Jacobian Iterative Method - Velocity Propagation (Method No. 2) Part 1 – Method Derivation

Jacobian Matrix - Derivation Methods

Explicit Method

Differentiation the
Forward Kinematics Eqs.
(Method 1)

Iterative Methods
Recursive Equations

Velocity
Propagation –
Base to EE
(Method 2)

Force/Torque
Propagation –
EE to Base
(Method 3)

Jacobian Matrix

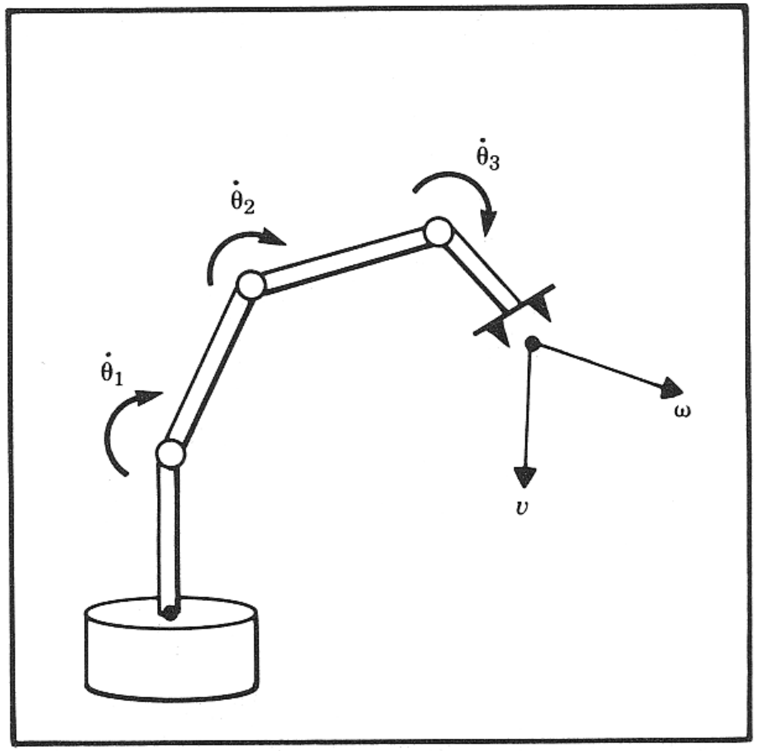
Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

$$\nu = \mathbf{J}\,(\Theta)\dot{\Theta}$$

$$\nu = \frac{d}{dt}[X] = \begin{bmatrix} [\,v_N] \\ [\,\omega_N] \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}$$

$$\dot{\Theta} = \mathbf{J}(\underline{\theta})^{-1}\nu$$



Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$$ {}^A V_Q = {}^A V_{BORG} + {}^A_B R\, {}^B V_Q + {}^A\Omega_B \times {}^A_B R\, {}^B P_Q $$

  - Matrix Form

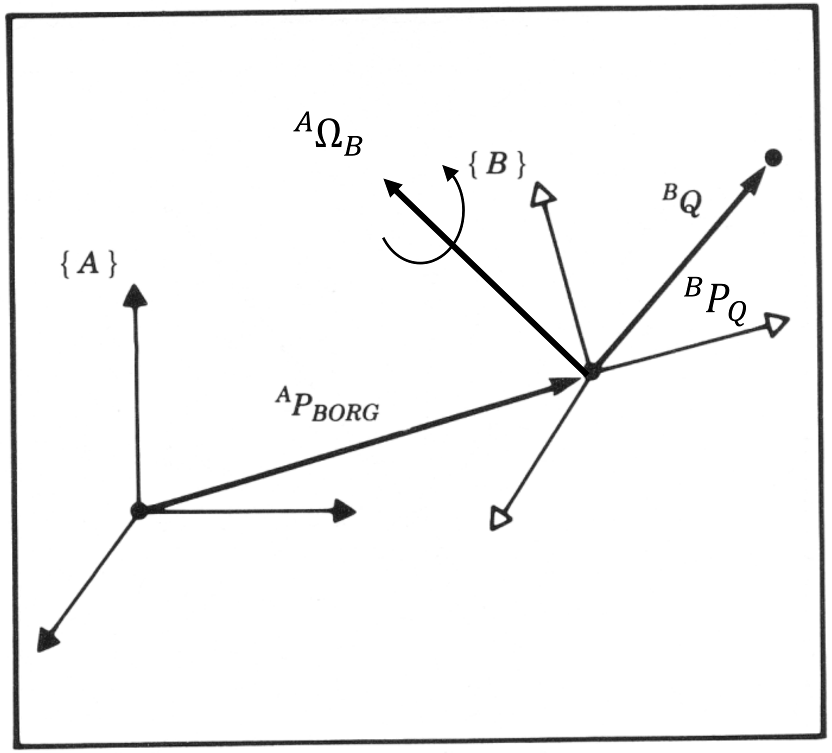
$$ {}^A V_Q = {}^A V_{BORG} + {}^A_B R\, {}^B V_Q + {}^A_B \dot{R}_\Omega \left({}^A_B R\, {}^B P_Q \right) $$

- Angular Velocity
  - Vector Form

$$ {}^A\Omega_C = {}^A\Omega_B + {}^A_B R\, {}^B\Omega_C $$

  - Matrix Form

$$ {}^A_C \dot{R}_\Omega = {}^A_B \dot{R}_\Omega + {}^A_B R\, {}^B_C \dot{R}_\Omega\, {}^A_B R^T $$



Velocity of Adjacent Links - Summary

- Angular Velocity

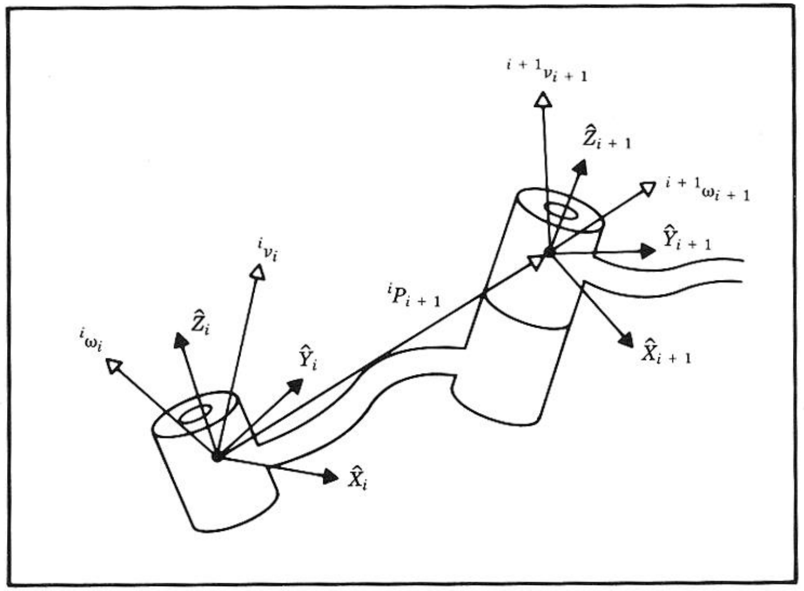
0 - Prismatic Joint

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

- Linear Velocity

0 - Revolute Joint

$$ {}^{i+1}v_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$



Representation / Reference Frame
Computed / Measured Frame

Frame Notation

Frame - Velocity

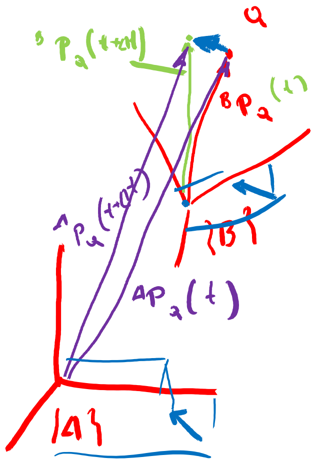
- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.

- A velocity vector **<u>computed</u>** in frame {B} and **<u>represented</u>** in frame {A} would be written

Represented (Reference Frame)

$$^{A}({}^{B}V_Q) = \frac{^{A}d}{dt}\,{}^{B}P_Q$$

Computed (Measured)



Position Propagation

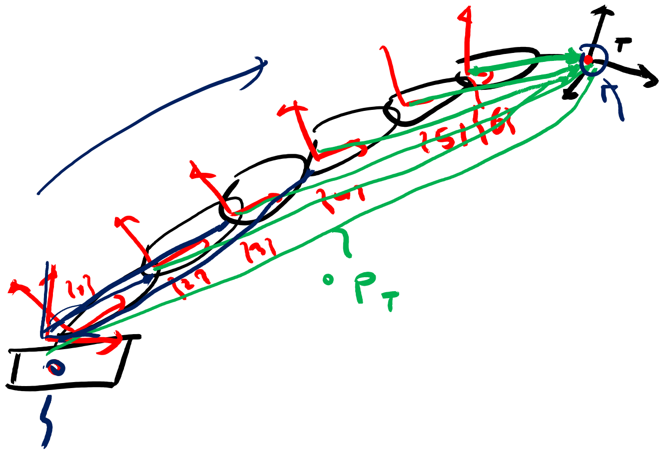
- The homogeneous transform matrix provides a complete description of the linear and angular position relationship between adjacent links.

- These descriptions may be combined together to describe the position of a link relative to the robot base frame {0}.

$${}^{o}_{i}T = {}^{o}_{1}T\,{}^{1}_{2}T \cdots {}^{i-1}_{i}T$$

- A similar description of the linear and angular velocities between adjacent links as well as the base frame would also be useful.

