Jacobian – Implications & Applications

Singularity & Performance & Design

Jacobian Methods – Reference Frame - Summary

Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)
Explicit (Diff. the Forward Kinematic Eq.)	${}^{0}J_N$	None
Iterative Velocity Eq.	${}^{N}J_N$	Transform Method 1: ${}^{0}v_N = {}^{0}_{N}R\,{}^{N}v_N$ ${}^{0}\omega_N = {}^{0}_{N}R\,{}^{N}\omega_N$ Transform Method 2: ${}^{0}J_N(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_N(\theta)$
Iterative Force Eq.	${}^{N}J_N^{T}$	Transpose ${}^{N}J_N = [{}^{N}J_N^{T}]^{T}$ Transform ${}^{0}J_N(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_N(\theta)$

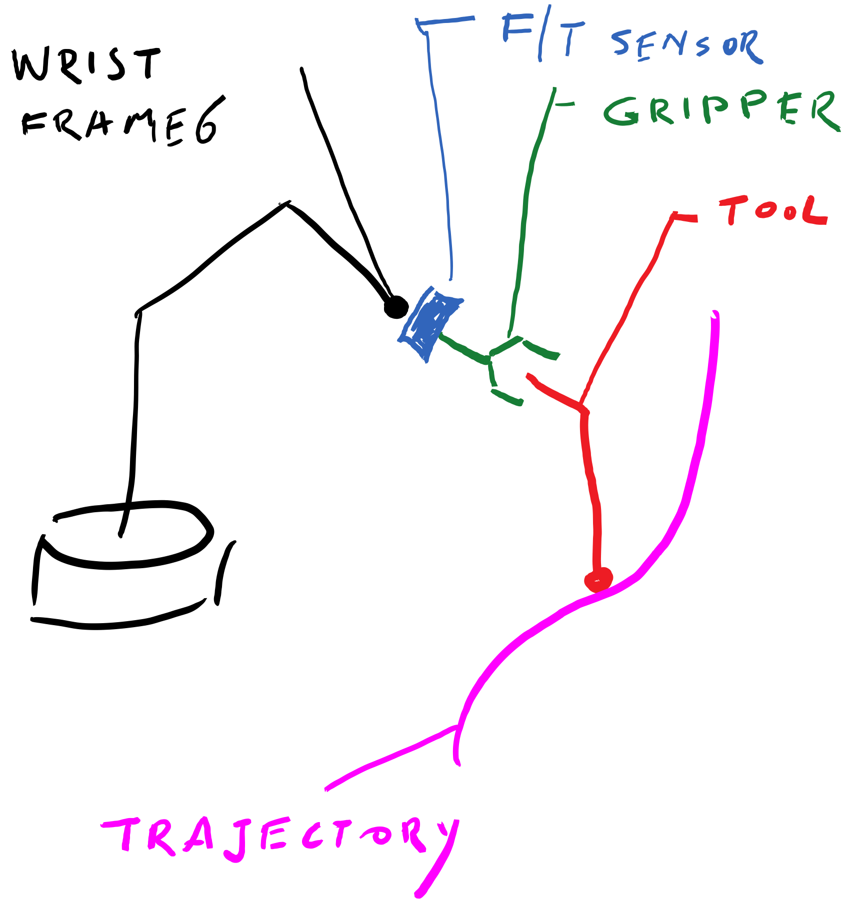
Propagation to the Tip of the Tool

Problem Defenition

Propagation to the Tip of the Tool – Problem Definition

- **Problem**
  - **Practical Configuration of a robotic arm** - The robotic arm typically includes the following
    - F/T Sensor
    - Gripper / End Effector
    - Tool
  - **Analysis** – The generic analysis of the robotic arm mapping position, velocities and forces / torques between the base and the wrist (last frame of the manipulators)
  - **Rational** –
    - **Generic Analysis versus task specific elements (F/T sensor, gripper tool)** - The analysis is conducted by the robot arm manufacturer however the F/T sensor, the gripper and the tool are task specific and selected by the user.
    - **Tool Change** – The same arm performing different tasks may need different tools that are changed during the course of its operation
  - **Need** – The need is typically to
    - Trace the position and orientation and velocities (linear and angular) of the tool tip as it follows a trajectory
    - Express force and torques applied on the environment by the tool tip and vice versa by a force sensor measuring these parameters in a different location

- **Solution** – Expressing position, velocity forces and torques from the last frame (Frame 6 at the wrist) to the tip of the tool



Propagation to the Tip of the Tool

Position

Jacobian Propagation to the Tip of the Tool – Position

- In a case where the tool tips follows a trajectory, the path defines the goal position and orientation

Known

$$\boxed{T_{path}} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T\,\boxed{{}^6_TT}$$

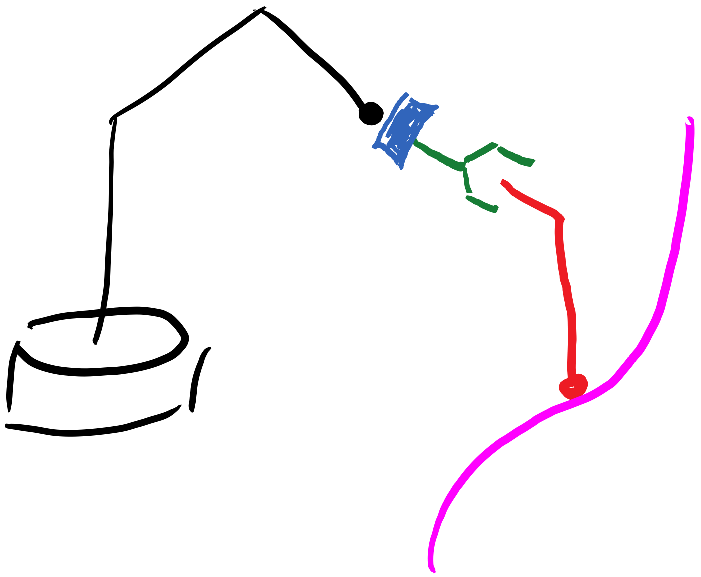
- Since the tool is attached to the end effector its position does not change as a function of time with respect to frame 6
- Multiply both sides of the equations by

$$({}^6_TT)^{-1} = {}^T_6T$$

$$T_{path}\left({}^6_TT\right)^{-1} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T\,{}^6_TT\left({}^6_TT\right)^{-1}$$

- Solve the Inverse Kinematics

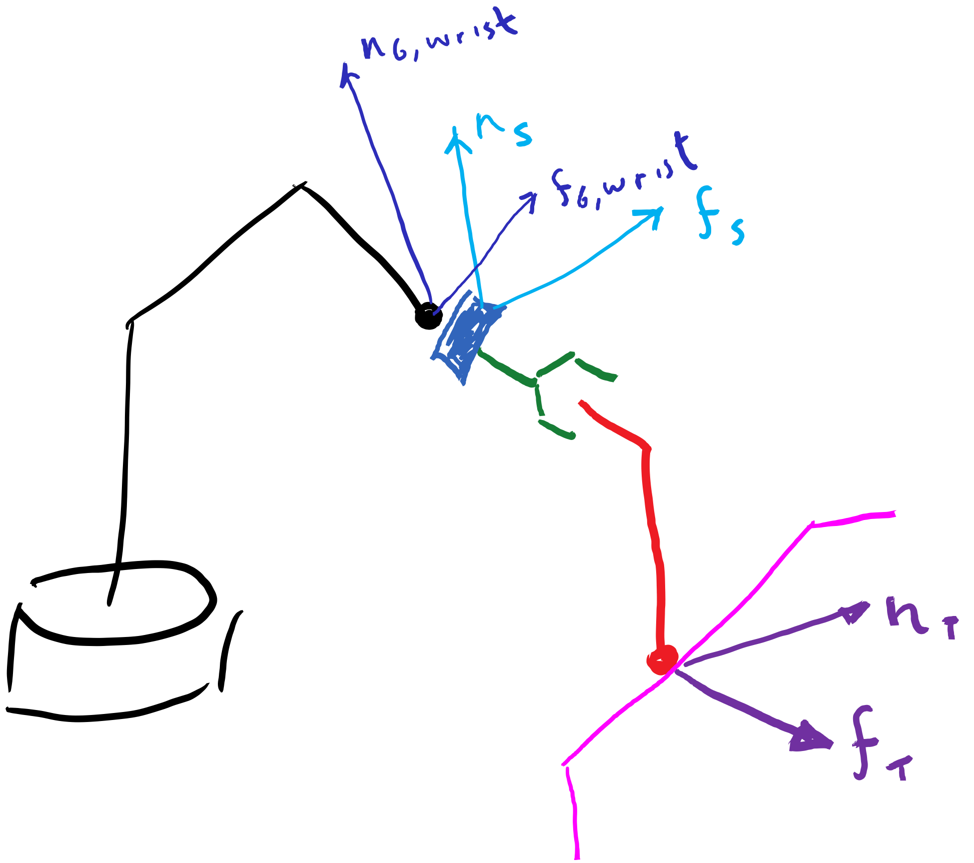
$$T_{path}\,{}^T_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T$$



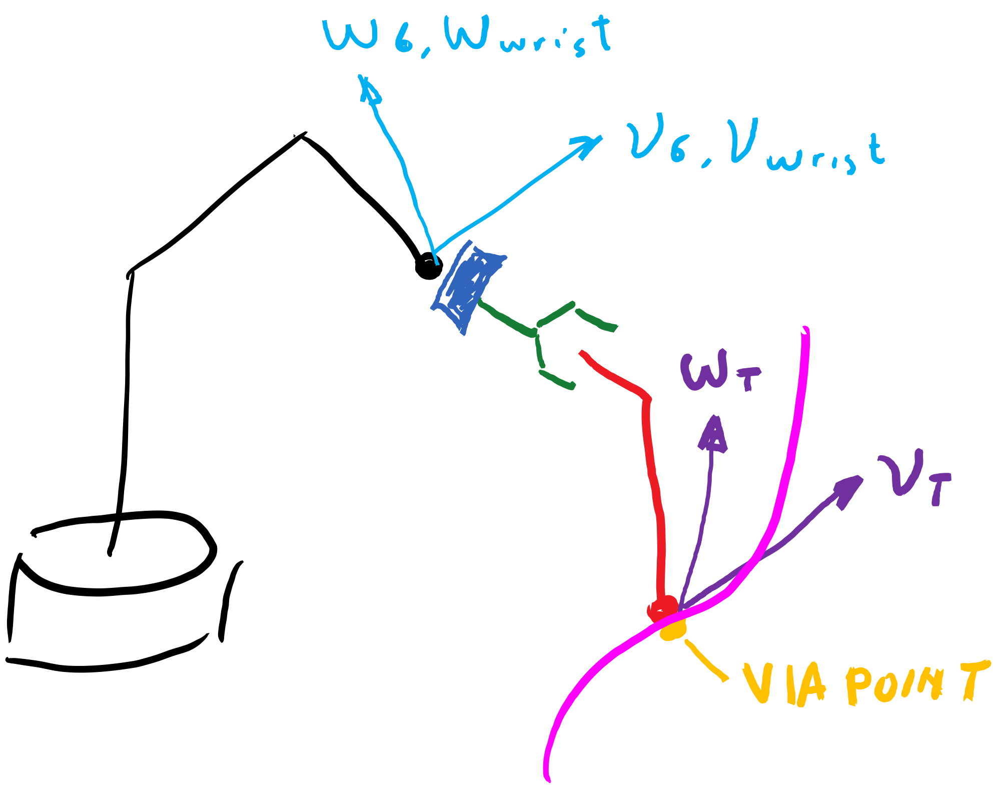
Jacobian Propagation to the Tip of the Tool

Jacobian Propagation to the Tip of the Tool

Forces/Torques



Velocities (Linear and Angular)



Jacobian Propagation to the Tip of the Tool

Velocity

Jacobian Propagation to the Tip of the Tool – Position

- Position

Vector Form

$$^{A}P_{Q} = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P_{Q}$$

$$^{i}P_{Q} = {}^{i}P_{i+1ORG} + {}^{i}_{i+1}R\,{}^{i+1}P_{Q}$$

Matrix Form

$$^{A}P_{Q} = \begin{bmatrix} \multicolumn{3}{c}{{}^{A}_{B}R} & {}^{A}P_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{B}P_{Q}$$

$$^{i}P_{Q} = \begin{bmatrix} \multicolumn{3}{c}{{}^{i}_{i+1}R} & {}^{i}P_{i+1ORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^{i+1}P_{Q}$$

Jacobian Propagation to the Tip of the Tool - Velocities

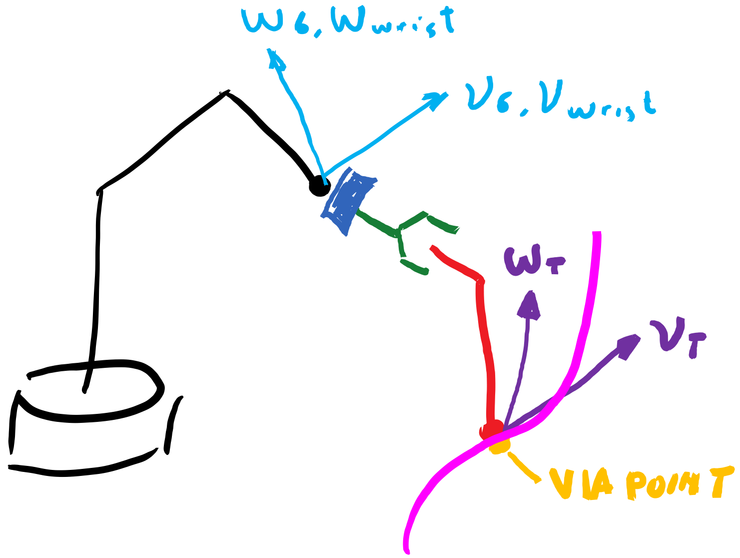
- Velocity of two rigidly connected frames (rigid body) $\begin{cases} i \rightarrow 6, W(Wrist) \\ i+1 \rightarrow T(Tool) \end{cases}$

- Vector Form

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \overset{0\ \textbf{(Rigid)}}{\cancel{\dot{\theta}_{i+1}}}\,{}^{i+1}Z_{i+1} $$
$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}v_{i} + {}^{i}\omega_{i} \times {}^{i}P_{i+1}\right) $$

- Matrix Form

$$ \begin{Bmatrix} {}^{i+1}v_{i+1} \\ {}^{i+1}\omega_{i+1} \end{Bmatrix} = \begin{bmatrix} {}^{i+1}_{i}R & {}^{i+1}_{i}R\left(\square \times {}^{i}P_{i+1}\right) \\ 0 & {}^{i+1}_{i}R \end{bmatrix} \begin{Bmatrix} {}^{i}v_{i} \\ {}^{i}\omega_{i} \end{Bmatrix} $$



- Note: The equations are formulated as a forward propagation i.e propagating from frame {i} to {i+1}. **Challenge:** We know the linear and angular velocities at the tip of the tool i.e. frame {T} and we wish to express the velocities by backwards propagation to the wrist i.e. frame {6} or frame {w}
- **Solution:** Find the expressions for ${}^{6}\omega_{6}$ ${}^{6}v_{6}$ as a function of ${}^{T}\omega_{T}$ ${}^{T}v_{T}$

