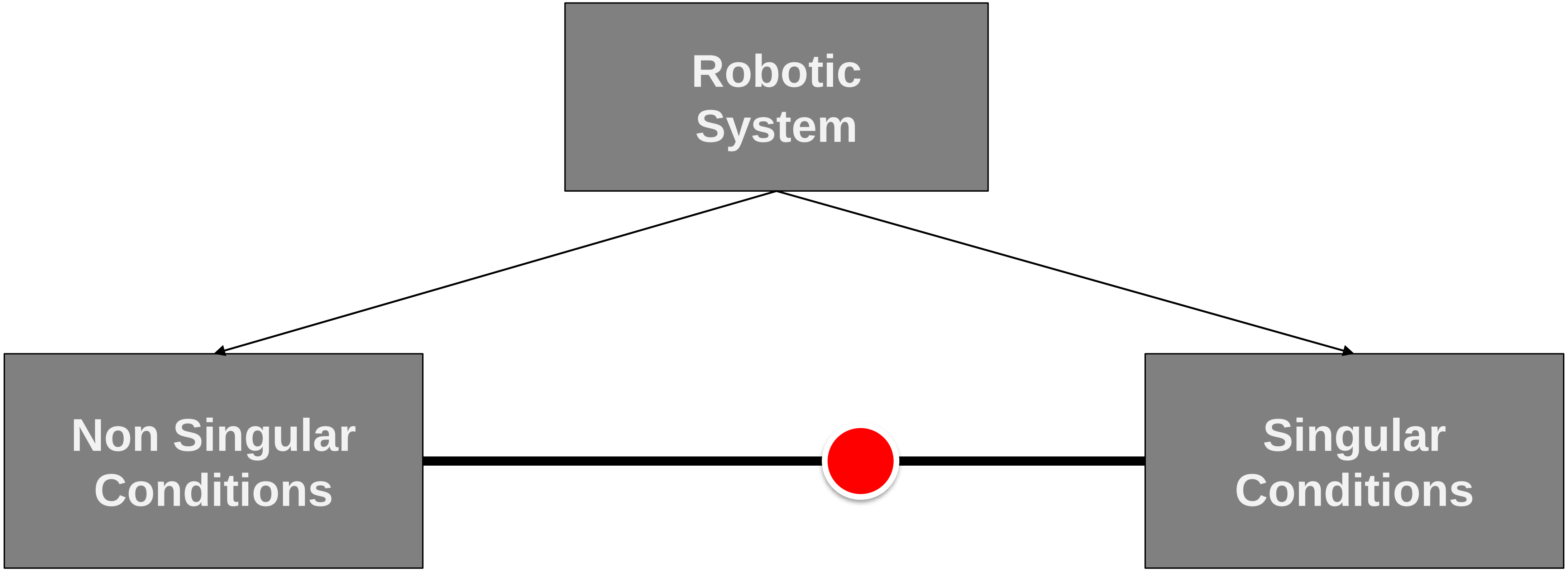
Part 2: Design - Manipulability Ellipsoid & Performance Index

Performance Index – Manipulability

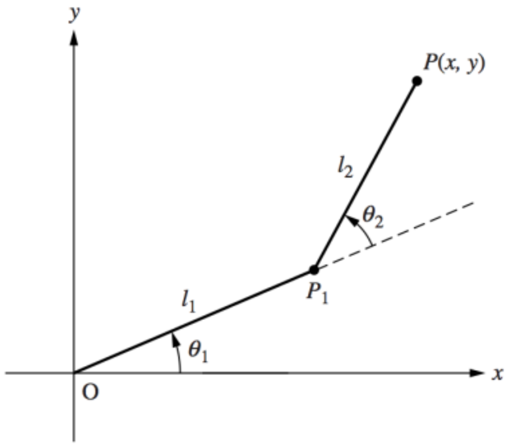
- **Kinematic Singularity** – The robot end effector loses its ability to translate or rotate in one or more directions
- **Kinematic Singularity – Binary** - A kinematic singularity presents a binary proposition – a particular configuration is either kinematically singular or it is not
- **Proximity to Singularity** - it is reasonable to ask if a nonsingular configuration is “close” to being singular.
- **Manipulability Ellipsoid -** The manipulability ellipsoid allows one to visualize geometrically the directions in which the end-effector moves with least effort or with greatest effort

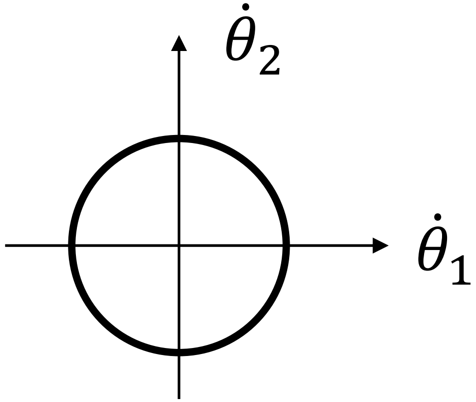


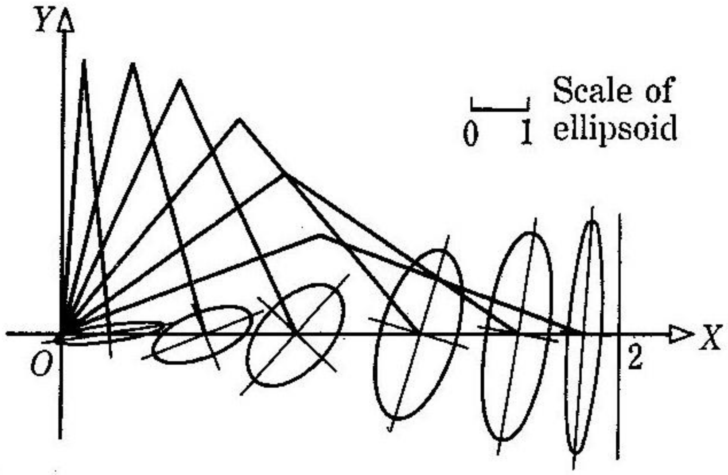
Jacobian – Singularity – Mathematical Introduction



Jacobian – Singularity – Mathematical Introduction







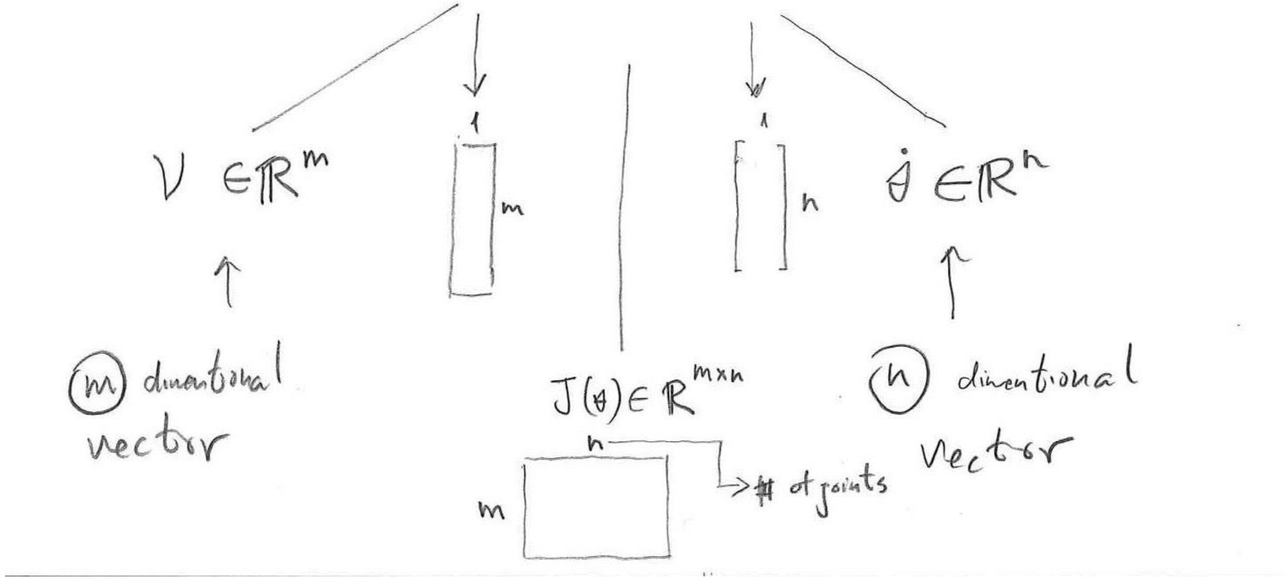
$$v = J(\theta)\dot{\theta}$$

$$\tau = J^T(\theta)F$$

Jacobian – Singularity – Mathematical Introduction

General expression of the end effector velocity ellipsoid

$$v = J(\theta)\dot{\theta}$$



Jacobian – Singularity – Mathematical Introduction

- Linear Algebra – Norm – Definition

  - **Norm P - L_p norm of x**

$$\|x\|_p = \left[\sum_i |x_i|^p\right]^{\frac{1}{p}} = \sqrt[\frac{1}{p}]{\sum_i |x_i|^p}$$

  - $|x_i|$ Calculate the absolute value of the i-th element
  - $|x_i|^p$ take its power p
  - $\sum_i |x_i|^p$ sum all these power absolute values
  - $\left[\sum_i |x_i|^p\right]^{\frac{1}{p}}$ take the power $\frac{1}{p}$ of the result

Jacobian – Singularity – Mathematical Introduction

– **Norm P=0 - L_0 norm of x**

$$\|x\|_0 = \left[\sum_i |x_i|^0\right]^{\frac{1}{0}}$$

- Using the power of 0 with an absolute values will get you
  - 1 for every non-zero value
  - 0 for every zero value
- This norm corresponds to the number of non-zero elements in the vector

– **Norm P=1 - L_1 norm of x**

$$\|x\|_1 = \left[\sum_i |x_i|^1\right]^{\frac{1}{1}} = \sum_i |x_i|$$

- The sum of the absolute values

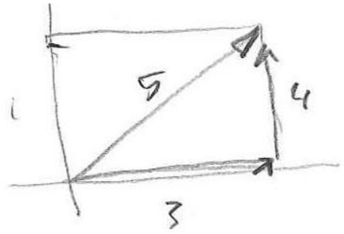
– **Norm P=2 - L_2 norm X (Euclidean Norm)**

- The absolute value is not needed anymore since x is squared
- Provide the length of the vector in Pythagorean theorem

$$\|x\|_2 = \left(\sum_i x_i^2\right)^{\frac{1}{2}} = \sqrt{\sum_i x_i^2}$$

- Example

$$u = \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$



$$\|u\|_2 = \sqrt{|3|^2 + |4|^2} = \sqrt{25} = 5$$

Jacobian – Singularity – Mathematical Introduction

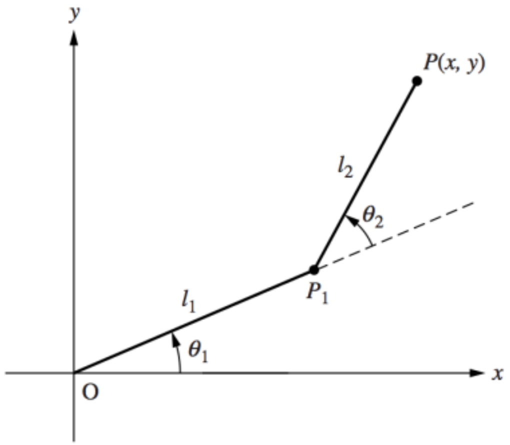
– **L_2^2 - Squared Euclidean norm (Squared L_2^2 norm)**

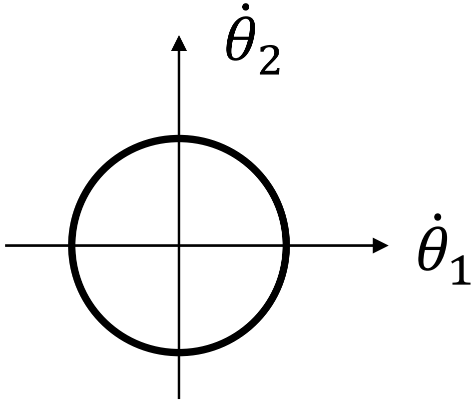
$$(\|u\|_2)^2 = \left(\sqrt{\sum_i x_i^2}\right)^2 = \sum_i x_i^2$$

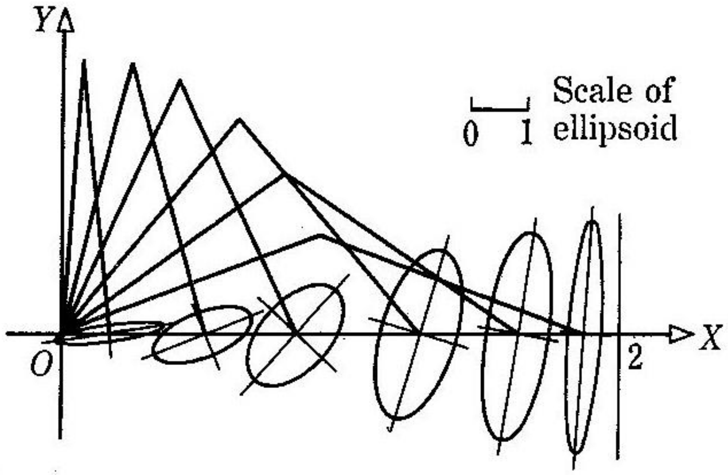
- Alternative expressions for L_2^2

$$(\|u\|_2)^2 = x \cdot x = x^T x$$

Jacobian – Singularity – Mathematical Introduction







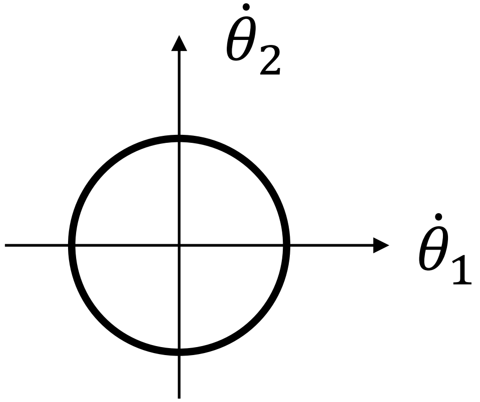
$$v = J(\theta)\dot{\theta}$$

$$\tau = J^T(\theta)F$$

Jacobian – Singularity – Mathematical Introduction

- A circle/sphere of joint velocities, like the circle shown here is defined by the equation

$$\dot{\theta}^T \dot{\theta} = 1$$

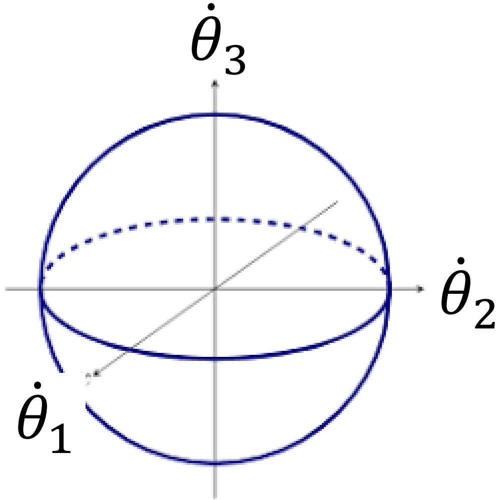


- Using the definition of the Jacobian

$$v = J(\theta)\dot{\theta}$$

$$\dot{\theta} = J^{-1} v$$

$$\dot{\theta}^T = (J^{-1} v)^T$$



- Assume that the Jacobian is invertible (not strictly necessary) the previous equation can be rewritten as

$$(J^{-1} v)^T (J^{-1} v) = 1$$

Jacobian – Singularity – Mathematical Introduction

- Based on Linear Algebra property $(Ax)^T = x^T A^T$
- The previous equations $(J^{-1}v)^T (J^{-1}v) = 1$
- Can be rewritten as $v^T (J^{-1})^T J^{-1} v = 1$

- Based on linear Algebra properties

$$(A^T)^{-1} = (A^{-1})^T \quad (\star)$$
$$(AB)^{-1} = B^{-1}A^{-1} \quad (\star\star)$$

- The previous equation can be rewritten as

$$\text{from } \star \rightarrow v^T (J^T)^{-1} J^{-1} v = 0$$
$$\text{from } \star\star \rightarrow v^T (JJ^T)^{-1} v = 0$$

Jacobian – Singularity – Mathematical Introduction

- Rewriting

$$v^T (JJ^T)^{-1} v = 0$$

$$v^T (A)^{-1} v = 0$$

- where

$$A = JJ^T$$

$$A \in \mathbb{R}^{m \times m}, J \in \mathbb{R}^{m \times n}, J^T \in \mathbb{R}^{n \times m}$$

$\mathrm{A}^{-1}, \mathrm{A}$ properties:

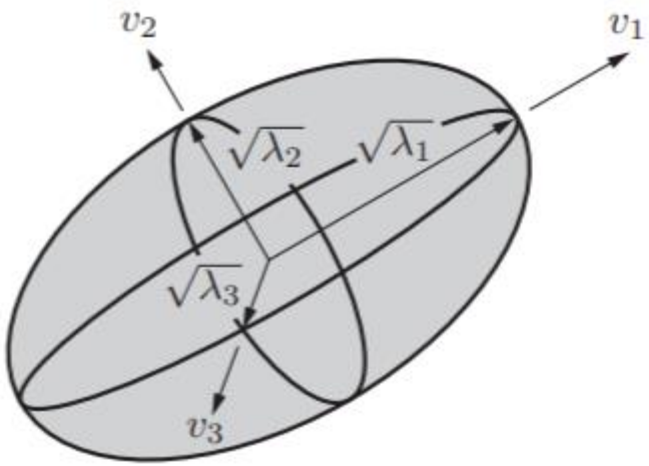
$\mathrm{A}^{-1}, \mathrm{A}$ – is positive

$\mathrm{A}^{-1}, \mathrm{A}$ – symmetric

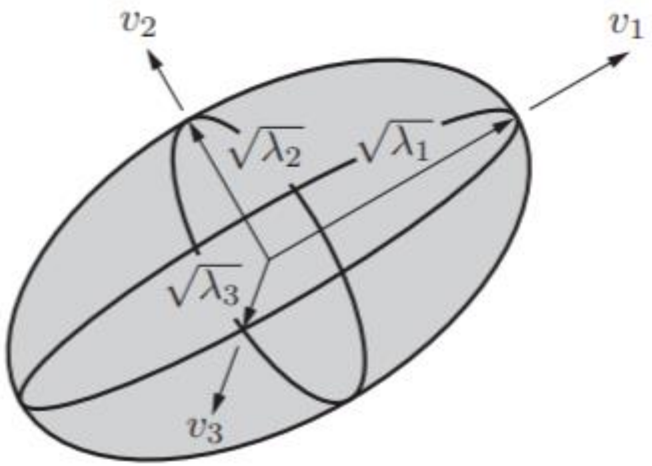
Performance Index – Manipulability

- Performing eigenvector/eigenvalue analysis of $A = JJ^T$ defining
  - Eigenvectors v_i
  - eigenvalues λ_i

- The directions of the principal axes of the ellipsoid are v_i and the lengths of the principal semi-axes are $\sqrt{\lambda_i}$



Jacobian – Singularity – Mathematical Introduction



- Replace v (velocity of the tip) by a vector x

$$x^T A^{-1} x = 0$$

- $A \in \mathbb{R}^{m \times m}$ (symmetric, positive definite)

Eigenvalues of A→ $\lambda_1, \lambda_2, \dots, \lambda_m$
Eigenvectors of A→ $v_1, v_2, \dots, v_m$

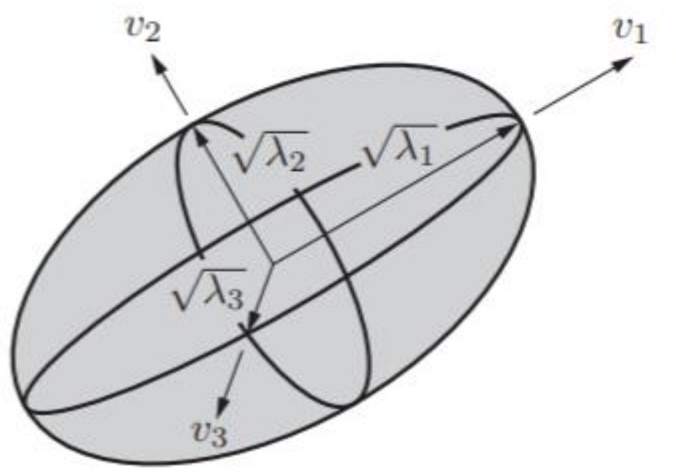
- The A matrix defines an ellipsoid of x values that satisfy the equation

- If $A = JJ^T$ $x = v_{tip}$
- Then Manipulability Ellipsoid Resulting from a unit sphere of joint velocity

Jacobian – Singularity – Mathematical Introduction

$$x^T A^{-1} x = 0$$

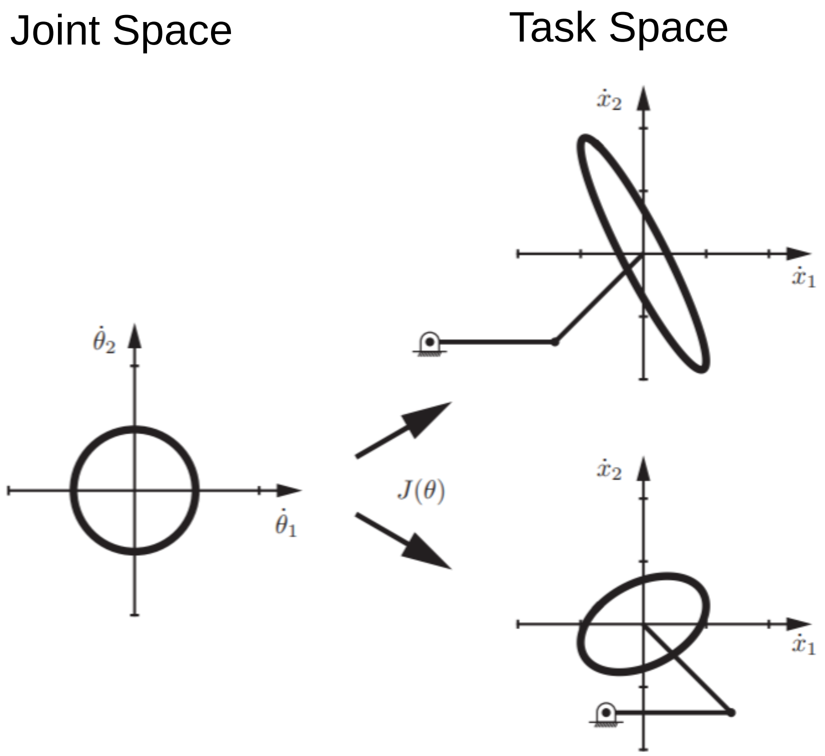
- If $A = JJ^T$ $x = v_{tip}$
- Then Manipulability Ellipsoid Resulting from a unit sphere of joint velocity
- If $A = (JJ^T)^{-1}$ $x = F_{tip}$
- Then Force Ellipsoid Resulting from a unit sphere of joint forces or torques

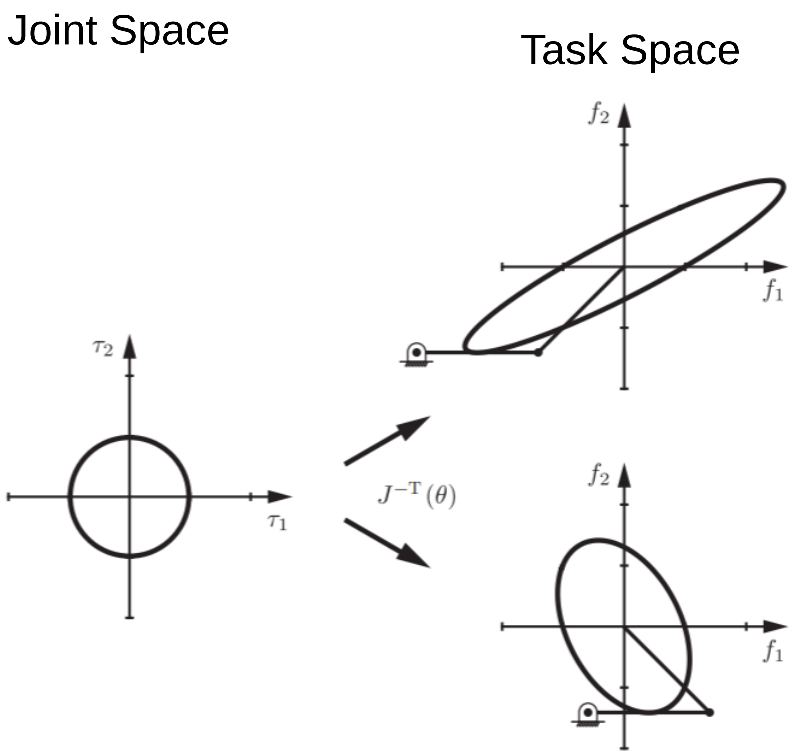


Performance Index – Manipulability

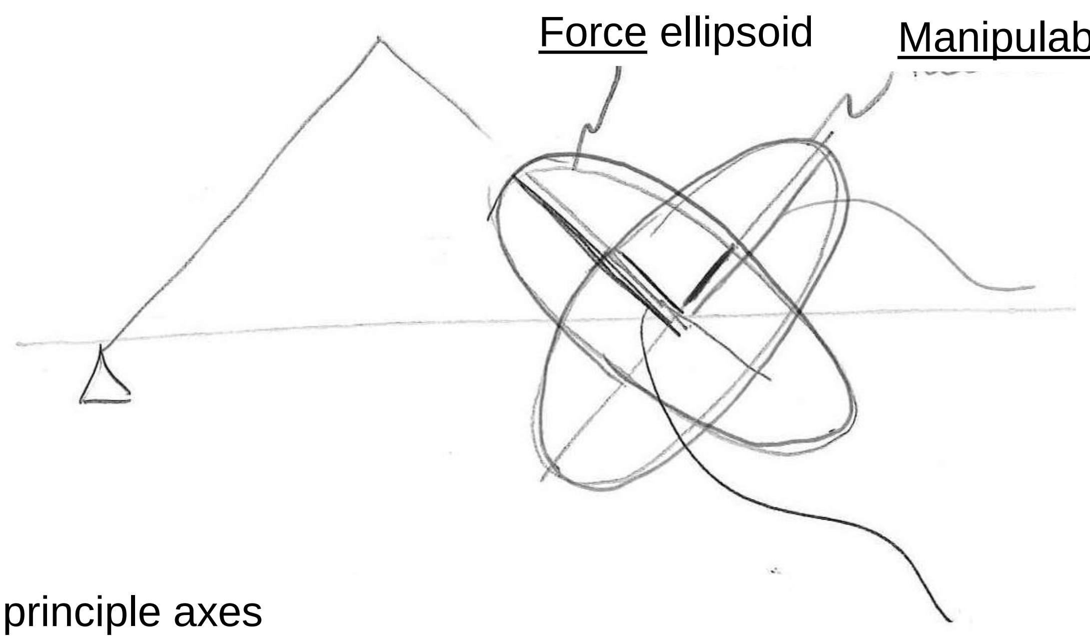
$$\dot{q} = \dot{X} = J\dot{\Theta}$$

$$\tau = J^T F$$
$$F = J^{-T}\tau$$





Jacobian – Singularity – Mathematical Introduction



Small forces can be applied in direction of large velocities

Small velocity can be applied in direction of large forces

- Same principle axes
- Same semi axis lengths

Jacobian – Singularity – Mathematical Introduction

- Assigning a single number representing how close the robot is to being a singular

OR

- Reducing the representation of the ellipsoid into a single number

↓

Performance Index

Performance Indices

Definition

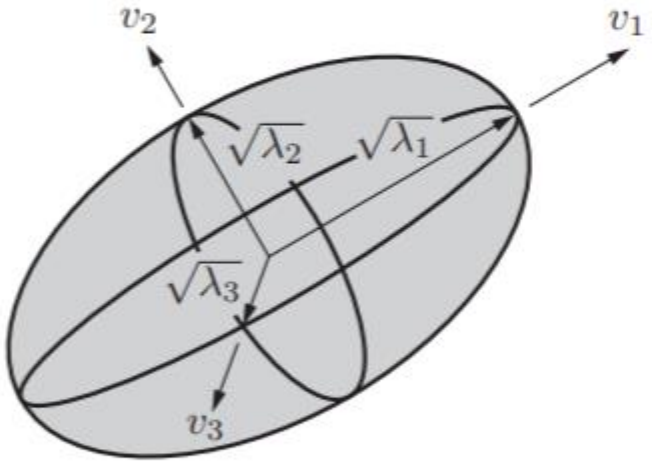
Performance Index Measure No.1 - Isotropy

- **Isotropy** – The ratio of the longest and shortest semi-axes of the manipulability ellipsoid

$$\mu_1(JJ^T) = \frac{\sqrt{\lambda_{\max}}}{\sqrt{\lambda_{\min}}} \geq 1$$

$$1 \leq \mu_1(JJ^T) \leq \infty$$

  - When $\mu_1(JJ^T) \rightarrow 1$ then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable
  - When $\mu_1(JJ^T) \rightarrow \infty$ the robot approaches a singularity



Performance Index Measure No.2 – Condition Number

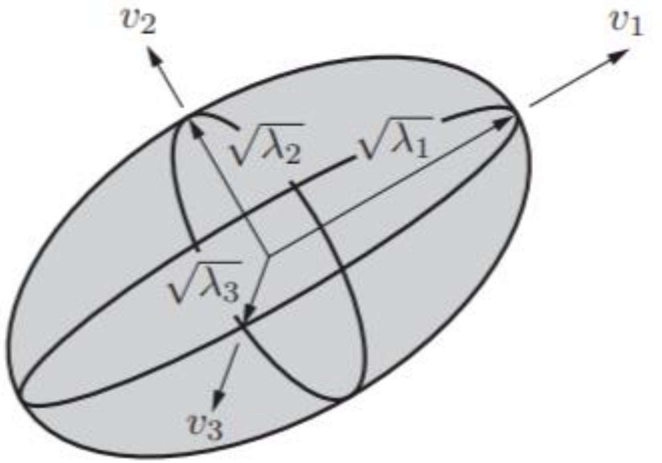
- **Manipulability Measure No. 2 – Condition Number** – Squaring the isotropy measure

$$\mu_2(JJ^T) = \left(\mu_1(JJ^T)\right)^2 = \frac{\lambda_{\max}}{\lambda_{\min}}$$

$$1 \le \mu_2(JJ^T) \le \infty$$

$$0 \ge \frac{1}{\mu_2(JJ^T)} = \frac{\lambda_{min}(A)}{\lambda_{max}(A)} \ge 1$$

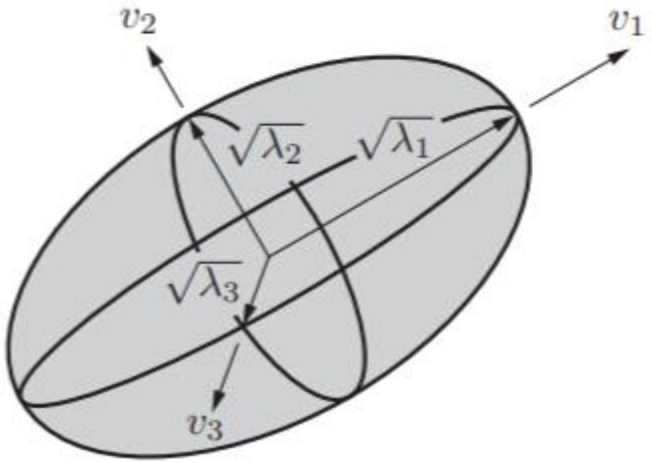
  - When $\frac{1}{\mu_2(JJ^T)} \to 1$ then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable
  - When $\frac{1}{\mu_2(JJ^T)} \to 0$ the robot approaches a singularity



Performance Index – Manipulability

- **Manipulability Measure No. 3 – Manipulability** – The volume V of the ellipsoid is proportional to the product of the semi-axis lengths

$$V \propto \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(JJ^T)}$$



- The Manipulability is defined as

$$\mu_3(JJ^T) = w = \sqrt{\lambda_1 \lambda_2 \cdots \lambda_m} = \sqrt{\det(JJ^T)}$$

$$0 \leq w < \infty$$

  – A good manipulator design has large area of characterized by high value of the manipulability (w)

Performance Index – Manipulability

- Given the structure of the Jacobian matrix, it makes sense to separate it into the two sub matrixes because the units of
  - J_v are linear velocities (m/s) and the unites of
  - J_ω are angular velocities (rad/s)

$$J(\theta) \in \mathbb{R}^{6\times n}$$

$$J(\theta) = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$J_v \in \mathbb{R}^{3\times n} \rightarrow$ Linear velocity/force ellipsoids

$J_w \in \mathbb{R}^{3\times n} \rightarrow$ Angular velocity/moment ellipsoids

- This leads to two three-dimensional manipulability ellipsoids, one for linear velocities and one for angular velocities.

$$J_v {J_v}^T$$

$$J_\omega {J_\omega}^T$$

Performance Index – Manipulability

When calculating the linear-velocity manipulability ellipsoid ($J_v {J_v}^T$),

it generally makes more sense to use the Jacobian expressed in the end effector space

$$^N J_v \, {^N J_v}^T$$

instead of the Base Frame

$$^0 J_v \, {^0 J_v}^T$$

since we are usually interested in the linear velocity of the end effector in its own coordinate system than a fixed frame at the base

Designing Well Conditioned Workspace – Rational

- **Challenge**
  - Difficulty in operating at
    - Workspace Boundaries
    - Neighborhood of singular point inside the workspace

- **Goal**
  - **Singularity** - The further the manipulator is away from singularities the better it moves uniformly and apply forces in all directions

- **Performance Criterion**
  - It is useful to assign a single scalar measure defining how easily the robot can move at a given posture.

