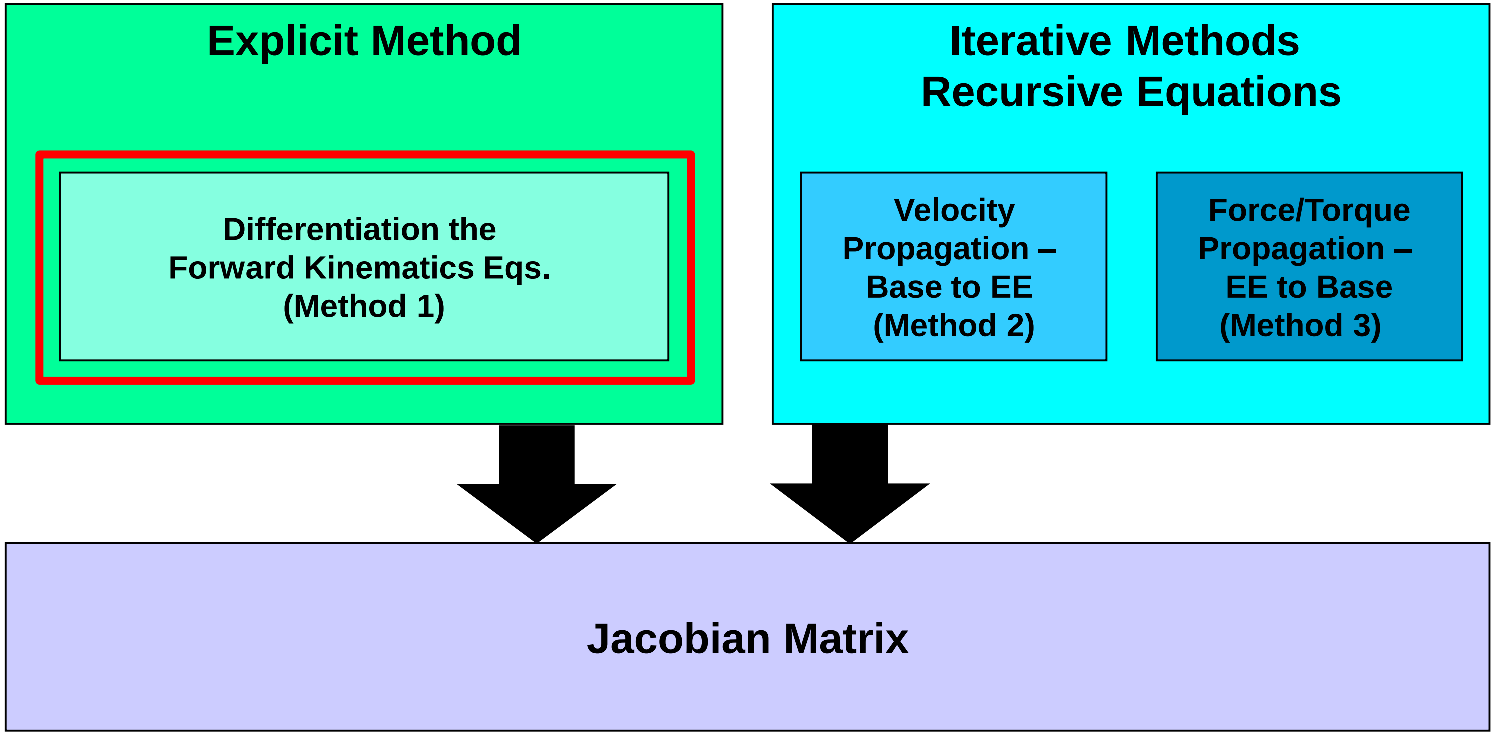
Advanced Kinematics
Linear and Angular Velocities