Jacobian Propagation to the Tip of the Tool - Velocities

- Expressing the equations in a back propagation fashion given the linear and angular velocities in frame {i} or frame {w} as a function of the linear and angular velocities in frame {i+1} or frame {T}
- Finding expressions for ${}^{i}\omega_i$ and ${}^{i}v_i$ → Multiply both sides of the equation by ${}_{i+1}^{i}R$

$$\boxed{{}_{i+1}^{i}R}\,{}^{i+1}\omega_{i+1} = \boxed{{}_{i+1}^{i}R}\,{}_{i}^{i+1}R\,{}^{i}\omega_i$$

$$\boxed{{}_{i+1}^{i}R}\,{}^{i+1}v_{i+1} = \boxed{{}_{i+1}^{i}R}\,{}_{i}^{i+1}R\left({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}\right)$$

- Resulting in vector form

$$\begin{aligned} {}^{i}\omega_i &= {}_{i+1}^{i}R\,{}^{i+1}\omega_{i+1} \\ {}^{i}v_i &= {}_{i+1}^{i}R\,{}^{i+1}v_{i+1} - {}^{i}\omega_i \times {}^{i}P_{i+1} \end{aligned}$$

- and a matrix form

$$\begin{Bmatrix} {}^{i}v_i \\ {}^{i}\omega_i \end{Bmatrix} = \begin{bmatrix} {}_{i+1}^{i}R & -\square \times {}^{i}P_{i+1} \\ 0 & {}_{i+1}^{i}R \end{bmatrix} \begin{Bmatrix} {}^{i+1}v_{i+1} \\ {}^{i+1}\omega_{i+1} \end{Bmatrix}$$

Jacobian Propagation to the Tip of the Tool – Velocities - Summary

- Preform the following replacement of indexes $\begin{cases} i \rightarrow 6 \; or \; W(Wrist) \\ i+1 \rightarrow T(Tool) \end{cases}$

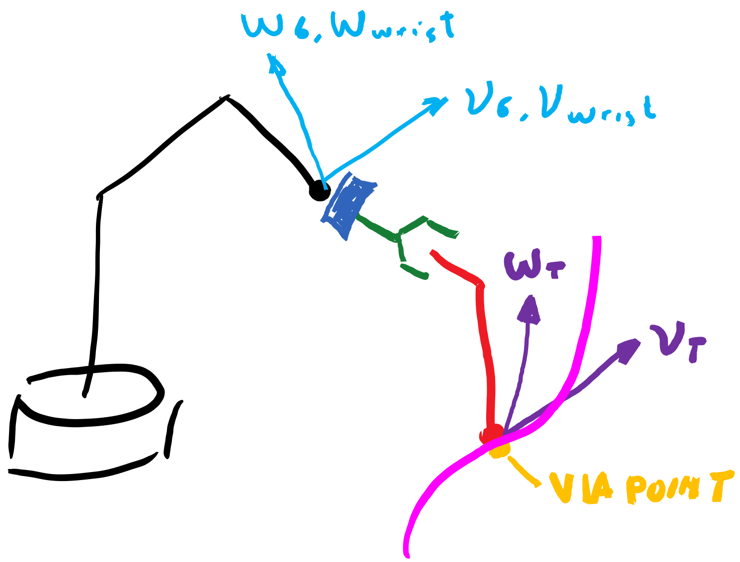
- Resulting in vector form

$$^{6}\omega_6 = {}^{6}_{T}R \; {}^{T}\omega_T$$

$$^{6}v_6 = {}^{6}_{T}R \; {}^{T}v_T - {}^{6}\omega_6 \times {}^{6}P_T$$

- and a matrix form

$$\begin{Bmatrix} {}^{6}v_6 \\ {}^{6}\omega_6 \end{Bmatrix} = \begin{bmatrix} {}^{6}_{T}R & -\square \times {}^{6}P_T \\ 0 & {}^{6}_{T}R \end{bmatrix} \begin{Bmatrix} {}^{T}v_T \\ {}^{T}\omega_T \end{Bmatrix}$$



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Jacobian Propagation to the Tip of the Tool – Forces/Torques

- Force and torque applied on a rigid body $\begin{cases} i \rightarrow 6, W(Wrist) \\ i+1 \rightarrow T(Tool) \end{cases}$

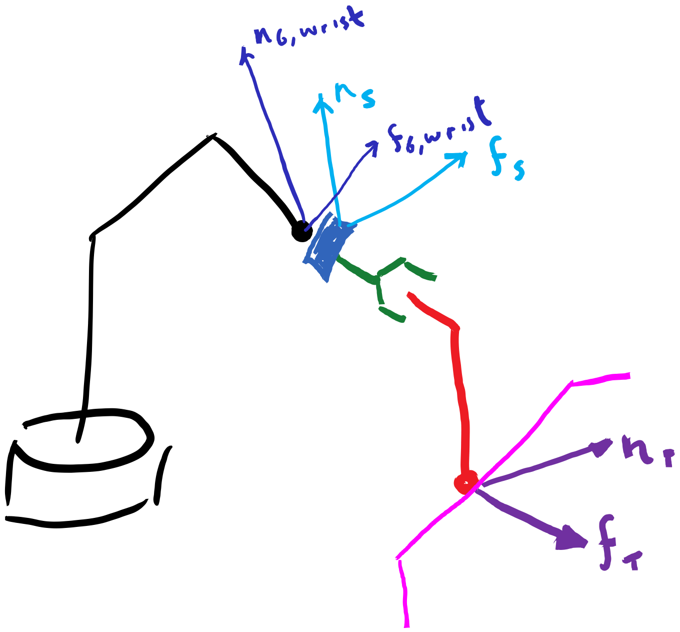
- Vector Form

$$ {}^{i}f_i = {}^{i}_{i+1}R\, {}^{i+1}f_{i+1} $$

$$ {}^{i}n_i = {}^{i}_{i+1}R\, {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} $$

- Matrix Form

$$ \begin{Bmatrix} {}^{i}f_i \\ {}^{i}n_i \end{Bmatrix} = \begin{bmatrix} {}^{i}_{i+1}R & 0 \\ {}^{i}P_{i+1} \times {}^{i}_{i+1}R\square & {}^{i}_{i+1}R \end{bmatrix} \begin{Bmatrix} {}^{i+1}f_{i+1} \\ {}^{i+1}n_{i+1} \end{Bmatrix} $$

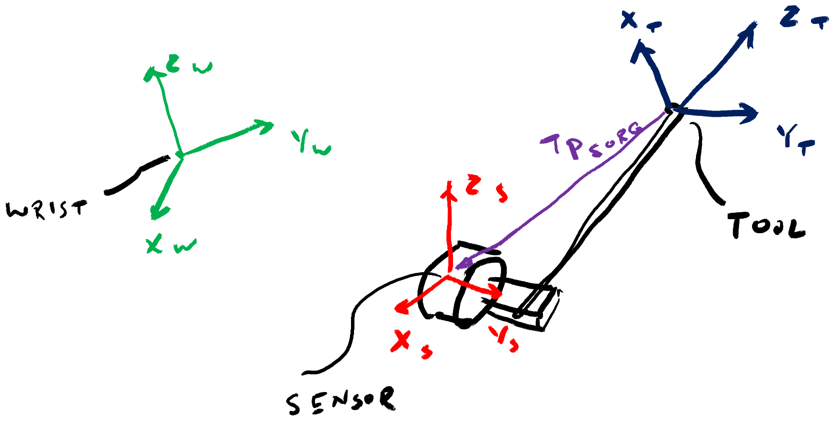


- Note: The equations are formulated as a <u>backward propagation</u> i.e propagating from frame {i+1} to {i}.
  Challenge: The Force/Torque (F/T) sensor i.e. frame {S} is located between the tool tip i.e. frame {T} and the wrist i.e. frame {6} or frame {w}
- **Solution:** Find an expression for the forces and torque applied on the tool tip but measured by the sensor

Jacobian Propagation to the Tip of the Tool

- The end effector holds a tool on which forces and torques are applied
- A force/torque sensor is attached close to the wrist that is expected to measure the forces and torque applied on the tool

$$\begin{Bmatrix} {}^T f_T \\ {}^T n_T \end{Bmatrix} = \begin{bmatrix} {}^T_S R & 0 \\ {}^T P_{SORG} \times {}^T_S R \; \square & {}^T_S R \end{bmatrix} \begin{Bmatrix} {}^S f_S \\ {}^S n_S \end{Bmatrix}$$

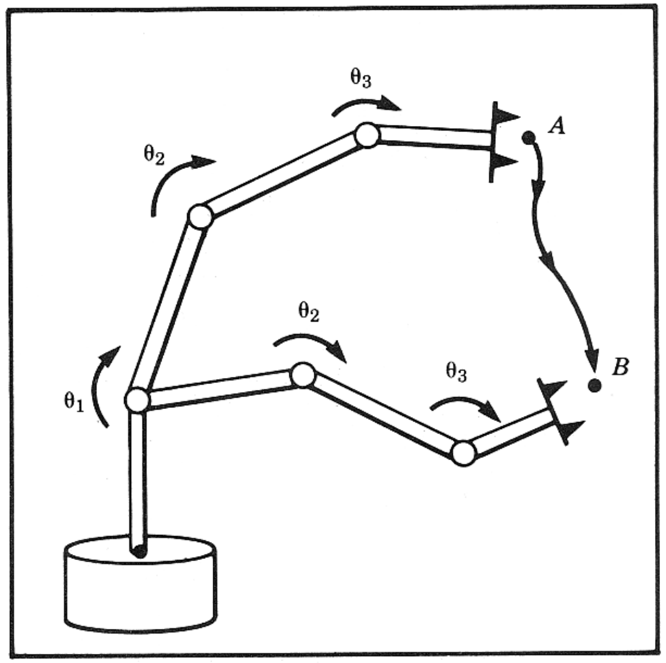


Jacobian – Singularity
Problem Definition

Inverse Jacobian

- ***Given***
  - Tool tip path (defined mathematically)
  - Tool tip position/orientation
  - Tool tip velocity
  - Jacobian Matrix

$$\underline{\dot{x}} = J(\theta)\, \underline{\dot{\theta}}$$



- ***Problem:*** Calculate the joint velocities

- ***Solution:***
  - Compute the inverse Jacobian matrix
  - Use the following equation to compute the joint velocity

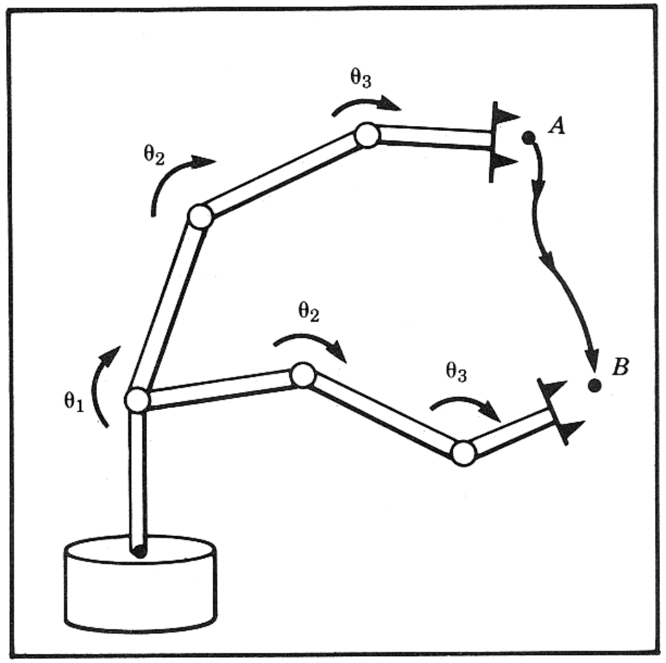
$$\underline{\dot{\theta}} = J(\theta)^{-1}\underline{\dot{x}}$$

Singularity - The Concept

- ***Motivation:*** We would like the hand of a robot (end effecror) to move with a certain velocity vector in Cartesian space. Using linear transformation relating the joint velocity to the Cartesian velocity we could calculate the necessary joint rates at each instance along the path.

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$

- ***Given:*** a linear transformation relating the joint velocity to the Cartesian velocity (usually the end effector)
- ***Questions:***
  - Is the Jacobian matrix invertible? (Or) Is it nonsingular?
  - Is the Jacobian invertible for all values of θ?
  - If not, where is it not invertible?



Inverse Jacobian

- Cases in which the Jacobian matrix $J(\theta)$ is not inevitable $(J(\theta)^{-1}$ does not exists).
- Non invertible matrix is called singular matrix

  – **Case 1** - The Jacobian matrix is not squared

    In general the 6xN Jacobian matrix may be non-square in which case the inverse is not defined

  – **Case 2** - The determinant $\det\big(J(\theta)\big)$ is equal to zero

Singularity - The Concept

- ***Answer (Conceptual):*** Most manipulator have values of where the Jacobian becomes singular . Such locations are called ***singularities of the mechanism*** or ***singularities*** for short

Singularities of the mechanism

Workspace boundary singularities

Workspace interior Singularities

- Stretched out
- Folded back

End Effector → **Workspace Boundary**

- Two or more joints are lining up

Singularity - The Concept

Manipulator

Singular Configuration

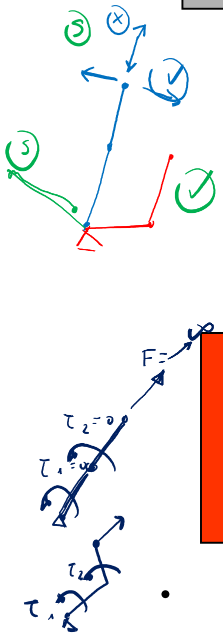
General Configuration

Losing One or More DOF

Problematic Load Balance

Problematic Joint Velocity

All DOF Are Available

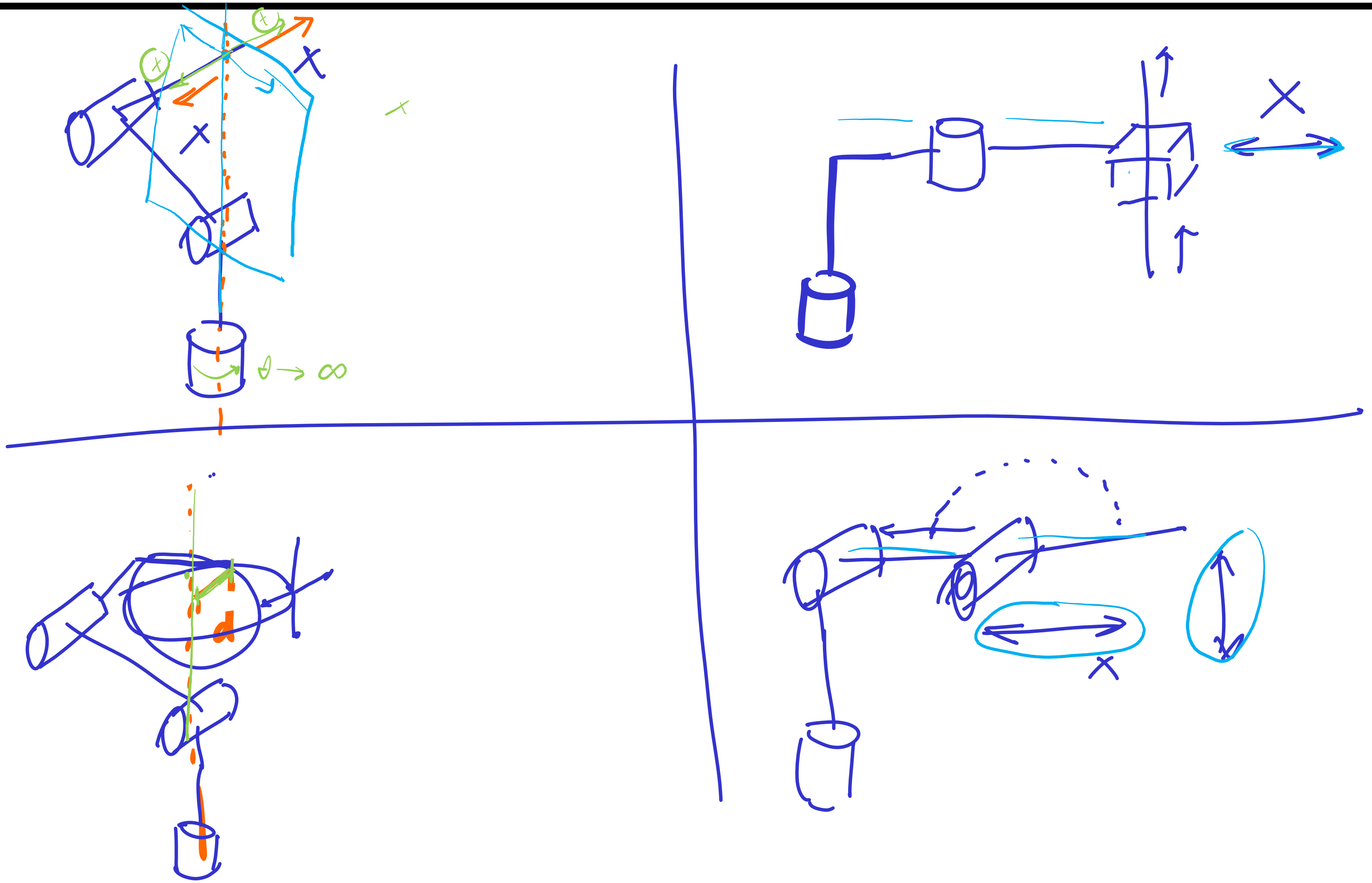


- **Lost of DOF** - Losing one or more DOF means that there is a some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot (end effector) no matter which joint rate are selected
- **Load Balance** – A finite force can be applied to the end effector that produces no torque at the robot's joints
- **Joint Velocity** – A zero end effector velocity will cause high joint velocity

