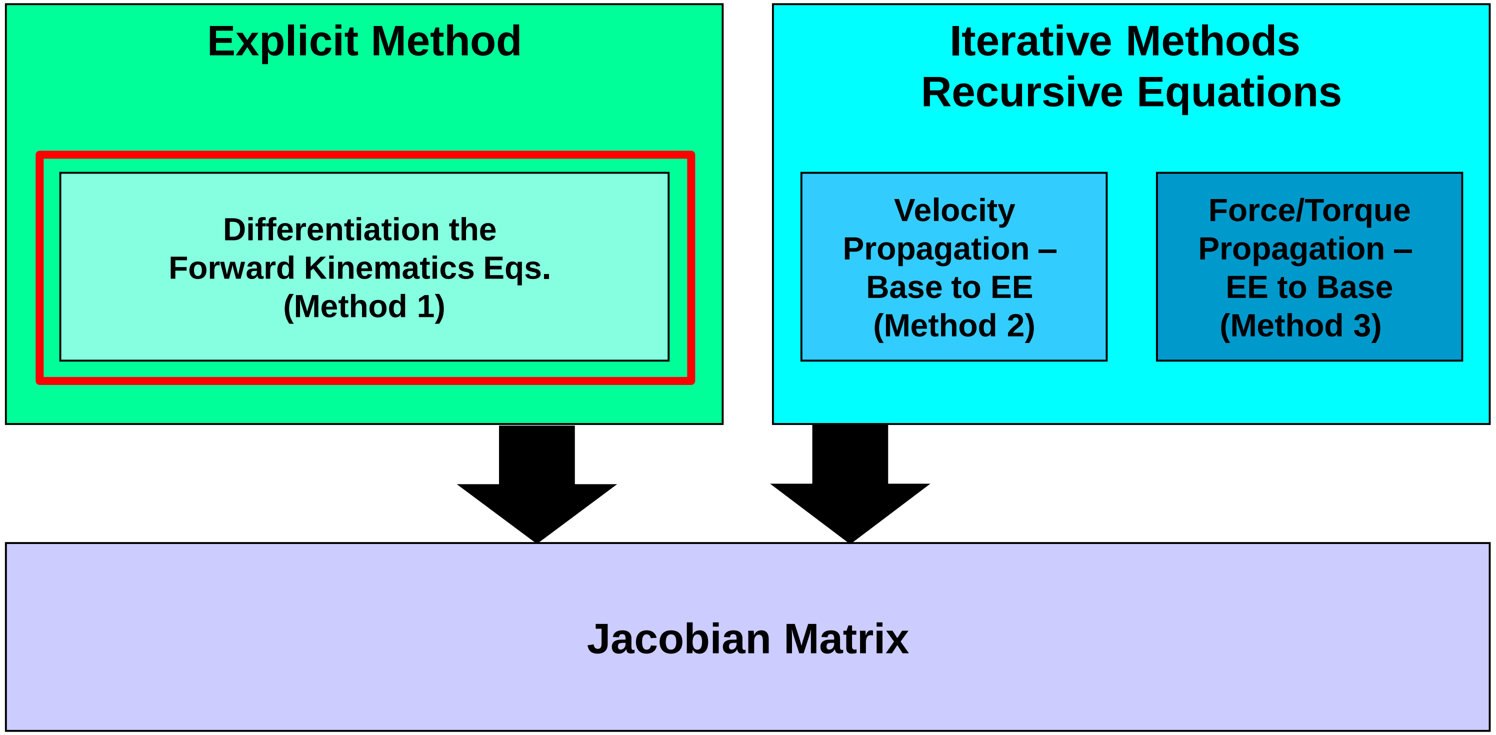
Jacobian Explicit Method - Differentiation the Forward Kinematics Eqs. (Method No. 1)

Jacobian Matrix - Derivation Methods



Jacobian – Explicit Form – Overview

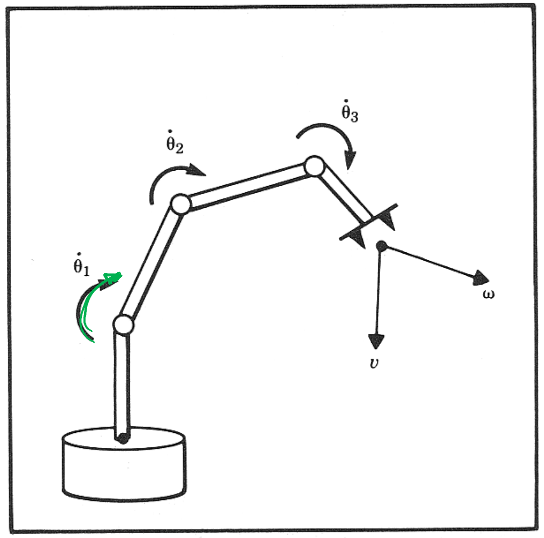
$$\begin{bmatrix} {}^0v \\ {}^0\omega \end{bmatrix} = \dot{X} = {}^0J(\theta)\dot{\Theta}$$

$$\begin{bmatrix} {}^0v_x \\ {}^0v_y \\ {}^0v_z \\ {}^0\omega_x \\ {}^0\omega_y \\ {}^0\omega_z \end{bmatrix} = \begin{bmatrix} J_\upsilon(\underline{\theta}) \\ J_\omega(\underline{\theta}) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 / \dot{d}_3 \\ \cdots \\ \\ \dot{\theta}_N \end{bmatrix}$$

Jacobian Matrix

- The meaning of **<u>each column</u>** (e.g. the first column) of the Jacobian matrix:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} J_{11} & & \\ J_{21} & J_{\upsilon}(\underline{\theta}) & \\ J_{31} & & \\ J_{41} & & \\ J_{51} & J_{\omega}(\underline{\theta}) & \\ J_{61} & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \dot{\theta}_N \end{bmatrix}$$



- The first column maps the contribution of the angular velocity of the first joint to the linear and angular velocities of the end effector along all the axis (x,y,z)

Jacobian Explicit Method

Technique

Jacobian – Explicit Form

i

$J_\upsilon(\underline{\theta})$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

OR

$${}^0_iZ \times ({}^0P_6 - {}^0P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

OR

$${}^0_iZ$$

$J_\omega(\underline{\theta})$

$\rho = 1$ $\quad \rho\, {}^0_iZ \quad$ $\rho = 0$

$${}^0_iT = \left[\begin{array}{ccc|c} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$${}^0_6T = \left[\begin{array}{ccc|c} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