Jacobian Matrix - Derivation Methods



Jacobian Matrix – Introduction - Velocity Transformation

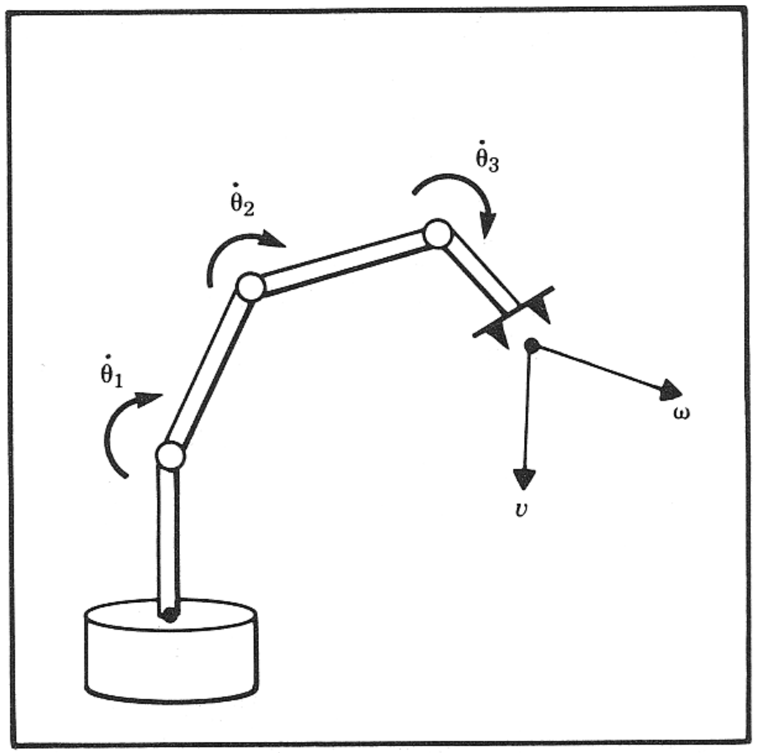
Problem

Given: Joint angles and velocities and links geometry along with the transformation matrixes between the joints.

Compute: The Jacobian matrix that maps between the joint velocities $\dot{\Theta}$ in the joint space to the end effector velocities ν in the Cartesian space or the end effector space

$$\nu = \mathbf{J}\,(\Theta)\dot{\Theta}$$

$$\dot{\Theta} = \mathbf{J}^{-1}(\Theta)\nu$$



$$\dot{\Theta} = \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ d_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{Bmatrix}$$

$$\nu = \begin{Bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

Solution – There are two approaches to the solution:

- **Velocity Propagation** - A velocity propagation approach is taken in which velocities are propagated stating form the stationary base all the way to the end effector. The Jacobian is then extracted from the velocities of the end effector as a function of the joint velocities.
- **Time derivative of the end effector position and ordinations** – The time derivative of the explicit positional and orientation is taken given the forward kinematics. The Jacobian is then extracted from the velocities of the end effector as a function of the joint velocities.

Notes:

Spatial Description – The matrix is a function of the joint angle.

Singularities - At certain points, called ***singularities***, this mapping is not invert-able and the Jacobian Matrix J loosing its rank and therefore this mathematical expression is no longer valid.

Jacobian Matrix – Introduction - Force Transformation

Problem

Given: *Joint angles, links geometry, transformation matrixes between the joints, along with the external loads (forces and moments) typically applied on the end effector*

Compute: *The transpose Jacobian matrix that maps between the external loads (forces and moments) typically applied at the end effector space*$\mathcal{F}$ joint torques at the *joint space* $\boldsymbol{\tau}$