Singularity – Physical Interpretation - Examples

- Type of Singularities
  - Wrist
  - Elbow
  - Shoulder

Singularity – Physical Interpretation - Examples



Singularities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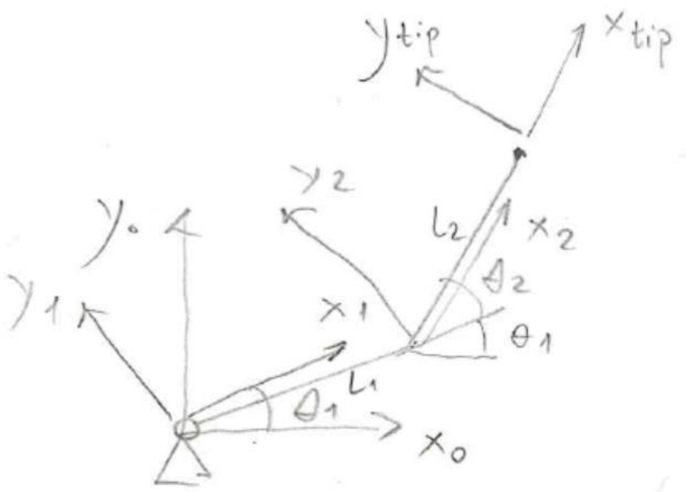
A1

Singularities

Jacobian – Singularity
Example 1 – 2R
Elbow Singularity
Singularity at the Edge of the Workspace

Jacobian Matrix by Differanciation - 3R - 1/4

- Consider the following 3 DOF Planar manipulator



Jacobian Matrix by Differanciation - 3R - 4/4

- Using a matrix form we get

$$\underline{\dot{x}} = {}^{0}J(\underline{\theta})\underline{\dot{\theta}}$$

$$\begin{Bmatrix} v_x \\ v_y \end{Bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & -L_2 s_{12} \\ L_1 c_1 + L_2 c_{12} & L_2 c_{12} \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}$$

- The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees a freedom. Before tackling these problems, consider this brief review of linear algebra.

Properties of the Jacobian - Velocity Mapping and Singularities

- ***Example:*** Planar 3R

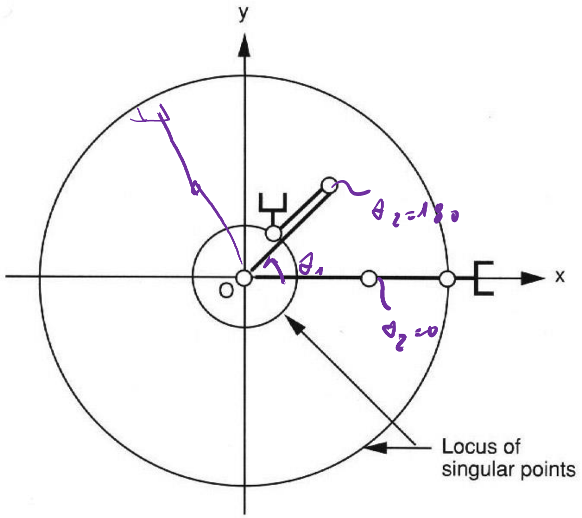
$$\det(J(\theta)) = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & -L_2 s_{12} \\ L_1 c_1 + L_2 c_{12} & L_2 c_{12} \end{bmatrix} = L_1 L_2 s_2$$

$$\det(J(\theta)) = L_1 L_2 s_2 = 0$$

- Note that $\det(J(\theta))$ is not a function of θ_1

Properties of the Jacobian - Velocity Mapping and Singularities

$$\text{singular configuration} \begin{cases} \theta_2 = 0 & \text{Stretched Out} \\ \theta_2 = \pi & \text{Fold Back} \end{cases}$$



- The manipulator loses 1 DOF. The end effector can only move along the tangent direction of the arm. Motion along the radial direction is not possible.

Properties of the Jacobian - Force Mapping and Singularities

- The relationship between joint torque and end effector force and moments is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

- The rank of $J(\theta)^T$ is equals the rank of $J(\theta)$

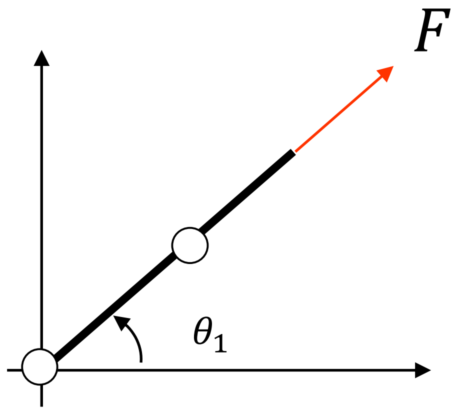
- At a singular configuration there exists a non trivial force $\underline{F}$ such that

$$J(\underline{\theta})^T \underline{F} = 0$$

- In other words, a finite force can be applied to the end effector that produces no torque at the robot’s joints. In the singular configuration, the manipulator can “lock up.”

Properties of the Jacobian - Force Mapping and Singularities

- ***Example:*** Planar 3R $\theta_1 = \theta; \quad \theta_2 = 0$



- In this case the force acting on the end effector (relative to the {0} frame) is given by

$$ {}^0F = \begin{bmatrix} Fc_1 \\ Fs_1 \end{bmatrix} $$

Properties of the Jacobian - Force Mapping and Singularities

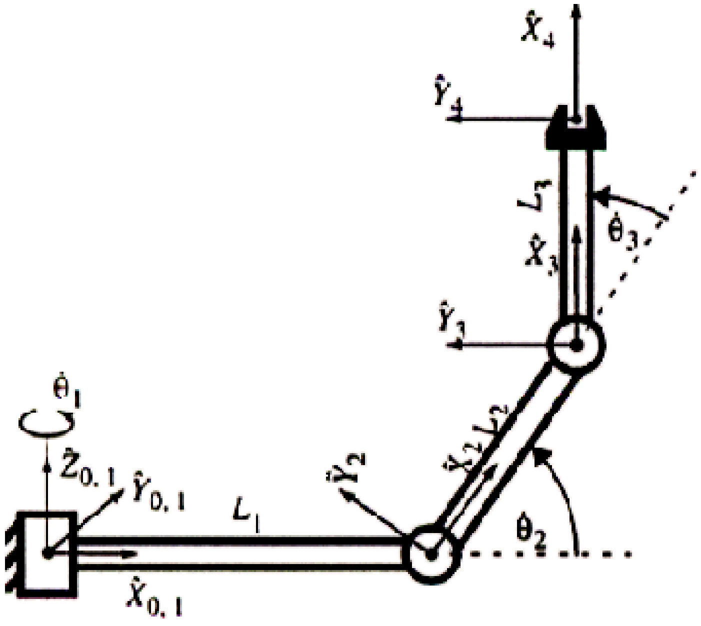
$$\underline{{}^{0}\tau} = {}^{0}J(\underline{\theta})^{T}\,\underline{{}^{0}F} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & L_1 c_1 + L_2 c_{12} \\ -L_2 s_{12} & L_2 c_{12} \end{bmatrix}$$

- For $\theta_1 = \theta; \quad \theta_2 = 0$ we get

$$\underline{{}^{0}\tau} = {}^{0}J(\underline{\theta})^{T}\,\underline{{}^{0}F} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} & L_1 c_1 + L_2 c_{12} \\ -L_2 s_{12} & L_2 c_{12} \end{bmatrix} \begin{bmatrix} F c_1 \\ F s_1 \end{bmatrix} =$$

$$\begin{bmatrix} -F s_1 c_1 (L_1 + L_2) + F s_1 c_1 (L_1 + L_2) \\ -F s_1 c_1 (L_2) + F s_1 c_1 (L_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Jacobian – Singularity
Example 2 – 3R
Shoulder Singulaity
Singularity Inside the Workspace



$\hat{X}_4$
$\hat{Y}_4$
L_3
$\dot{\theta}_3$
$\hat{X}_3$
$\hat{Y}_3$
L_2
$\hat{X}_2$
$\hat{Y}_2$
$\dot{\theta}_1$
$\hat{Z}_{0,1}$
$\hat{Y}_{0,1}$
L_1
θ_2
$\hat{X}_{0,1}$

Jacobian: Singular Configuration - 3R Example

- If we want to use the inverse Jacobian to compute the joint angular velocities we need to first find out at what points the inverse exists.

$$\underline{\dot{x}} = {}^4J(\underline{\theta})\underline{\dot{\theta}}$$

- Considering the 3R robot

$${}^4J(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

- The determinate of the Jacobian is defined as follows

$$|{}^4J(\theta)| = -(L1 + L2c2 + L3c23)(L2s3)L3$$

Jacobian: Singular Configuration - 3R Example

$$|{}^{4}J\,(\theta)| = -(L1 + L2c2 + L3c23)(L2s3)L3$$

- The reduced Jacbian matrix is singular when it determinate is equal to zero

$$-(L1 + L2c2 + L3c23)(L2s3)L3 = 0$$

- The singular condition occur when either of the following are true

$$s3 = 0$$

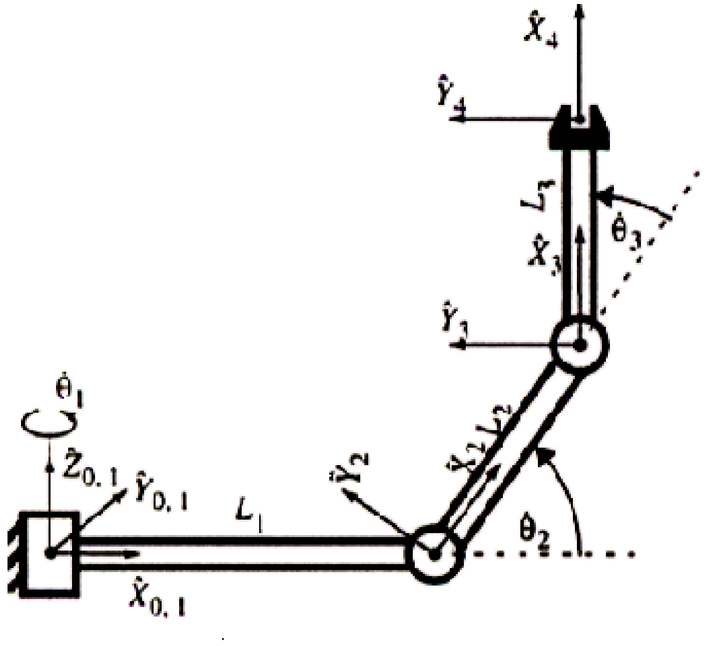
$$-L1 - L2c2 - L3c23 = 0$$

Jacobian: Singular Configuration - 3R Example

- Case 1: $s3 = 0$

$$s3 = 0 \Rightarrow \begin{cases} \theta_3 = 0^o \\ \theta_3 = 180^o \end{cases}$$

$${}^4J\,(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$



- The first row of the Jacobian is zero
- The 3R robot is loosing one DOF.
- The robot can no longer move along the X-axis of frame {4}