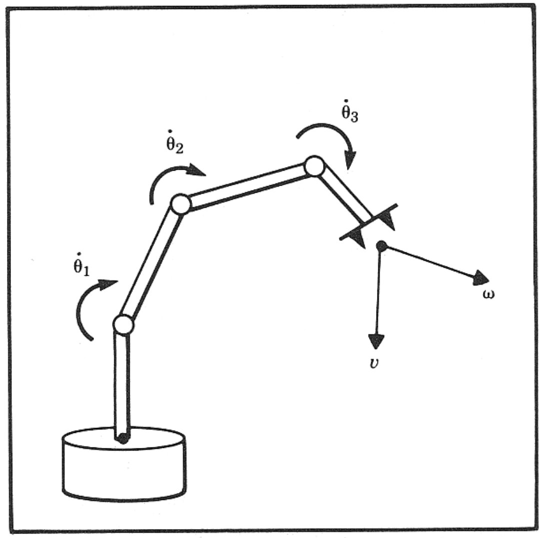
Jacobian Explicit Method – Rational

Intuitive Explanation

Jacobian Matrix

- The meaning of **each column** of the Jacobian matrix:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix}=\begin{bmatrix} J_{\upsilon}(\underline{\theta}) & \cdots & J_{1i} & \cdots \\ & & J_{2i} & \\ & & J_{3i} & \\ J_{\omega}(\underline{\theta}) & \cdots & J_{4i} & \cdots \\ & & J_{5i} & \\ & & J_{6i} & \end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \cdots\\ \dot{\theta}_N\end{bmatrix}$$



- The i'th column maps the contribution of the angular velocity of the i'th joint to the linear and angular velocities of the end effector along all the axis (x,y,z)

Jacobian – Explicit Form

Column i:

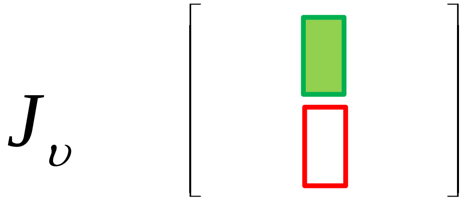
$$\begin{bmatrix} J_{\upsilon}(\underline{\theta}) \\ J_{\omega}(\underline{\theta}) \end{bmatrix}$$

	Revolute Joint	i	Prismatic Joint
$J_{\upsilon}(\underline{\theta})$	$\frac{\partial {}^{0}P_{6}}{\partial \theta_i}$ — OR — ${}^{0}_{i}Z \times ({}^{0}P_{6} - {}^{0}P_{i})$		$\frac{\partial {}^{0}P_{6}}{\partial d_i}$ — OR — ${}^{0}_{i}Z$
$J_{\omega}(\underline{\theta})$	$\rho = 1$	$\rho\, {}^{0}_{i}Z$	$\rho = 0$

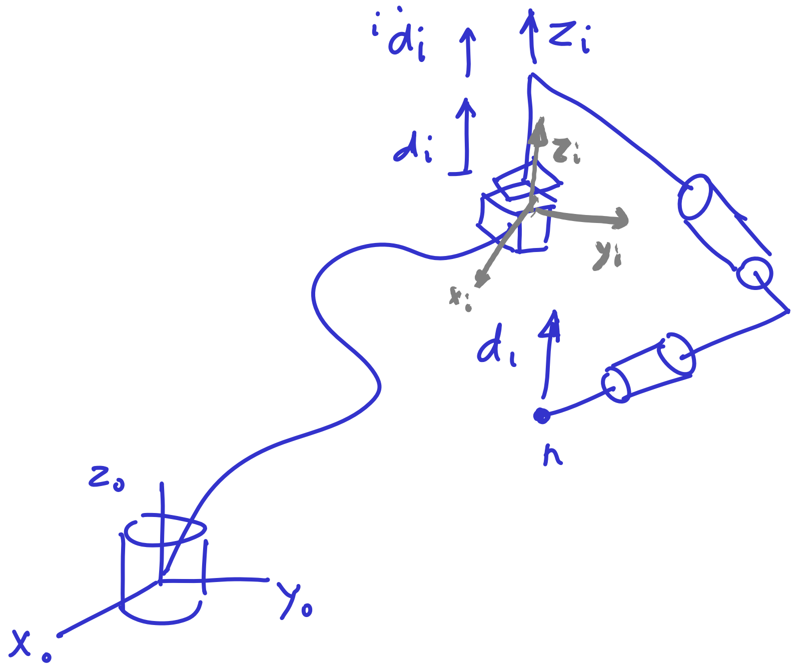
$$ {}^{0}_{i}T = \left[\begin{array}{ccc|c} * & * & {}^{0}_{i}Z_x & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_y & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_z & {}^{0}P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

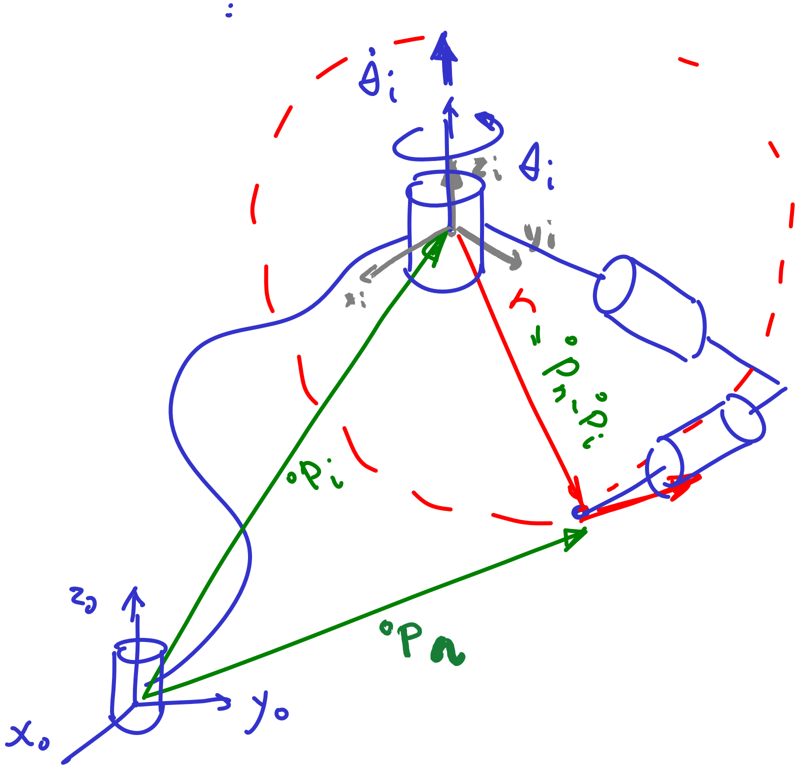
$$ {}^{0}_{6}T = \left[\begin{array}{ccc|c} * & * & * & {}^{0}P_{6x} \\ * & * & * & {}^{0}P_{6y} \\ * & * & * & {}^{0}P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Jacobian – Explicit Form – Linear Velocity
Two Cases



- The i'th column of the Jacobian can be generated by holding all the joints fixed but the i'th and actuating the i'th at a unite velocity





Jacobian – Explicit Form

i

$J_\upsilon(\underline{\theta})$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

OR

$${}^0_iZ \times ({}^0P_6 - {}^0P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

