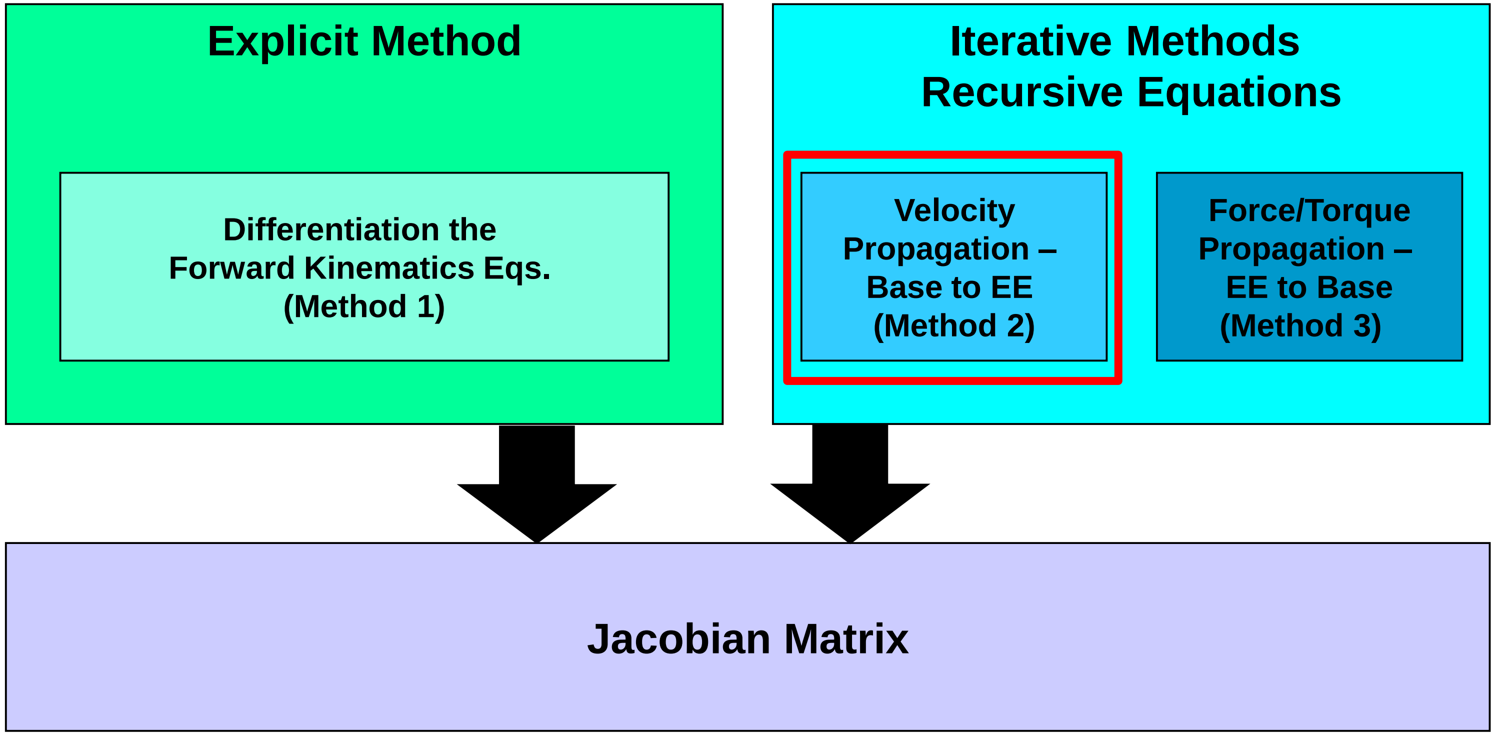
Jacobian Iterative Method - Velocity Propagation (Method No. 2) Part 2 – Reference Frame

Jacobian Matrix - Derivation Methods



Jacobian: Velocity propagation

- The recursive expressions for the adjacent joint linear and angular velocities describe a relationship between the joint angle rates ($\dot{\Theta}$) and the transnational and rotational velocities of the end effector (ν):