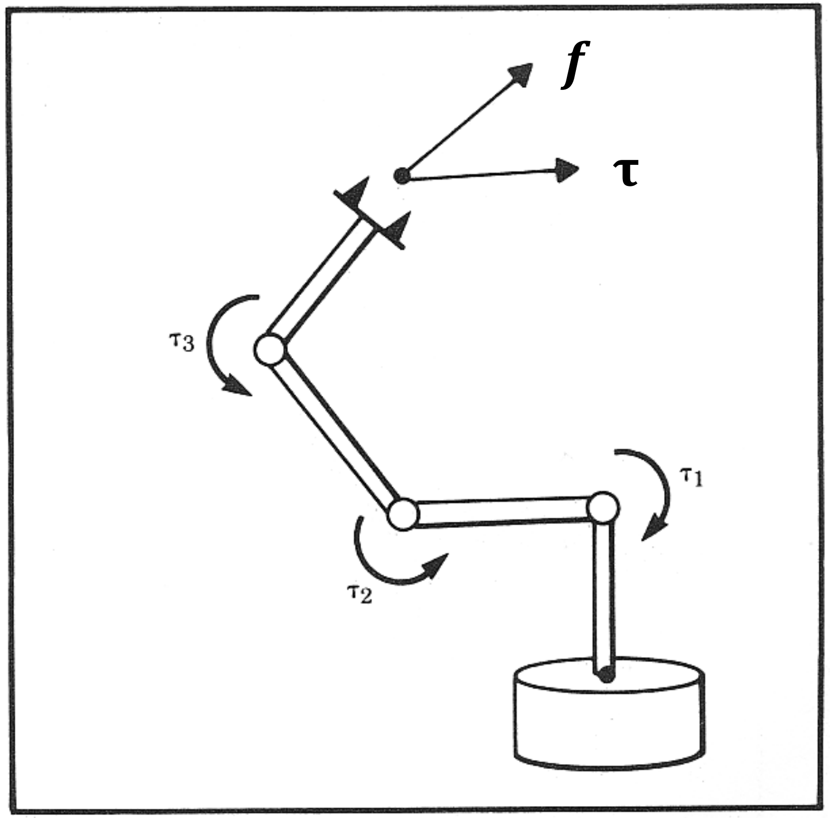
$$\tau = \mathbf{J}^T \mathcal{F}$$

Solution

- **Force/Moment Propagation** - A force/moment propagation approach is taken in which forces and moments are propagated stating form the end effector where they can be measured by a F/T sensor attached between the gripper and the arm all the way to the base of the arm. The Jacobian transposed is then extracted from the joint torques as a function of the force/moment applied on the end effector

Note

- **Conditions:** Static or quasi static conditions



$$\dot{\Theta} = \begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ f_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix}$$

$$\mathcal{F} = \begin{Bmatrix} f_x \\ f_y \\ f_z \\ \tau_x \\ \tau_y \\ \tau_z \end{Bmatrix}$$

Sensation 1 – Simultaneous Linear and Rotational Velocities

- Given - Two frames i.e frame {A} and frame {B} as well as point Q.

- Three actions take place simultaneously

  - The origin of frame B moves as a function of time with respect to the origin of frame A
  - Point Q moves with respect to frame B
  - Frame B rotates with respect to frame A along an axis defined by ${}^{A}\Omega_{B}$

- **Challenge** – Express the velocity of point Q

Scenarios 1 – Implications for Serial Manipulators

Sensation 2 – Two Consecutive Rotation

- Given - Three frames i.e frame {A} and frame {B} frame {C} all sharing the same origin. Frame {A} is stationary and Frames {B} and {C} rotate

- Two actions take place simultaneously

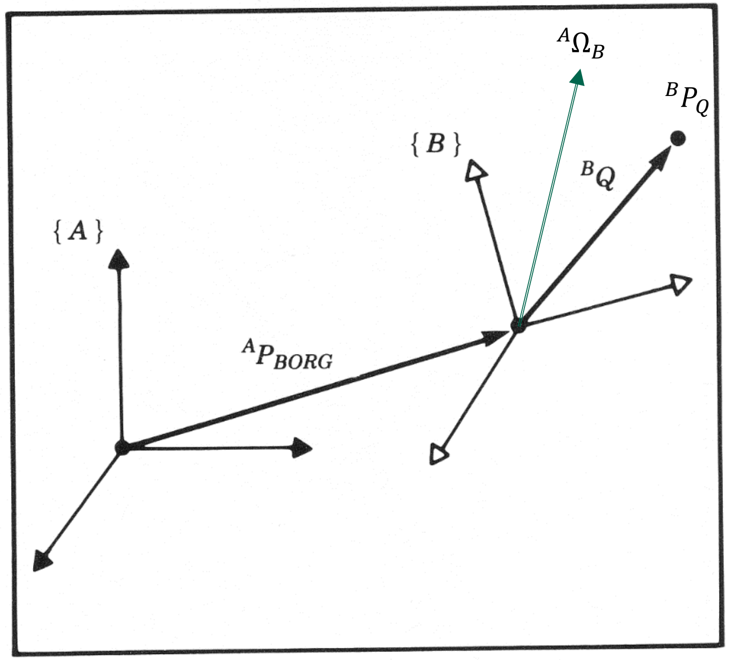
  – Frame B rotates with respect to frame A along an axis defined by the vector ${}^{A}\Omega_{B}$
  – Frame C rotates with respect to frame B along an axis defined by the vector ${}^{B}\Omega_{C}$

- **Challenge** – Express the rotation of frame {C} with respect to frame {A} or alternatively express the vector ${}^{A}\Omega_{C}$