OR

$${}^0_iZ$$

$${}^0_iR\hat{k}\rho_i\dot{\theta}_i$$

$J_\omega(\underline{\theta})$

$\rho = 1$ $\qquad \rho\, {}^0_iZ \qquad$ $\rho = 0$

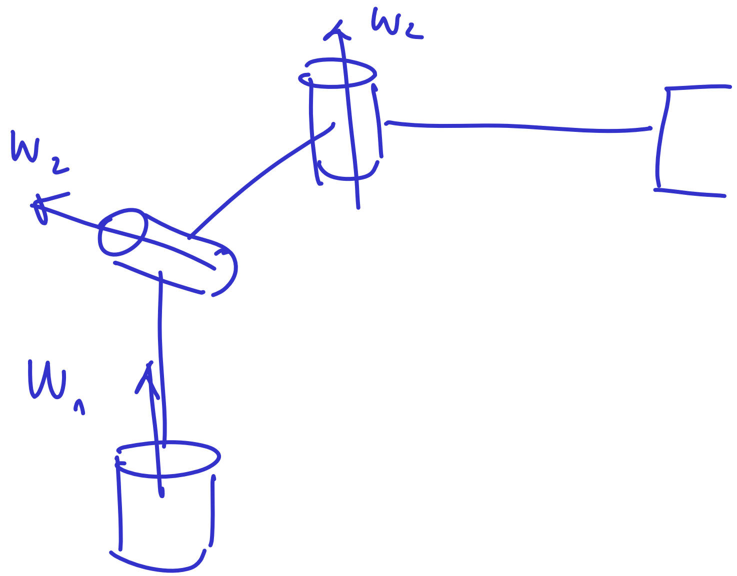
$$ {}^0_iT = \left[\begin{array}{ccc|c} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^0_6T = \left[\begin{array}{ccc|c} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

Jacobian – Explicit Form – Angular Velocity J_ω

$$
\begin{array}{c} J_\upsilon(\underline{\theta}) \\ \\ J_\omega(\underline{\theta}) \end{array}
\left[\begin{array}{ccccc}
 & & & & \\
\hline
\rho_1\, {}_1^0Z & \rho_2\, {}_2^0Z & \cdots & \rho_{n-1}\, {}_{n-1}^{\;\;0}Z & \rho_n\, {}_n^0Z
\end{array}\right]
$$

Jacobian – Explicit Form – Angular Velocity J_{ω}



Jacobian Explicit Method – Rational

Geometrical / Analytical Explanation Based on Velocity Propagation Method

Jacobian – Explicit Form

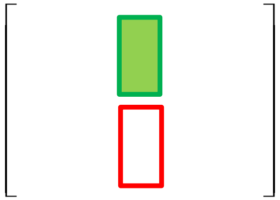
i

$$\begin{bmatrix} J_\upsilon(\underline{\theta}) & & \text{Revolute Joint: } \dfrac{\partial\, {}^0P_6}{\partial \theta_i} \text{ OR } {}^0_iZ \times ({}^0P_6 - {}^0P_i) & & \text{Prismatic Joint: } \dfrac{\partial\, {}^0P_6}{\partial d_i} \text{ OR } {}^0_iZ \\ \hline J_\omega(\underline{\theta}) & & \rho = 1 \quad \rho\, {}^0_iZ & & \rho = 0 \end{bmatrix}$$

$${}^0_iT = \left[\begin{array}{ccc|c} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$${}^0_6T = \left[\begin{array}{ccc|c} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Jacobian – Explicit Form – Linear Velocity J_υ



$$y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$\vdots$$
$$y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)$$

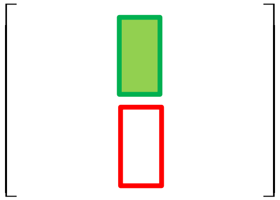
$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6$$
$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6$$
$$\vdots$$
$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

$$Y = F(X)$$

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

Jacobian – Explicit Form – Linear Velocity J_υ



$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

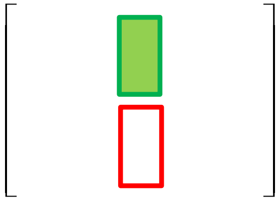
$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

$$^0\dot{P}_{6i} = \sum_i \frac{\partial\, ^0P_6}{\partial \theta_i}\dot{q}_i \qquad \dot{q}_i = \begin{cases}\dot{\theta}_i \\ \dot{d}_i\end{cases}$$

$$J_{vi} = \frac{\partial\, ^0P_6}{\partial \theta_i}$$

Jacobian – Explicit Form – Linear Velocity Derivative

J_υ



- The linear velocity of the end effector is ${}^{0}\dot{P}_n$. By the chain rule for differentiation

$$ {}^{0}\dot{P}_n = \sum_{i}^{n} \frac{\partial\, {}^{0}P_n}{\partial q_i} \dot{q}_i $$

- Where q_i is the generalized notation for both the angle (revolute joint) and displacement (prismatic joint)

$$ q_i = \begin{cases} \theta_i \\ d_i \end{cases} $$

- Thus the i'th column of $J_\upsilon(\underline{\Theta})$ which denoted as $J_{\upsilon i}$ is given by

$$ \frac{\partial\, {}^{0}P_n}{\partial q_i} $$

- This expression is just the linear velocity of the end effector that would result if $q_i = 1$ and all the others $q_j = 0$

Jacobian – Explicit Form

i

$$J_\upsilon(\underline{\theta})$$

Revolute Joint

$$\frac{\partial\, {}^{0}P_6}{\partial \theta_i}$$

OR

$${}^{0}_{i}Z \times ({}^{0}P_6 - {}^{0}P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^{0}P_6}{\partial d_i}$$

OR

$${}^{0}_{i}Z$$

$$J_\omega(\underline{\theta})$$

$$\rho = 1 \qquad \rho\, {}^{0}_{i}Z \qquad \rho = 0$$

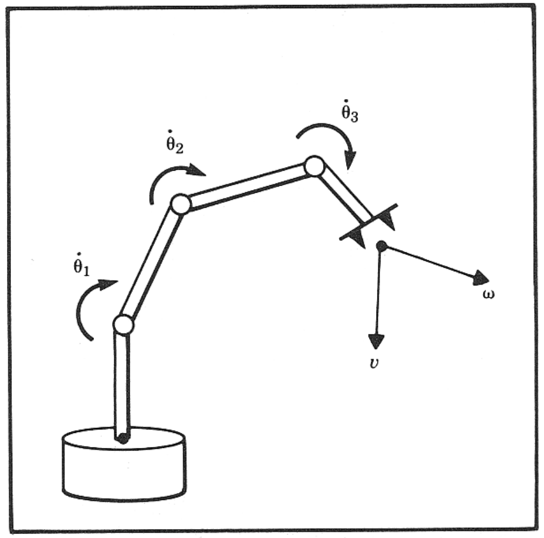
$${}^{0}_{i}T = \begin{bmatrix} * & * & {}^{0}_{i}Z_x & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_y & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_z & {}^{0}P_{ix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{6}T = \begin{bmatrix} * & * & * & {}^{0}P_{6x} \\ * & * & * & {}^{0}P_{6y} \\ * & * & * & {}^{0}P_{6z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Jacobian Matrix

- The meaning of **<u>each column</u>** of the Jacobian matrix:

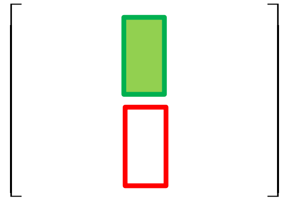
$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}
=
\begin{bmatrix} J_\upsilon(\underline{\theta}) & \cdots & \begin{matrix} J_{1i} \\ J_{2i} \\ J_{3i} \end{matrix} & \cdots \\ J_\omega(\underline{\theta}) & \cdots & \begin{matrix} J_{4i} \\ J_{5i} \\ J_{6i} \end{matrix} & \cdots \end{bmatrix}
\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}
$$