Recap

Performance Index – Manipulability

- **Manipulabity Ellipsoid** - For a general n-joint serial (open chain) and a task space with coordinates the q manipulability ellipsoid corresponds to the end-effector velocities for joint rates $\dot{q}$

$$\Theta = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \\ d_i \\ \\ \theta_n \end{Bmatrix} \quad q = X = \begin{Bmatrix} x \\ y \\ Z \\ \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix} \qquad \dot{\Theta} = \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \\ \dot{d}_i \\ \\ \dot{\theta}_n \end{Bmatrix} \quad \dot{q} = \dot{X} = \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{Z} \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix} \qquad \begin{aligned} \dot{q} &= J\dot{\Theta} \\ \dot{\Theta} &= J^{-1}\dot{q} \end{aligned}$$

satisfying the norm of $\dot{\Theta}$ to be equal to 1

$$\|\dot{\Theta}\| = \dot{\Theta}^T\dot{\Theta} = 1$$

representing a unite sphere in the n-th dimensional joint velocity space

Performance Index – Manipulability

Assuming J is invertible, the unit joint-velocity condition can be written

$$1 = \dot{\Theta}^T \dot{\Theta}$$
$$1 = (J^{-1}\dot{q})^T (J^{-1}\dot{q})$$
$$1 = \dot{q}^T (J^{-1})^T J^{-1}\dot{q}$$
$$= \dot{q}^T J^{-T} J^{-1}\dot{q}$$
$$1 = \dot{q}^T (JJ^T)^{-1}\dot{q}$$

If J is full rank the matrix JJ^T and $(JJ^T)^{-1}$ are

- square,
- symmetric
- positive definite

Performance Index – Manipulability

For any symmetric positive-definite JJ^T , the set of vectors $\dot{q}$ satisfying

$$\dot{q}^T (JJ^T)^{-1} \dot{q} = 1$$

defines an ellipsoid in the m-dimensional space.

Recap

- Represent an circle / sphere $\dot{\Theta}^T \dot{\Theta} = 1$
- Represent a ellipse / ellipsoid $\dot{q}^T (JJ^T)^{-1} \dot{q} = 1$

Performance Indices - Design

Optimization Approach

Jacobian – Design - Performance Index – Optimization

Jacobian – Design - Performance Index – Optimization – Pseudo Code

FOR $L_1: 0 \rightarrow 500 + +\Delta L_1$

FOR $L_2: 0 \rightarrow 500 + +\Delta L_2$

Check if the link lengths allows the tip to reach all the points in the workspace by solving the IK for every point in the workspace

IF NOT → select a new pair of L_1, L_2

FOR $x: x + d \rightarrow x + d + w \quad + +\Delta x$

FOR $y: y - \frac{h}{2} \rightarrow y + \frac{h}{2} \quad + +\Delta y$

[Calculate the angles θ_1, θ_2 using IK]

[Calculate J and J^T]

[Calculate the eigenvalues of JJ^T]

[Calculate $\sqrt{\frac{\lambda_{min}}{\lambda_{max}}}$]

[Populate κ]

END (FOR y)

END (FOR x)

[Calculate $\sum \kappa_i$]

[Calculate κ_{min}]

[Calculate $C = \frac{\sum \kappa_i \kappa_{min}}{L_1^3 + L_2^3}$]

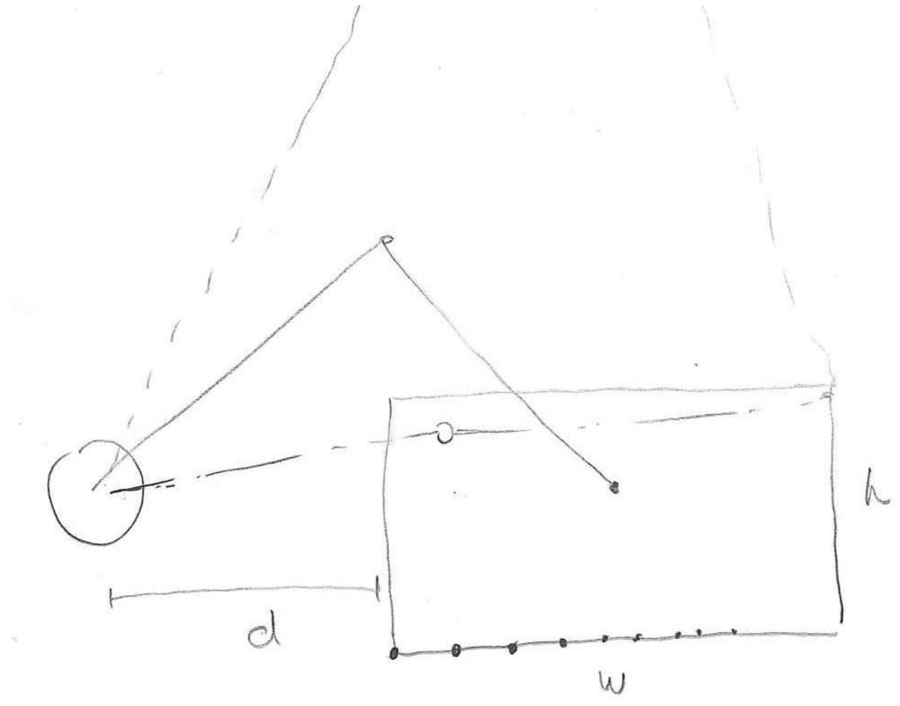
[Calculate optimal $L_1, L_2, \sum \kappa_i, \kappa_{min}, C$]

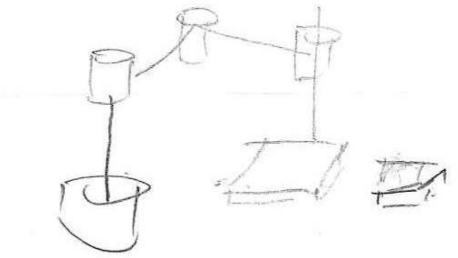
END (FOR L_2)

END (FOR L_1)

[Select max $C \rightarrow \mathrm{L}_1, \mathrm{L}_2$]

Jacobian – Design - Performance Index – Optimization

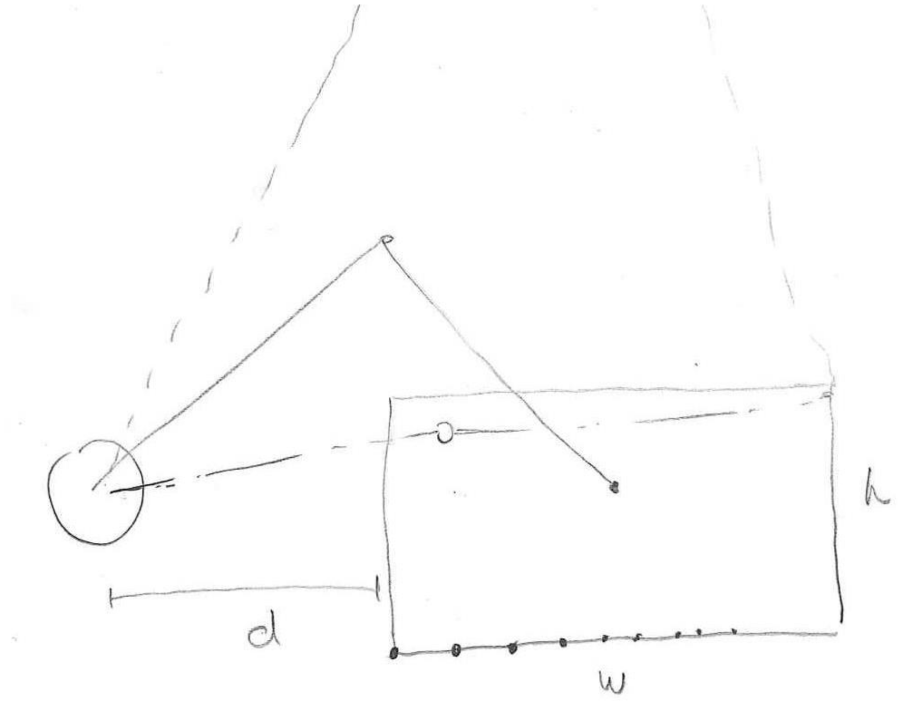


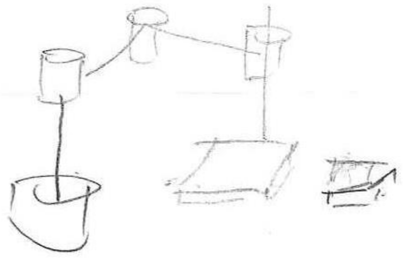


$$C_{l_1, l_2} \quad \frac{\left(\sum_i K_i\right) K_{min}}{L_1^3 + L_2^2}$$

$$K = \sqrt{\frac{\lambda_{min}^2}{J_{max}^2}}$$

Jacobian – Design - Performance Index – Optimization



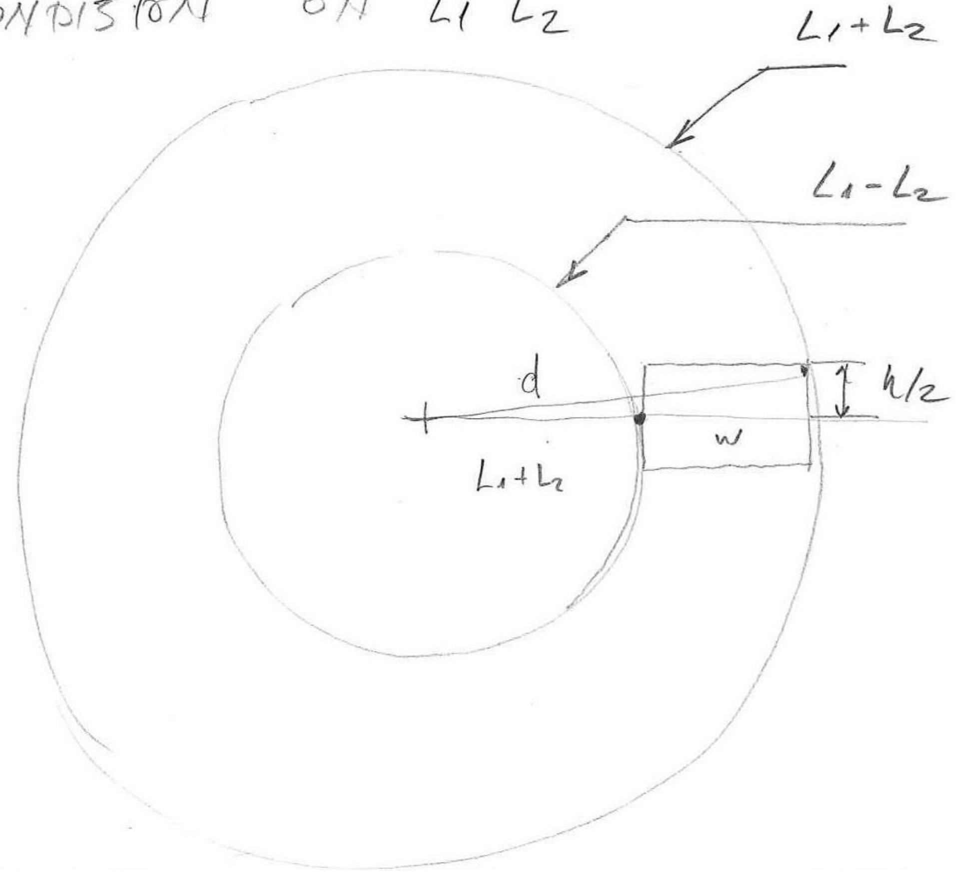


$$C_{L_1,L_2} = \frac{(\sum_i \kappa_i)\kappa_{min}}{L_1^3 + L_2^2}$$

$$\kappa = \sqrt{\frac{\lambda_{min}^2}{\lambda_{max}^2}}$$

Jacobian – Design - Performance Index – Optimization

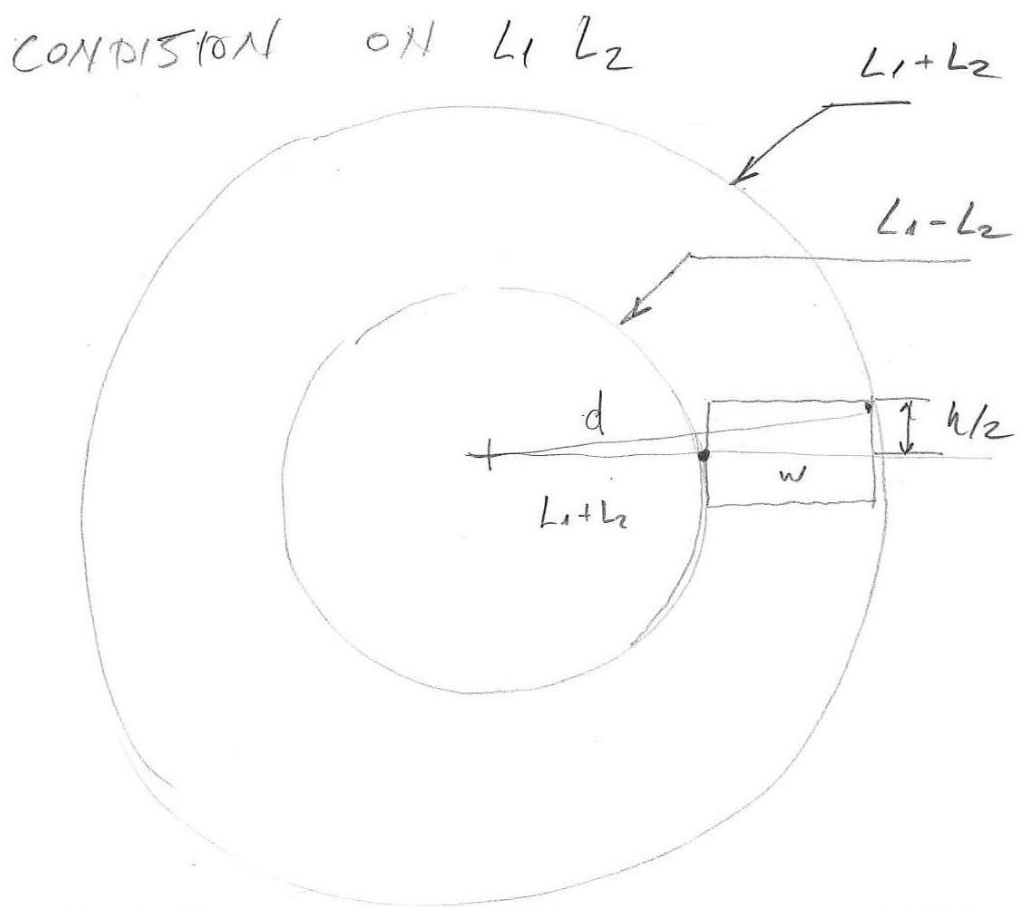
CONDITION ON L_1 L_2



$$L_1 + L_2 < d$$

$$L_1 + L_2 < \sqrt{(d+w)^2 + (h/2)^2}$$

Jacobian – Design - Performance Index – Optimization



$$L_1 + L_2 < d$$

$$L_1 + L_2 < \sqrt{(d+w)^2 + \left(\frac{h}{2}\right)^2}$$

Jacobian – Design - Performance Index – Optimization

For $l_1 : 0 \longrightarrow 500$ $+\Delta L_1$

for $l_2 : 0 \longrightarrow 500$ ΔL_2

Check if the link lengths allows the tip to reach all points of the work space

IF $l_1 + l_2 < d$

IF $l_1 + l_2 < \sqrt{(d+w)^2 + (h/2)^2}$

For $x : x+d \longrightarrow x+d+w$ Δx

For $y : y-\frac{h}{2} \longrightarrow y+\frac{h}{2}$ Δy

calculate the θ_1, θ_2 using the IK

Jacobian – Design - Performance Index – Optimization

FOR $L_1: 0 \rightarrow 500 ++\Delta L_1$

FOR $L_2: 0 \rightarrow 500 ++\Delta L_2$

Check if the link lengths allows the tip to reach all the points in the workspace by solving the IK for every point in the workspace

IF $L_1 + L_2 < d$

IF $L_1 + L_2 < \sqrt{(d+w)^2 + \left(\frac{h}{2}\right)^2}$

FOR $x: x + d \rightarrow x + d + w \quad ++\Delta x$

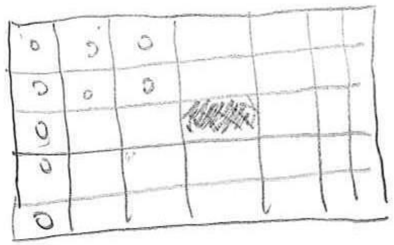
FOR $y: y - \frac{h}{2} \rightarrow y + \frac{h}{2} \quad ++\Delta y$

[Calculate the angles θ_1, θ_2 using IK]

Calculate J and J^T

Calculate the eigenvalues of JJ^T

calculate $\sqrt{\frac{\lambda_{min}^2}{\lambda_{max}^2}}$

$K =$ 

END (for y)

END (for x)

Jacobian – Design - Performance Index – Optimization

[Calculate J and J^T]
[Calculate the eigenvalues of JJ^T]
[Calculate $\sqrt{\frac{\lambda_{min}^2}{\lambda_{max}^2}}$]
[Populate κ]
END (FOR y)
END (FOR x)