Scenarios 2 – Implications for Serial Manipulators

Simultaneous Linear and Rotational Velocity - Scenario No.1

$$ {}^{A}V_{Q} = f\left({}^{B}P_{Q},\ {}^{B}V_{Q},\ {}^{A}V_{BORG},\ {}^{A}\Omega_{B},\ {}^{A}_{B}R\right) $$



- Vector Form (Method No. 1)

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form (Method No. 2)

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

- Matrix Formulation – Homogeneous Transformation Form – Method No. 3

$$ \begin{bmatrix} \left[{}^{A}V_{Q}\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega} \cdot {}^{A}_{B}R\right] & \left[{}^{A}V_{B\ org}\right] \\ 0\ \ 0\ \ 0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_{Q}\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\ org}\right] \\ 0\ \ 0\ \ 0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_{Q}\right] \\ 0 \end{bmatrix} $$

Angular Velocity – Changing the Frame of Representation – Scenario No.2

- Angular Velocity Representation in Various Frames
  - Vector Form $$ {}^A\Omega_C = {}^A\Omega_B + {}^A_B R\, {}^B\Omega_C $$
  - Matrix Form $$ {}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_B R\, {}^B_C\dot{R}_\Omega\, {}^A_B R^T $$

Scenarios 1
Simultaneous Linear and Rotational Velocity

Linear & Angular Velocity – Derivation Method No. 1 & 2

Vector Form

Matrix Form

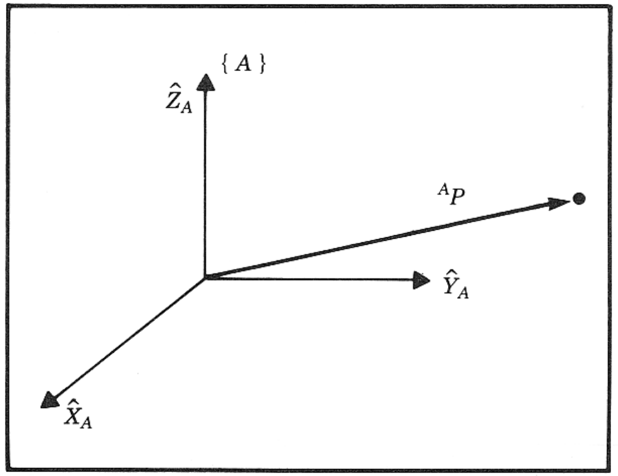
$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Linear Velocity

- ***Linear velocity*** - The instantaneous rate of change in linear position of a point relative to some frame.

$$^{A}P_Q(t+\Delta t) = {}^{A}P_Q(t) + {}^{A}V_Q \Delta t$$

$$^{A}V_Q = \frac{d}{dt}\,{}^{A}P_Q \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_Q(t+\Delta t) - {}^{A}P_Q(t)}{\Delta t}$$



$$ {}^A V_Q = {}^A V_{BORG} + {}^A_B R\, {}^B V_Q + {}^A\Omega_B \times {}^A_B R\, {}^B P_Q \qquad {}^A V_Q = {}^A V_{BORG} + {}^A_B R\, {}^B V_Q + {}^A_B \dot{R}_\Omega \left({}^A_B R\, {}^B P_Q \right) $$

Definitions - Linear Velocity

- The position of point Q in frame {A} is represented by the ***linear position vector***

$$ {}^A P_Q = \begin{bmatrix} {}^A P_{Qx} \\ {}^A P_{Qy} \\ {}^A P_{Qz} \end{bmatrix} $$

- The velocity of a point Q relative to frame {A} is represented by the ***linear velocity vector***