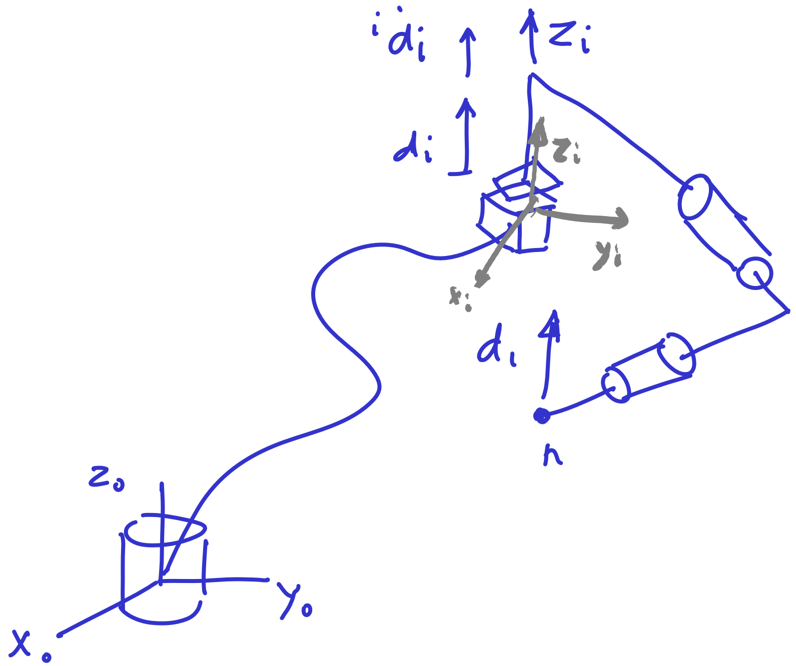
- The i’th column maps the contribution of the angular velocity of the i’th joint to the linear and angular velocities of the end effector along all the axis (x,y,z)

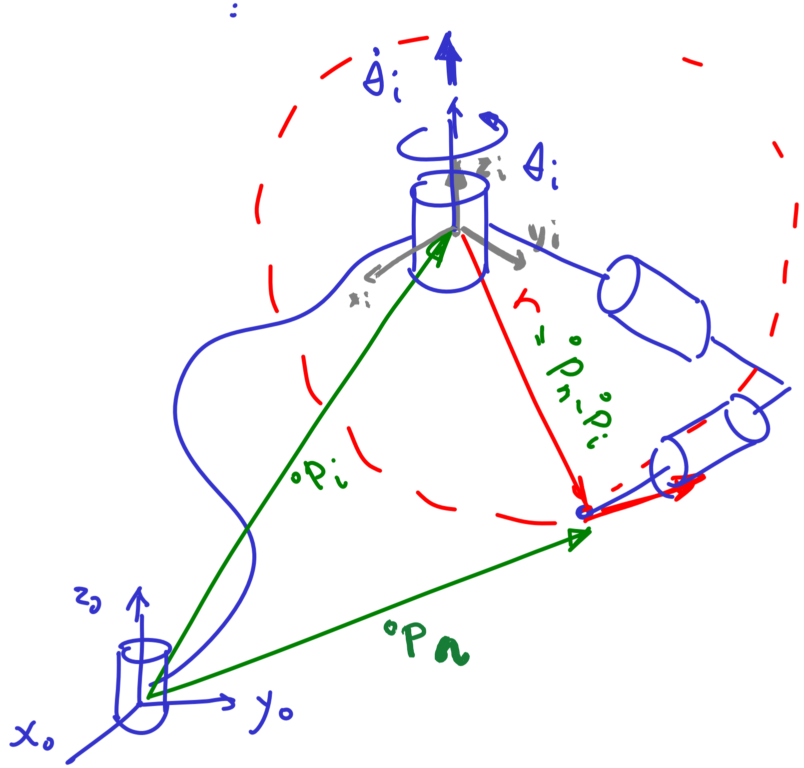
Jacobian – Explicit Form – Linear Velocity
Two Cases

J_υ



- The i’th column of the Jacobian can be generated by holding all the joints fixed but the i’th and actuating the i’th at a unite velocity





Jacobian – Explicit Form

i

$$\begin{bmatrix} J_\upsilon(\underline{\theta}) \\ \hline J_\omega(\underline{\theta}) \end{bmatrix}$$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

— OR —

$${}^0_iZ \times ({}^0P_6 - {}^0P_i)$$

$$\rho = 1$$

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

— OR —

$${}^0_iZ$$

$$\rho = 0$$

Column i (lower block): $\rho\, {}^0_iZ$

$${}^0_iT = \left[\begin{array}{ccc|c} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$${}^0_6T = \left[\begin{array}{ccc|c} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Jacobian – Explicit Form – Linear Velocity
Case 1 – Revolute Joint

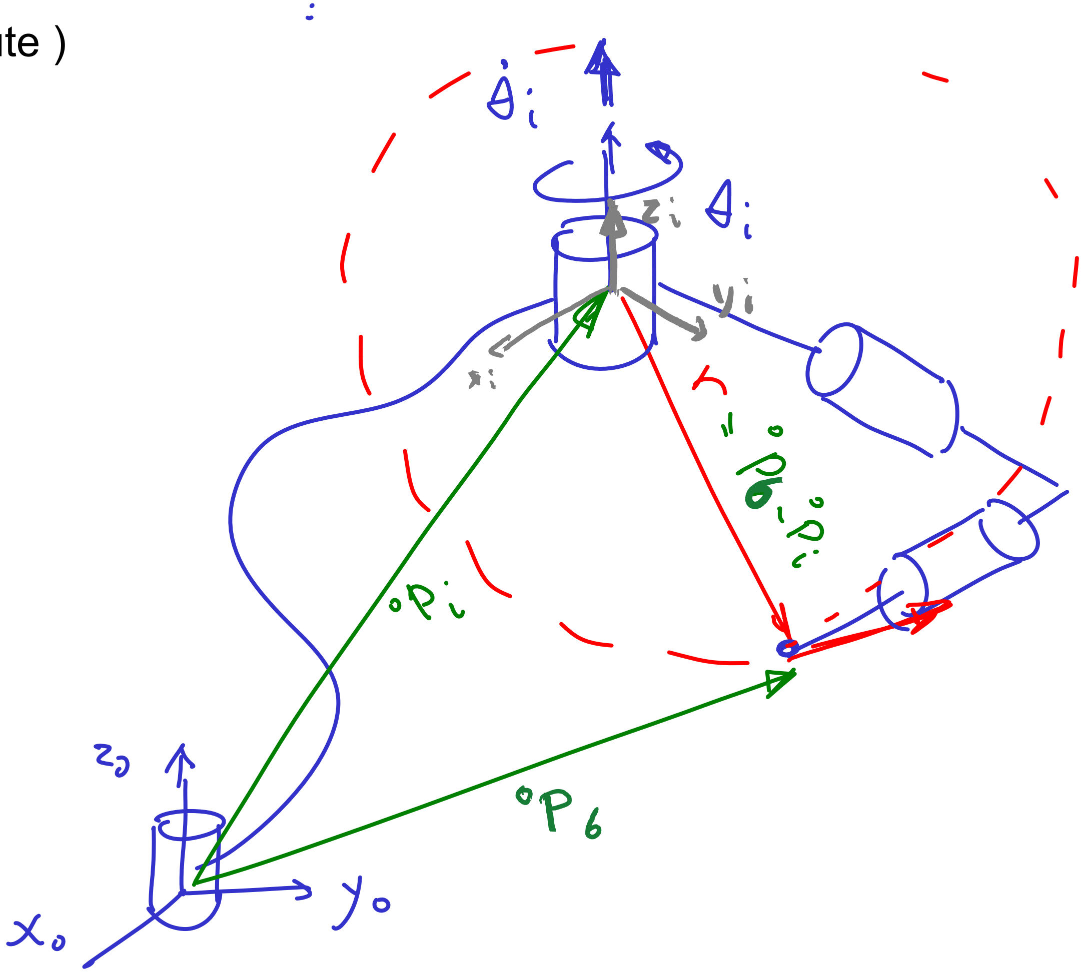
J_υ

- Freeze all DOF Except the i'th DOF (Revolute)