Jacobian: Singular Configuration - 3R Example

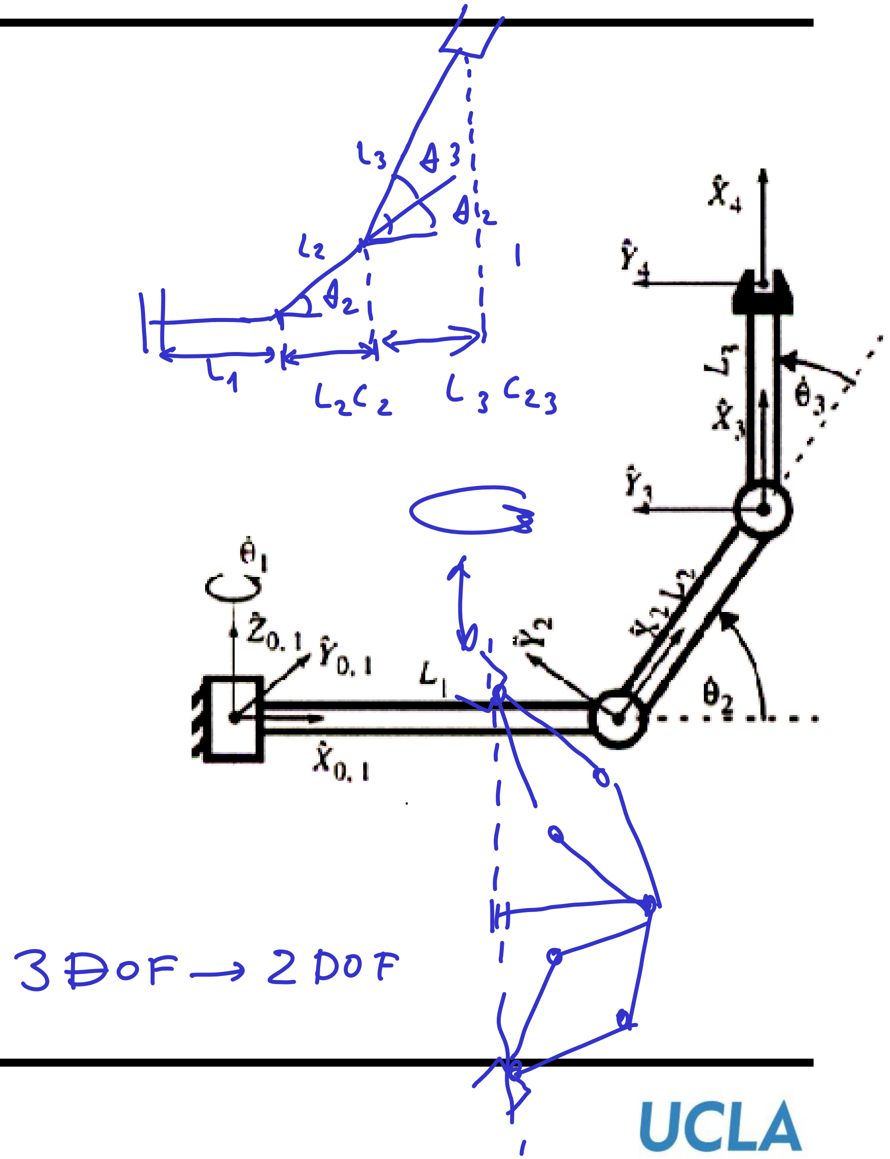
- Case 2: $-L1 - L2c2 - L3c23 = 0$

$$L1 = -L2c2 - L3c23$$

- Occur only if $L2 + L3 \geq L1$

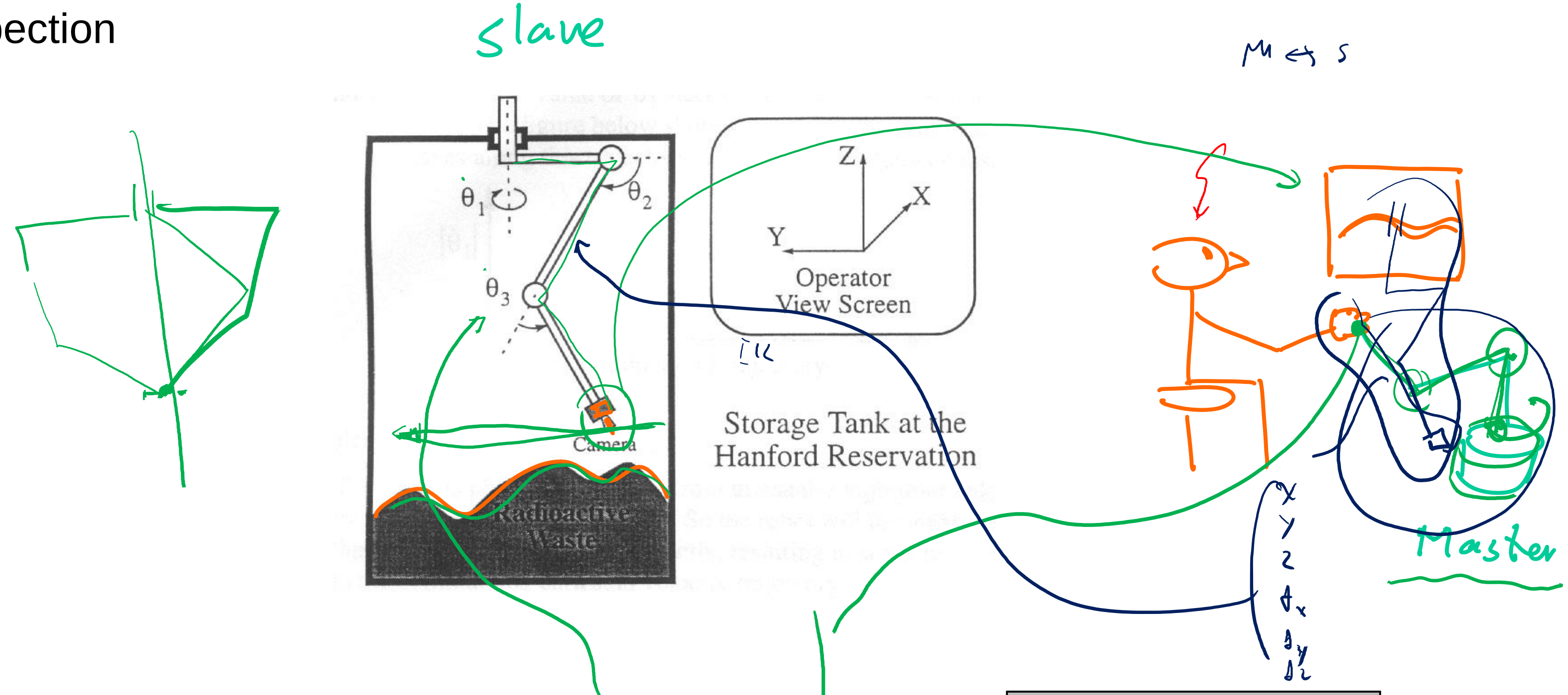
$$^{4}J(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

- The third row of the Jacobian is zero
- The origin of frame {4} intersects the Z-axis of frame {1}
- The 3R robot is loosing one DOF.
- The robot can no longer move along the Z-axis of frame {4}



Joint Velocity Near Singular Position - 3R Example

- Robot : 3R robot
- Task: Visual inspection



- Control

Operator → $\dot{\underline{\theta}} = J(\theta)^{-1}\dot{\underline{x}}$ **Control** → **Robot**

Joint Velocity Near Singular Position - 3R Example

- ***Singularity (Case 2)-*** The origin of frame {4} intersects the Z-axis of frame {1}

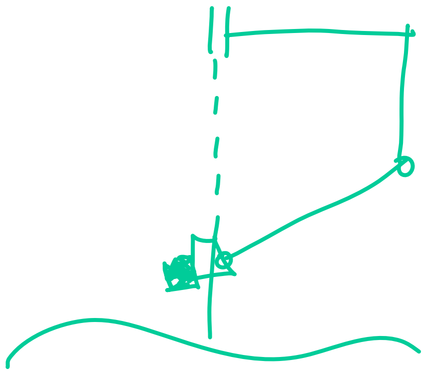
$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix} \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\end{bmatrix}$$

- Solve for $\dot{\theta}_1$ in terms of $\dot{z}$ we find

$$\dot{\theta}_1 = \frac{\dot{z}}{-L1 - L2c2 - L3c23}$$

$$-L1 - L2c2 - L3c23 = 0$$

$$\dot{\theta}_1 \rightarrow \infty$$



Joint Velocity Near Singular Position - 3R Example

- Singularity -

$$\det\big(J(\theta)\big) = 0$$

- Problems:
  - **Motor Constrains** - The robot is physically limited from moving in unusual high joint velocities by motor power constrains. Therefore, the robot will be unable to track the required joint velocity trajectory exactly resulting in some perturbation to the commanded Cartesian velocity trajectory.
  - **Gears and Shafts** - The derivative of the angular velocity is the angular acceleration. The high acceleration of the joint resulting form approaching too close to a singularity may cause damage to the gear/shafts.
  - **DOF** - At a singular configuration (specific point in space) the manipulator loses one or more DOF.

- Consequences – Certain tasks can not be performed at a singular configuration

Jacobian – Singularity
Example 3 – 3R
Wrist Singularity
Singularity Inside the Workspace

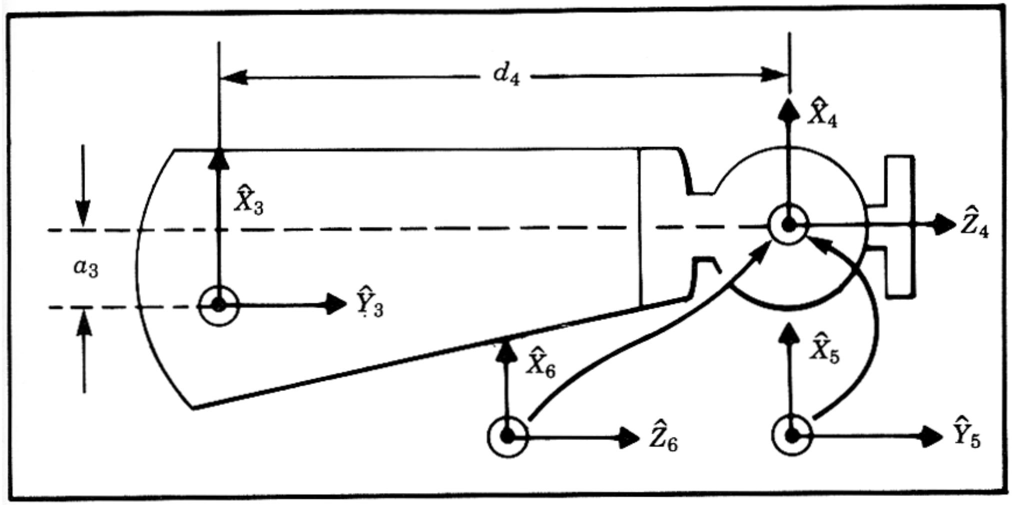
Mapping - Rotated Frames - Z-Y-Z Euler Angles

Start with frame {4}.

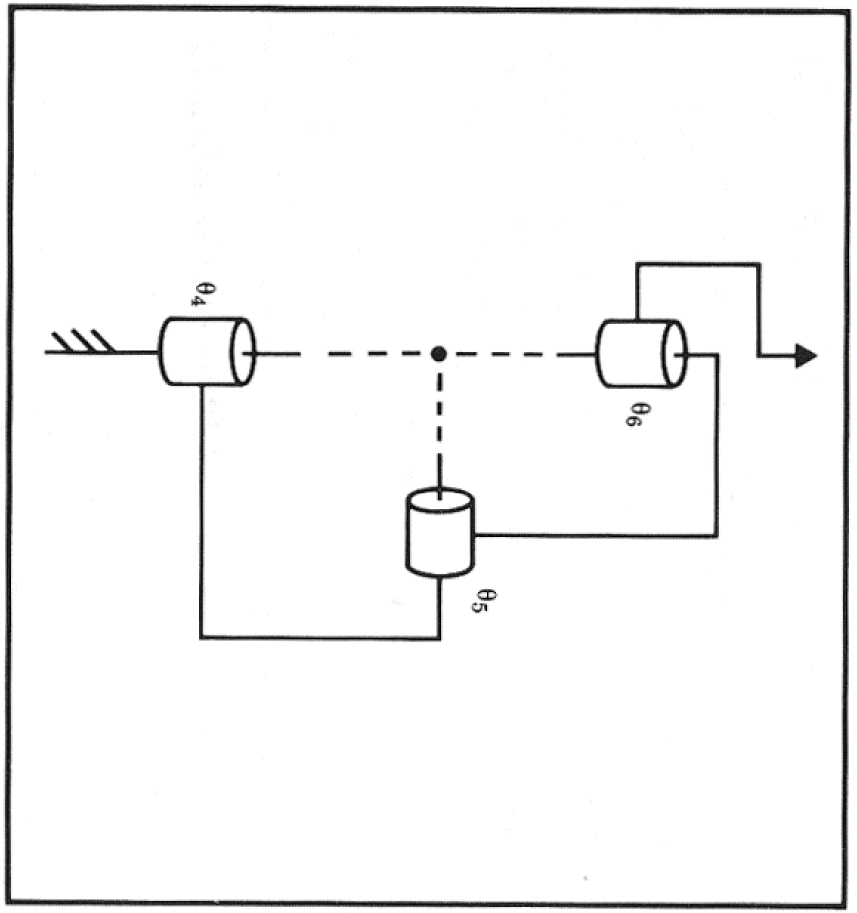
- Rotate frame {4} about $\hat{Z}_4$ by an angle α
- Rotate frame {4} about $\hat{Y}_B$ by an angle β
- Rotate frame {4} about $\hat{Z}_4$ by an angle γ

Euler Angles

Note *- Each rotation is preformed about an axis of the moving reference frame, rather then a fixed reference.*

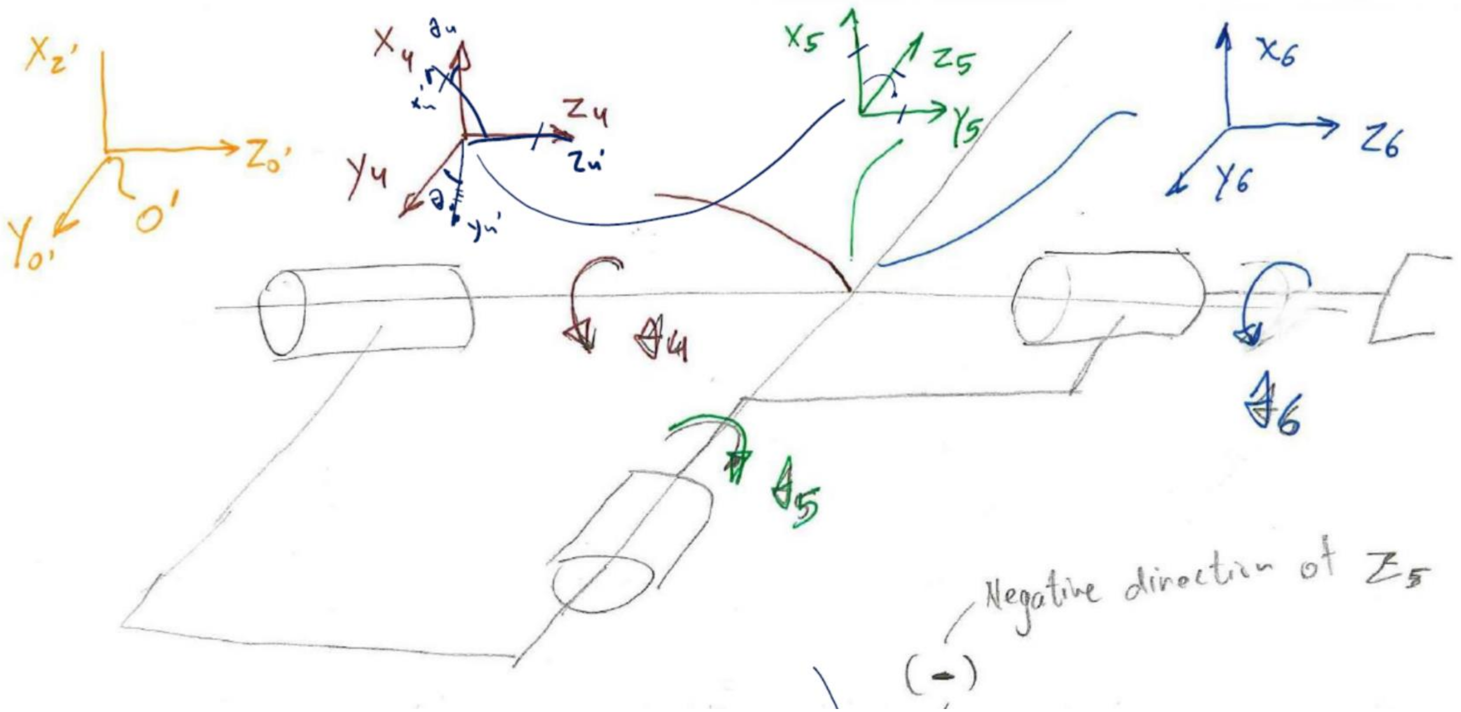


UCLA



θ4
θ6
θ5

Mapping - Rotated Frames - Z-Y-Z Euler Angles



$$R_{Euler} = R_{z4}(\theta_4)R_{y4}(-\theta_5)R_{z4}(\theta_6)$$

Mapping - Rotated Frames - Z-Y-Z Euler Angles

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$${}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = {}^{4}_{6}R_{\theta_4=0}$$