Position Propagation



${}^4P_T$

${}^5P_T$

${}^6P_T$

$$ {}^0_T T = {}^0_1 T \; {}^1_2 T \; {}^2_3 T \; {}^3_4 T \; {}^4_5 T \; {}^5_6 T \; {}^6_T T $$

Motion of the Link of a Robot

- In considering the motion of a robot link we will always use link frame {0} as the reference frame (Computed AND Represented). However any frame can be used as the reference (represented) frame including the link's own frame (i)

Where: v_i - is the linear velocity of the origin of link frame (i) with respect to frame {0} (Computed AND Represented)

ω_i - is the angular velocity of the origin of link frame (i) with respect to frame {0} (Computed AND Represented)

- Expressing the velocity of a frame {i} (associated with link i) relative to the robot base (frame {0}) using our previous notation is defined as follows:

$$v_i \equiv {}^0\left[{}^0V_i\right] = \left[{}^0V_i\right]$$

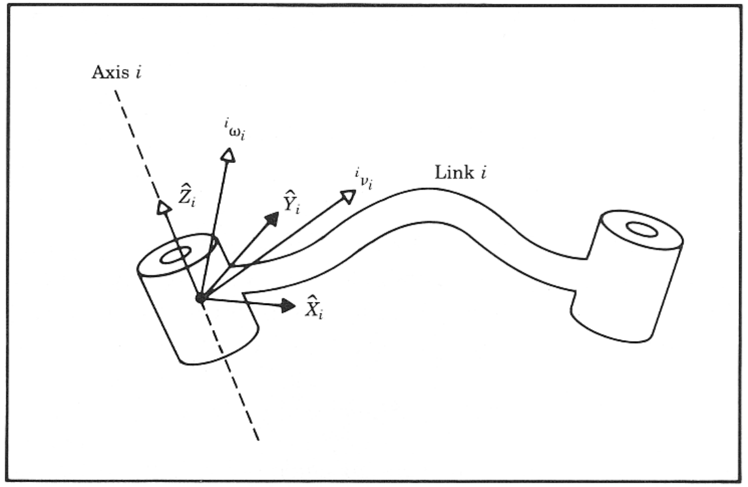
$$\omega_i \equiv {}^0\left[{}^0\Omega_i\right] = \left[{}^0\Omega_i\right]$$

Velocities - Frame & Notation

- The velocities differentiate (computed) relative to the base frame $\{0\}$ are often represented relative to other frames $\{k\}$. The following notation is used for this conditions

$$^{k}v_i \equiv {}^{k}\left[{}^{0}V_i\right] = {}^{k}_{0}R\left[{}^{0}V_i\right] = {}^{k}_{0}R \cdot v_i$$

$$^{k}\omega_i \equiv {}^{k}\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R\left[{}^{0}\Omega_i\right] = {}^{k}_{0}R \cdot \omega_i$$



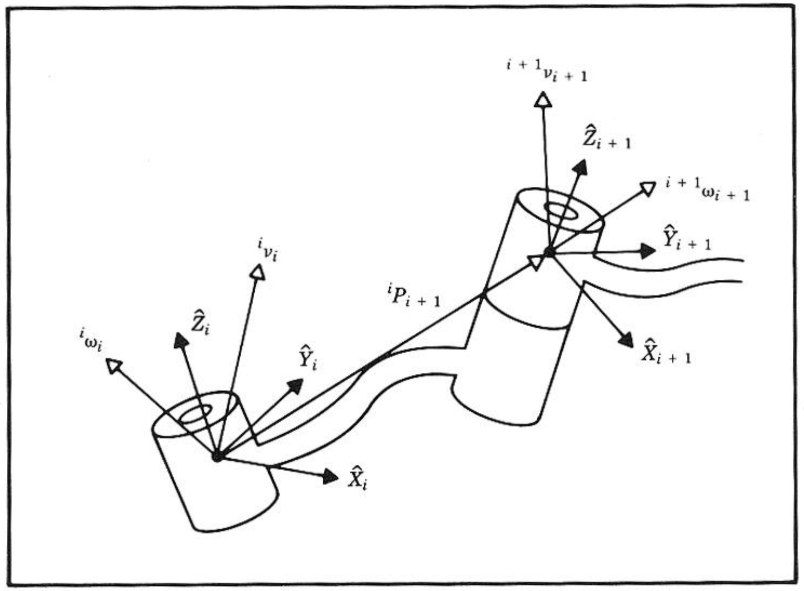
Velocity Propagation

- ***Given:*** A manipulator - A chain of rigid bodies each one capable of moving relative to its neighbor

- ***Problem:*** Calculate the linear and angular velocities of the link of a robot

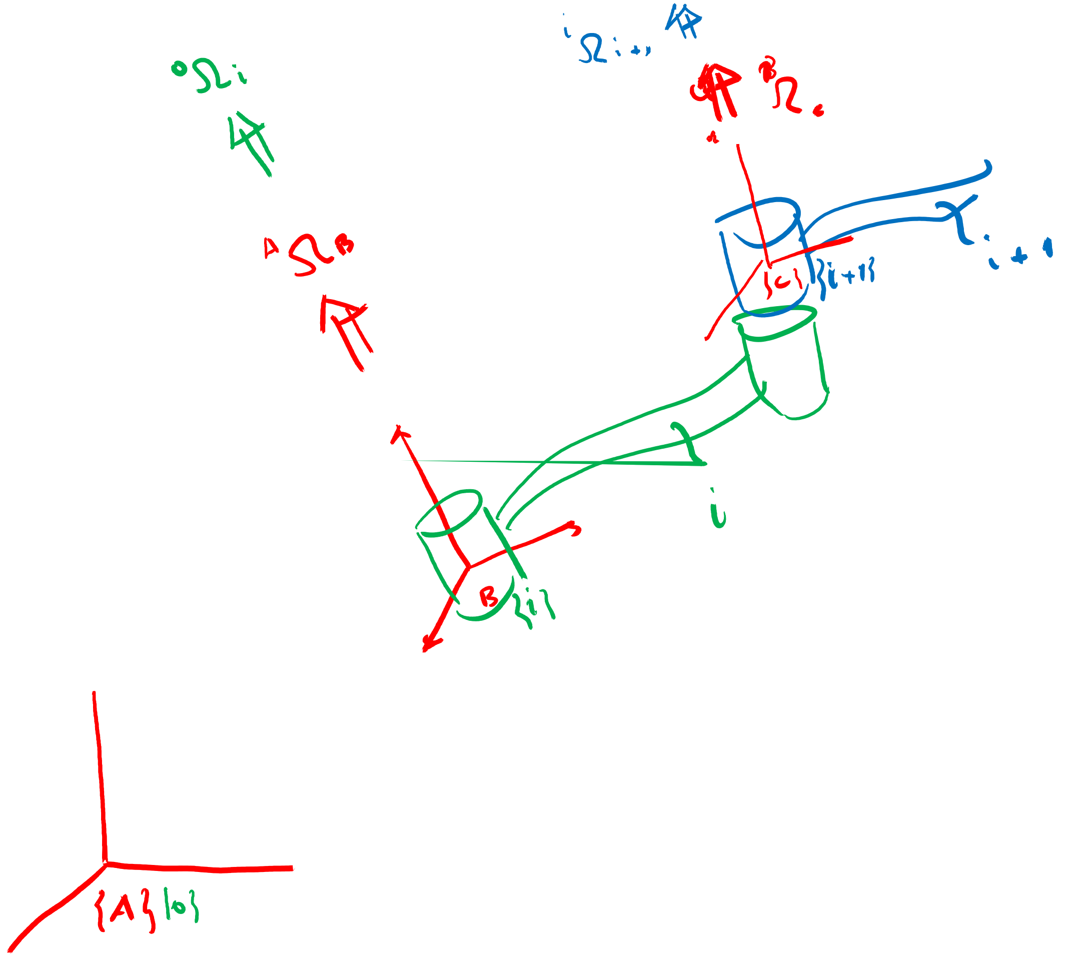
- ***Solution (Concept):*** Due to the robot structure (serial mechanism) we can **compute the velocities** of each link in order **starting from the base**.