$$v = \omega \times r$$

$$\left|\theta_i\right| {}^0_iZ \times ({}^0P_6 - {}^0P_i) = J_{vi} \left|\theta_i\right|$$

$$ {}^0_iT = \begin{bmatrix} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^0_6T = \begin{bmatrix} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Jacobian – Explicit Form

i

$$J_\upsilon(\underline{\theta})$$

Revolute Joint

$$\frac{\partial\, {}^{0}P_6}{\partial \theta_i}$$

OR

$${}^{0}_{i}Z \times ({}^{0}P_6 - {}^{0}P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^{0}P_6}{\partial d_i}$$

OR

$${}^{0}_{i}Z$$

$$J_\omega(\underline{\theta})$$

$$\rho = 1 \qquad \rho\, {}^{0}_{i}Z \qquad \rho = 0$$

$${}^{0}_{i}T = \begin{bmatrix} * & * & {}^{0}_{i}Z_x & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_y & {}^{0}P_{ix} \\ * & * & {}^{0}_{i}Z_z & {}^{0}P_{ix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{6}T = \begin{bmatrix} * & * & * & {}^{0}P_{6x} \\ * & * & * & {}^{0}P_{6y} \\ * & * & * & {}^{0}P_{6z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

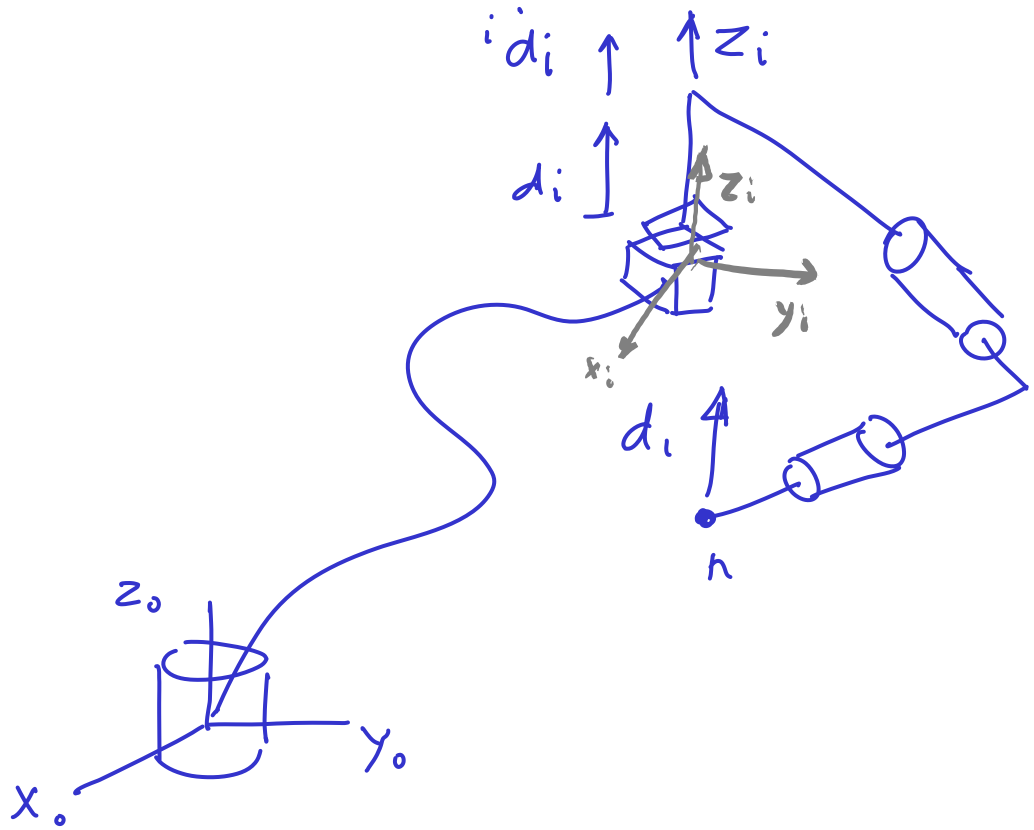
Jacobian – Explicit Form – Linear Velocity
Case 2- Prismatic Joint

J_υ

- Freeze all DOF Except the i'th DOF (Prismatic)

$$^0\dot{P}_i = \left|\dot{d}_i\right| {}^0_iR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left|\dot{d}_i\right| \begin{bmatrix} {}^0_iZ_x \\ {}^0_iZ_y \\ {}^0_iZ_z \end{bmatrix} = \left|\dot{d}_i\right| {}^0_iZ$$

$$^0_iT = \begin{bmatrix} * & * & {}^0_iZ_x & * \\ * & * & {}^0_iZ_y & * \\ * & * & {}^0_iZ_z & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