Jacobian – Design - Performance Index – Optimization

CALCULAT $\sum_i K_i =$

$K_{i\,min}$

$$C = \frac{\sum K_i \; K_{min}}{L_1^3 + L_2^3}$$

OPT =

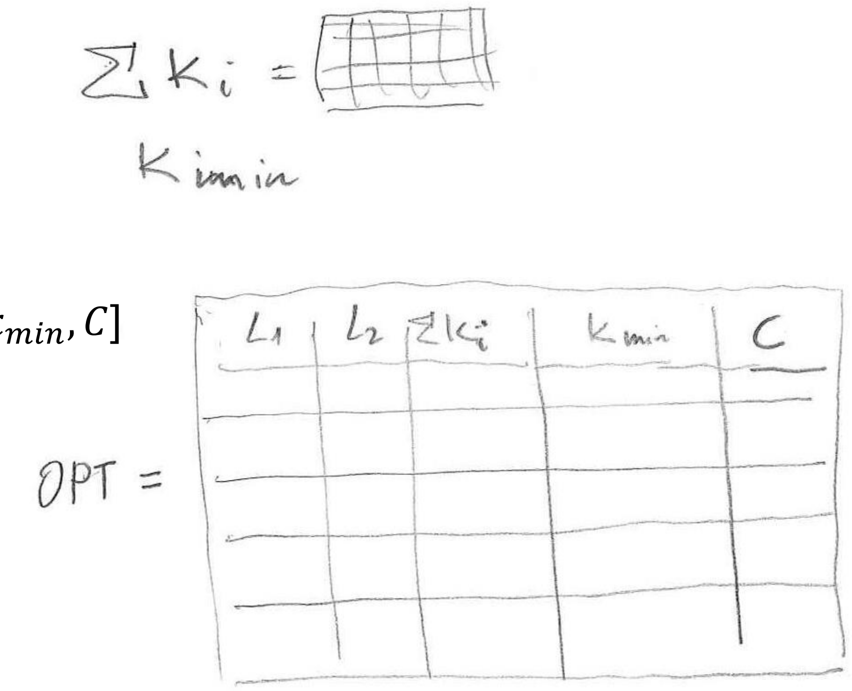
L_1	L_2	$\sum K_i$	K_{min}	C

END (OF L_2)

END (OF L_1)

Jacobian – Design - Performance Index – Optimization

[Calculate $\sum \kappa_i$]
[Calculate κ_{min}]
[Calculate $C = \frac{\sum \kappa_i \kappa_{min}}{L_1^3 + L_2^3}$]
[Calculate optimal $L_1, L_2, \sum \kappa_i, \kappa_{min}, C$]
END (OF L_2)
END (OF L_1)



Jacobian – Design - Performance Index – Optimization

- SEARCH FOR MAX C in OPT
- Find L_1, L_2

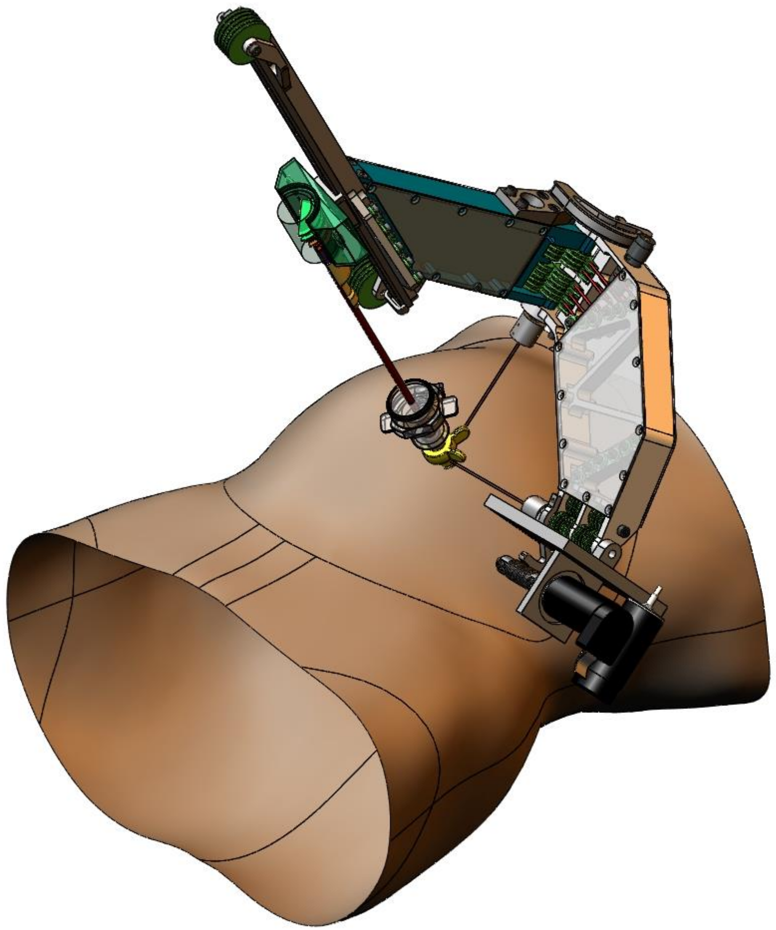
Jacobian – Design - Performance Index – Optimization

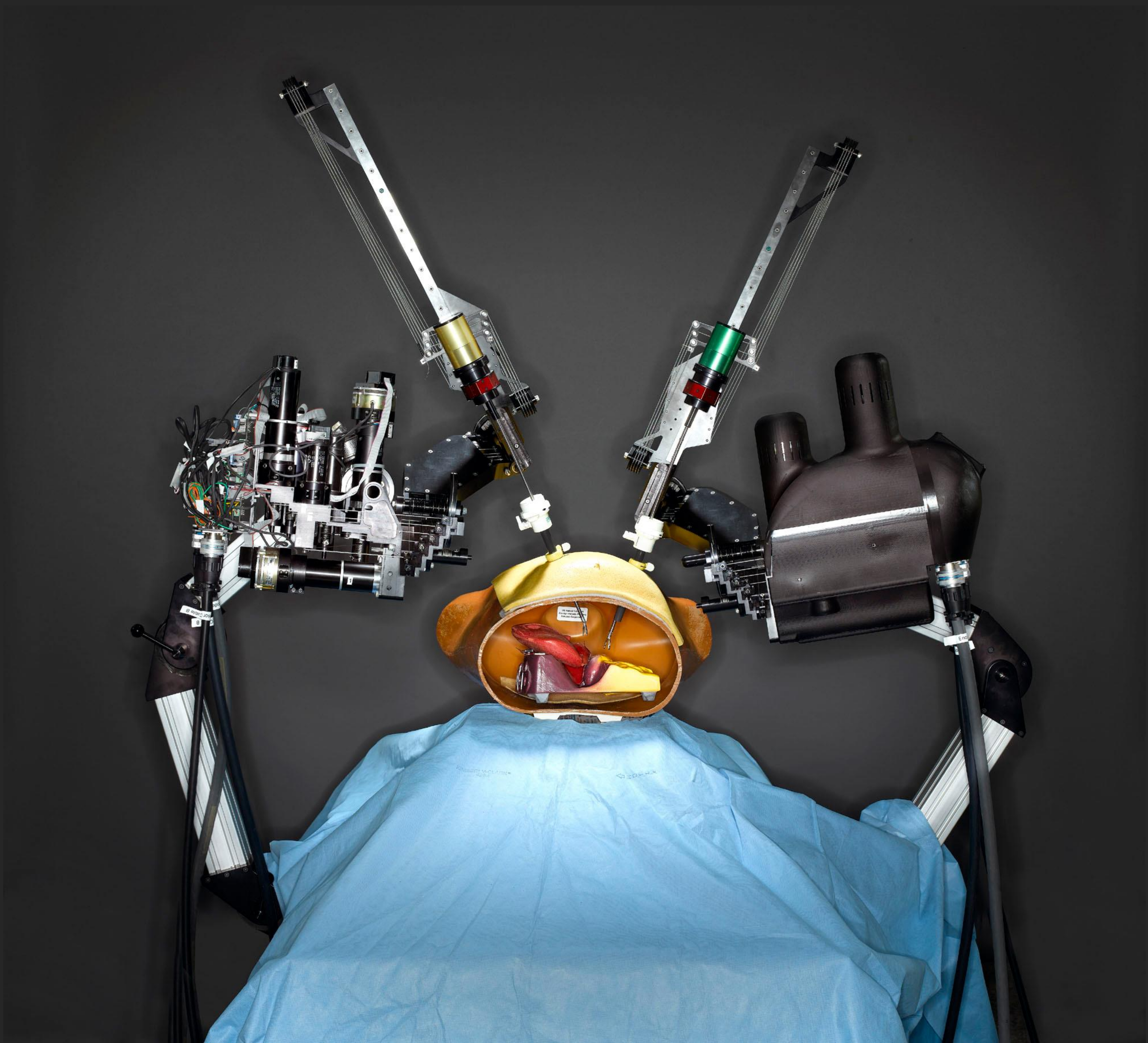
- Search for MAX C in OPT
- Find L_1, L_2