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- The velocity propagation method provided us the linear and angular velocities of the end effector differentiated with respect to the base frame {0} but expressed in the end effector frame – frame {4}

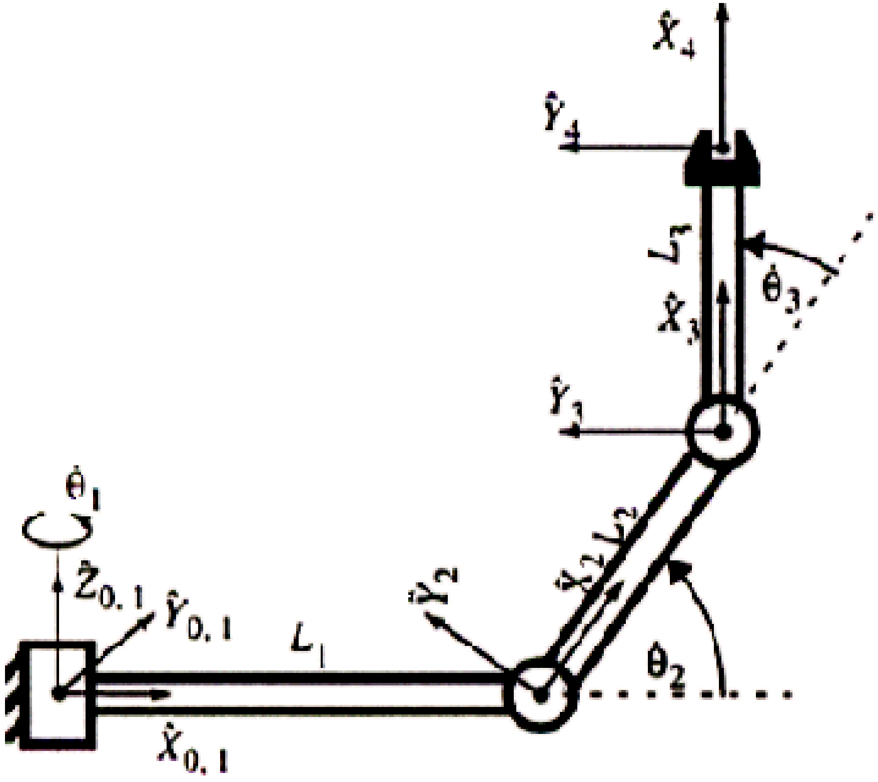
$$^{4}(^{0}V_4) \equiv {}^{4}v_4 = \begin{bmatrix} L_2 S_3 \dot{\theta}_2 \\ (L_2 C_3 + L_3)\dot{\theta}_2 + L_3 \dot{\theta}_3 \\ (-L_1 - L_2 C_2 - L_3 C_{23})\dot{\theta}_1 \end{bmatrix}$$

$$^{4}(^{0}\Omega_4) \equiv {}^{4}\omega_4 = \begin{bmatrix} S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

- We wish to express the linear and angular velocities in the same base that the were differentiated i.e frame {0}

$$^{0}(^{0}V_4) \equiv {}^{0}v_4 \equiv v_4$$

$$^{0}(^{0}\Omega_4) \equiv {}^{0}\omega_4 \equiv \omega_4$$



Jacobian Expression
Frame of Reference

Frame Notation

Jacobian: Velocity propagation

- The recursive expressions for the adjacent joint linear and angular velocities defines the Jacobian in the end effector frame {N}

$$ {}^{N}\nu = {}^{N}\mathbf{J}\,(\Theta)\dot{\Theta} $$

- This equation can be expanded to:

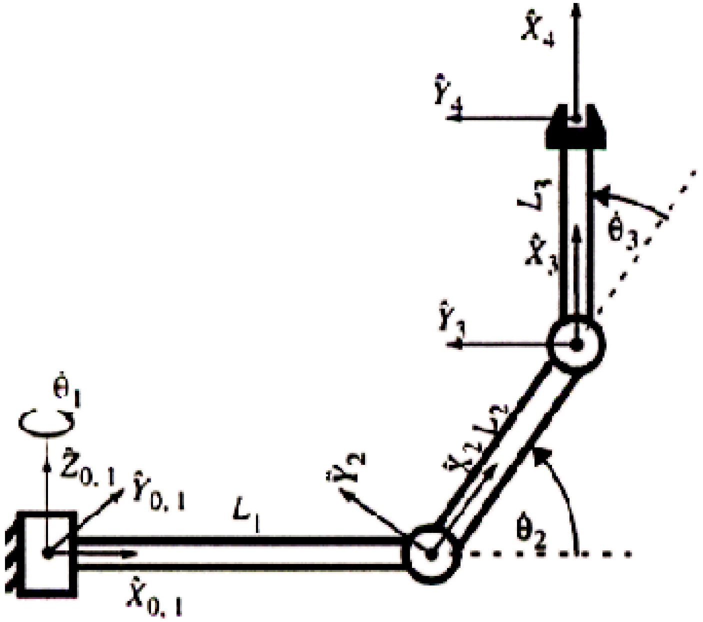
$$ {}^{N}\nu = {}^{N}\frac{d}{dt}[X] = {}^{N}\begin{bmatrix} [\, v_N] \\ [\, \omega_N] \end{bmatrix} = {}^{N}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{N}\begin{bmatrix} \quad \mathbf{J}\,(\Theta) \quad \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} $$

Jacobian - 3R - Example

- The linear angular velocities of the end effector (N=4)

$$
{}^{4}v_{4} = \begin{bmatrix} L_2 S_3 \dot{\theta}_2 \\ (L_2 C_3 + L_3)\dot{\theta}_2 + L_3 \dot{\theta}_3 \\ (-L_1 - L_2 C_2 - L_3 C_{23})\dot{\theta}_1 \end{bmatrix}
$$

$$
{}^{4}\omega_{4} = \begin{bmatrix} S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}
$$



Jacobian - 3R - Example

- Re-arranged to previous two terms gives an expression that encapsulates

$$^{4}v = {}^{4}\mathbf{J}\,(\Theta)\dot{\Theta}$$

$$^{4}v = \begin{bmatrix} ^{4}v_4 \\ ^{4}\omega_4 \end{bmatrix} = {}^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{4}\begin{bmatrix} & \mathbf{J}\,(\Theta) & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix} = \begin{bmatrix} L_2 S_3 \dot{\theta}_2 \\ (L_2 C_3 + L_3)\dot{\theta}_2 + L_3 \dot{\theta}_3 \\ (-L_1 - L_2 C_2 - L_3 C_{23})\dot{\theta}_1 \\ S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

- We can now factor out the joint velocities vector $\dot{\theta} = [\dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3]^T$ from the above vector to formulate the Jacobian matrix

$$^{4}\mathbf{J}\,(\Theta)$$

Jacobian - 3R - Example

$$^{4}v = {}^{4}\mathbf{J}(\Theta)\dot{\Theta}$$

$$^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2S_3 & 0 \\ 0 & L_2C_3 + L_3 & L_3 \\ -L_1 - L_2C_2 - L_3C_{23} & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- The equations for $^{N}v_N$ and $^{N}\omega_N$ are always a linear combination of the joint velocities, so they can always be used to find the 6xN Jacobian matrix ($^{N}J(\theta)$) for any robot manipulator.
- Note that the Jacobian matrix is expressed in frame {4}
- **Problems**
  - Dimensions of the Jacobian (Example: Current 6x3 reduce to 3x3)
  - Frame of Reference / Representation (Example: Frame 4 – Move to Frame 0)