  The velocity of link $i+1$ will be that of link i , plus whatever new velocity components were added by joint $i+1$



Angular Velocity Propagation

Velocity of Adjacent Links - Angular Velocity 0/5



$$\left\{ \begin{array}{l} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{array} \right.$$

Velocity of Adjacent Links - Angular Velocity 1/5

- From the relationship developed previously

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

- we can reassign link names to calculate the velocity of any link i relative to the base frame {0}

$$ \begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases} $$

$$ {}^{0}\Omega_{i+1} = {}^{0}\Omega_{i} + {}^{0}_{i}R\,{}^{i}\Omega_{i+1} $$

- We can convert the frame of reference from the base {0} to frame {i+1} by pre-multiplying both sides of the equation by ${}^{i+1}_{0}R$,we can convert the frame of reference for the base {0} to frame {i+1}

Velocity of Adjacent Links - Angular Velocity 2/5

$$ {}^{i+1}_{\ 0}R\, {}^{0}\Omega_{i+1} = {}^{i+1}_{\ 0}R\, {}^{0}\Omega_{i} + {}^{i+1}_{\ 0}R\, {}^{0}_{i}R\, {}^{i}\Omega_{i+1} $$

- Using the recently defined notation, we have

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}\omega_{i} + {}^{i+1}_{\ i}R\, {}^{i}\Omega_{i+1} $$

${}^{i+1}\omega_{i+1}$ - Angular velocity of frame {i+1} measured relative to the robot base, and expressed in frame {i+1}

Recall the car example ${}^{c}\left[{}^{w}V_c\right] = {}^{c}v_c$

${}^{i+1}\omega_{i}$ - Angular velocity of frame {i} measured relative to the robot base, and expressed in frame {i+1}

${}^{i+1}_{\ i}R\, {}^{i}\Omega_{i+1}$ - Angular velocity of frame {i+1} measured relative to frame {i} and expressed in frame {i+1}

Velocity of Adjacent Links - Angular Velocity 3/5

$$ {}^{i+1}\omega_{i+1} = \boxed{{}^{i+1}\omega_i} + {}^{i+1}_{\;\;i}R\, {}^{i}\Omega_{i+1} $$

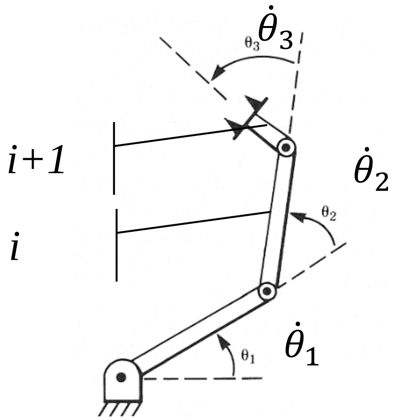
- Angular velocity of frame $\{i\}$ measured relative to the robot base, expressed in frame $\{i+1\}$

$$ {}^{i+1}\omega_i = {}^{i+1}_{\;\;i}R\, {}^{i}\omega_i $$

Velocity of Adjacent Links - Angular Velocity 4/5

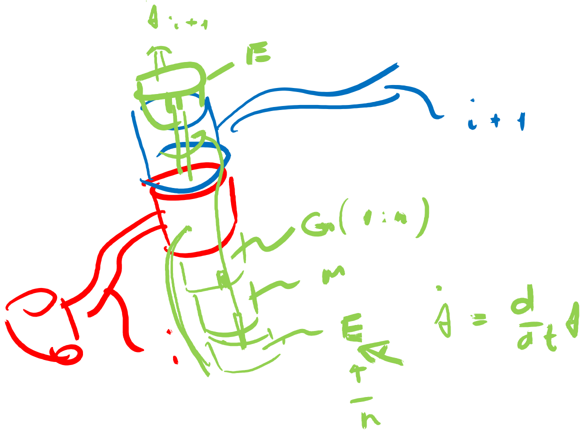
$$ {}^{i+1}\omega_{i+1} = {}^{i+1}\omega_i + \boxed{{}^{i+1}_{\ \ i}R\,{}^{i}\Omega_{i+1}} $$

- Angular velocity of frame {*i+1*} measured (differentiate) in frame {*i*} and represented (expressed) in frame {*i*+1}
- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (***angular velocity***) or prismatic joint (Linear velocity).
- Based on the frame attachment convention in which we assign the Z axis pointing along the *i+1* joint axis such that the two are coincide (rotations of a link is preformed only along its Z- axis) we can rewrite this term as follows:



$$ {}^{i+1}_{\ \ i}R\,{}^{i}\Omega_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_{i+1} \end{bmatrix} $$

E



Velocity of Adjacent Links - Angular Velocity 5/5

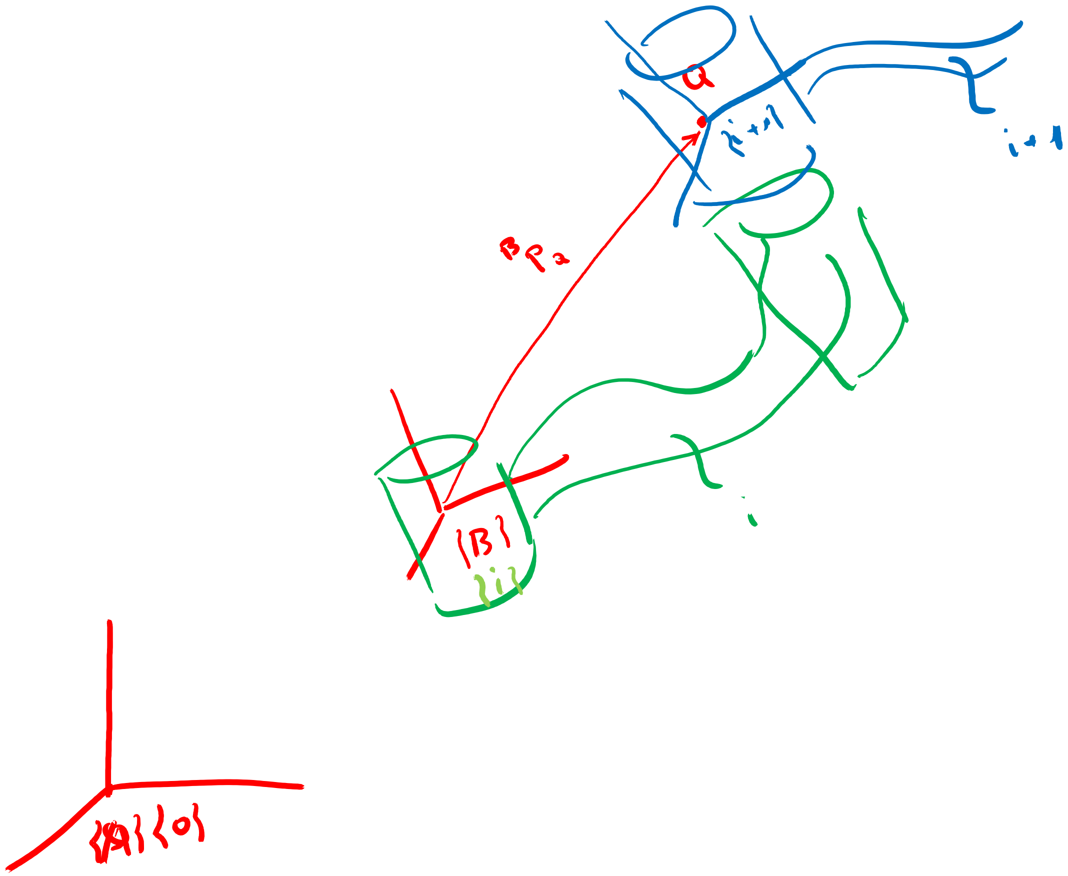
- The result is a ***recursive equation*** that shows the angular velocity of one link in terms of the angular velocity of the previous link plus the relative motion of the two links.

$$\boxed{{}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}}$$

- Since the term ${}^{i+1}\omega_{i+1}$ depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.

Linear Velocity Propagation

Velocity of Adjacent Links - Linear Velocity 0/6



$$\begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases}$$

Velocity of Adjacent Links - Linear Velocity 1/6

- Simultaneous Linear and Rotational Velocity

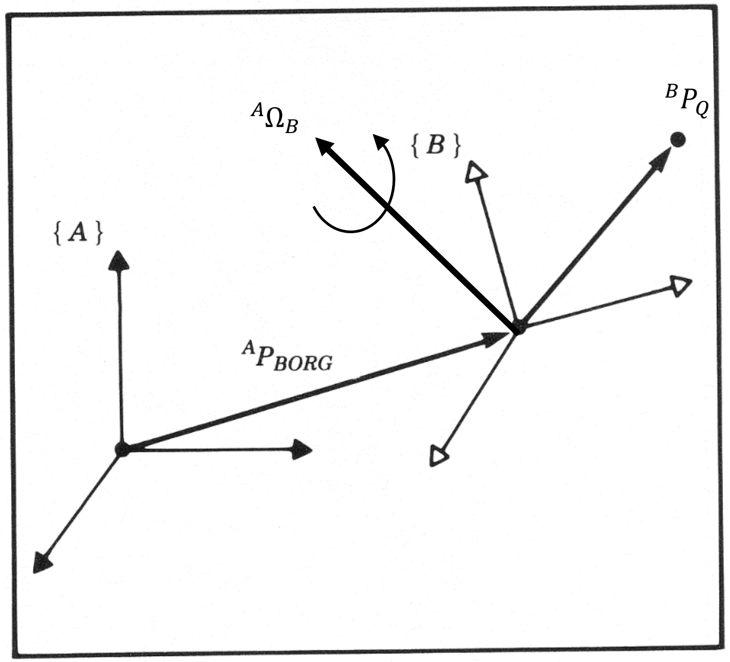
- The derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame

- Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



Velocity of Adjacent Links - Linear Velocity 2/6

- From the relationship developed previously (matrix form)

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

- we re-assign link frames for adjacent links (***i*** and ***i*** +1) with the velocity computed relative to the robot base frame {0}

$$ \begin{cases} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{cases} $$

$$ {}^{0}V_{i+1} = {}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{0}V_{i} + {}^{0}_{i}R\,{}^{i}V_{i+1} $$

- We can convert the frame of reference from frame {0} to frame {i+1} by pre-multiplying both sides of the equation by ${}^{i+1}_{0}R$

Velocity of Adjacent Links - Linear Velocity 3/6

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = {}^{i+1}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{i+1}_{0}R\,{}^{0}V_{i} + \boxed{{}^{i+1}_{0}R\,{}^{0}_{i}R\,{}^{i}V_{i+1}} $$