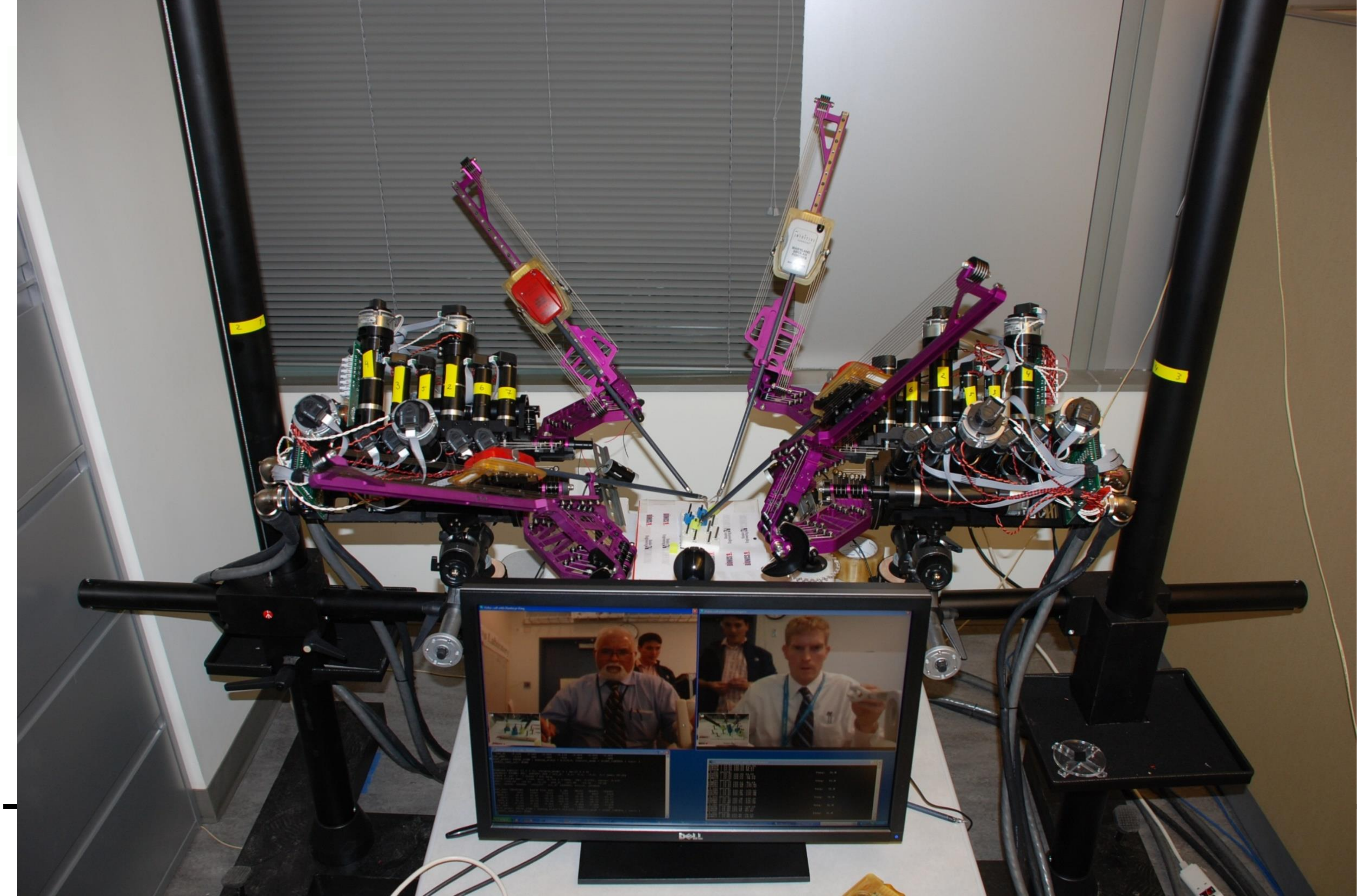
Design – Example

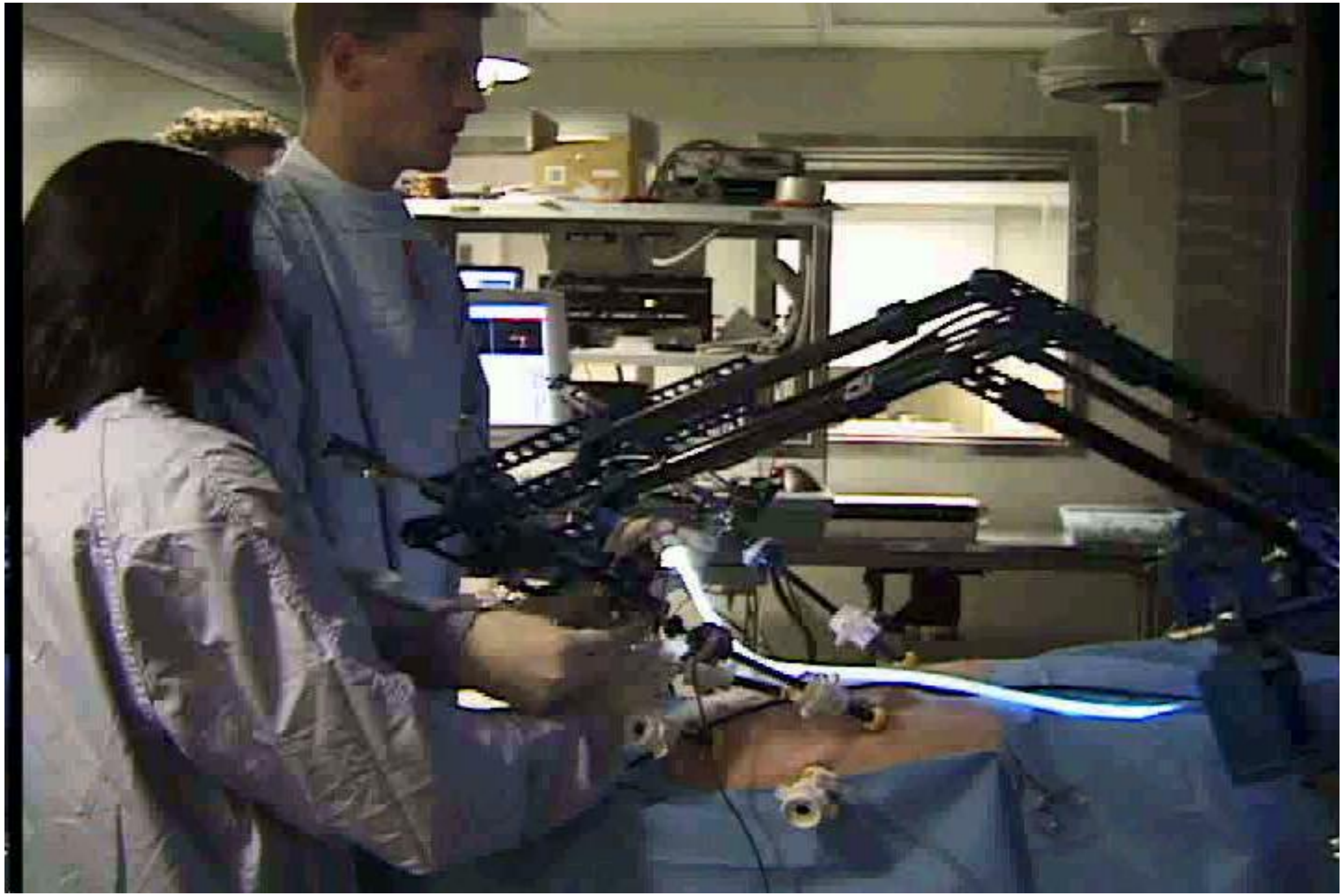
RAVEN – A SURGICAL ROBTICS SYSTEM

DESIGN – SPECIFICATIONS





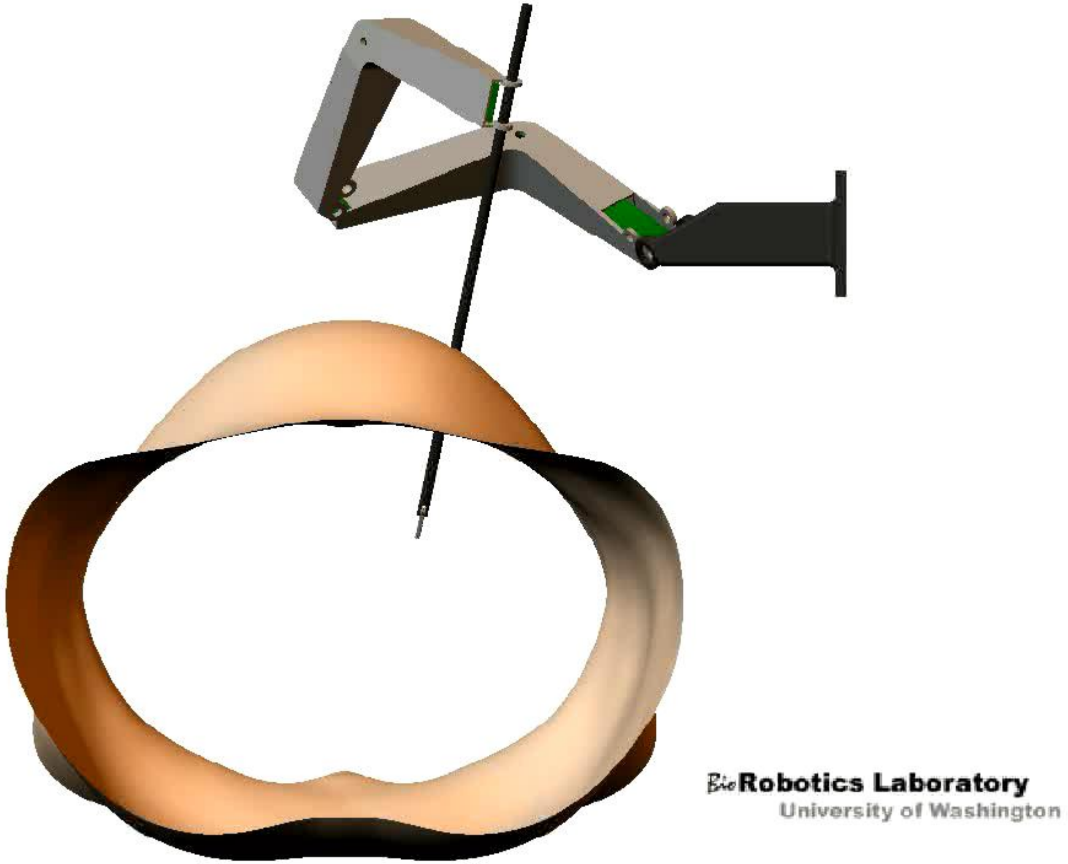




Engineering Specifications - BlueDRAGON

Device				DRAGON	UC Berkeley	UC Berkeley	UC Berkeley	DeVinchi	Zeus
Generation				R1 - E (95%)		1	2		
Referance					Measured	Traget	Obtained		
Base	Overall Geomtery	Shaft Diameter	[m]			0.01 - 0.015	0.01 - 0.015	0.01	0.005
	Position / Oriantataion	Delta Theta x	[Deg]	53.8047				+/-60	
		Delta Theta y	[Deg]	36.3807				+/-80	
		Delta Theta z	[Deg]	148.0986	90	180-270	720	+/-180	
		R	[m]	0.1027				0.2	
		Grasping Jaw s	[Deg]	24.0819				200	
		Grasping Jaw s	[m]	*	0.006	0.002-0.003	0.008 min		
		Delta X	[m]	0.1026					
		Delta Y	[m]	0.0815					
		Delta Z	[m]	0.0877					
	Velocity (Angular Linear)	Wx	[Rad/sec]	0.432					
		Wy	[Rad/sec]	0.486					
		Wz	[Rad/sec]	1.053			9.4 min		
		VR	[m/sec]	0.072					
		Wg	[Rad/sec]	0.0468					
	Force	Fx	[N]	14.7299					
		Fy	[N]	13.1981					
		Fz	[N]	184.3919					
		Fg	[N]	41.6085	15	5 min	40 min		
	Torque	Tx	[Nm]	2.3941					
		Ty	[Nm]	1.6011					
		Tz	[Nm]	0.0464	0.088	0.022			

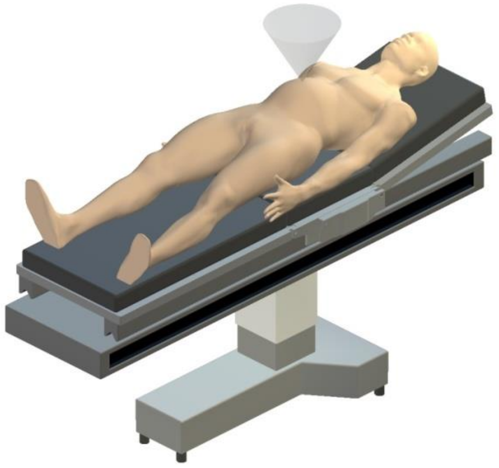
Kinematic Analysis –
Playback Simulation using Measured Data

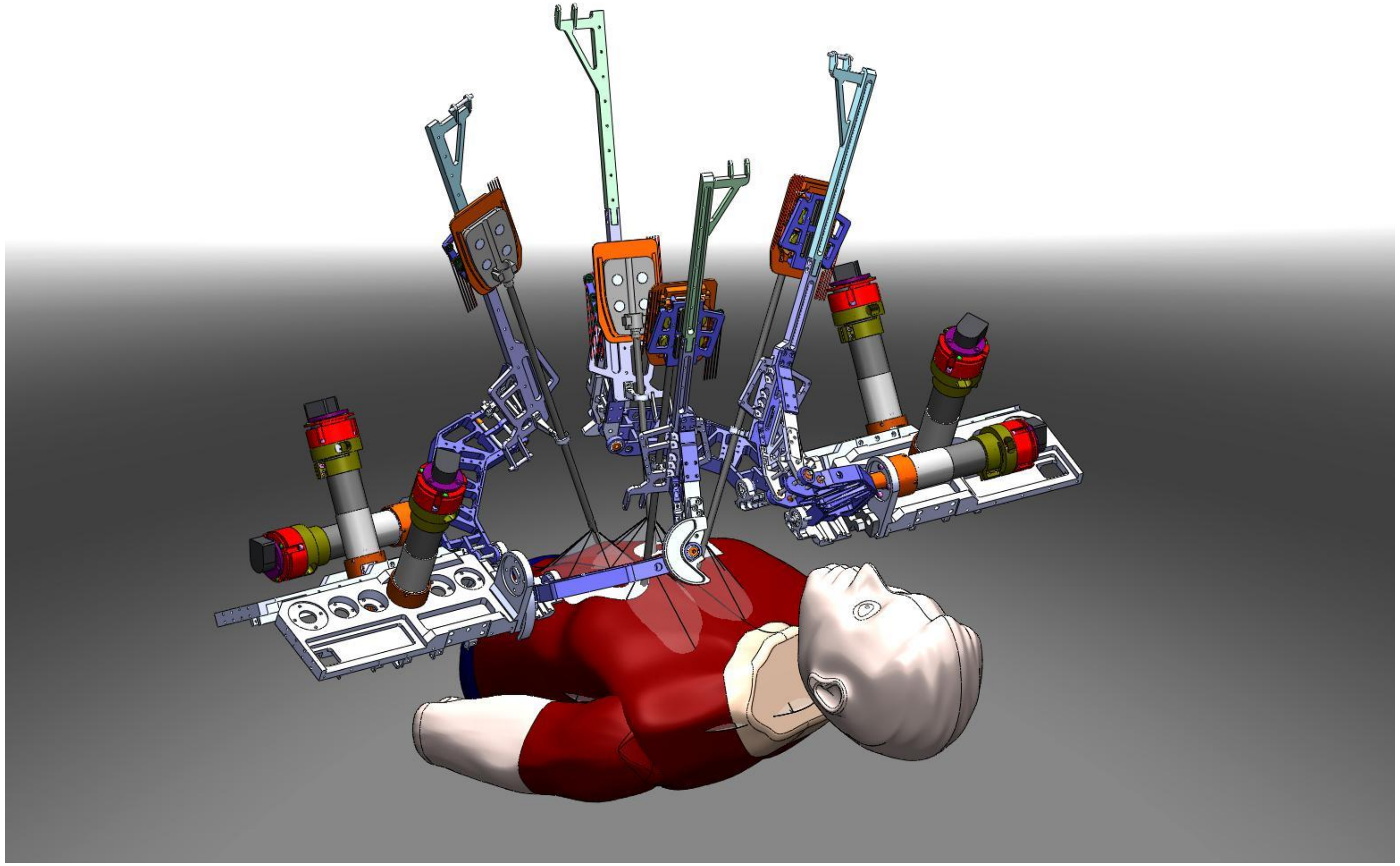


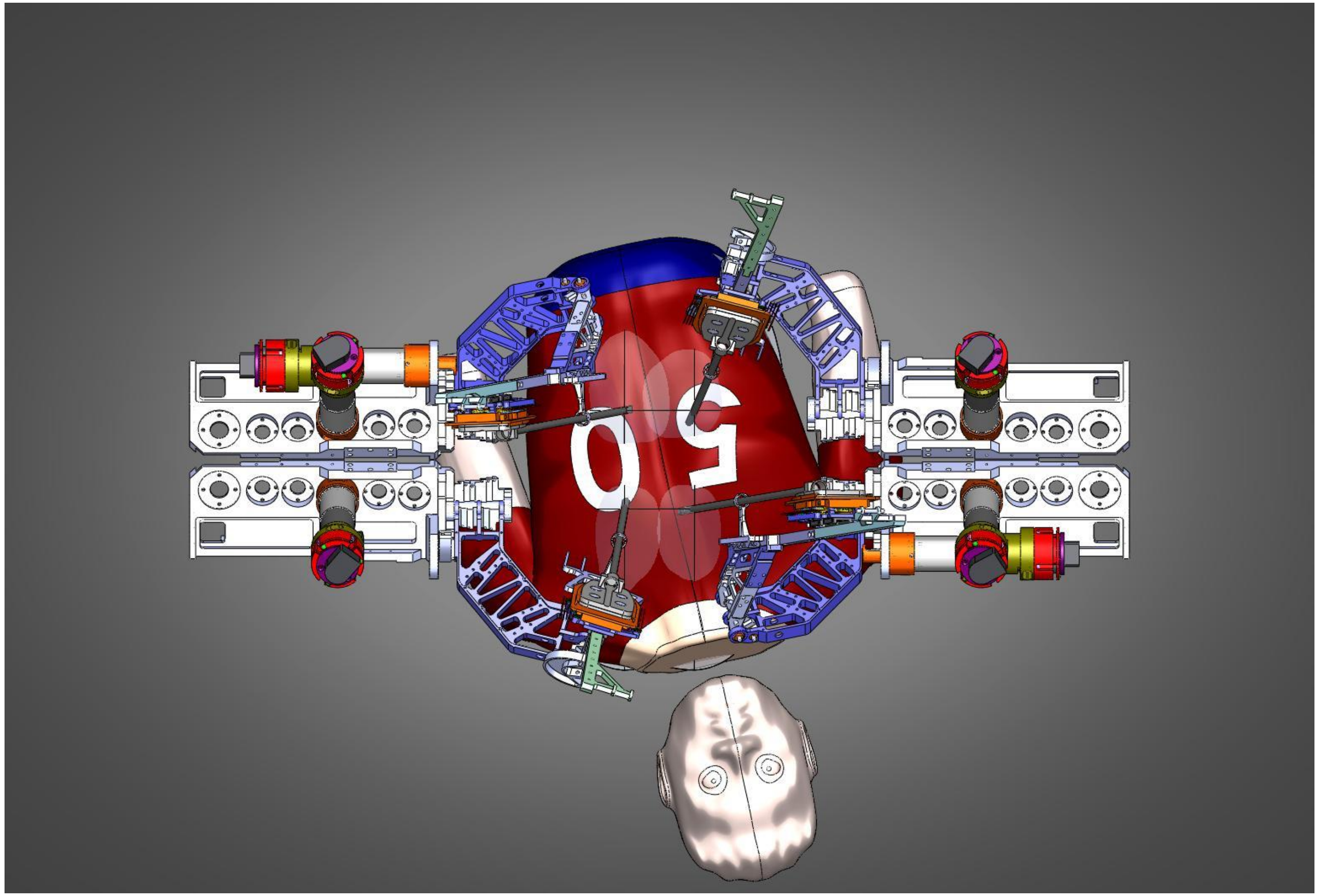
Robot Optimization - Workspace

- Dexterous Workspace (DWS)
  - High dexterity region defined by a right circular cone with a vertex angel of 60º
  - Contains 95% of the tool motions based on *in-vivo* measurements.