Three consecutive Axes Intersect - wrist

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

Goal **Direct Kinematics**

Three consecutive Axes Intersect - wrist

- Solve for β
- using element r_{31}, r_{32}, r_{33}

$$r_{31} = -s\beta c\alpha$$
$$r_{32} = s\beta s\alpha$$
$$r_{33} = c\beta$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$$r_{31}^2 + r_{32}^2 = s\beta^2(c\alpha^2 + s\alpha^2)$$
$$r_{33} = c\beta$$

$$s\beta = \pm\sqrt{r_{31}^2 + r_{32}^2}$$

- Using the Atan2 function, we find

$$\beta = \mathrm{Atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

Three consecutive Axes Intersect - wrist

- Solve for α using elements r_{23}, r_{13}

$$r_{13} = c\alpha s\beta$$

$$r_{23} = s\alpha s\beta$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$$\alpha = \mathrm{Atan2}(r_{23}/s\beta, r_{13}/s\beta)$$

Three consecutive Axes Intersect - wrist

- Solve for γ using elements r_{32}, r_{31}

$$r_{32} = s\beta s\gamma$$

$$r_{31} = -s\beta c\gamma$$

$$\gamma = \mathrm{Atan2}(r_{32}/s\beta, -r_{31}/s\beta)$$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

Three consecutive Axes Intersect - wrist

- Note: Two answers exist for angle β which will result in two answers each for angles α and γ

$$\beta = \text{Atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

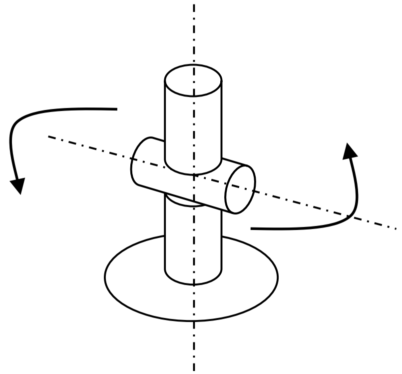
$$\alpha = \text{Atan2}(r_{23}/s\beta, r_{13}/s\beta)$$

$$\gamma = \text{Atan2}(r_{32}/s\beta, -r_{31}/s\beta)$$

- If $\beta = 0^o, \beta = 180^o \Rightarrow s\beta = 0$ the solution degenerates

Three consecutive Axes Intersect - wrist

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$



$\beta = 0^o$

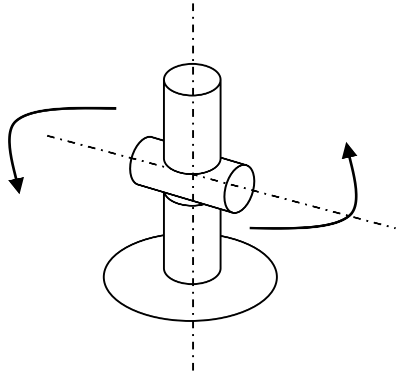
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\gamma - s\alpha s\gamma & -c\alpha s\gamma - s\alpha c\gamma & 0 \\ s\alpha c\gamma + c\alpha s\gamma & -s\alpha s\gamma + c\alpha c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c(\alpha+\gamma) & -s(\alpha+\gamma) & 0 \\ s(\alpha+\gamma) & c(\alpha+\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We are left with $(\gamma + \alpha)$ for every case. This means we can't solve for either, just their sum.

Three consecutive Axes Intersect - wrist

- One possible convention is to choose $\alpha = 0^o$

- The solution can be calculated to be

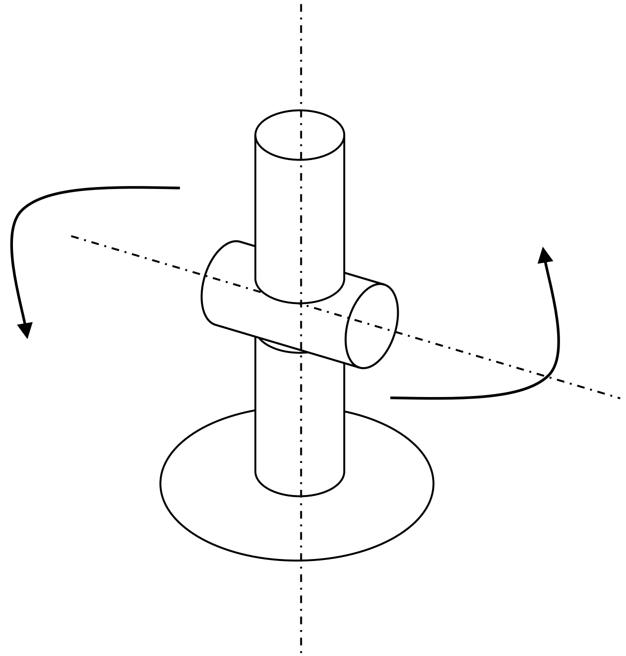


$$\beta = 0$$
$$\alpha = 0$$
$$\gamma = \text{Atan2}(-r_{12}, r_{11}) = \text{Atan2}(s\gamma, c\gamma)$$

$$\beta = 180$$
$$\alpha = 0$$
$$\gamma = \text{Atan2}(r_{12}, -r_{11}) = \text{Atan2}(s\gamma, c\gamma)$$

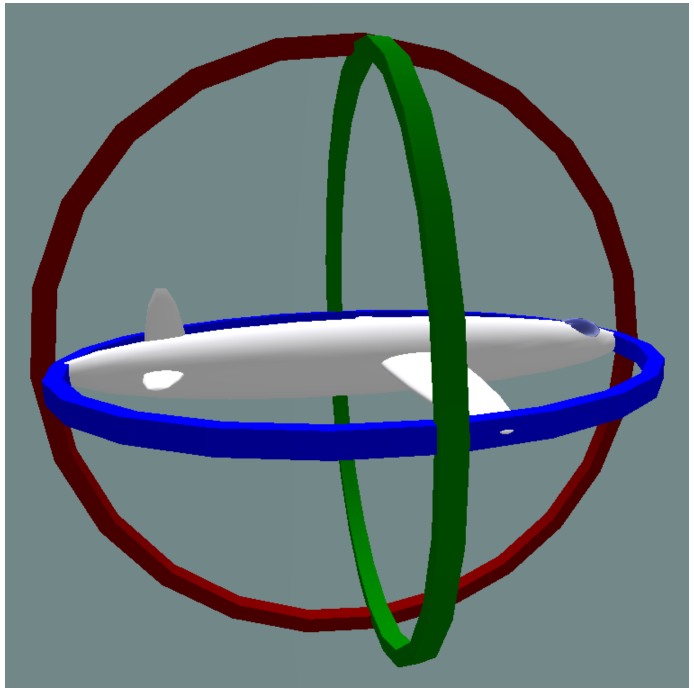
Three consecutive Axes Intersect - wrist

- For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector's orientation never changes.

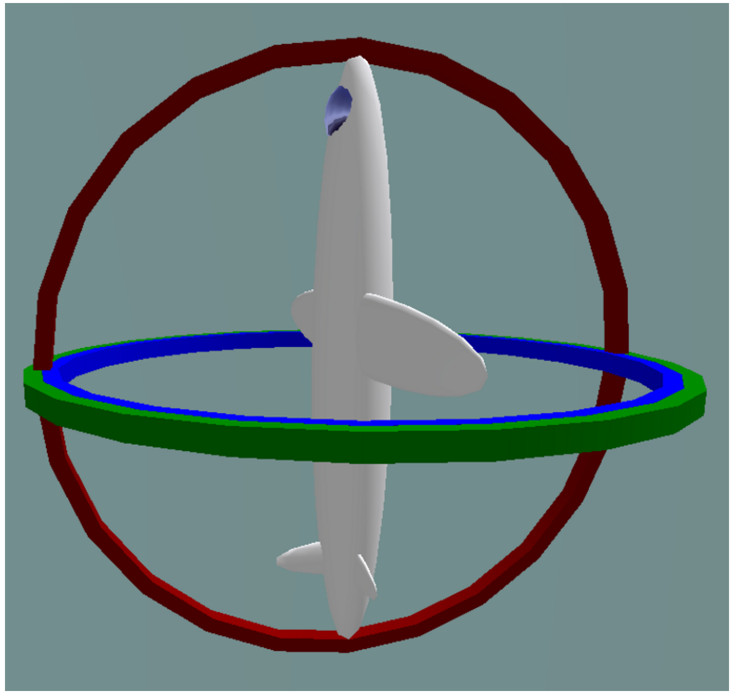


Gimbal Lock

http://youtu.be/zc8b2Jo7mno



Normal situation
The three gimbals are independent



Gimbal lock:
Two out of the three gimbals are in the same plane, one degree of freedom is lost

Gimbal Lock – Robotics

- In robotics, gimbal lock is commonly referred to as "wrist flip", due to the use of a "triple-roll wrist" in robotic arms, where three axes of the wrist, controlling yaw, pitch, and roll, all pass through a common point.

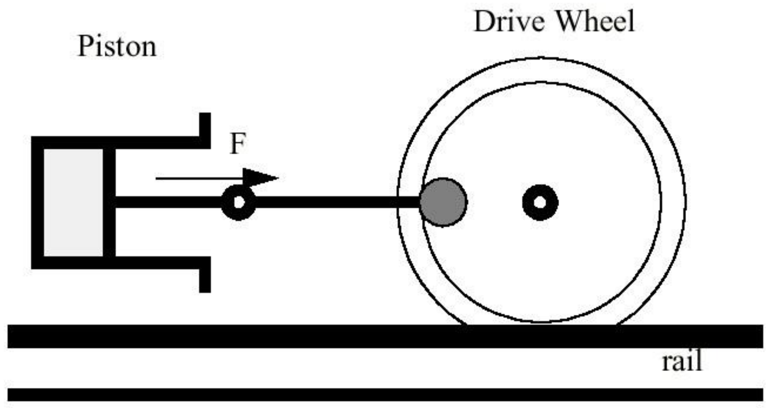
- An example of a wrist flip, also called a wrist singularity, is when the path through which the robot is traveling causes the first and third axes of the robot's wrist to line up. The second wrist axis then attempts to spin 180° in zero time to maintain the orientation of the end effector. The result of a singularity can be quite dramatic and can have adverse effects on the robot arm, the end effector, and the process.

- The importance of non-singularities in robotics has led the American National Standard for Industrial Robots and Robot Systems — Safety Requirements to define it as "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities".

Properties of the Jacobian - Force Mapping and Singularities

- This situation is an old and famous one in mechanical engineering.

- For example, in the steam locomotive, “top dead center” refers to the following condition



- The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.

Properties of the Jacobian -
Velocity Mapping and Singularities

- We have shown the relationship between joint space velocity and end effector velocity, given by

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

- It is interesting to determine the inverse of this relationship, namely

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

- Consider the square 6x6 case for $J(\underline{\theta})$.

- If rank < 6 ($Det\left(J(\underline{\theta})\right)=0$) , then there is no solution to the inverse equation (see Brief Linear Algebra Review - 1,7).

$$Rank\left(J(\underline{\theta})\right)<6$$

$$\underline{\dot{\theta}}=J(\underline{\theta})^{-1}\underline{\dot{x}}$$

- However, if the rank = 5, then there is at least one non-trivial solution to the forward equation (see Brief Linear Algebra Review - 7). That is, for

$$\underline{\dot{x}}=J(\underline{\theta})\underline{\dot{\theta}}=0$$

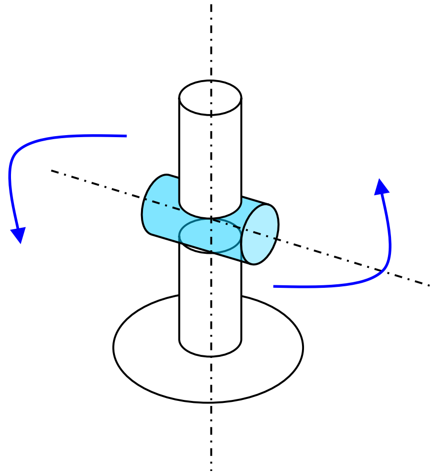
Properties of the Jacobian - Velocity Mapping and Singularities

- The solution is a direction $(\underline{\theta})$ in the in joint velocity space for which joint motion produces no end effector motion.

- We call any joint configuration $\underline{\theta} = Q$ for which

$$Rank\left(J(\underline{\theta})\right) < 6$$

a ***singular configuration***.



Properties of the Jacobian - Velocity Mapping and Singularities

- For certain directions of end effector motion , $\underline{\dot{x}}i$ $1 \le i \le 6$

$$\underline{\dot{x}} = J(\theta)\underline{\dot{\theta}} = \lambda_i\left(\underline{\theta}\right)\underline{\omega_i}$$

where:

- λ_i are the eigenvalues of $J(\theta)$
- $\underline{\omega}_i$ are the eigenvectors of $J(\theta)$

- If $J(\theta)$ is fully ranked (see Brief Linear Algebra Review - 6/), we have

$$\underline{\omega}i = J(\theta)^{-1}\underline{\dot{x}} = \lambda_i\left(\underline{\theta}\right)^{-1}\underline{\dot{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

- As the joint approach a singular configuration $\underline{\theta} = Q$ there is at least one eigenvalue for which $\lambda_i \to 0$. This results in

$$\underline{\omega} i = \frac{\dot{\underline{x}}}{\lambda_i(\underline{\theta})} \to \frac{\dot{\underline{x}}}{0} \to \infty$$

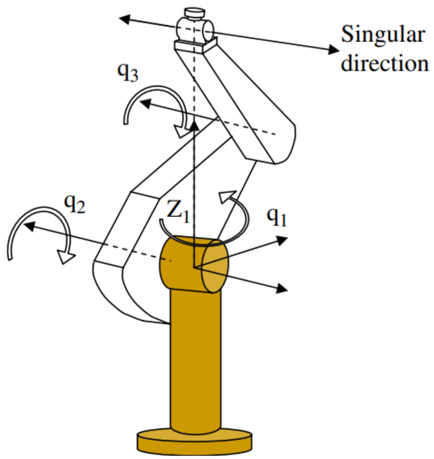
- In other word, as the joints approach the singular configuration, the end effector motion in a particular task direction $\dot{\underline{x}}$ causes the joint velocities to approach infinity. However, there are task velocities that can have solutions.

- If $J(\underline{\theta})$ loses rank by only one, then there are n-1 eigenvectors in the task velocity space ($\dot{\underline{x}}$) for which solutions do exist. However, there can be multiple solutions.