- Which simplifies to

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = \boxed{{}^{i+1}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\left({}^{0}_{i}R\,{}^{i}P_{i+1}\right) + {}^{i+1}_{0}R\,{}^{0}V_{i}} + \boxed{{}^{i+1}_{i}R\,{}^{i}V_{i+1}} $$

- Factoring out ${}^{i+1}_{i}R$ from the blue term

$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = \boxed{{}^{i+1}_{i}R\left({}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i}_{0}R\,{}^{0}V_{i}\right)} + \boxed{{}^{i+1}_{i}R\,{}^{i}V_{i+1}} $$

Velocity of Adjacent Links - Linear Velocity 4/6

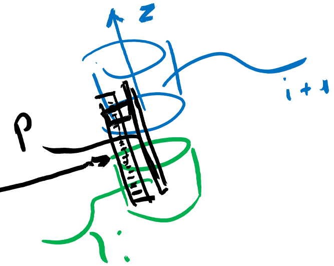
$$ {}^{i+1}_{0}R\,{}^{0}V_{i+1} = {}^{i+1}_{i}R\left({}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R\,{}^{i}P_{i+1} + {}^{i}_{0}R\,{}^{0}V_{i}\right) + \boxed{{}^{i+1}_{i}R\,{}^{i}V_{i+1}} $$

${}^{i+1}_{i}R\,{}^{i}V_{i+1}$ - Linear velocity of frame {$i+1$} measured relative to frame {i} and expressed in frame {$i+1$}

- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (angular velocity) or prismatic joint (Linear velocity).

- Based on the frame attachment convention in which we assign the Z axis pointing along the $i+1$ joint axis such that the two are coincide (translation of a link is preformed only along its Z- axis) we can rewrite this term as follows:

$$ {}^{i+1}_{i}R\,{}^{i}V_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} $$



$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}\Omega_B \times} {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}_{B}\dot{R}_{\Omega}(}{}^{A}_{B}R\,{}^{B}P_Q) $$

Angular Velocity - Matrix & Vector Forms

	Matrix Form	Vector Form
Definition	${}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	${}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$
Multiply by Constant	$k\ \left[{}^{A}_{B}\dot{R}_{\Omega}\right]$	$k\ \left[{}^{A}\Omega_B\right]$
Multiply by Vector	$\left[{}^{A}_{B}\dot{R}_{\Omega}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	${}^{A}\Omega_B \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \omega \times r$
Multiply by Matrix	$\left[\ {}^{s}_{t}R\ \right] \left[\ {}^{A}_{B}\dot{R}_{\Omega}\ \right] \left[\ {}^{s}_{t}R\ \right]^T$	$\left[\ {}^{s}_{t}R\ \right] \left[\ {}^{A}\Omega_B\ \right]$

Velocity of Adjacent Links - Linear Velocity 5/6

$$\boxed{{}^{i+1}_{0}R\,{}^{0}V_{i+1}} = {}^{i+1}_{i}R\left(\boxed{{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R}\,{}^{i}P_{i+1} + \boxed{{}^{i}_{0}R\,{}^{0}V_{i}}\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

$$\boxed{{}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{0}_{i}R} = {}^{i}_{0}R\,{}^{0}_{i}\dot{R}_{\Omega}\,{}^{i}_{0}R^{T} = {}^{i}_{0}R\,{}^{0}\Omega_{i} = {}^{i}_{0}R\,\omega_{i} = {}^{i}\omega_{i}$$

Multiply by Matrix

Definition

$$\boxed{{}^{i+1}_{0}R\,{}^{0}V_{i+1}} = {}^{i+1}v_{i+1}$$

$$\boxed{{}^{i}_{0}R\,{}^{0}V_{i}} = {}^{i}v_{i}$$

Definition

Definition

Velocity of Adjacent Links - Linear Velocity 6/6

- The result is a ***recursive equation*** that shows the linear velocity of one link in terms of the previous link plus the relative motion of the two links.

$$\boxed{{}^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega_i \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}}$$

- Since the term ${}^{i+1}v_{i+1}$ depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.

Velocity of Adjacent Links - Summary

- Angular Velocity

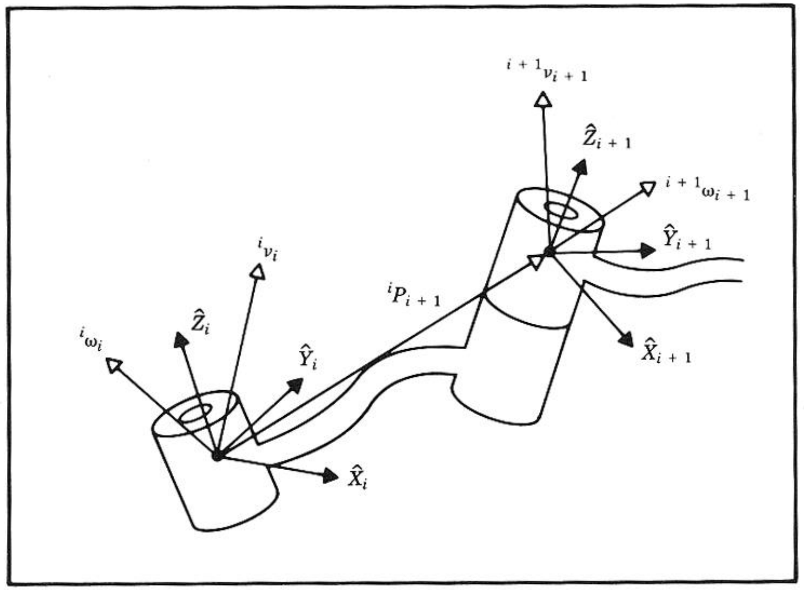
0 - Prismatic Joint

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \cancel{\dot{\theta}_{i+1}} \end{bmatrix}$$

- Linear Velocity

0 - Revolute Joint

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \begin{bmatrix} 0 \\ 0 \\ \cancel{\dot{d}_{i+1}} \end{bmatrix}$$

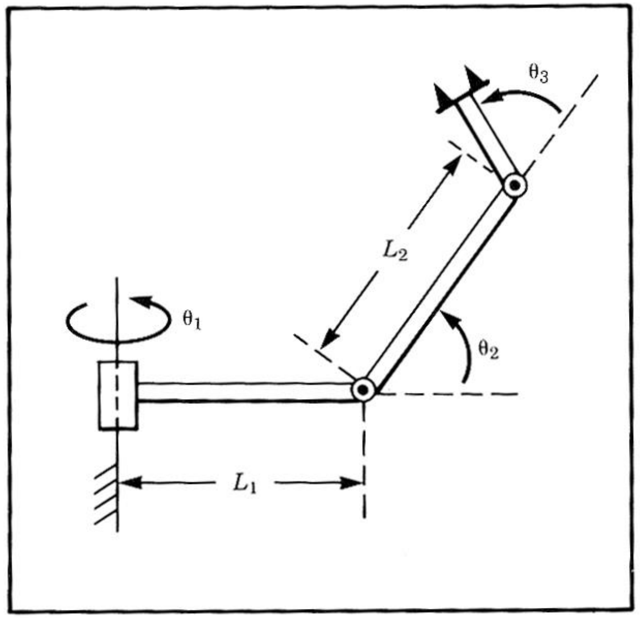


3R – Example

Analytical Approach

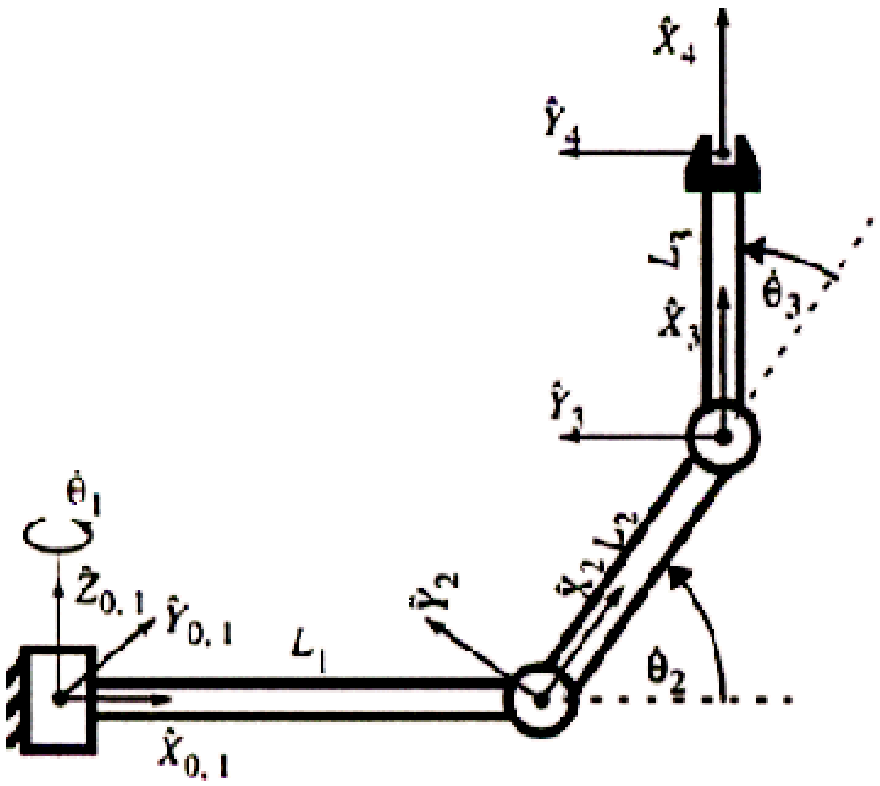
Angular and Linear Velocities - 3R Robot - Example

- For the manipulator shown in the figure, compute the angular and linear velocity of the "tool" frame relative to the base frame expressed in the "tool" frame (that is, calculate ${}^{4}\omega_{4}$ and ${}^{4}v_{4}$).