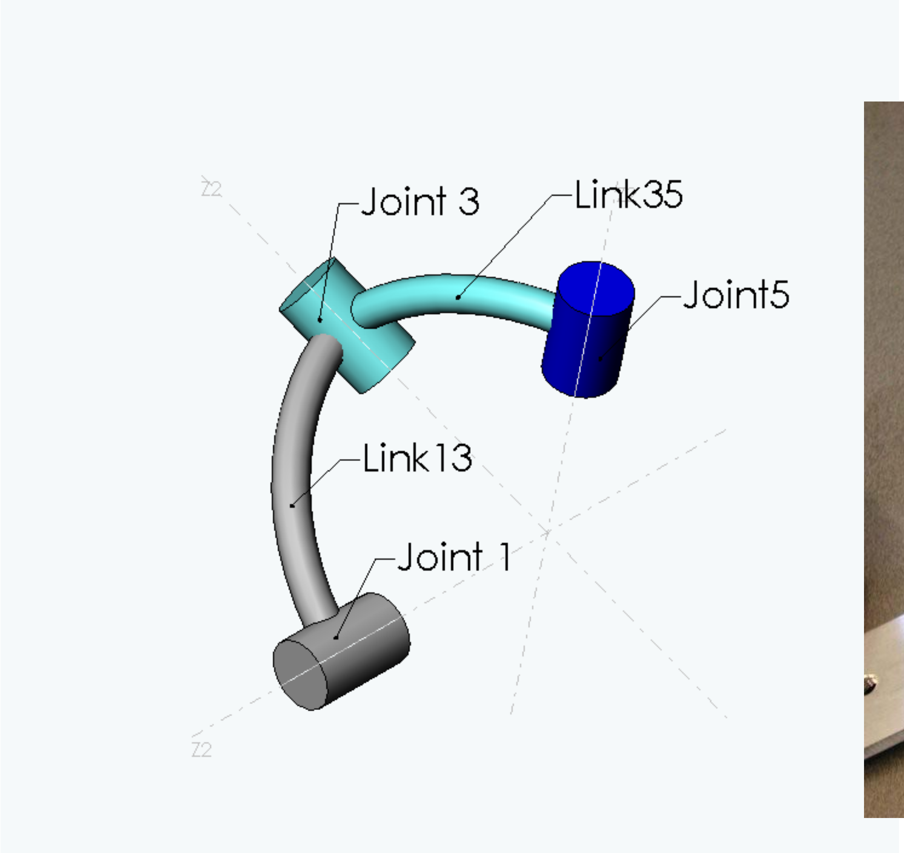
60º- 60º

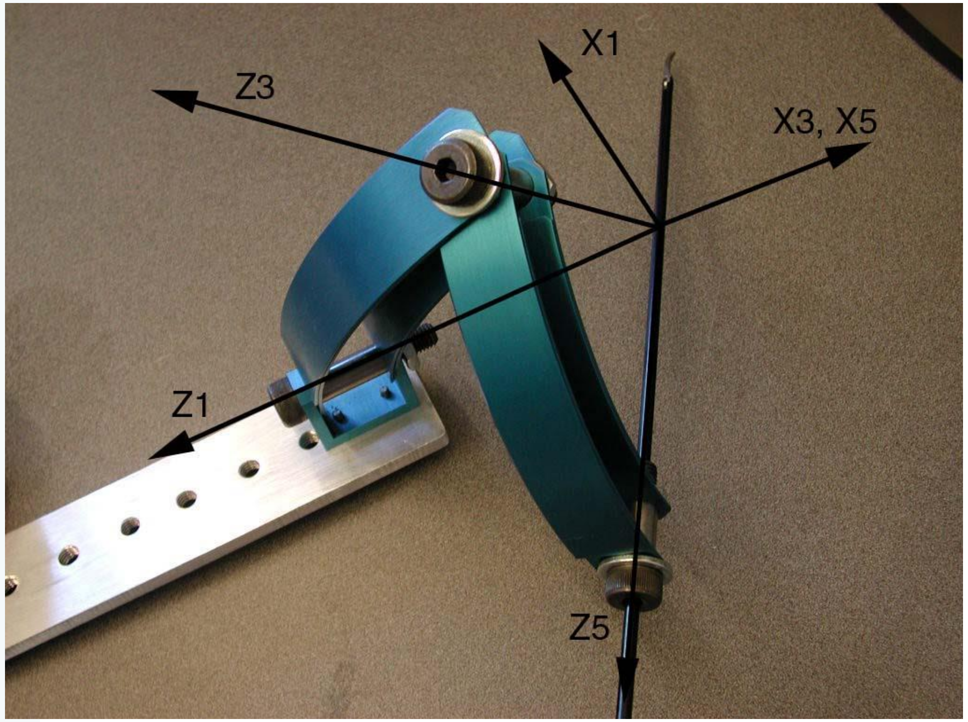




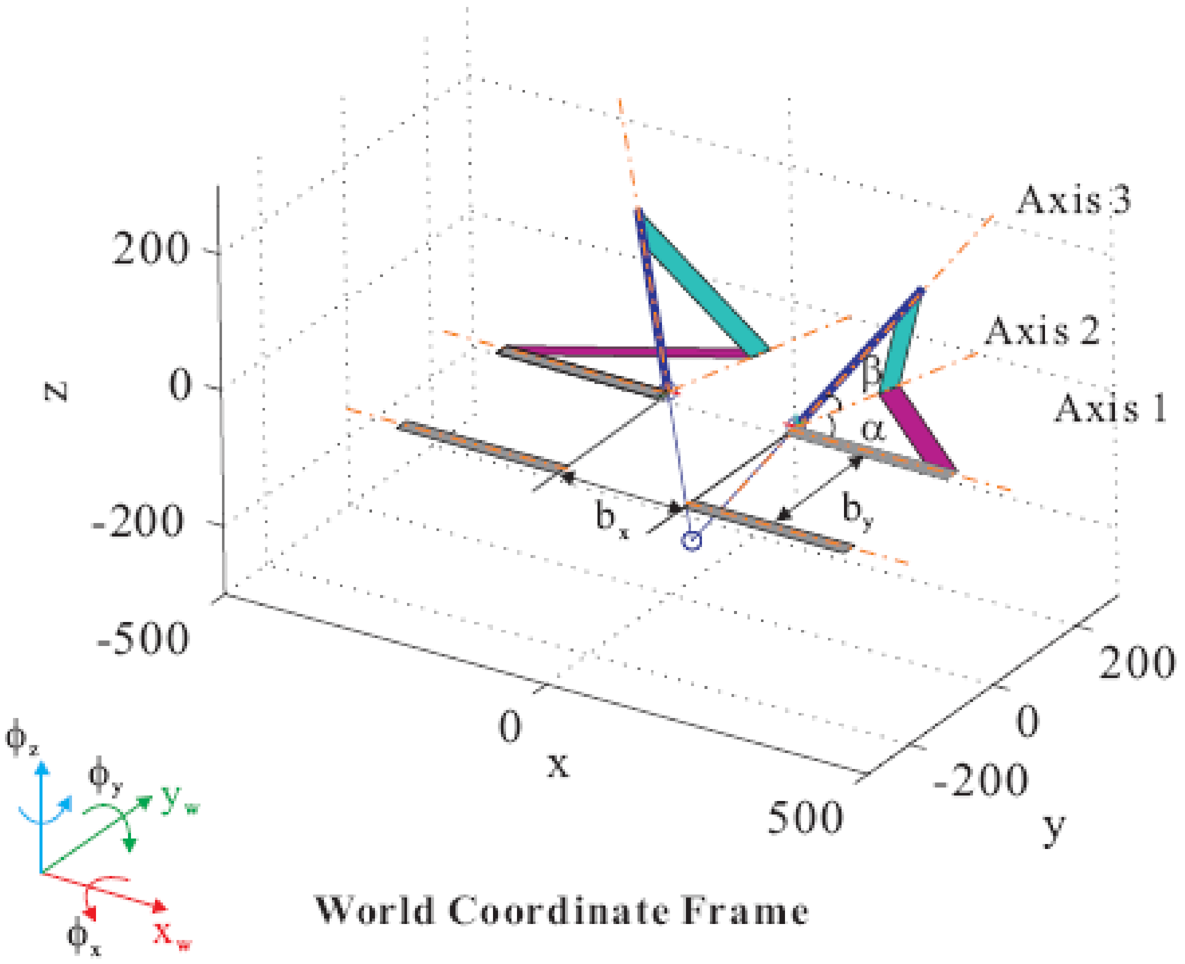


Spherical Mechanism - Robot Optimization

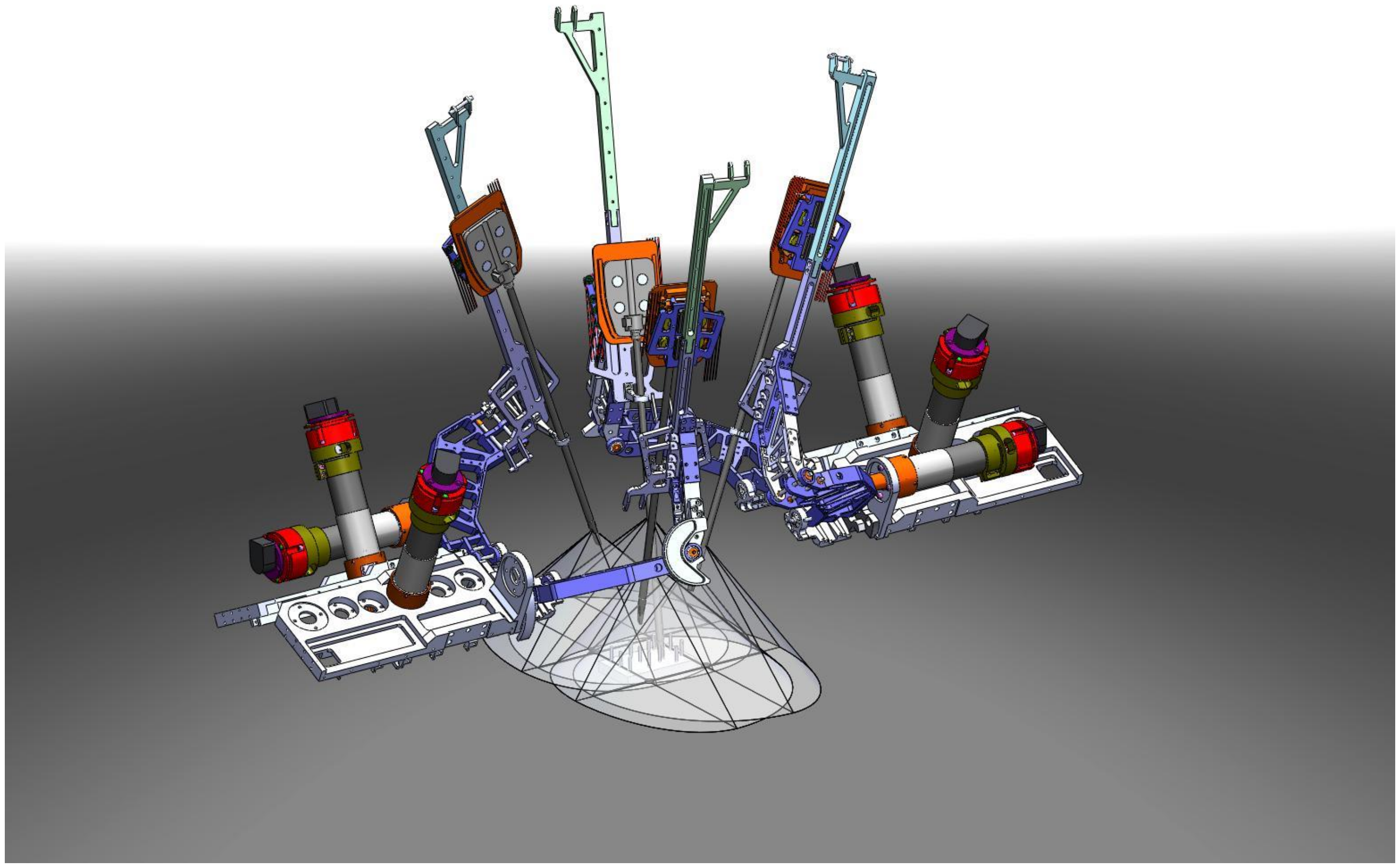




Optimization of Raven IV – Problem & Parameters (7) Definitions



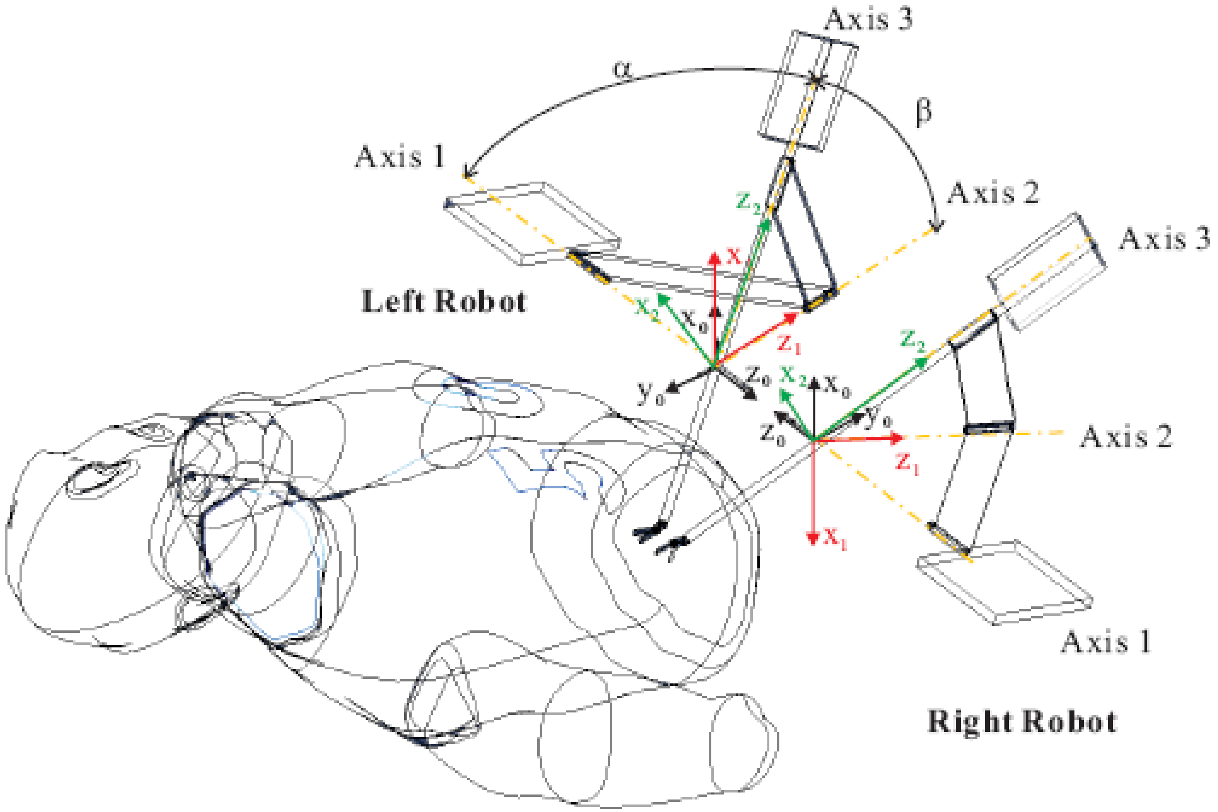
World Coordinate Frame



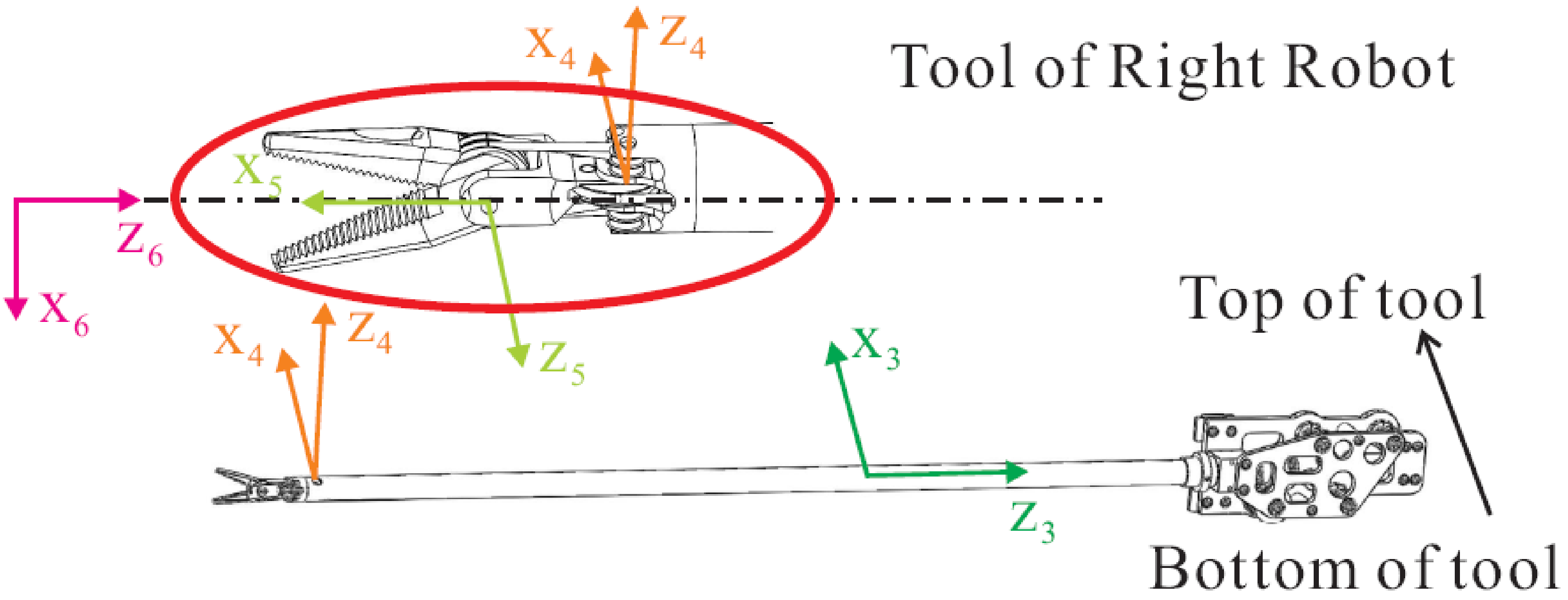
RAVEN – A SURGICAL ROBTICS SYSTEM

DESIGN – KINEMATIC ANALYSIS & OPTIMIZATION

Direct Kinematics –
Coordinate Systems Assignment



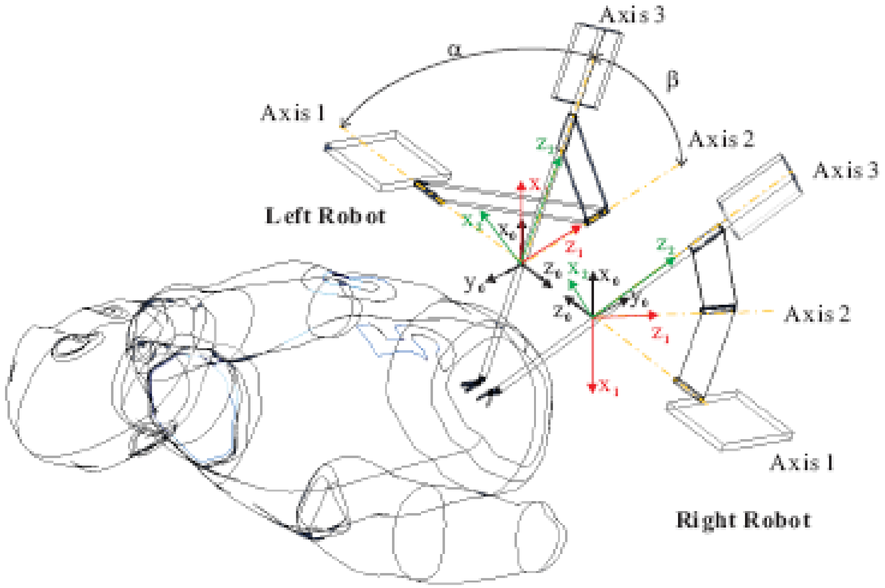
Direct Kinematics –
Coordinate Systems Assignment



Direct Kinematics:
DH Parameters - Left and Right Robot

Robot	$i-1$	i	α_i	a_i	d_i	θ_i
Left Robot (1,3)	0	1	$\pi-\alpha$	0	0	$\theta_1(t)$
	1	2	$-\beta$	0	0	$-\theta_2(t)$
	2	3	0	0	0	$\pi/2-\theta_3(t)$
	3	4	$-\pi/2$	0	$d_4(t)$	0
	4	5	$\pi/2$	a_5	0	$\pi/2-\theta_5$
	5	6	$-\pi/2$	0	0	$\pi/2+\theta_6$
Right Robot (2,4)	0	1	$\pi-\alpha$	0	0	$\pi-\theta_1(t)$
	1	2	$-\beta$	0	0	$\theta_2(t)$
	2	3	0	0	0	$\pi/2+\pi+\theta_3(t)$
	3	4	$-\pi/2$	0	$d_4(t)$	0
	4	5	$-\pi/2$	a_5	0	$\pi/2+\theta_5$
	5	6	$-\pi/2$	0	0	$\pi/2-\theta_6$

Direct Kinematics:
Transform Matrix for Left Robot



$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 c\alpha & s_1 s\alpha & 0 \\ s_1 & c_1 c\alpha & -c_1 s\alpha & 0 \\ 0 & s\alpha & c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 c\beta & s_2 s\beta & 0 \\ s_2 & c_2 c\beta & -c_2 s\beta & 0 \\ 0 & s\beta & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