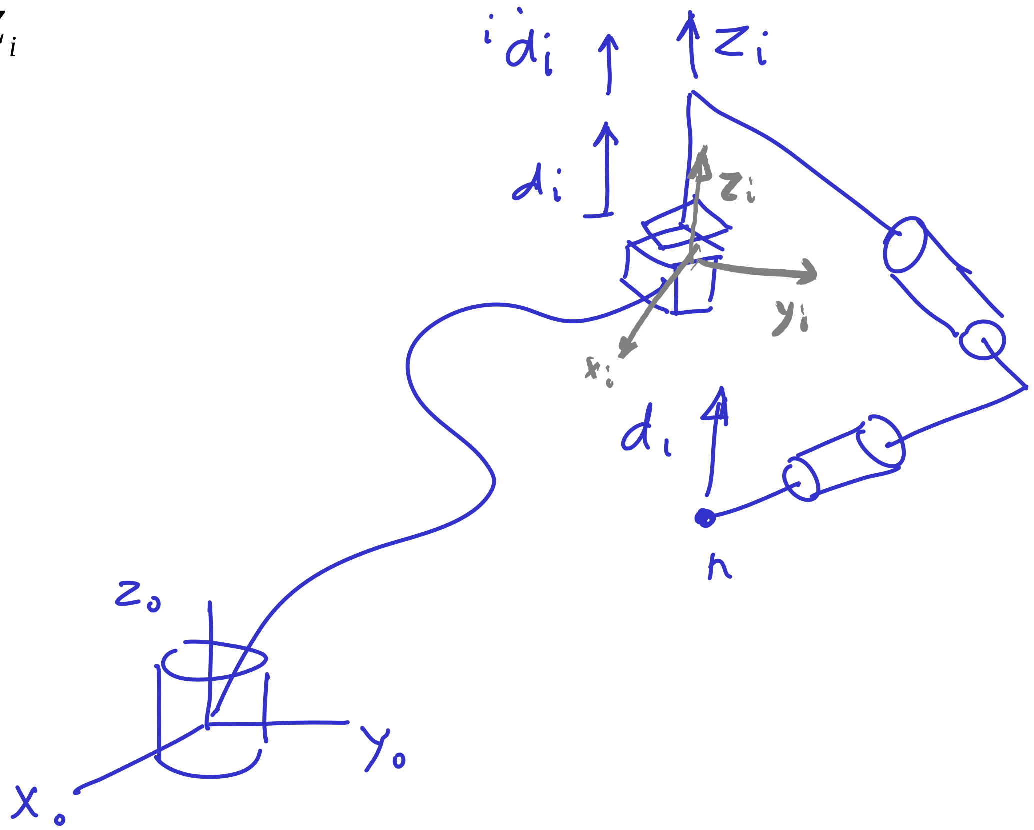


Jacobian – Explicit Form – Linear Velocity J_υ
Case 2- Prismatic Joint

- All the joints are fixed except a single prismatic joint.
- The i'th prismatic joint generates pure translation of the end effector
- The direction of the translation is parallel to the axis Z_i

$$ {}^0\dot{P}_i = |\dot{d}_i| \, {}^0_iR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = |\dot{d}_i| \begin{bmatrix} {}^0_iZ_x \\ {}^0_iZ_y \\ {}^0_iZ_z \end{bmatrix} = |\dot{d}_i| \, {}^0_iZ $$

$$ {}^0_iT = \begin{bmatrix} * & * & {}^0_iZ_x & * \\ * & * & {}^0_iZ_y & * \\ * & * & {}^0_iZ_z & * \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



Jacobian – Explicit Form

i

$J_\upsilon(\underline{\theta})$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

OR

$${}^0_iZ \times ({}^0P_6 - {}^0P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

OR

$${}^0_iZ$$

$${}^0_iR\hat{k}\rho_i\dot{\theta}_i$$

$J_\omega(\underline{\theta})$

$\rho = 1$

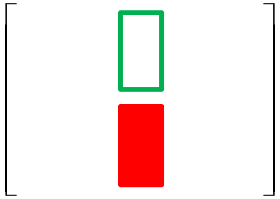
$$\rho\, {}^0_iZ$$

$\rho = 0$

$${}^0_iT = \left[\begin{array}{ccc|c} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$${}^0_6T = \left[\begin{array}{ccc|c} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Jacobian – Explicit Form – Angular Velocity J_ω



$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \rho \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

For ***i=0***

$$ {}^{1}\omega_1 = {}^{1}_{0}R\,\cancel{{}^{0}\omega_0} + \rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad ({}^{0}\omega_0 \to 0) $$

For ***i=1***

$$ {}^{2}\omega_2 = {}^{2}_{1}R\,{}^{1}\omega_1 + \rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = {}^{2}_{1}R\rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} $$

For ***i=2***

$$ {}^{3}\omega_3 = {}^{3}_{2}R\,{}^{2}\omega_2 + \rho_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = {}^{3}_{2}R \left[{}^{2}_{1}R\rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} \right] + \rho_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = {}^{3}_{1}R\rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + {}^{3}_{2}R\rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} + \rho_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} $$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \rho \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} $$

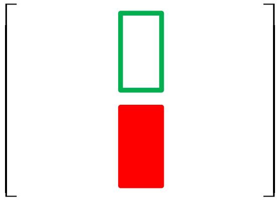
For ***i=n-1***

$$ {}^{n}\omega_{n} = {}^{n}_{n-1}R\,{}^{n-1}\omega_{n-1} + \rho_n \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{3} \end{bmatrix} = {}^{n}_{1}R\rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} + {}^{n}_{2}R\rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{2} \end{bmatrix} + {}^{n}_{3}R\rho_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{3} \end{bmatrix} + \cdots \underbrace{{}^{n}_{n}R}_{I}\rho_n \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{n} \end{bmatrix} $$

- Multiply both side of the equations by ${}^{0}_{n}R$

$$ {}^{0}_{n}R\,{}^{n}\omega_{n} = {}^{0}_{n}R \left[{}^{n}_{n-1}R\,{}^{n-1}\omega_{n-1} + \rho_n \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{3} \end{bmatrix} \right] = {}^{0}_{n}R\,{}^{n}_{1}R\rho_1 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{1} \end{bmatrix} + {}^{0}_{n}R\,{}^{n}_{2}R\rho_2 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{2} \end{bmatrix} + {}^{0}_{n}R\,{}^{n}_{3}R\rho_3 \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{3} \end{bmatrix} + \cdots + {}^{0}_{n}R\,{}^{n}_{n}R\rho_n \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{n} \end{bmatrix} $$

Jacobian – Explicit Form – Angular Velocity J_ω



$$ {}^0_nR\,{}^n\omega_n = {}^0_nR\left[{}^n_{n-1}R\,{}^{n-1}\omega_{n-1} + \rho_n \begin{bmatrix}0\\0\\\dot{\theta}_3\end{bmatrix} \right] = {}^0_nR\,{}^n_1R\rho_1 \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} + {}^0_nR\,{}^n_2R\rho_2 \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} + {}^0_nR\,{}^n_3R\rho_3 \begin{bmatrix}0\\0\\\dot{\theta}_3\end{bmatrix} + \cdots + {}^0_nR\,{}^n_nR\rho_n \begin{bmatrix}0\\0\\\dot{\theta}_n\end{bmatrix} $$

$$ {}^0\omega_n = \quad \cdots \quad = {}^0_1R\rho_1 \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} + {}^0_2R\rho_2 \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} + {}^0_3R\rho_3 \begin{bmatrix}0\\0\\\dot{\theta}_3\end{bmatrix} + \cdots + {}^0_nR\rho_n \begin{bmatrix}0\\0\\\dot{\theta}_n\end{bmatrix} $$

$$ {}^0\omega_n = \sum_{i=1}^{n} {}^0_iR\rho_i \begin{bmatrix}0\\0\\\dot{\theta}_i\end{bmatrix} = \sum_{i=1}^{n} {}^0_iR\hat{k}\rho_i\dot{\theta}_i = \sum_{i=1}^{n} {}^0_iZ\rho_i\dot{\theta}_i $$