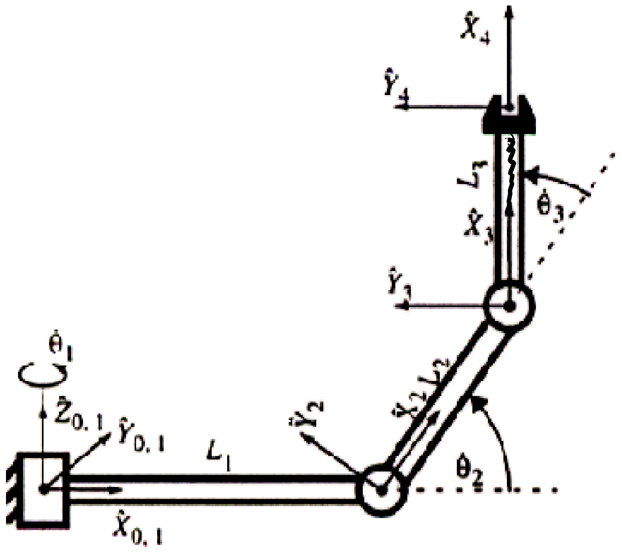
Angular and Linear Velocities - 3R Robot - Example

- Frame attachment



Angular and Linear Velocities - 3R Robot - Example

- DH Parameters



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90	L1	0	θ_2
3	0	L2	0	θ_3
4	0	L3	0	0

Angular and Linear Velocities - 3R Robot - Example

- From the DH parameter table, we can specify the homogeneous transform matrix for each adjacent link pair:

$$ {}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- The homogeneous transform matrix from frame 0 to each one of the joints (1,2,3,4)

$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}T = {}^{0}_{1}T\,{}^{1}_{2}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c2 & -c1s1 & s1 & L1c1 \\ s1c2 & -s1s2 & -c1 & L1s1 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c23 & -c1s23 & s1 & c1(L1 + L2c2) \\ s1c23 & -s1s23 & -c1 & s1(L1 + L2c2) \\ s23 & c23 & 0 & L2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{4}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c23 & -c1s23 & s1 & c1(L1 + L3c23 + L2c2) \\ s1c23 & -s1s23 & -c1 & s1(L1 + L3c23 + L2c2) \\ s23 & c23 & 0 & L3s23 + L2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- Compute the angular velocity of the end effector frame relative to the base frame expressed at the end effector frame.

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,^{i}\omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

- For ***i=0***

$$^{1}\omega_1 = {}^{1}_{0}R\,^{0}\omega_0 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=1***

$$^{2}\omega_2 = {}^{2}_{1}R\,{}^{1}\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

- For ***i=2***

$$^{3}\omega_3 = {}^{3}_{2}R\,{}^{2}\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

- For ***i=3***

$$^{4}\omega_4 = {}^{4}_{3}R\,{}^{3}\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

- Note $\quad {}^{3}\omega_3 = {}^{4}\omega_4$

Angular and Linear Velocities - 3R Robot - Example

- Compute the linear velocity of the end effector frame relative to the base frame expressed at the end effector frame.

- Note that the term involving the prismatic joint has been dropped from the equation (it is equal to zero).

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_i\right) + \cancelto{0}{\begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}}$$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=0***

$$ {}^{1}v_{1} = {}^{1}_{0}R\{{}^{0}\omega_{0} \times {}^{0}P_{1} + {}^{0}v_{0}\} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

- For ***i=1***

$$ {}^{2}v_{2} = {}^{2}_{1}R\{{}^{1}\omega_{1} \times {}^{1}P_{2} + {}^{1}v_{1}\} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} \times \begin{bmatrix} L1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ -L_{1}\dot{\theta}_{1} \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- For $\boldsymbol{i=3}$

$$ {}^{3}v_{3} = {}^{3}_{2}R\{{}^{2}\omega_{2} \times {}^{2}P_{3} + {}^{2}v_{2}\} = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c30 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -L1\dot{\theta}_1 \end{bmatrix} \right\} $$

$$ = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ L2\dot{\theta}_1 \\ -L2c2\dot{\theta}_1 - L1\dot{\theta}_1 \end{bmatrix} \right\} = \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

- For ***i=4***

$$^4v_4 = {}^4_3R\{{}^3\omega_3 \times {}^3P_4 + {}^3v_3\}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \times \begin{bmatrix} L3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- Note that the linear and angular velocities (${}^4\omega_4, {}^4v_4$) of the end effector where differentiate (measured) in frame {0} however represented (expressed) in frame {4}

- In the car example: Observer sitting in the “Car” ${}^C\left[{}^W V_C\right]$

  Observer sitting in the “World” ${}^W\left[{}^W V_C\right]$

$$ {}^k v_i \equiv {}^k\left[{}^0 V_i\right] = {}^k_0 R \left[{}^0 V_i\right] = {}^k_0 R \cdot v_i $$

$$ {}^k \omega_i \equiv {}^k\left[{}^0 \Omega_i\right] = {}^k_0 R \left[{}^0 \Omega_i\right] = {}^k_0 R \cdot \omega_i $$

Solve for v_4 and ω_4 by multiply both side of the questions from the left by ${}^4_0 R^{-1}$

$$ {}^4 v_4 = {}^4_0 R \cdot v_4 $$

$$ {}^4 \omega_4 = {}^4_0 R \cdot \omega_4 $$

Angular and Linear Velocities - 3R Robot - Example

- Multiply both sides of the equation by the inverse transformation matrix, we finally get the linear and angular velocities expressed and measured in the stationary frame {0}

$$v_4 = {}^4_0R^{-1} \cdot {}^4v_4 = {}^4_0R^T \cdot {}^4v_4 = {}^0_4R \cdot {}^4v_4$$

$$\omega_4 = {}^4_0R^{-1} \cdot {}^4\omega_4 = {}^4_0R^T \cdot {}^4\omega_4 = {}^0_4R \cdot {}^4\omega_4$$

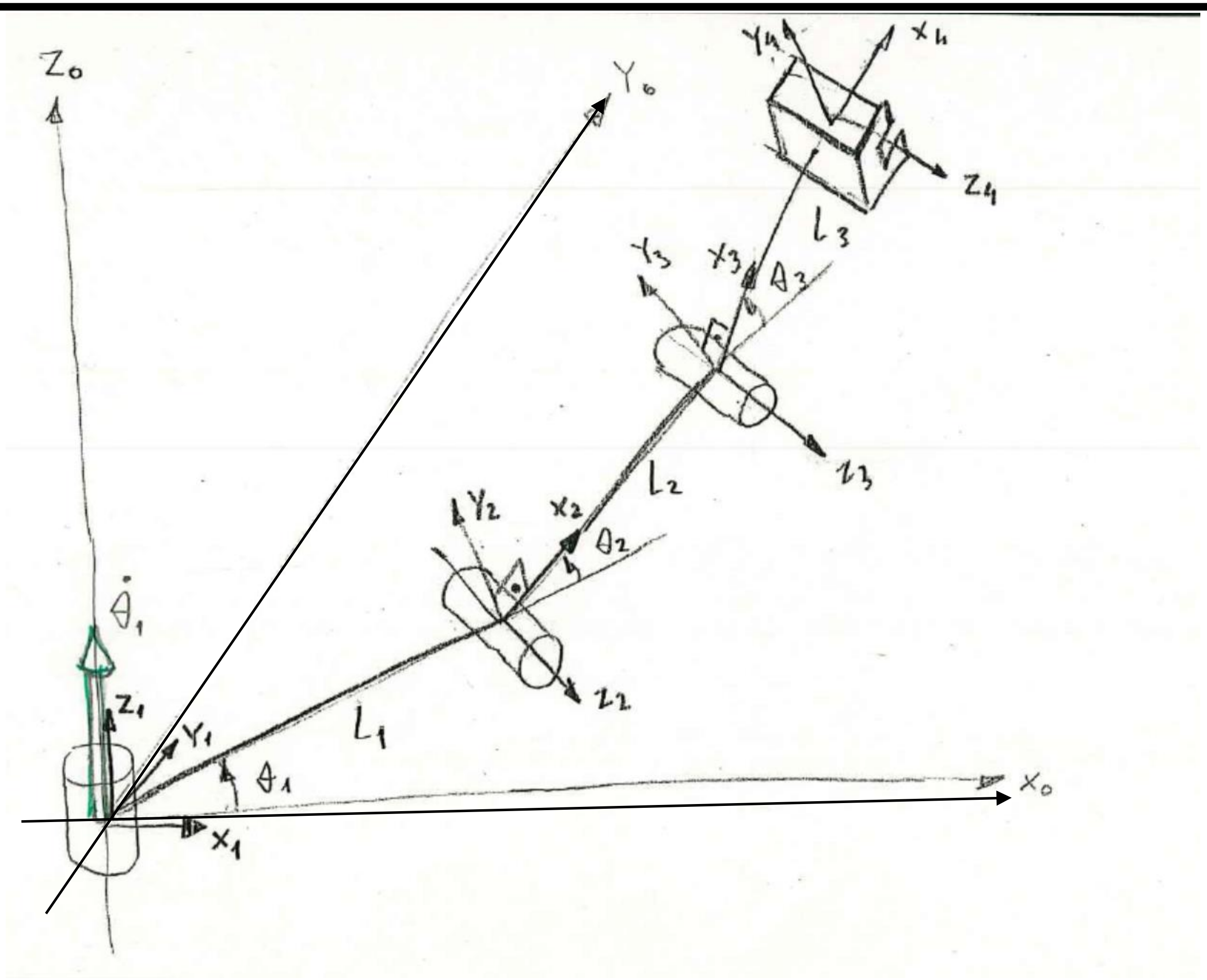
$${}^0_4T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T$$