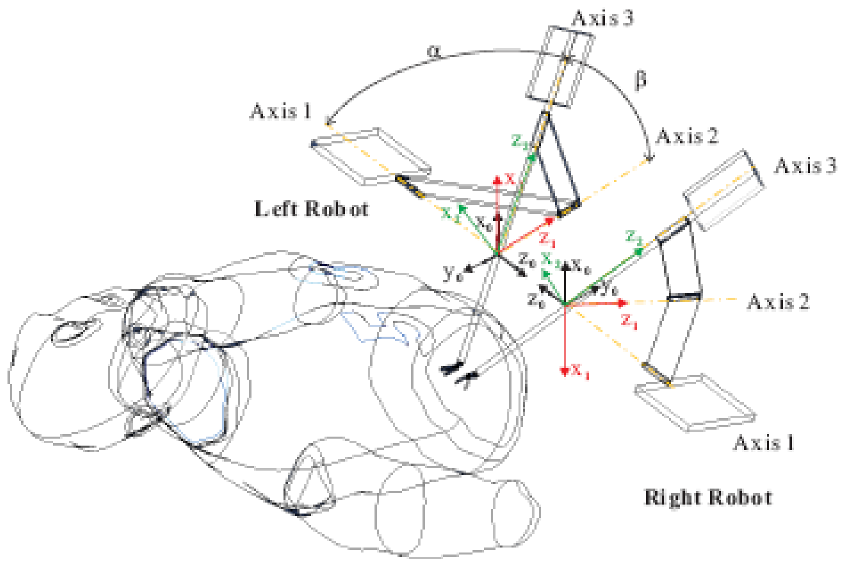
$$
{}^{2}_{3}T = \begin{bmatrix} s_3 & -c_3 & 0 & 0 \\ c_3 & s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}_{5}T = \begin{bmatrix} s_5 & 0 & c_5 & a_5 s_5 \\ c_5 & 0 & -s_5 & a_5 c_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}_{6}T = \begin{bmatrix} -s_6 & 0 & -c_6 & 0 \\ c_6 & 0 & -s_6 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Direct Kinematics:
Transform Matrix for Right Robot



$$ {}^{0}_{1}T = \begin{bmatrix} -c_1 & s_1 c\alpha & s_1 s\alpha & 0 \\ s_1 & c_1 c\alpha & c_1 s\alpha & 0 \\ 0 & s\alpha & -c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 c\beta & -s_2 s\beta & 0 \\ s_2 & c_2 c\beta & c_2 s\beta & 0 \\ 0 & -s\beta & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} s_3 & c_3 & 0 & 0 \\ -c_3 & s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} -s_5 & 0 & -c_5 & -a_5 s_5 \\ c_5 & 0 & -s_5 & a_5 c_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} s_6 & 0 & -c_6 & 0 \\ c_6 & 0 & s_6 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Direct Kinematics: Solution

$$ {}^{0}_{1}T = \begin{bmatrix} -c_1 & s_1 c\alpha & s_1 s\alpha & 0 \\ s_1 & c_1 c\alpha & c_1 s\alpha & 0 \\ 0 & s\alpha & -c\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 c\beta & -s_2 s\beta & 0 \\ s_2 & c_2 c\beta & c_2 s\beta & 0 \\ 0 & -s\beta & c\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} s_3 & c_3 & 0 & 0 \\ -c_3 & s_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} -s_5 & 0 & -c_5 & -a_5 s_5 \\ c_5 & 0 & -s_5 & a_5 c_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} s_6 & 0 & -c_6 & 0 \\ c_6 & 0 & s_6 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse Kinematics

- 6 DOFs for positioning and orienting → Inverse Kinematics
- 1 DOF for the opening and closing of the grasper → Redundancy
- Joint Limit Range

θ_i	range	sin	cos
θ_1	$[0°, 90°]$	$+$	$+$
θ_2	$[20°, 140°]$	$+$	$+/-$
θ_3	$[-86°, 86°]$	$+/-$	$+$
d_4	$[-250, -0]$ mm	N/A	N/A
θ_5	$[-86°, 86°]$	$+/-$	$+$
θ_6	$[-86°, 86°]$	$+/-$	$+$

Inverse Kinematics: Homogeneous Transformation Matrix and Its Inverse

- **Homogenous Transform Matrix → Inverse**

$$ {}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \quad {}^6_0T = [{}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T]^{-1} = \begin{bmatrix} r'_{11} & r'_{12} & r'_{13} & P_{xinv} \\ r'_{21} & r'_{22} & r'_{23} & P_{yinv} \\ r'_{31} & r'_{32} & r'_{33} & P_{zinv} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- **For the left robot,**

$$ P_{xinv} = (-d_4c_5 + a_5)c_6 $$
$$ P_{yinv} = s_5d_4 $$
$$ P_{zinv} = (-d_4c_5 + a_5)s_6 $$

- **For the right robot,**

$$ P_{xinv} = (d_4c_5 - a_5)c_6 $$
$$ P_{yinv} = s_5d_4 $$
$$ P_{zinv} = -(d_4c_5 - a_5)s_6 $$

- **Define**

$$ P_{inv}^2 = P_{xinv}^2 + P_{yinv}^2 + P_{zinv}^2 $$
$$ = (a_5 - d_4c_5)^2c_6^2 + (a_5 - d_4c_5)s_6^2 + s_5^2d_4^2 $$
$$ \Rightarrow $$
$$ P_{inv}^2 = (a_5 - d_4c_5)^2 + s_5^2d_4^2 = a_5^2 - 2a_5d_4c_5 + d_4^2c_5^2 + d_4^2s_5^2 $$
$$ \Rightarrow $$
$$ P_{inv}^2 = a_5^2 - 2a_5d_4c_5 + d_4^2 $$

- **Which gives**

$$ c_5^2 = \left(\frac{a_5^2 + d_4^2 - P_{inv}^2}{2a_5d_4}\right)^2 $$

Inverse Kinematics

- **For the left robot,**

$$P_{xinv} = (-d_4 c_5 + a_5) c_6$$
$$P_{yinv} = s_5 d_4$$
$$P_{zinv} = (-d_4 c_5 + a_5) s_6$$

$$\Rightarrow \quad s_5 = \frac{P_{yinv}}{d_4}$$

- **For the right robot,**

$$P_{xinv} = (d_4 c_5 - a_5) c_6$$
$$P_{yinv} = s_5 d_4$$
$$P_{zinv} = -(d_4 c_5 - a_5) s_6$$

$$c_5^2 = \left(\frac{a_5^2 + d_4^2 - P_{inv}^2}{2 a_5 d_4}\right)^2$$

$$c_5^2 + s_5^2 = 1$$

$$\Rightarrow \quad d_4$$

Inverse Kinematics

- **Four Possible Solutions of** d_4

$$\left\{\begin{matrix} d_4 = \sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = -\sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = \sqrt{a_5^2 + P_{inv}^2 + 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \\ d_4 = -\sqrt{a_5^2 + P_{inv}^2 - 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}} \end{matrix}\right\} \Rightarrow$$

$$d_4 = -\sqrt{a_5^2 + P_{inv}^2 - 2a_5 + \sqrt{P_{inv}^2 - P_{zinv}^2}}$$

- **Resolve** θ_6

$$c_6 = \frac{P_{zinv}}{(-c_5 d_4 + a_5)}$$

For the left robot,

$$s_6 = \frac{P_{xinv}}{(-c_5 d_4 + a_5)}$$

For the right robot,

$$s_6 = \frac{-P_{xinv}}{(-c_5 d_4 + a_5)}$$

$$\theta_6 = A\tan 2\,(s_6, c_6)$$

- **Resolve** θ_5

$$s_6 = \frac{P_{yinv}}{d_4}$$

$$c_6 = \sqrt{1 - s_6^2}$$

$$\theta_6 = A\tan 2\,(s_6, c_6)$$

Inverse Kinematics

- **With resolved** d_4, θ_5 **and** θ_6

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = {}^{6}_{0}T\,[{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T]^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_x \\ a_{21} & a_{22} & a_{23} & a_y \\ a_{31} & a_{32} & a_{33} & a_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- **Where**

$$ a_{32} = s_2 s_\alpha c_3 + (c_2 s_\alpha c_\beta + c_\alpha s_\beta) s_3 $$
$$ a_{33} = c_2 s_\alpha c_\beta - c_\alpha s_\beta $$

- **Resolve** θ_2

$$ c_2 = \frac{c_\alpha s_\beta + a_{33}}{s_\alpha c_\beta} $$
$$ s_2 = \sqrt{1 - c_2^2} $$

$$ \theta_2 = A\tan 2\,(s_2, c_2) $$

- **Define**

$$ a =_2 s_\alpha $$
$$ b = c_2 s_\alpha c_\beta + c_\alpha s_\beta $$

- **We have**

$$ a_{32} = a c_3 + b s_3 $$

- **According to [1]**

$$ \theta_3 = 2A\tan\left(\frac{b + \sqrt{a^2 + b^2 - a_{32}}}{a + a_{32}}\right) $$

Inverse Kinematics

Check a_{13} **to select between the two solution of** θ_3

For the left robot, $a_{13} = -s_2 s_\alpha s_3 + c_2 s_\alpha c_3 c_\beta + s_\alpha c_3 s_\beta$

For the right robot, $a_{13} = s_2 s_\alpha s_3 - c_2 s_\alpha c_3 c_\beta - s_\alpha c_3 s_\beta$

- **With resolved** θ_5, θ_5, d_4, $\theta_3 = 2A\tan\left(\frac{b + \sqrt{a^2 + b^2 - a_{32}^2}}{a + a_{32}}\right)$ **and** θ_6

$$ {}^0_1T = {}^6_0T[{}^3_4T\,{}^4_5T\,{}^5_6T]^{-1}[{}^1_2T\,{}^2_3T]^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_x \\ b_{21} & b_{22} & b_{23} & b_y \\ b_{31} & b_{32} & b_{33} & b_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- **Where**

For the left robot, $s_1 = b_{11}, c_1 = b_{21}$ $\}$

$\Rightarrow$ $\theta_1 = A\tan 2\,(s_1, c_1)$

For the right robot, $s_1 = b_{11}, c_1 = -b_{21}$

Jacobian & Isotropy

- The mechanism isotropy is determined by the eigen-values of **Jacobian matrix**, which can be derived by **velocity propagation**

- General equations for velocity propagation: $\dot{X} = J\dot{\Theta}$

For the angular velocity, $${}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+2}\hat{Z}_{i+1}$$

For the linear velocity, $${}^{i+1}v_{i+1} = {}^{i+1}_{i}R({}^{i}\omega_i \times P_{i+1} + {}^{i}v_i) + \dot{d}_{i+2}\hat{Z}_{i+1}$$

For the revolute joint, $$\dot{\theta}_{i+2} = 0$$

For the prismatic joint, $$\dot{d}_{i+2} = 0$$

Jacobian & Isotropy

- **Initial Condition**

Link 1 is rotating at $\dot{\theta}_1$ about z_0: $\quad {}^0\omega_0 = [0,0,\dot{\theta}_1]^T \quad {}^0v_0 = [0,0,0]^T$

Link 2 is rotating at $\dot{\theta}_2$ about z_1

Link 3 is frozen with $\dot{\theta}_3 = 0$

Translation in homogeneous transformation matrix: ${}^0P_1 = {}^1P_2 = {}^2P_3 = [0,0,0]^T$

Link 4 is translating at $\dot{d}_4$ along z_3

- Rotation Matrices ${}^{i+1}_{i}R = {}^{i}_{i+1}R^T$, which leads to

For the left robot $\quad {}^1_0R = \begin{bmatrix} c_1 & s_1 & 0 \\ s_1c\alpha & -c_1c\alpha & s\alpha \\ s_1s\alpha & -c_1s\alpha & -c\alpha \end{bmatrix} \quad {}^2_1R = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2c\beta & c_1c\beta & -s\beta \\ s_2s\beta & c_1s\beta & c\beta \end{bmatrix} \quad {}^3_2R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For the right robot $\quad {}^1_0R = \begin{bmatrix} -c_1 & s_1 & 0 \\ s_1c\alpha & c_1c\alpha & s\alpha \\ s_1s\alpha & c_1s\alpha & -c\alpha \end{bmatrix} \quad {}^2_1R = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2c\beta & c_1c\beta & -s\beta \\ -s_2s\beta & c_1s\beta & c\beta \end{bmatrix} \quad {}^3_2R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Jacobian & Isotropy

- Angular velocity propagation

$$ {}^{1}\omega_{1} = {}^{1}_{0}R\,{}^{0}\omega_{0} + \dot{\theta}_{2}\hat{Z}_{1} $$

$$ {}^{2}\omega_{2} = {}^{2}_{1}R\,{}^{1}\omega_{1} $$

$$ {}^{3}\omega_{3} = {}^{3}_{2}R\,{}^{2}\omega_{2} $$

- Linear velocity propagation

$$ {}^{1}v_{1} = {}^{1}_{0}R({}^{0}\omega_{0} \times P_{1} + {}^{0}v_{0}) $$

$$ {}^{2}v_{2} = {}^{2}_{1}R({}^{1}\omega_{1} \times P_{2} + {}^{1}v_{1}) $$

$$ {}^{3}v_{3} = {}^{3}_{2}R({}^{2}\omega_{2} \times P_{3} + {}^{2}v_{2}) + \dot{d}_{4}\hat{Z}_{3} $$

Jacobian & Isotropy

- Hence, the velocity of the end-point of Link 3 is with reference to Frame 3 is

Angular Velocity

For the left robot

$$ {}^{3}_{3}\omega = \begin{bmatrix} c_2 c\beta s\alpha \dot{\theta}_1 + s\beta c\alpha \dot{\theta}_1 - s\beta \dot{\theta}_2 \\ s_2 s\alpha \dot{\theta}_1 \\ c_2 s\beta s\alpha \dot{\theta}_1 - c\beta c\alpha \dot{\theta}_1 + c\beta \dot{\theta}_2 \end{bmatrix} $$

For the right robot

$$ {}^{3}_{3}\omega = \begin{bmatrix} -c_2 c\beta s\alpha \dot{\theta}_1 - s\beta c\alpha \dot{\theta}_1 + s\beta \dot{\theta}_2 \\ s_2 s\alpha \dot{\theta}_1 \\ c_2 s\beta s\alpha \dot{\theta}_1 - c\beta c\alpha \dot{\theta}_1 + c\beta \dot{\theta}_2 \end{bmatrix} $$

Linear Velocity

For both the left robot and right robot

$$ {}^{3}_{3}v = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_4 \end{bmatrix} $$

Jacobian & Isotropy

- Hence, the Jacobian Matrix is

For the left robot

$$\begin{bmatrix} {}^3\omega_x \\ {}^3\omega_y \\ {}^3\dot{d}_z \end{bmatrix} = {}^3J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_4 \end{bmatrix} = \begin{bmatrix} c_2 c\beta s\alpha + s\beta c\alpha & -s\beta & 0 \\ s_2 s\alpha & 0 & 0 \\ c_2 s\beta s\alpha - c\beta c\alpha & c\beta & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_4 \end{bmatrix}$$

For the right robot

$$\begin{bmatrix} {}^3\omega_x \\ {}^3\omega_y \\ {}^3\dot{d}_z \end{bmatrix} = {}^3J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_4 \end{bmatrix} = \begin{bmatrix} -(c_2 c\beta s\alpha + s\beta c\alpha) & -s\beta & 0 \\ s_2 s\alpha & 0 & 0 \\ c_2 s\beta s\alpha - c\beta c\alpha & c\beta & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_4 \end{bmatrix}$$

- The mechanism isotropy only depends on the 2X2 sub-matrix to the left corner

$${}^3J_s = \begin{bmatrix} \pm(c_2 c\beta s\alpha + s\beta c\alpha) & -s\beta \\ s_2 s\alpha & 0 \end{bmatrix}$$

Mechanism Isotropy

- **Mechanism isotropy -** the end-effector's ability of moving in all direction given a specific manipulator configuration.