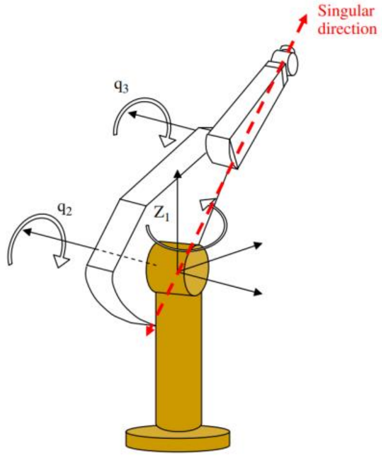
Physical Expressions of Singularity – Summary

Physical Expressions of Singularity

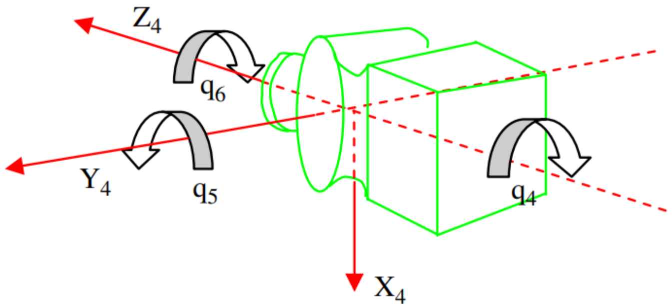
- **DOF**
  - **Phenomena** -
  - **Kinematic Deficiency** –
  - **Mechanical Risk** -



Shoulder



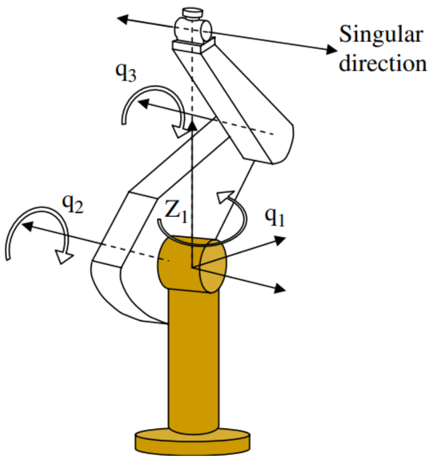
Elbow



Wrist

Physical Expressions of Singularity

- **DOF**
  - **Phenomena** - Losing one or more DOF
  - **Kinematic Deficiency** – Inability to move in a specific direction
  - **Mechanical Risk** - None



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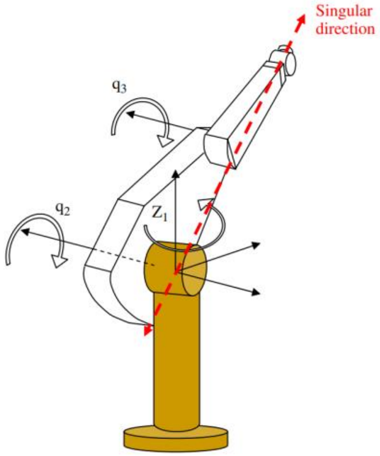

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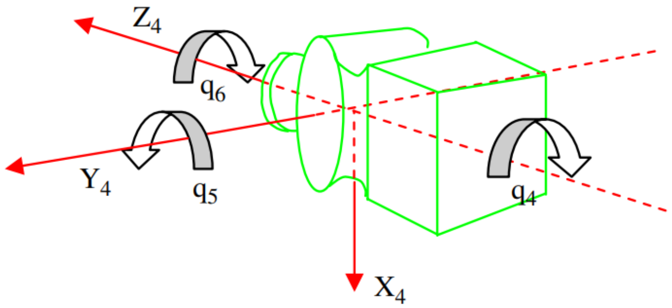
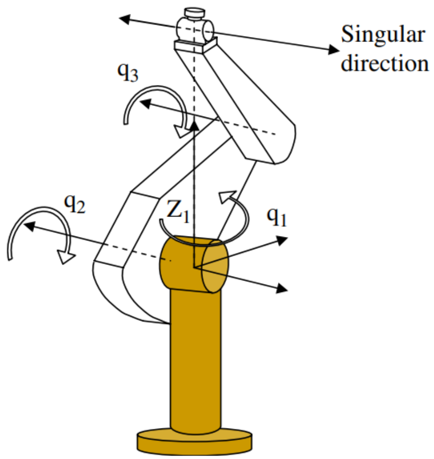


Wrist

Physical Expressions of Singularity

- **Velocity**
  - **Phenomena** -
  - **Kinematic Deficiency –**
  - **Mechanical Risk** -



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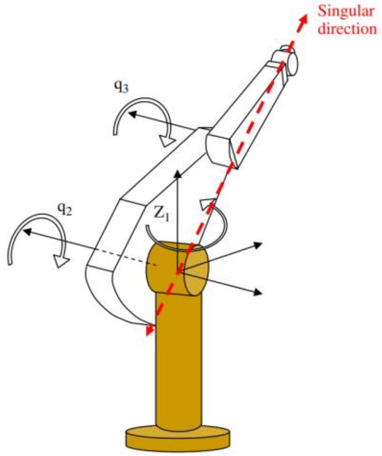

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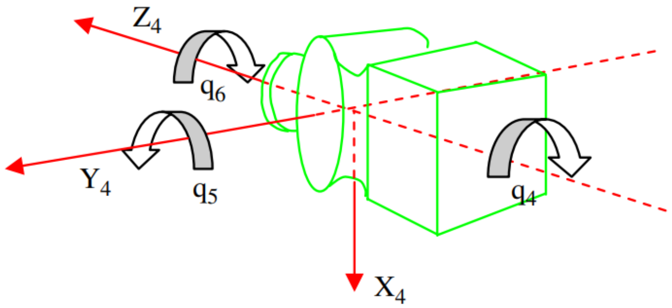
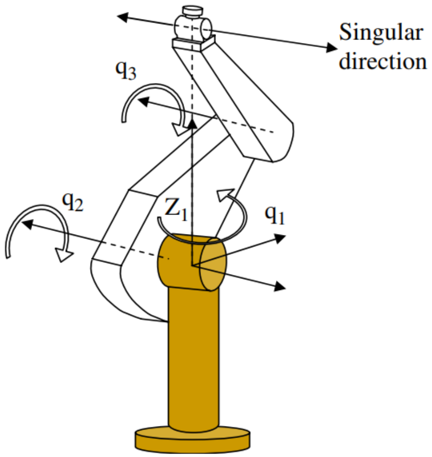


Wrist

Physical Expressions of Singularity

- **Velocity**
  - **Phenomena** - High velocities (theoretically infinity) of some joints, and zero velocities in other directions
  - **Kinematic Deficiency –** Inability to provide the required joint velocity to support the end effector velocity
  - **Mechanical Risk** - High demand of joint velocities may damage the actuator/gear/joint



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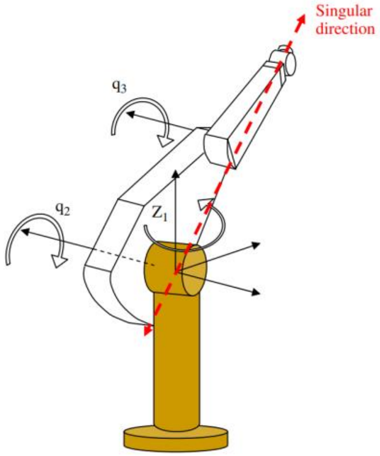

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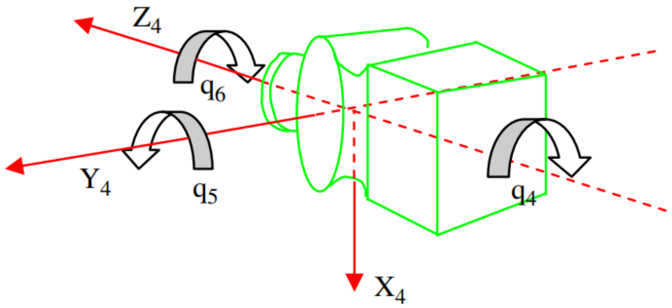
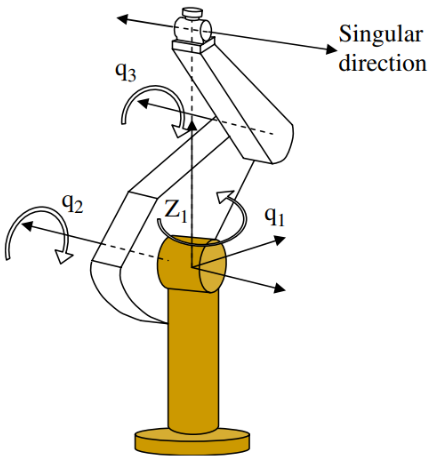


Wrist

Physical Expressions of Singularity

- **Forces / Torques**
  - **Phenomena** -
  - **Kinematic Deficiency –**
  - **Mechanical Risk** -



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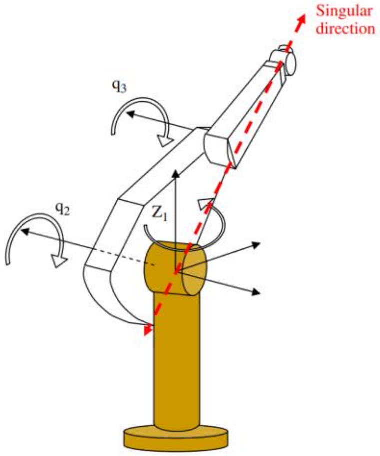

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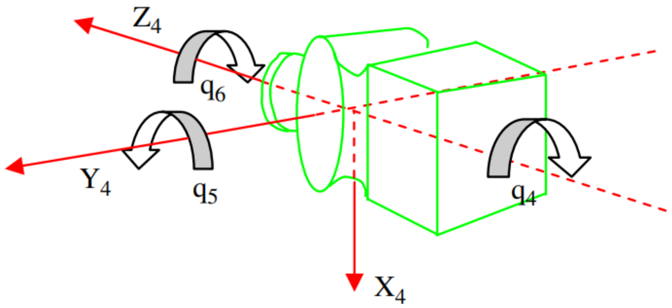
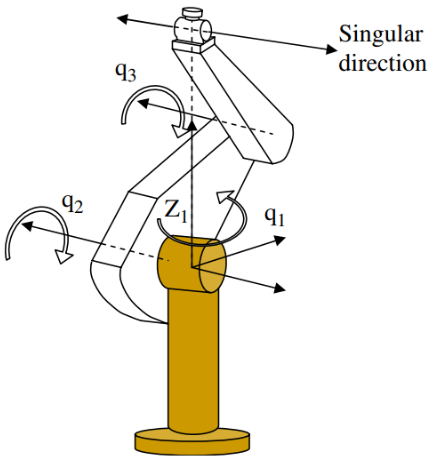


Wrist

Physical Expressions of Singularity

- **Forces / Torques**
  - **Phenomena** - High external loads applied on the end effector resulted with zero joint torques generated at the joint.
  - **Kinematic Deficiency -** None
  - **Mechanical Risk** - The high loads deform the links and can potential damage them



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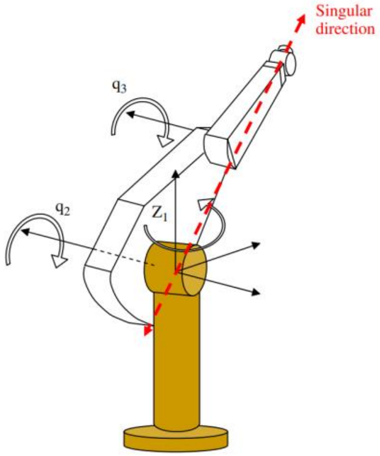

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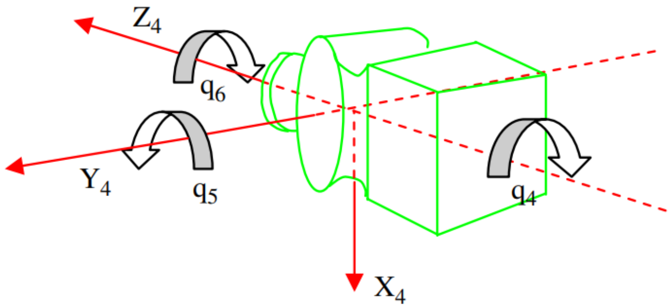
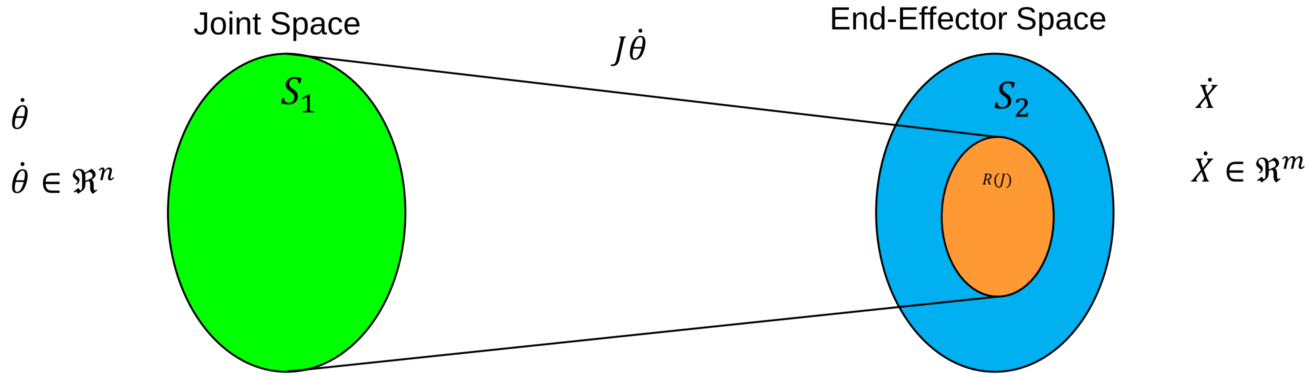