3R – Example

Analytical Approach – Graphical Interpretation

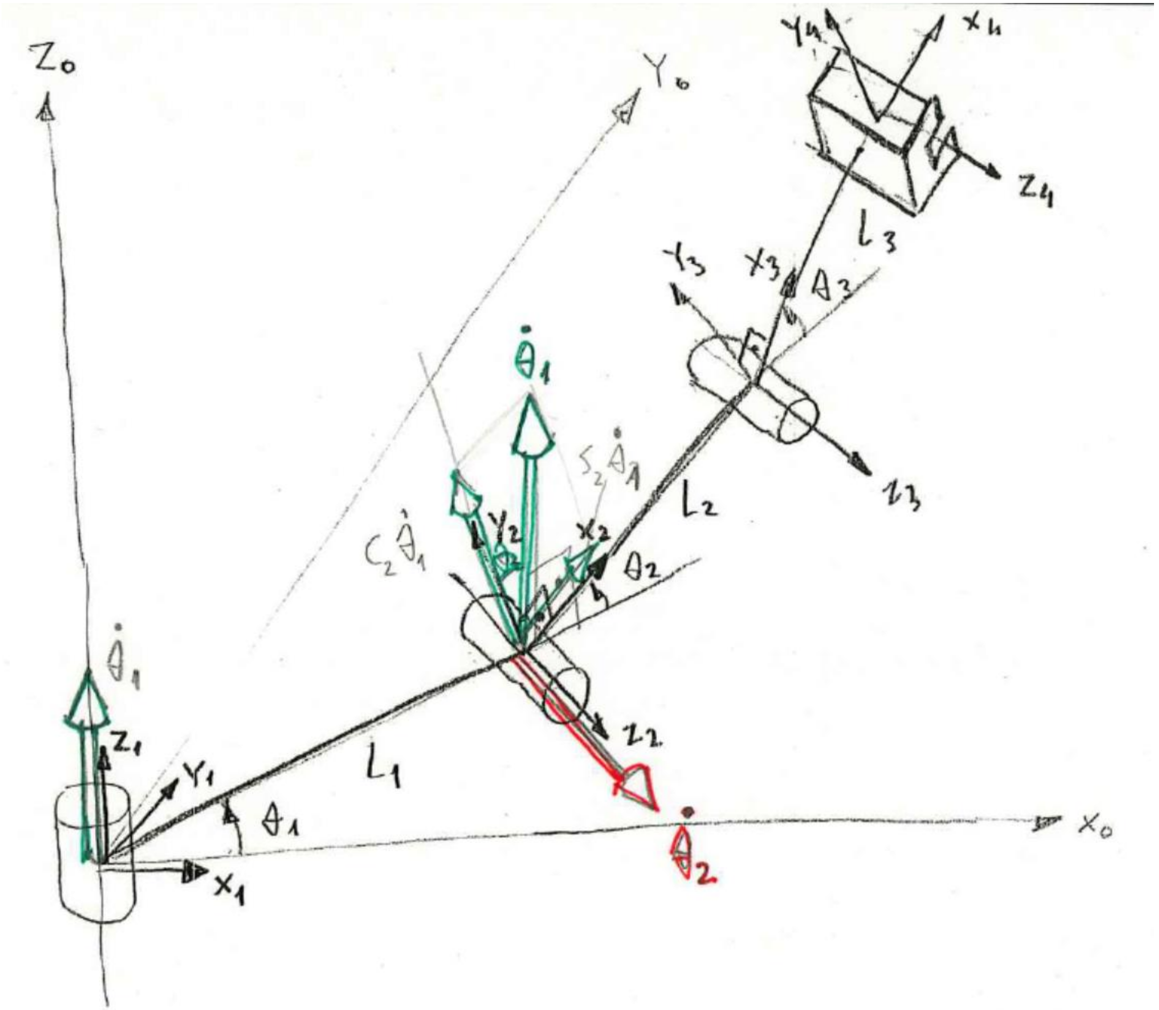
Angular and Linear Velocities - 3R Robot - Example

$$^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$



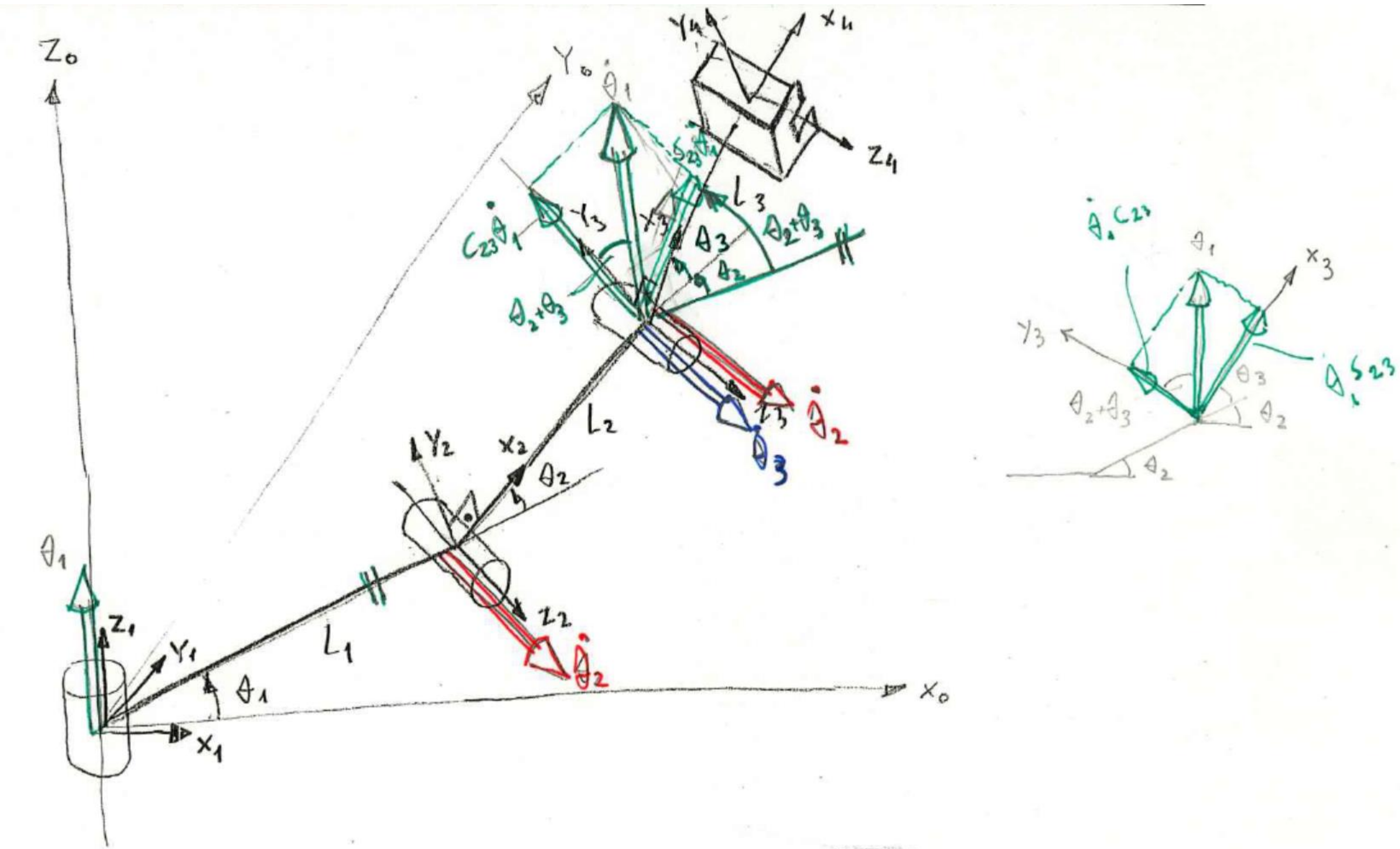
Angular and Linear Velocities - 3R Robot - Example