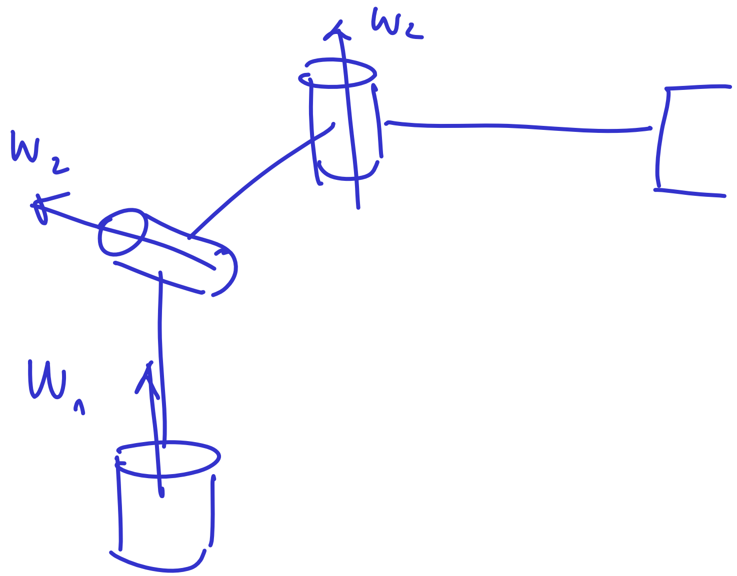
Jacobian – Explicit Form – Angular Velocity J_ω

$J_\upsilon(\underline{\theta})$

$J_\omega(\underline{\theta})$

$$\begin{bmatrix} & & & & \\ \hline \rho_1 \, {}_1^0Z & \rho_2 \, {}_2^0Z & \cdots & \rho_{n-1} \, {}_{n-1}^0Z & \rho_n \, {}_n^0Z \end{bmatrix}$$

Jacobian – Explicit Form – Angular Velocity J_{ω}

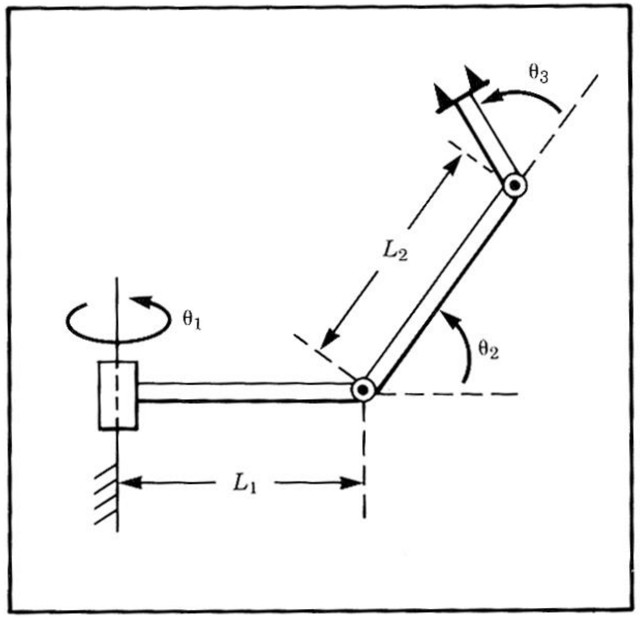


3R – Example

Explicit Methods

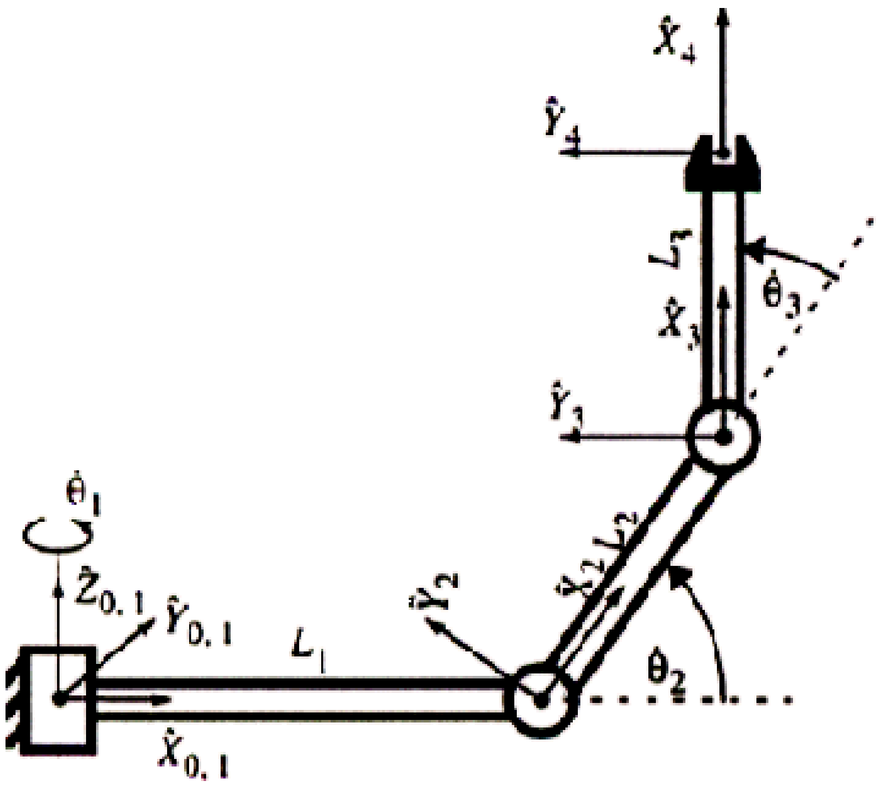
Angular and Linear Velocities - 3R Robot - Example

- For the manipulator shown in the figure, compute the angular and linear velocity of the "tool" frame relative to the base frame expressed in the "tool" frame (that is, calculate ${}^4\omega_4$ and ${}^4v_4$).



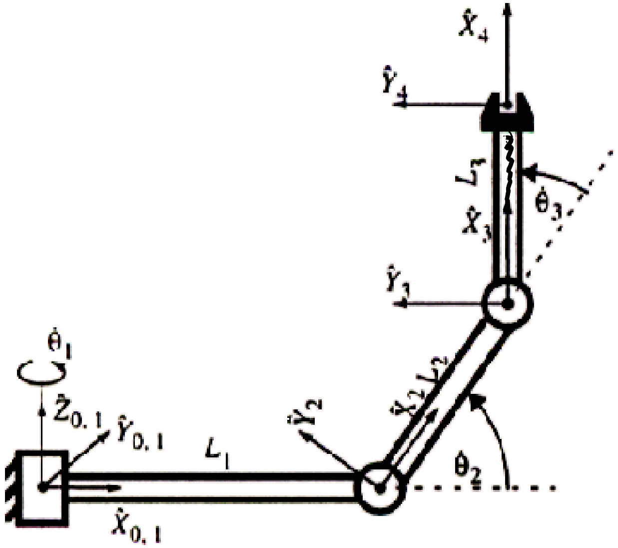
Angular and Linear Velocities - 3R Robot - Example

- Frame attachment



Angular and Linear Velocities - 3R Robot - Example

- DH Parameters



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90	L1	0	θ_2
3	0	L2	0	θ_3
4	0	L3	0	0

Angular and Linear Velocities - 3R Robot - Example

- From the DH parameter table, we can specify the homogeneous transform matrix for each adjacent link pair:

$$
{}^{i-1}_{i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}T = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Jacobian – Explicit Form

i

$J_\upsilon(\underline{\theta})$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