- **Definition**

$$Iso = \frac{\lambda_{\min}}{\lambda_{\max}}$$

- **Range**

$$0 \leq Iso \leq 1$$

Jacobian & Isotropy

- The eigen-values of the Jacobian matrix can be found by solving

$$\det\left({}^{3}J_{s}\ {}^{3}J_{s}^{T} - \lambda I_{2\times 2} \right) = 0$$

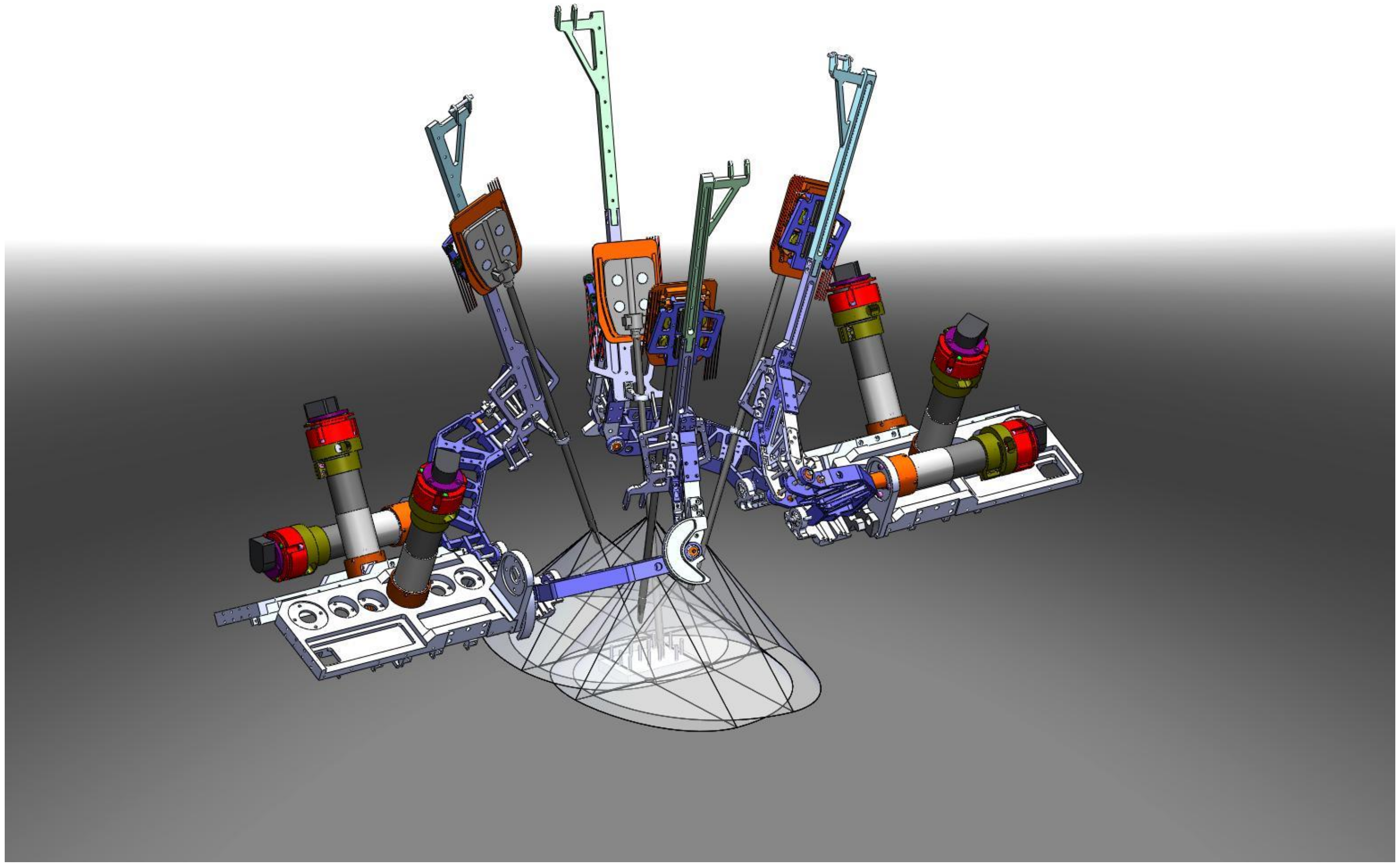
- Which gives

$$\det\left({}^{3}J_{s}\ {}^{3}J_{s}^{T} - \lambda I_{2\times 2} \right) = \lambda^2 - \left[(c_2 c\beta s\alpha + s\beta c\alpha)^2 + (s\beta)^2 \right]\lambda - (c_2 c\beta s\alpha + s\beta c\alpha)^2 (s_2 s\alpha)^2$$

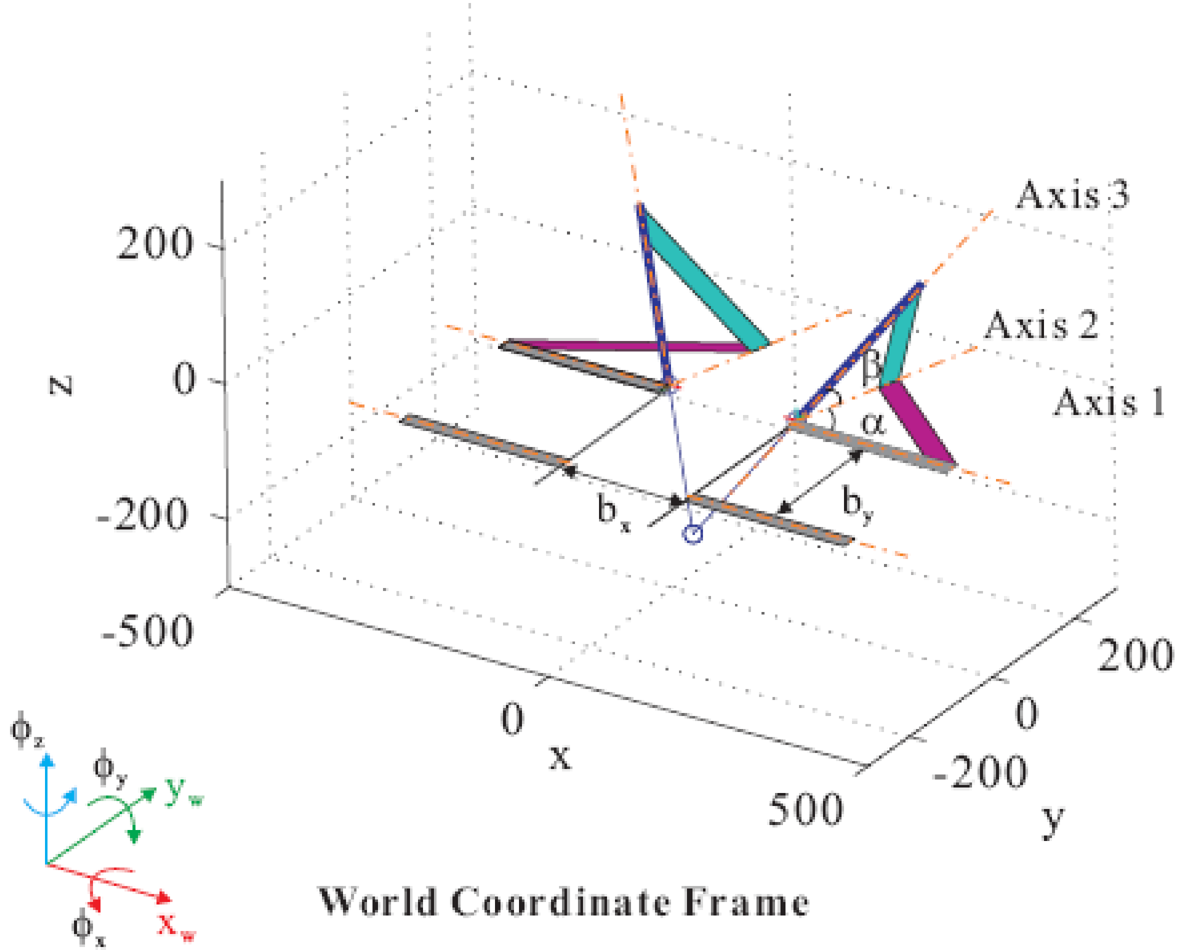
- Define

$$B = (c_2 c\beta s\alpha + s\beta c\alpha)^2 + (s\beta)^2$$
$$C = -(c_2 c\beta s\alpha + s\beta c\alpha)^2 (s_2 s\alpha)^2$$

$$Iso = \frac{\lambda_{\min}}{\lambda_{\max}} = \frac{B - \sqrt{B^2 - 4C}}{B + \sqrt{B^2 - 4C}} = 1 - \frac{2\sqrt{B^2 - 4C}}{B + \sqrt{B^2 - 4C}}$$



Optimization of Raven IV –
Problem & Parameters (7) Definitions



World Coordinate Frame

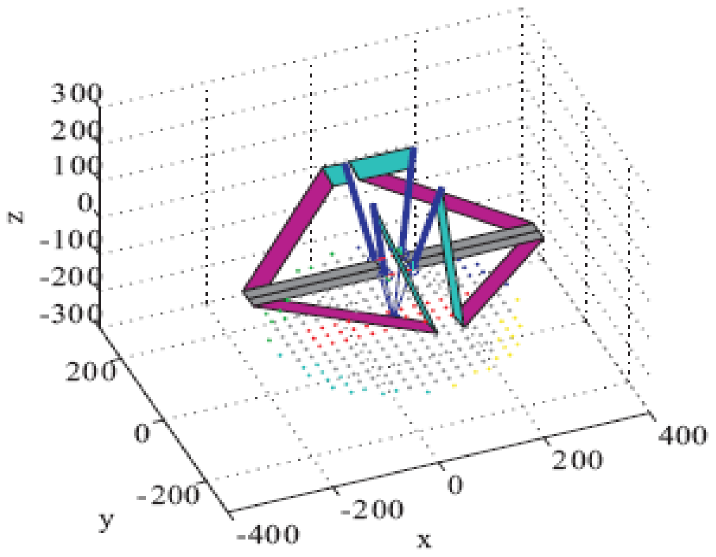
Optimization of Raven IV – Cost Function

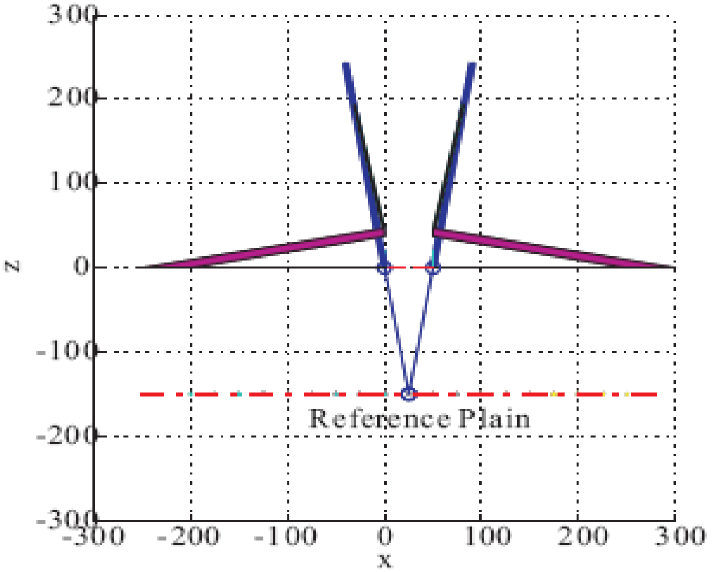
- **Cost Function**
  - **Geometry** - Largest circular common workspace (Area Circumference Ratio)
  - **Manipulations** - Best Isotropy
    - Across the common workspace
    - Worst case value (min/max problem)
  - **Mechanics** - Stiff mechanism (Smallest Mechanism)

$$C = \max_{(\alpha,\beta,\varphi_x,\varphi_y,\varphi_z,b_x,b_y)} \left\{ \varsigma \cdot \frac{\sum Iso \cdot Iso_{\min}}{\alpha^2 + \beta^2} \right\}$$

- **Method**
  - Brute force search across all the free parameters

Common Workspace – Reference Plane





Area-Circumference Ratio

- **Definition**

$$\varsigma = \frac{Area}{Circumference}$$

- According to the **Isoperimetric Inequality**, a circle has the largest possible area among all the figures with the length of boundary

$$\varsigma_c = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$$

Effect of Limiting Minimum Isotropy Performance

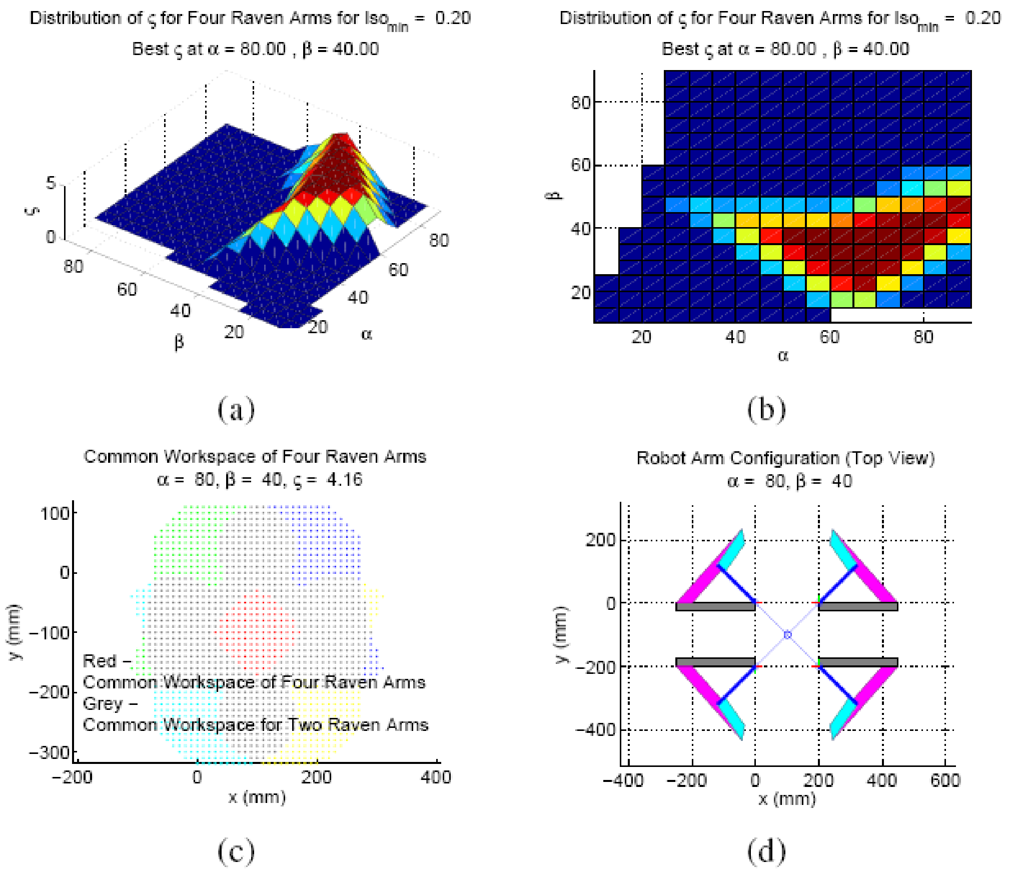
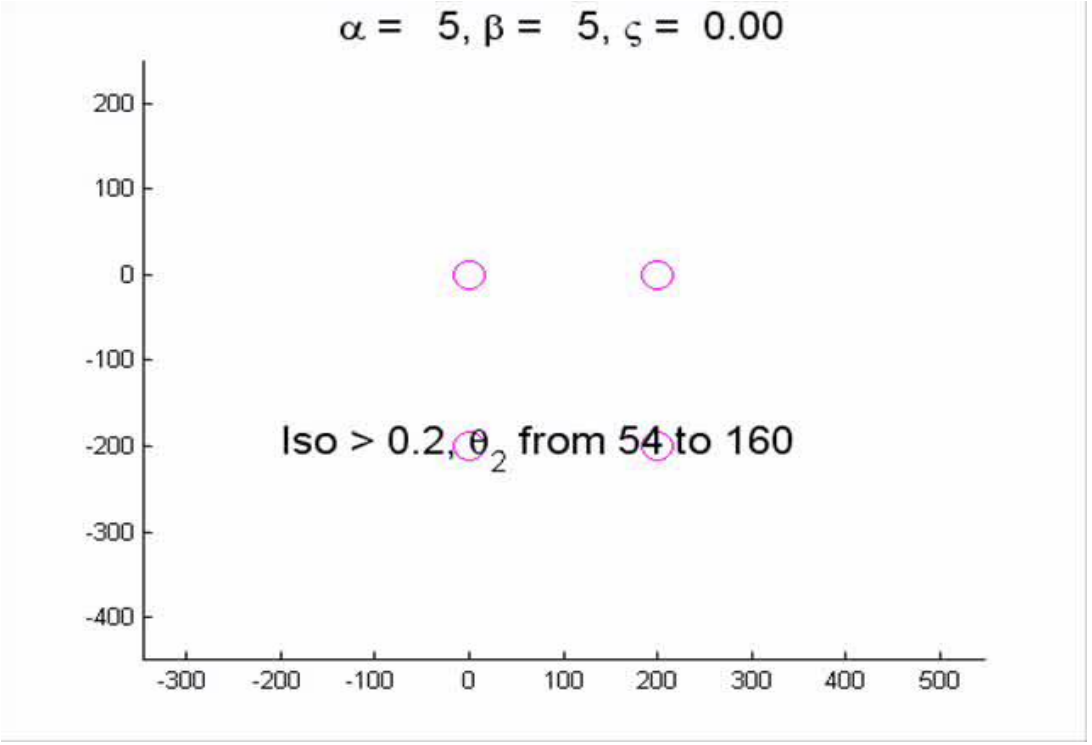


Fig. 13. Four Raven Arms: Distribution of ς for different α and β, $Iso_{min} = 0.2$.

Optimization of Raven IV surgical System
Effect of Limiting Minimum Isotropy Performance

- **Workspace propagation** – Minimum Mechanism Isotropy = **0.2**



Optimization of Raven IV surgical System
Overall simulation result

- Parameter ranges, resolutions and optimal values

	Range	Optimal Value	Resolution
α	$[5°, 90°]$	$85°$	$20°$
β	$[5°, 90°]$	$65°$	$20°$
ϕ_x	$[-20°, 20°]$	$20°$	$10°$
ϕ_y	$[-20°, 20°]$	$10°$	$10°$
ϕ_z	$[-20°, 20°]$	$-20°$	$10°$
b_x	$[50, 200]$ (mm)	100 (mm)	50 (mm)
b_y	$[50, 200]$ (mm)	50 (mm)	50 (mm)
Iso_{min}	$[0.1, 0.9]$	0.5	0.2
Result	$C_{max} = 526.3$ for Iso_{min} =0.5		

Optimization of Raven IV - Conclusion