$$
{}^{2}\omega_{2} = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$



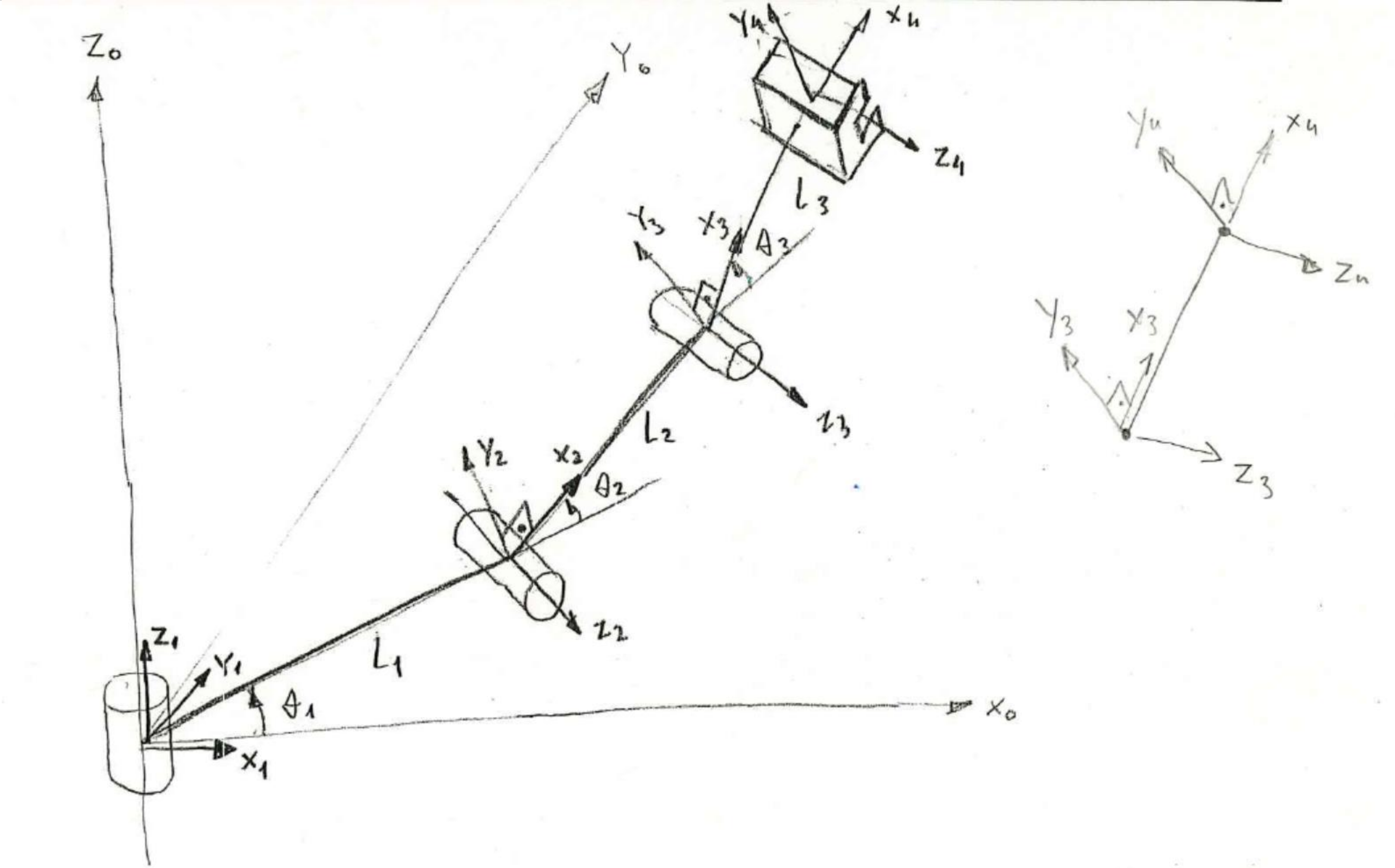
Angular and Linear Velocities - 3R Robot - Example

$$^{3}\omega_3 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$



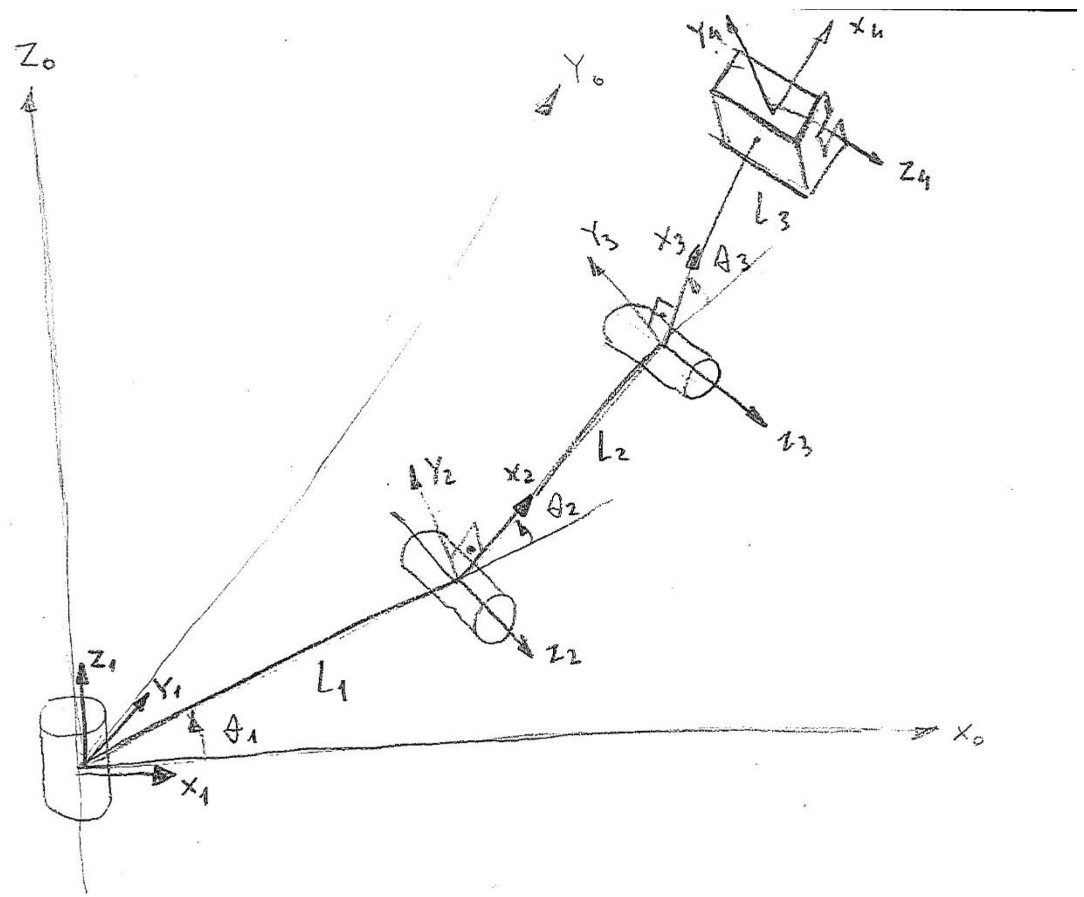
Angular and Linear Velocities - 3R Robot - Example

$$^3\omega_3 = {}^4\omega_4$$



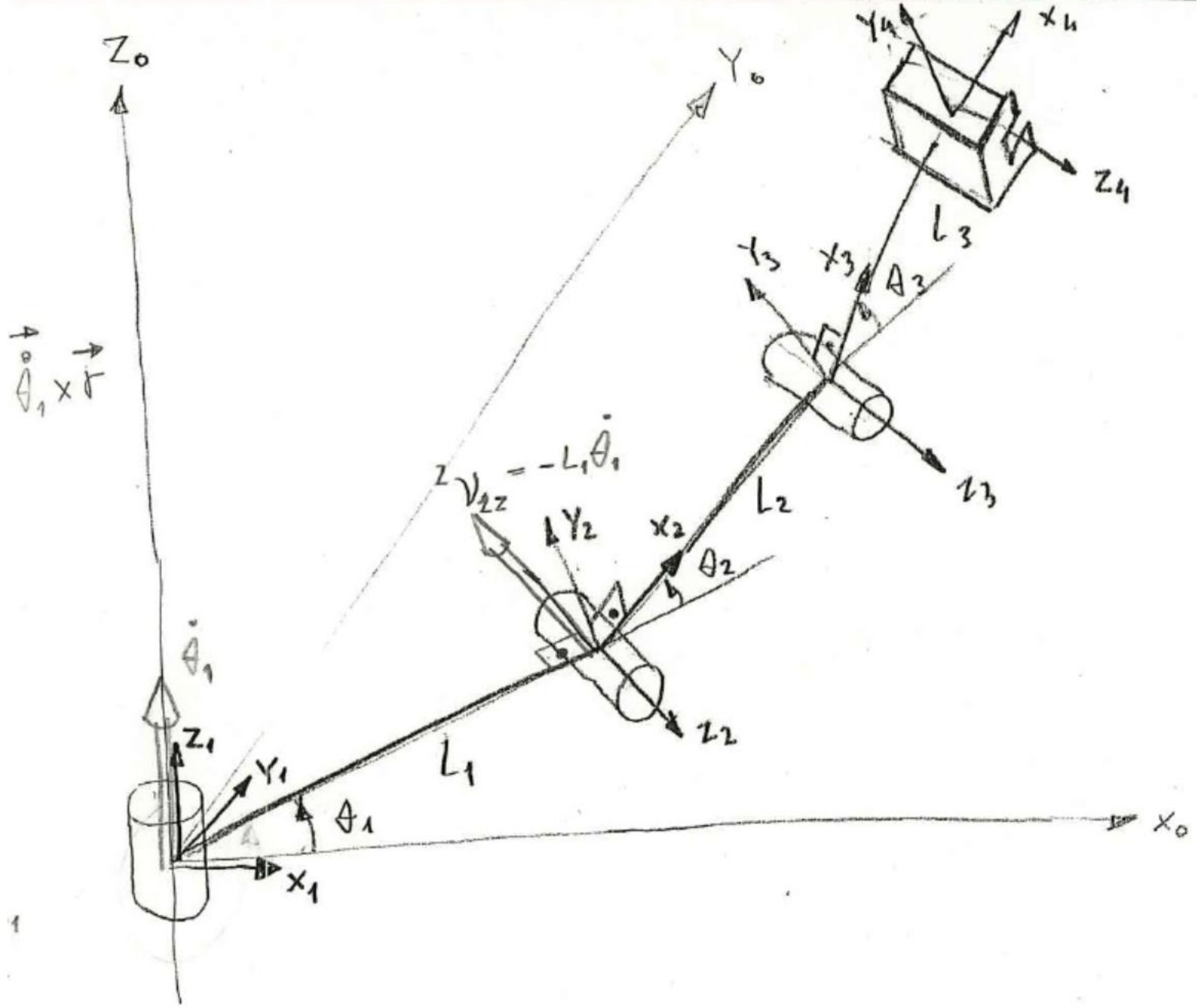
Angular and Linear Velocities - 3R Robot - Example