OR

$${}^0_iZ \times ({}^0P_6 - {}^0P_i)$$

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

OR

$${}^0_iZ$$

$J_\omega(\underline{\theta})$

$\rho = 1$ $\quad \rho\, {}^0_iZ \quad$ $\rho = 0$

$${}^0_iT = \begin{bmatrix} * & * & {}^0_iZ_x & {}^0P_{ix} \\ * & * & {}^0_iZ_y & {}^0P_{ix} \\ * & * & {}^0_iZ_z & {}^0P_{ix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^0_6T = \begin{bmatrix} * & * & * & {}^0P_{6x} \\ * & * & * & {}^0P_{6y} \\ * & * & * & {}^0P_{6z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- The homogeneous transform matrix from frame 0 to each one of the joints (1,2,3,4)

$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{2}T = {}^{0}_{1}T\,{}^{1}_{2}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c2 & -c1s1 & s1 & L1c1 \\ s1c2 & -s1s2 & -c1 & L1s1 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{3}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c23 & -c1s23 & s1 & c1(L1+L2c2) \\ s1c23 & -s1s23 & -c1 & s1(L1+L2c2) \\ s23 & c23 & 0 & L2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{4}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c1c23 & -c1s23 & s1 & c1(L1+L3c23+L2c2) \\ s1c23 & -s1s23 & -c1 & s1(L1+L3c23+L2c2) \\ s23 & c23 & 0 & L3s23+L2s2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Angular and Linear Velocities - 3R Robot - Example

$J_v(\Theta)$

- Expressing the columns of $J_v(\Theta)$. Since all the joints are revolute joints i.e. $\rho = 1$ and the last frame is frame 4 the generic expression for the columns of $J_v(\Theta)$ is ${}^0_iZ \times ({}^0P_4 - {}^0P_i)$ for $i = 1,2,3,4$

$For\ i = 1$

$${}^0_1Z \times ({}^0P_4 - {}^0P_1) = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \left[\begin{bmatrix}c1(L1 + L3c23 + L2c2)\\s1(L1 + L3c23 + L2c2)\\L3s23 + L2s2\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix}\right] = \begin{vmatrix}i & j & k\\0 & 0 & 1\\c1(L1 + L3c23 + L2c2) & s1(L1 + L3c23 + L2c2) & L3s23 + L2s2\end{vmatrix} = \begin{bmatrix}-s1(L1 + L3c23 + L2c2\\c1(L1 + L3c23 + L2c2\\0\end{bmatrix}$$

$For\ i = 2$

$${}^0_2Z \times ({}^0P_4 - {}^0P_2) = \begin{bmatrix}s1\\-c1\\0\end{bmatrix} \times \left[\begin{bmatrix}c1(L1 + L3c23 + L2c2)\\s1(L1 + L3c23 + L2c2)\\L3s23 + L2s2\end{bmatrix} - \begin{bmatrix}L1c1\\L1s1\\0\end{bmatrix}\right] = \begin{vmatrix}i & j & k\\s1 & -c1 & 0\\c1(L1 + L3c23 + L2c2) & s1(L1 + L3c23 + L2c2) & L3s23 + L2s2\end{vmatrix} = \begin{bmatrix}-c1 * (L3s23 + L2s2)\\-s1(L3s23 + L2s2)\\L3c23 + L2c2\end{bmatrix}$$

$For\ i = 3$

$${}^0_3Z \times ({}^0P_4 - {}^0P_3) = \begin{bmatrix}s1\\-c1\\0\end{bmatrix} \times \left[\begin{bmatrix}c1(L1 + L3c23 + L2c2)\\s1(L1 + L3c23 + L2c2)\\L3s23 + L2s2\end{bmatrix} - \begin{bmatrix}c1(L1 + L2c2)\\s1(L1 + L2c2)\\L2s2\end{bmatrix}\right] = \begin{vmatrix}i & j & k\\c1(L1 + L2c2) & s1(L1 + L2c2) & L2s2\\c1(L1 + L3c23 + L2c2) & s1(L1 + L3c23 + L2c2) & L3s23 + L2s2\end{vmatrix} = \begin{bmatrix}-L3c1s23\\-L3s1s23\\L3c23\end{bmatrix}$$

$For\ i = 4$

$${}^0_4Z \times ({}^0P_4 - {}^0P_4) = \begin{bmatrix}s1\\-c1\\0\end{bmatrix} \times \left[\begin{bmatrix}c1(L1 + L3c23 + L2c2)\\s1(L1 + L3c23 + L2c2)\\L3s23 + L2s2\end{bmatrix} - \begin{bmatrix}c1(L1 + L3c23 + L2c2)\\s1(L1 + L3c23 + L2c2)\\L3s23 + L2s2\end{bmatrix}\right] = \begin{vmatrix}i & j & k\\s1 & -c1 & 0\\0 & 0 & 0\end{vmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

$J_\omega(\Theta)$

- Expressing the columns of $J_\omega(\Theta)$. Since all the joints are revolute joints i.e. $\rho = 1$ and the last frame is frame 4 the generic expression for the columns of $J_\omega(\Theta)$ is ${}^0_iZ$ for $i = 1,2,3,4$

$$For\ i = 1 \quad {}^0_1Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$For\ i = 2 \quad {}^0_2Z = \begin{bmatrix} s1 \\ -c1 \\ 0 \end{bmatrix}$$

$$For\ i = 3 \quad {}^0_3Z = \begin{bmatrix} s1 \\ -c1 \\ 0 \end{bmatrix}$$

$$For\ i = 4 \quad {}^0_4Z = \begin{bmatrix} s1 \\ -c1 \\ 0 \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- Compiling the results into a single matrix

$$j = \begin{bmatrix} -s1(L1 + L3c23 + L2c2 & -c1 * (L3s23 + L2s2) & -L3c1s23 \\ c1(L1 + L3c23 + L2c2 & -s1(L3s23 + L2s2) & -L3s1s23 \\ 0 & L3c23 + L2c2 & L3c23 \\ 0 & s1 & s1 \\ 0 & -c1 & -c1 \\ 1 & 0 & 0 \end{bmatrix}$$

Angular and Linear Velocities - 3R Robot - Example

- With the end results mapping the joint space velocities to task space velocities

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -s1(L1 + L3c23 + L2c2 & -c1 * (L3s23 + L2s2) & -L3c1s23 \\ c1(L1 + L3c23 + L2c2 & -s1(L3s23 + L2s2) & -L3s1s23 \\ 0 & L3c23 + L2c2 & L3c23 \\ 0 & s1 & s1 \\ 0 & -c1 & -c1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- **Note 1- Size** – The size of the matrix is 6x3. Can we control all six velocities of the end effector with only three DOF (Hint: The dimension of the Jacobian matrix will have to be reduced)
- **Note 2 – Frame of Reference** – The explicit method produce an expression of the Jacobian matrix in frame zero

Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)
Explicit (Diff. the Forward Kinematic Eq.)	${}^{0}J_{N}$	None
Iterative Velocity Eq.	${}^{N}J_{N}$	Transform Method 1: ${}^{0}v_{N} = {}^{0}_{N}R\,{}^{N}v_{N}$; ${}^{0}\omega_{N} = {}^{0}_{N}R\,{}^{N}\omega_{N}$ Transform Method 2: ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$
Iterative Force Eq.	${}^{N}J_{N}^{T}$	Transpose: ${}^{N}J_{N} = [{}^{N}J_{N}^{T}]^{T}$ Transform: ${}^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$