Wrist

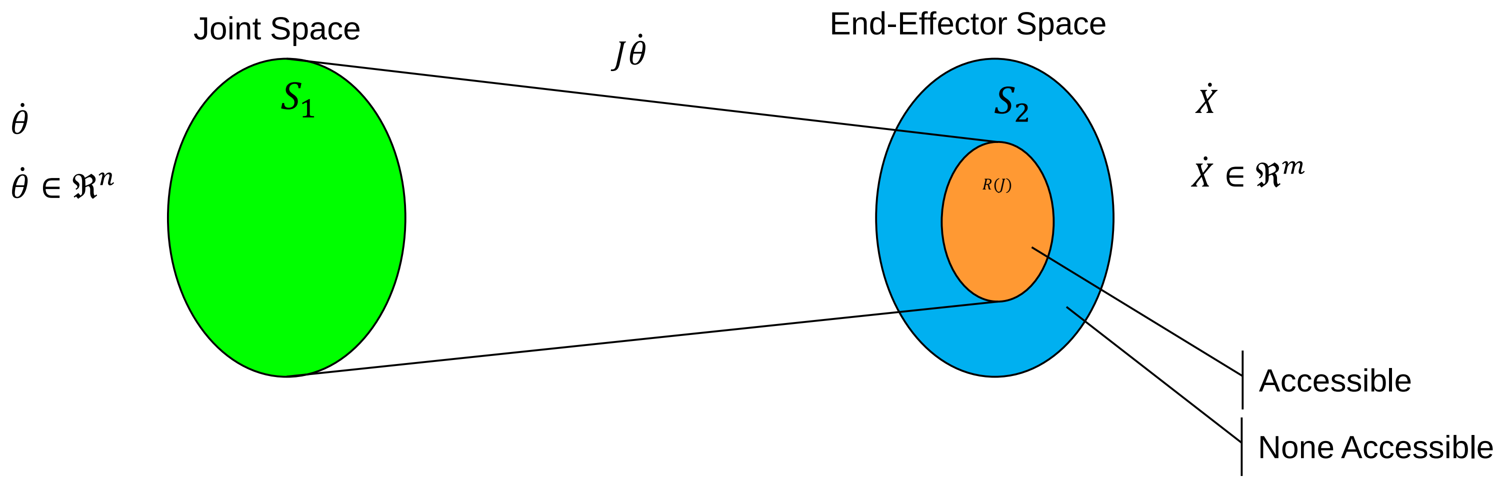
Jacobian – Duality

Jacobian – Duality



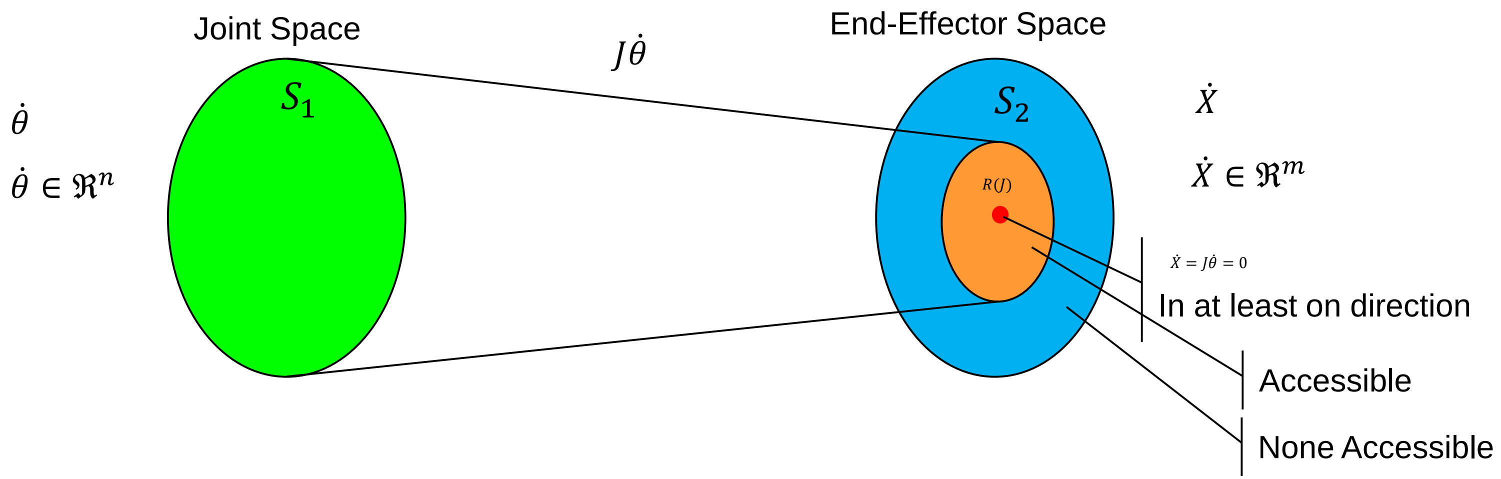
$\dot{X} = J\dot{\theta}$ is a linear mapping of the joint space velocities $\dot{\theta}$ which is a n - dimensional vector space $\dot{\theta} \in \Re^n$ to the end effector velocities $\dot{X}$ which is a m – dimensional vector space $\dot{X} \in \Re^m$

Jacobian – Duality



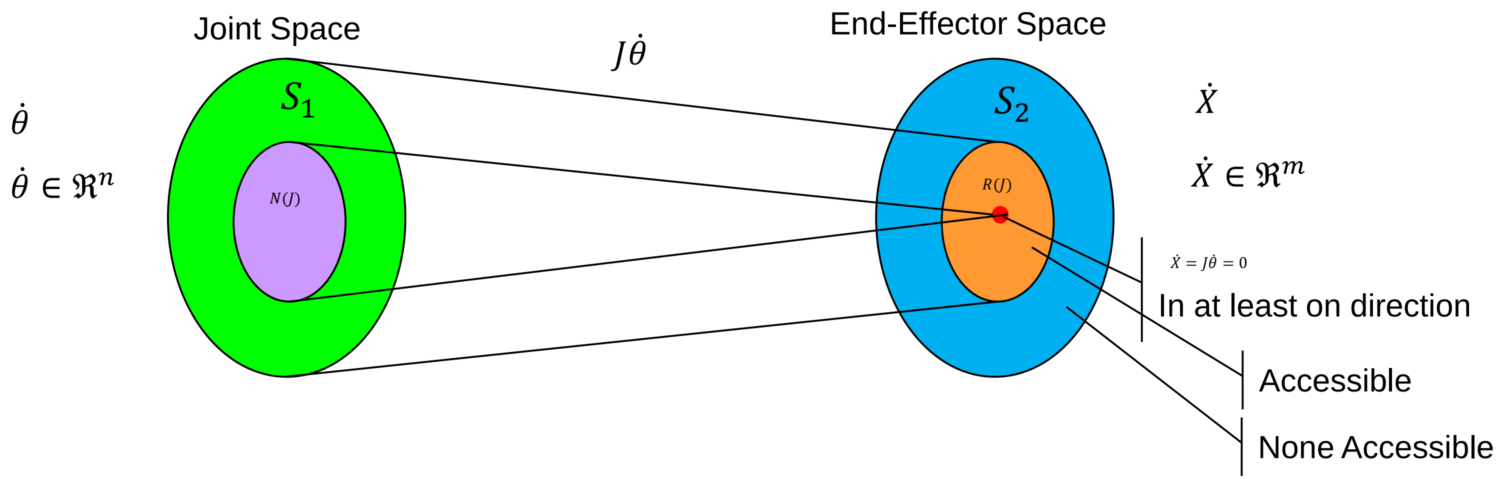
The subset of all the end effector velocities $\dot{X}$ resulting from the mapping $\dot{X} = J\dot{\theta}$ represents all the possible end effector velocities that can be generated by the n joints given the arm configuration

Jacobian – Duality



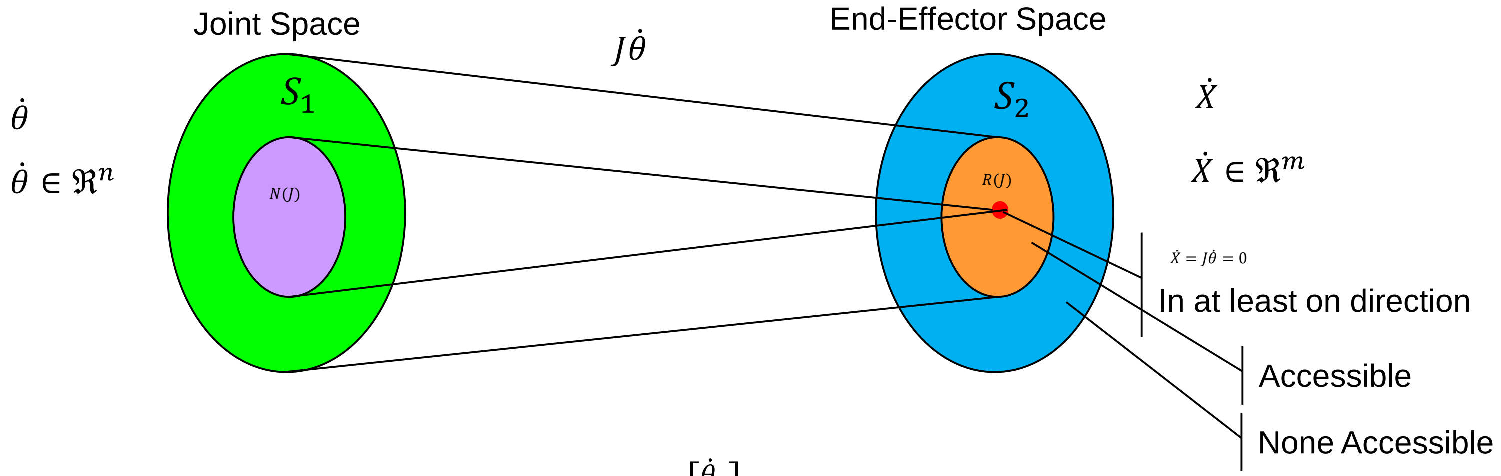
If the rank of the Jacobian matrix J is at full of row rank (square matrix) the joint space $\dot{\theta}$ covers the entire end effector vector $\dot{X}$ otherwise there is at least one direction in which the end effector can not be moved

Jacobian – Duality



The subset $\boxed{N(J)}$ is the null space of the linear mapping. Any element in this subspace is mapped into a zero vector in $\Re^m$ such that $\dot{X} = J\dot{\theta} = 0$ therefore any joint velocity vector $\dot{\theta}$ that belongs to the null space does not produce any velocity at the end effector

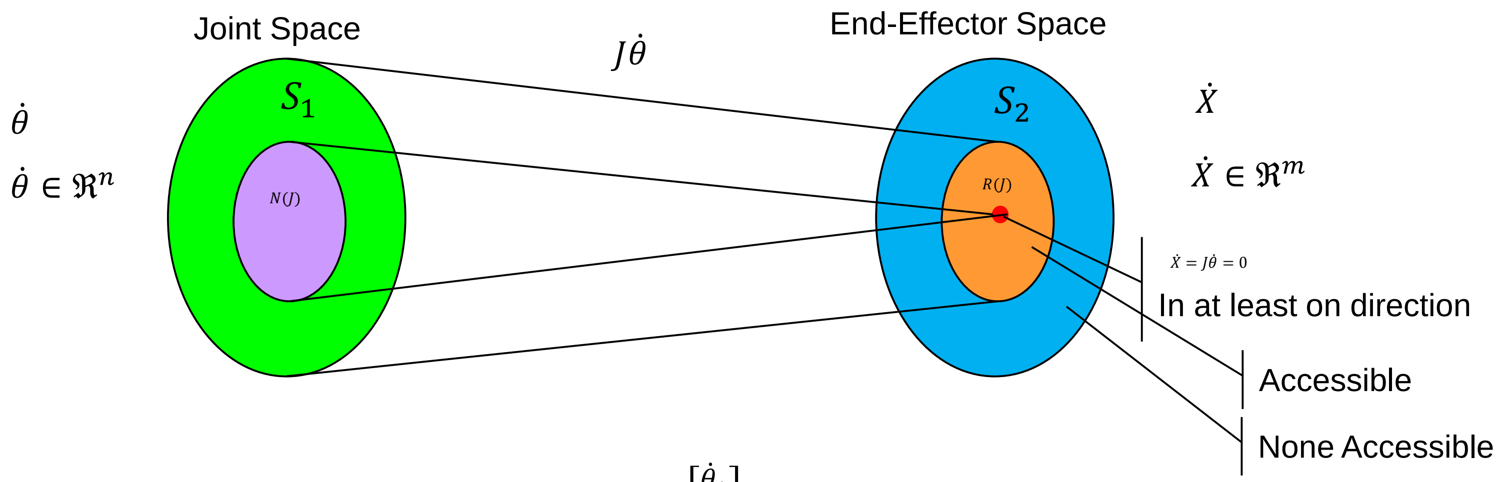
Jacobian – Duality



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & 6X7 & \\ & J(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \\ \dot{\theta}_7 \end{bmatrix}$$

If the Jacobian of a manipulator is full rank the dimension of the null space $\boxed{\dim(N(J))}$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the hand may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

Jacobian – Duality



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & 6X7 & \\ & J(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \\ \dot{\theta}_7 \end{bmatrix}$$

If the Jacobian of a manipulator is full rank (i.e. n>m full row rank where the rows are linearly independent) the dimension of the null space $\boxed{\dim(N(J))}$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the hand may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

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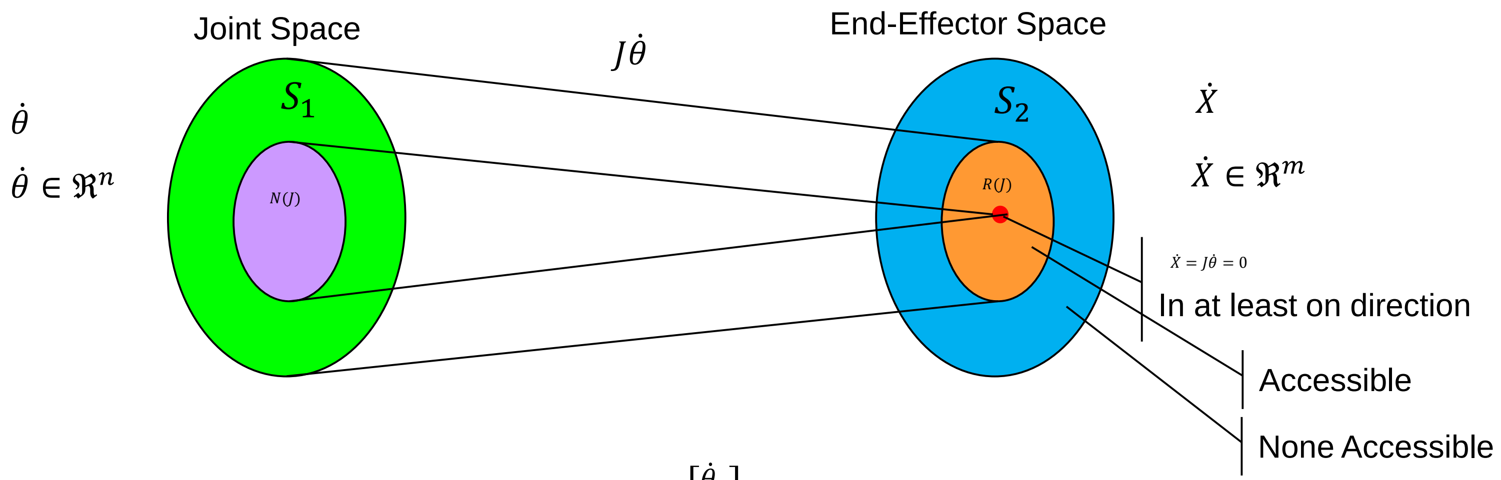

$$
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$$

If the Jacobian of a manipulator is full rank (i.e. for redundant manipulator n>m full row rank where the rows are linearly independent) the dimension of the null space $\dim(N(J))$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the end effector (hand) may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

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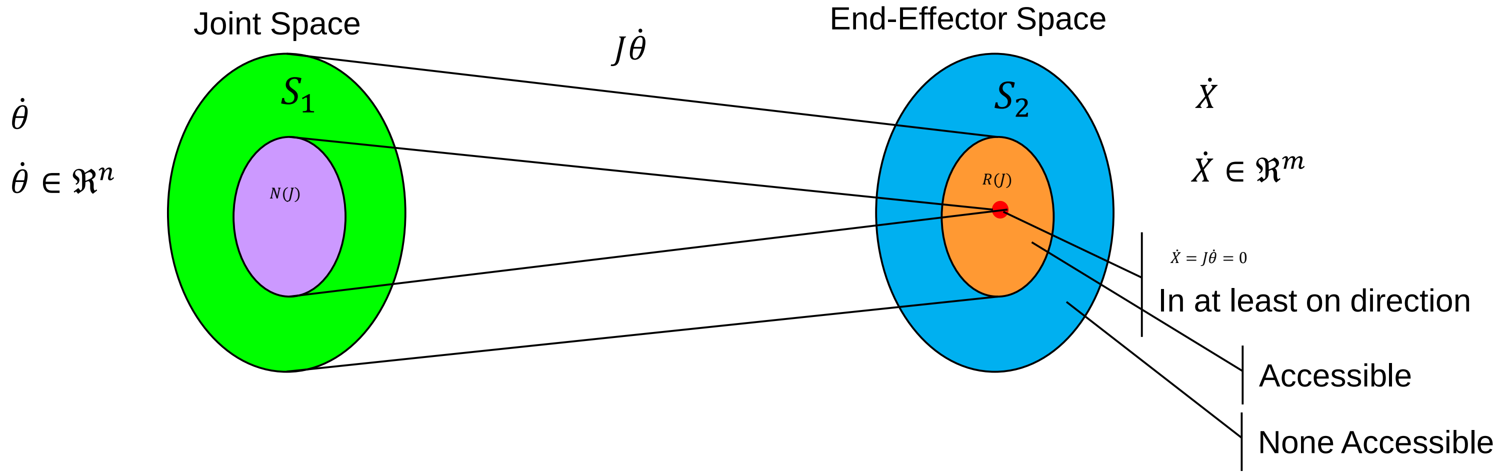