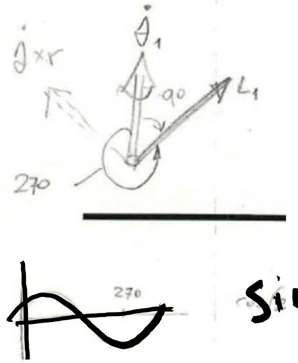
$$^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



Angular and Linear Velocities - 3R Robot - Example

$$^2v_2 = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}$$

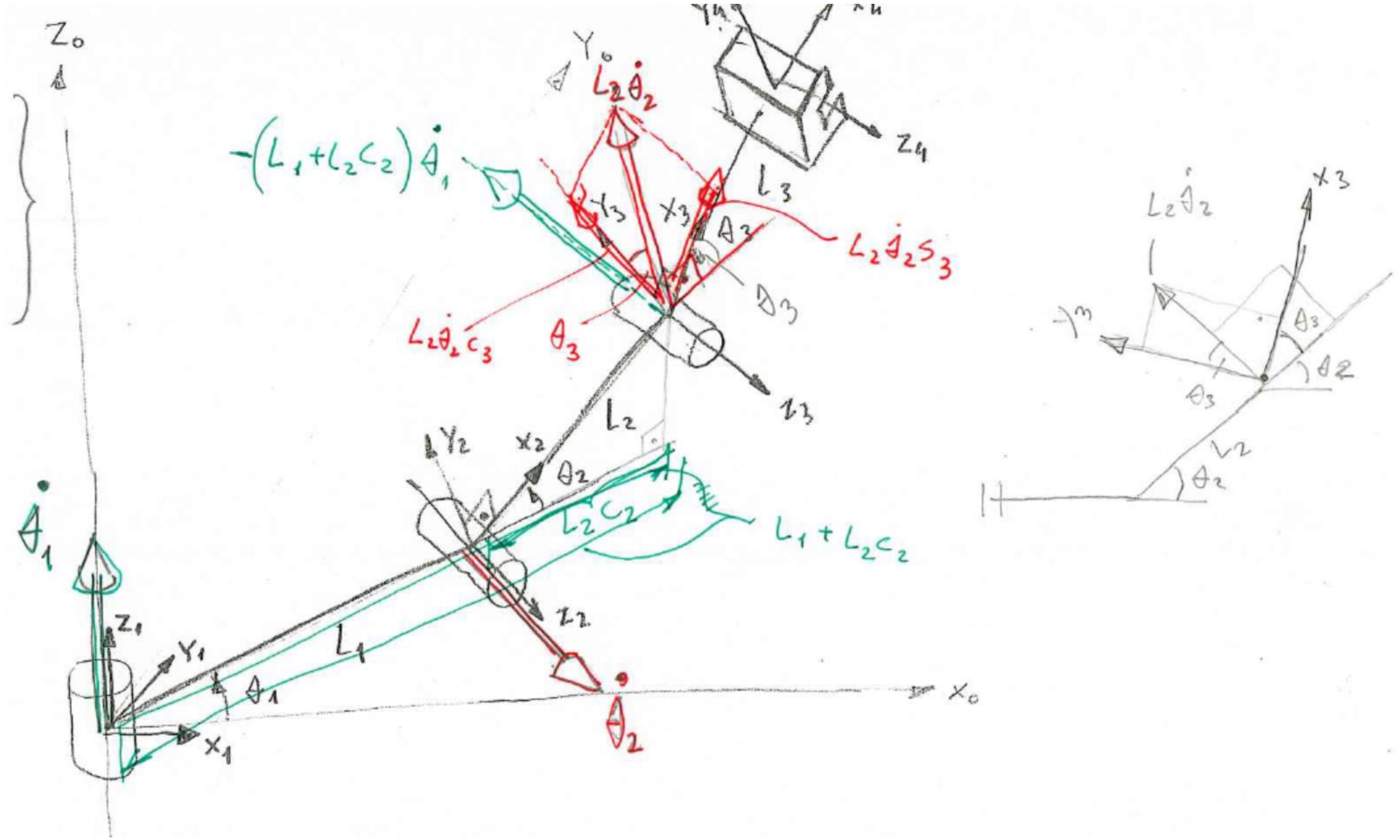




$\sin(270°) = -1$

Angular and Linear Velocities - 3R Robot - Example

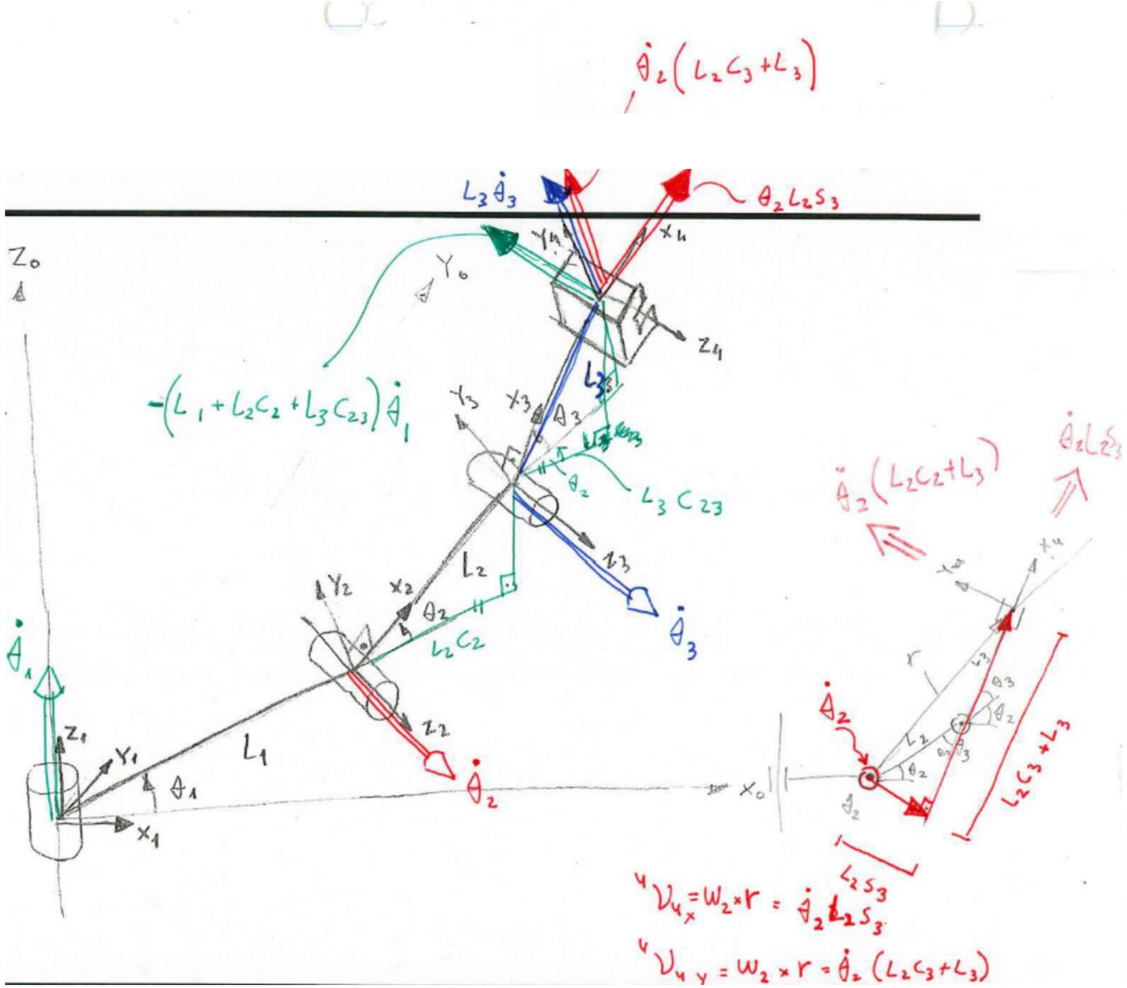
$$
{}^{3}v_3 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix}
$$



UCLA

Angular and Linear Velocities - 3R Robot - Example

$$
{}^{4}v_{4} = \begin{bmatrix} \boxed{L2s3\dot{\theta}_2} \\ \boxed{(L2c3 + L3)\dot{\theta}_2} + \boxed{L3\dot{\theta}_3} \\ \boxed{-(L1 + L2c2 + L3c23)\dot{\theta}_1} \end{bmatrix}
$$



Angular and Linear Velocities - 3R Robot - Example