Jacobian Expression
Frame of Reference / Representation

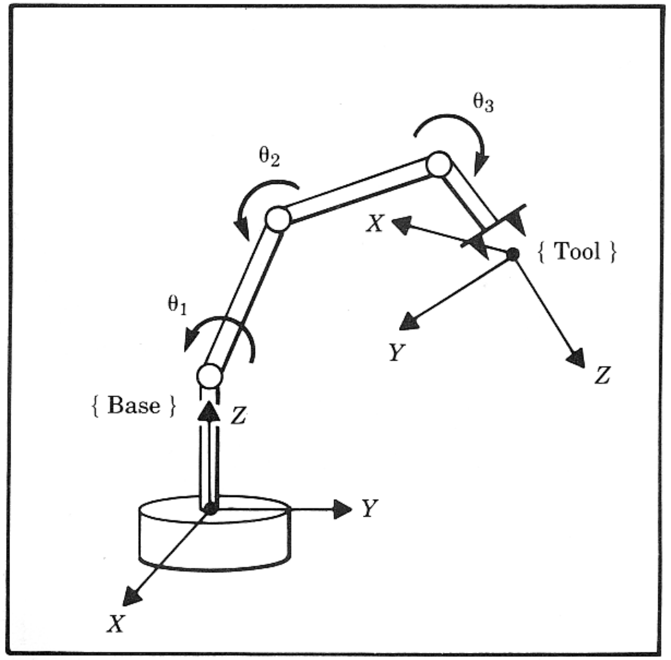
Jacobian: Frame of Representation

- Using the **velocity propagation method** we expressed the relationship between the velocity of the robot end effector measured relative to the robot base frame {0} and expressed in the end effector frame {N}.

$$^{N}\mathcal{V} = {}^{N}\mathbf{J}\,(\Theta)\dot{\Theta}$$

- Occasionally, it may be desirable to express (represent) the end effector velocities in another frame (e.g. frame {0}, in which case we will need a method to provide the transformation.

$$^{0}\mathcal{V} = {}^{0}\mathbf{J}\,(\Theta)\dot{\Theta}$$



Jacobian: Frame of Representation

- There are two methods to change the references frame (frame of representation) of the Jacobian Matrix
  - **Method 1 (Before The Jacobian Matrix Is Formulated)**
    - Transforming the linear and angular velocities to the new frame prior to formulating the Jabobian matrix.
  - **Method 2 (Before The Jacobian Matrix Is Formulated)**
    - Transforming the Jacobian matrix from it existing frame to the new frame after it was formulated.

Jacobian Expression
Frame of Reference

Method No. 1
Transform the Velocity Vectors

Jacobian: Frame of Representation – Method 1

- Consider the velocities in a different frame {B}

$$ {}^{\boldsymbol{B}}\nu = \begin{bmatrix} {}^{B}v_N \\ {}^{B}\omega_N \end{bmatrix} $$

- We may use the rotation matrix to find the velocities in frame {A}:

$$ {}^{\boldsymbol{A}}\nu = \begin{bmatrix} {}^{A}v_N \\ {}^{A}\omega_N \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R\,{}^{B}v_N \\ {}^{A}_{B}R\,{}^{B}\omega_N \end{bmatrix} $$

Jacobian: Frame of Representation – Method 1

- Example: Analyzing a 6 DOF manipulator while utilizing velocity propagation method results in an expressing the end effector (frame 6) linear and angular velocities.

$$^{6}\nu = \begin{bmatrix} ^{6}v_{6} \\ ^{6}\omega_{6} \end{bmatrix}$$

- Using the forward kinematics formulation the rotation matrix from frame 0 to frame 6 can be defined as

$$^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T = \begin{bmatrix} & {}^{0}_{6}R & & {}^{0}P_{6ORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The linear and angular velocities can than be expressed in frame 0 prior to extracting the Jacobian in frame 0

$$^{0}\nu = \begin{bmatrix} ^{0}v_{6} \\ ^{0}\omega_{6} \end{bmatrix} = \begin{bmatrix} {}^{0}_{6}R\,{}^{6}v_{6} \\ {}^{0}_{6}R\,{}^{6}\omega_{6} \end{bmatrix}$$

Jacobian Expression
Frame of Reference

Method No. 2
Transform the Jacobian Matrix

Jacobian: Frame of Representation – Method 2

- It is possible to define a Jacobian transformation matrix ${}^{A}_{B}R_J$ that can transform the Jacobian from frame B to frame A

$$ {}^{A}\nu = {}^{A}\mathbf{J}\,(\Theta)\dot{\Theta} = {}^{A}_{B}R_J\ {}^{B}\mathbf{J}\,(\Theta)\dot{\Theta} $$