When the Jacobian matrix degenerates (i.e. not full rank e.g. due to singularity) the dimension of the range space $\boxed{\dim(R(J))}$ decreases at the same time as the dimension of the null space increases $\boxed{\dim(N(J))}$ by the same amount. The sum of the two is always equal to n $\dim(R(J)) + \dim(N(J)) = n$

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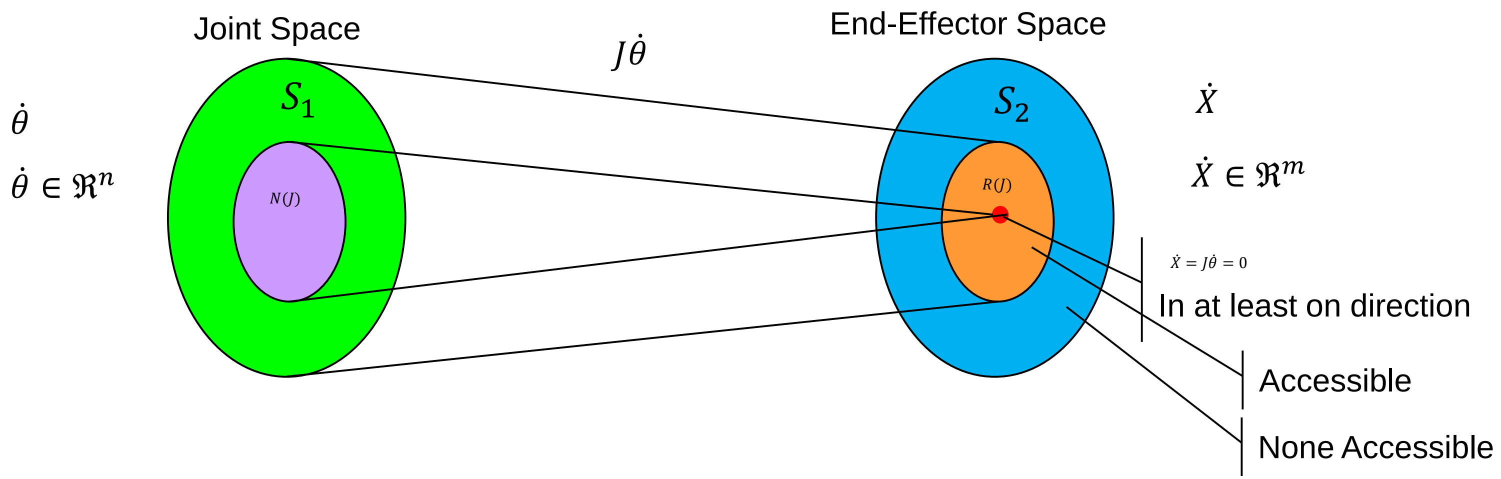

If the null space is not empty set, the instantaneous kinematic equation has an infinite number of solutions that cause the same end effector velocities (recall the 3 axis end effector)

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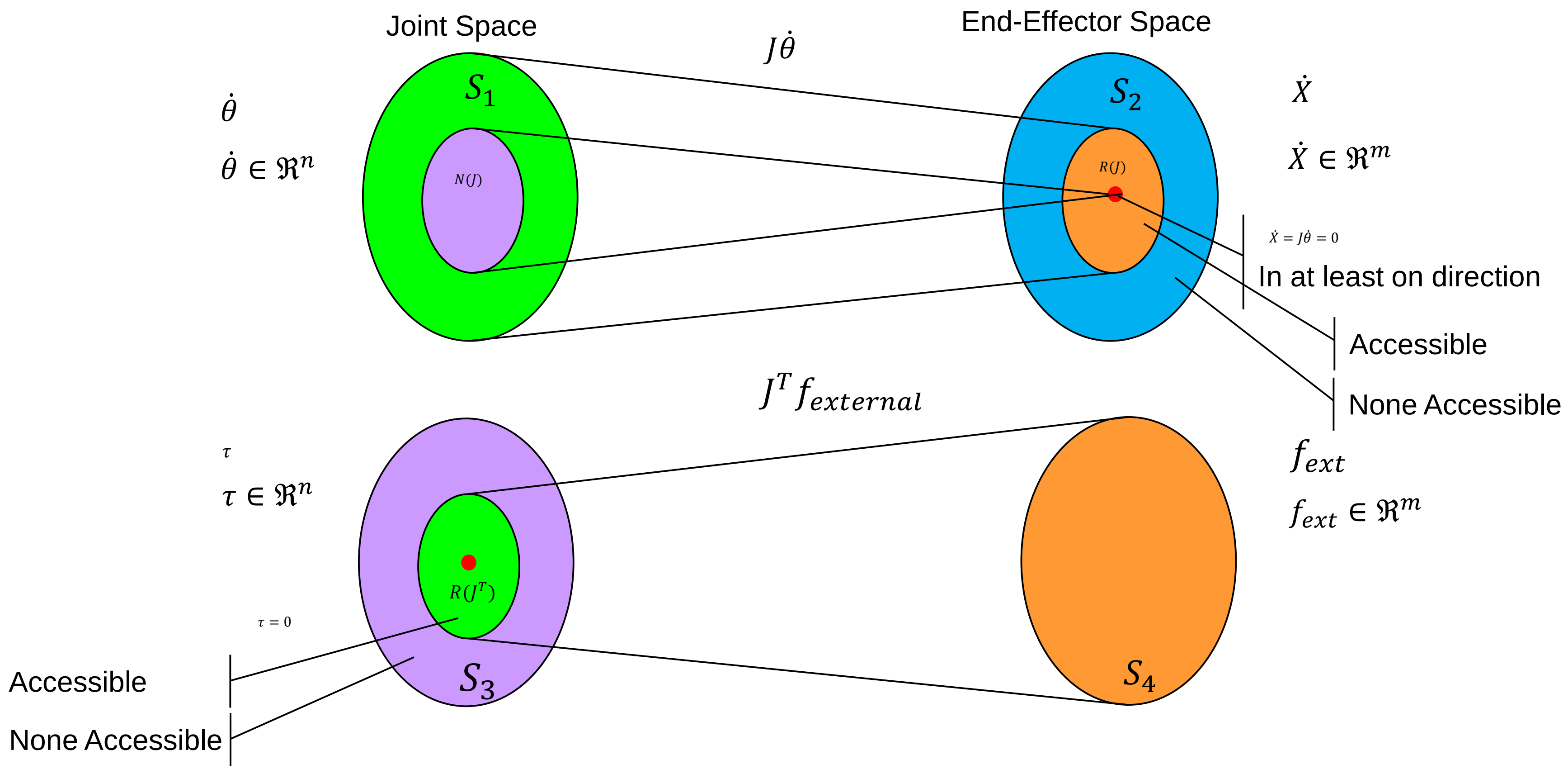

Unlike the mapping of the instantaneous kinematics the mapping of the static external forces is from the m-th vector space $\boxed{f_{ext} \in \Re^m}$ associated with the end effector coordinates to the n-th dimensional vector space $\boxed{\tau \in \Re^n}$ associated with the torques at the joint space. Therefore the joint torque are always determined uniquely from any end effector point force

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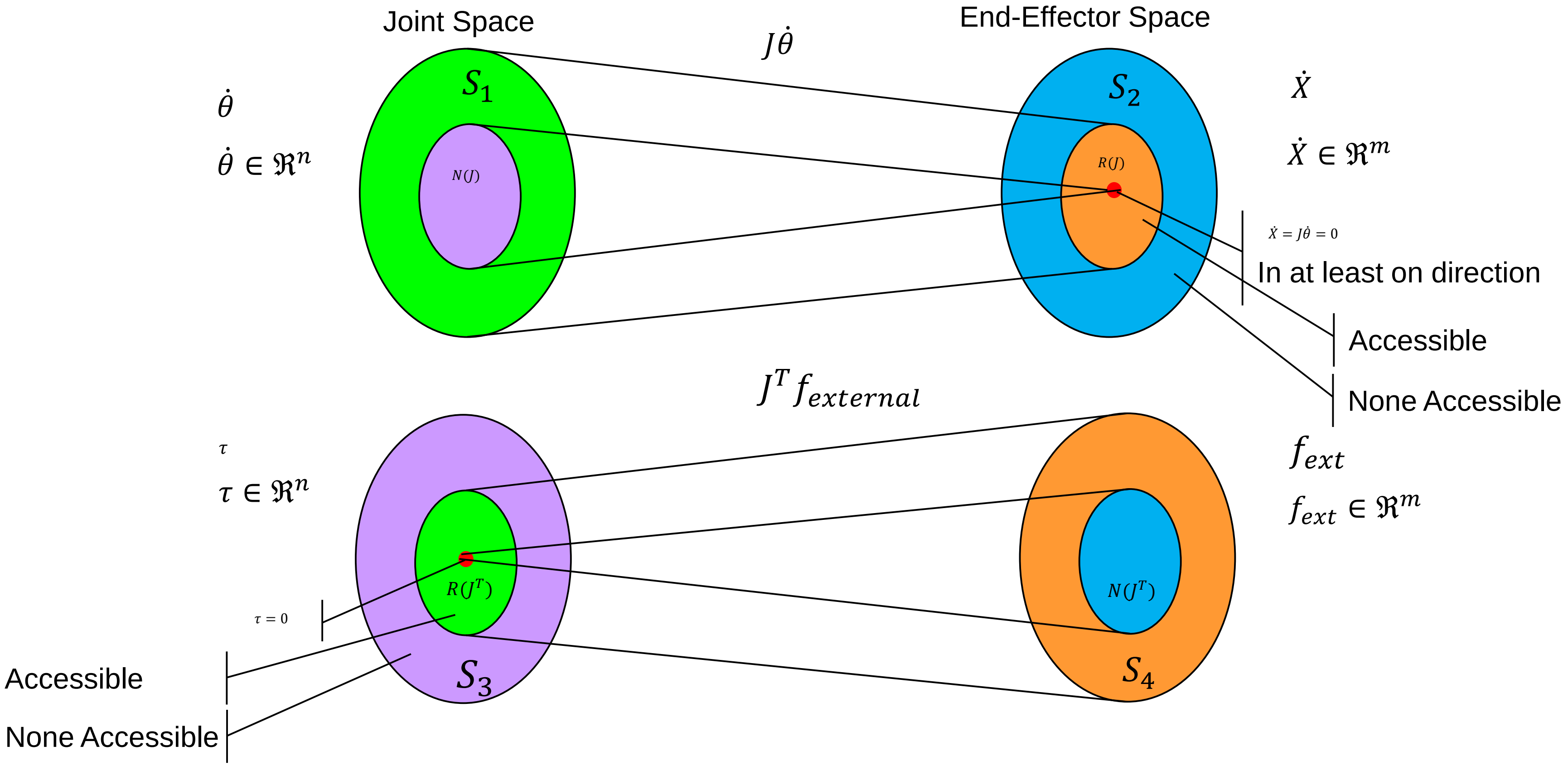

The null space $N(J^T)$ represents the set of all end point forces that do not require any torques at the joints to bear the corresponding load ● (e.g. 2R fully stretched or collapsed elbow).

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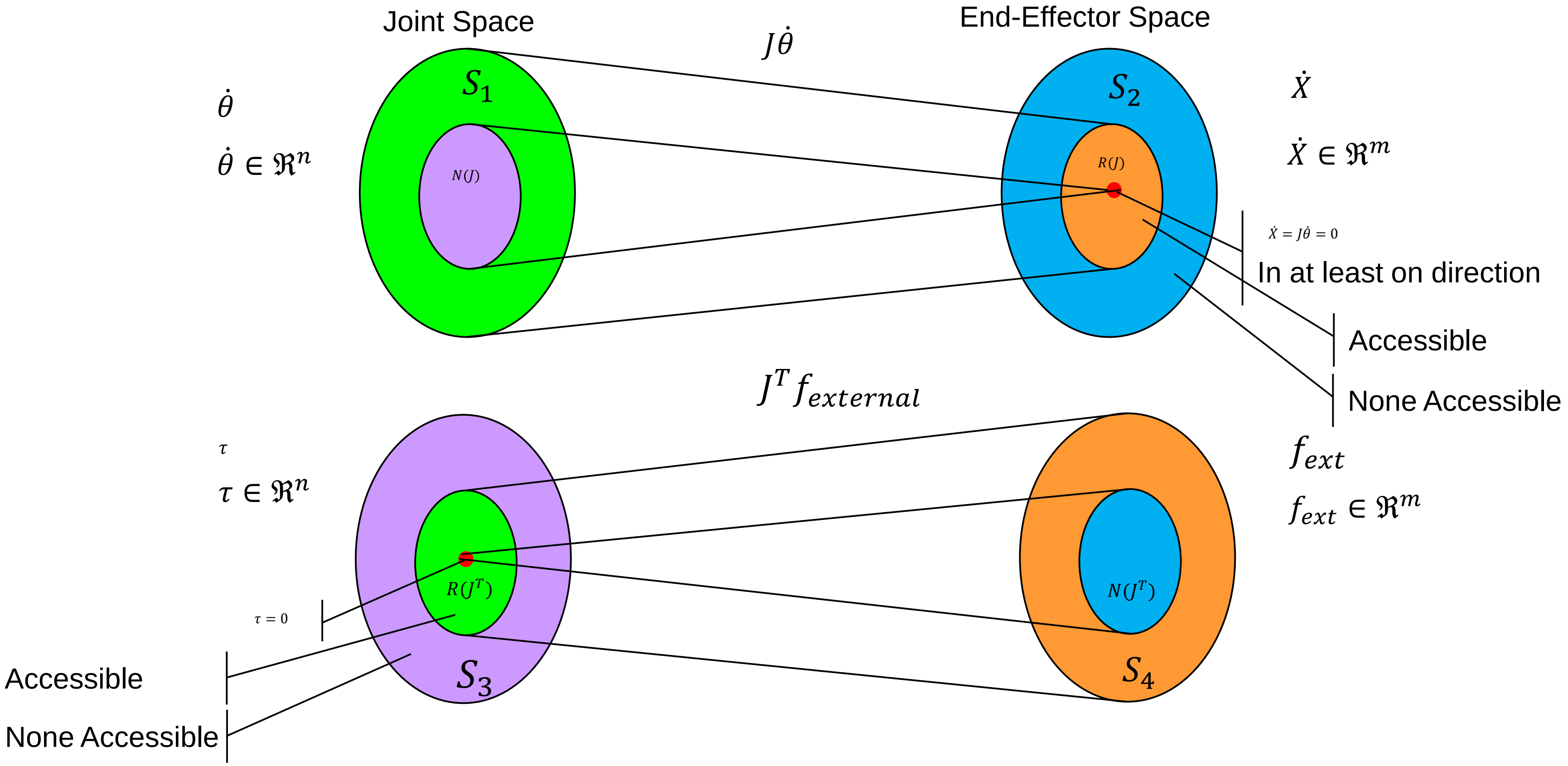

When the Jacobian matrix is degenerated or the arm is in a singular configuration external endpoint force is borne entirely by the structure and not by the joint torque.