- The Jacobian rotation matrix ${}^{A}_{B}R_J$ is given by

$$ {}^{A}_{B}R_J = \begin{bmatrix} \left[{}^{A}_{B}R\right] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \left[{}^{A}_{B}R\right] \end{bmatrix} $$

Jacobian: Frame of Representation

- or equivalently,

$$
{}^{A}\mathbf{J}(\Theta) = \begin{bmatrix} \begin{bmatrix} {}^{A}_{B}R \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} {}^{A}_{B}R \end{bmatrix} \end{bmatrix} {}^{B}\mathbf{J}(\Theta)
$$

Jacobian: Frame of Representation - 3R Example

$$ {}^{0}\mathbf{J}\,(\Theta) \;=\; {}^{0}_{4}R_J \;\, {}^{4}\mathbf{J}\,(\Theta) $$

$$ {}^{0}J(\theta) = \begin{bmatrix} \left[{}^{0}_{4}R\right] & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \left[{}^{0}_{4}R\right] \end{bmatrix} {}^{4}J(\theta) = \begin{bmatrix} \begin{bmatrix} c1c23 & -c1s23 & s1 \\ s1c23 & -s1s23 & -c1 \\ s23 & c23 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} c1c23 & -c1s23 & s1 \\ s1c23 & -s1s23 & -c1 \\ s23 & c23 & 0 \end{bmatrix} \end{bmatrix} {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} $$

- The rotation matrix (${}^{0}_{4}R$) can be calculated base on the direct kinematics given by

$$ {}^{0}_{4}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T = \begin{bmatrix} & {}^{0}_{4}R & & {}^{0}P_{4ORG} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)	
Explicit (Diff. the Forward Kinematic Eq.)	${}^{0}J_{N}$	None	
Iterative Velocity Eq.	${}^{N}J_{N}$	Transform Method 1:	${}^{0}v_{N} = {}^{0}_{N}R\,{}^{N}v_{N}$ ${}^{0}\omega_{N} = {}^{0}_{N}R\,{}^{N}\omega_{N}$
		Transform Method 2:	${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$
Iterative Force Eq.	${}^{N}J_{N}^{T}$	Transpose	${}^{N}J_{N} = [{}^{N}J_{N}^{T}]^{T}$
		Transform	${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$

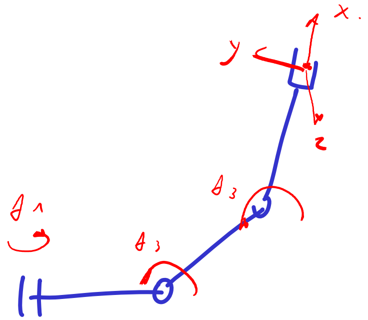
Dimension of the Jacobian Expression

Dimension Reduction

Inverse Jacobian - Reduced Jacobian

- ***Problem***
  - When the number of joints (N) is less than 6, the manipulator does not have the necessary degrees of freedom to achieve independent control of all six velocities components.

$$ {}^{4}v = {}^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} $$



- ***Solution***
  - We can reduce the number of rows in the original Jacobian to describe a reduced Cartesian vector.

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 1

$$
{}^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ -L_1 - L_2 C_2 - L_3 C_{23} & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Column of zeroes
- The determinate is equal to zero
- Only two out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 2

$$
{}^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ -L_1 - L_2 C_2 - L_3 C_{23} & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Two columns of zeroes
- The determinate is equal to zero
- Only one out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 3

$$
{}^{4}\begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ -L_1 - L_2 C_2 - L_3 C_{23} & 0 & 0 \\ S_{23} & 0 & 0 \\ C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- The resulting reduced Jacobian will be square (the number of independent rows in the Jacobian are equal to the number of unknown variables) and can be inverted unless in a singular configuration.

Jacobian: Singular Configuration - 3R Example

$$
{}^{4}J_r(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}
$$