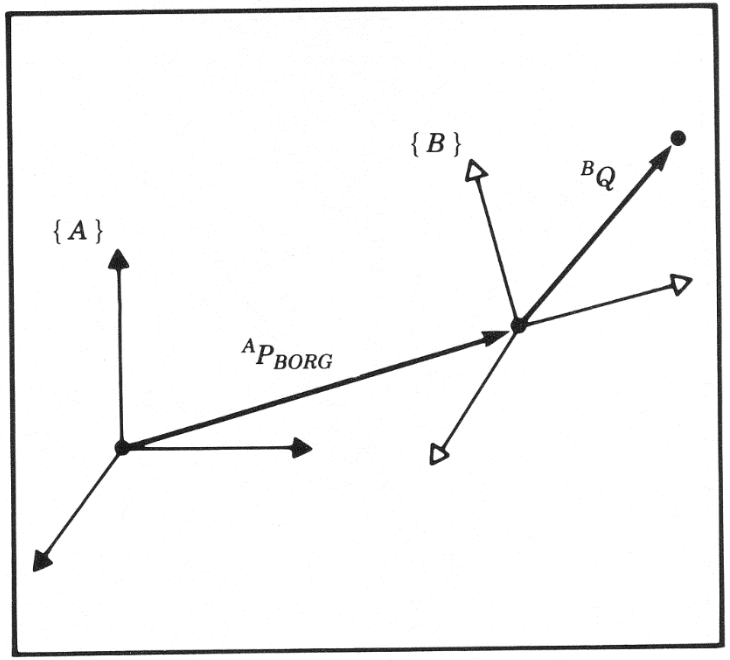
$$ {}^A V_Q = \frac{{}^A d}{dt} \begin{bmatrix} {}^A P_{Qx} \\ {}^A P_{Qy} \\ {}^A P_{Qz} \end{bmatrix} = \begin{bmatrix} {}^A \dot{P}_{Qx} \\ {}^A \dot{P}_{Qy} \\ {}^A \dot{P}_{Qz} \end{bmatrix} $$

Linear Velocity – Translation (No Rotation)- Problem 1 Derivation

- **Problem No. 1** – Change in a position of Point Q
- Conditions
  - Point Q is fixed in frame {B}
  - Frame {B} translates with respect to Frame {A}

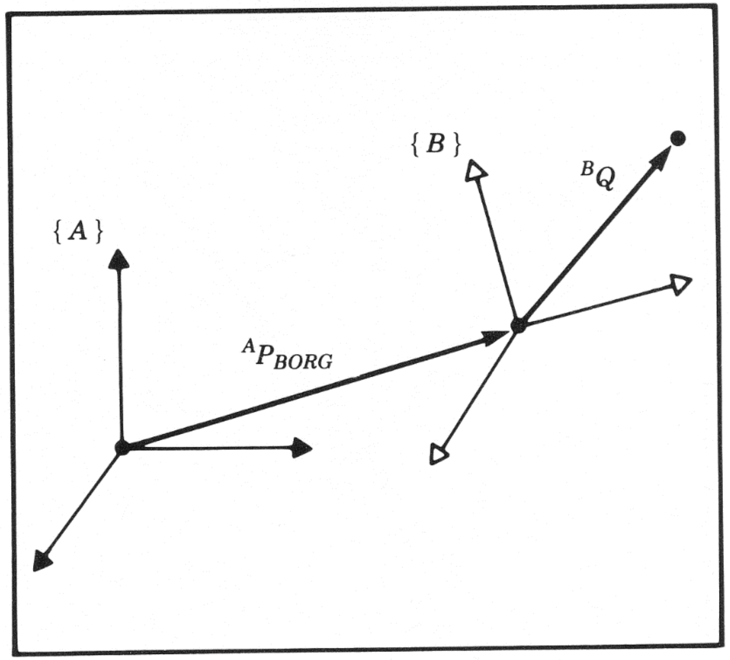
$$\frac{{}^{B}d}{dt}\left({}^{B}P_Q\right) \approx \lim_{\Delta t \to 0} {}^{B}\left(\frac{\overbrace{{}^{A}P_Q(t+\Delta t) - {}^{A}P_Q(t)}^{=0}}{\Delta t}\right) = {}^{B}\left({}^{B}V_Q\right) = 0$$

$$\frac{{}^{A}d}{dt}\left({}^{A}P_Q\right) \approx \lim_{\Delta t \to 0} {}^{A}\left(\frac{{}^{A}P_Q(t+\Delta t) - {}^{A}P_Q(t)}{\Delta t}\right) = {}^{A}\left({}^{A}V_Q\right) = {}^{A}V_Q = {}^{A}V_{B\,ORG}$$



Linear Velocity – Translation (No Rotation) – Problem 2 Derivation

- **Problem No. 2** – Translation of frame {B}
- Conditions
  - Point Q is fixed in frame {B}
  - Frame {B} translates with respect to Frame {A}



$$\frac{{}^{A}d}{dt}\left(\overbrace{{}^{A}P_{B\,ORG}}^{Const}\right) \approx \lim_{\Delta t \to 0} {}^{A}\left(\frac{\overbrace{{}^{A}P_{B\,ORG}(t+\Delta t) - {}^{A}P_{B\,ORG}(t)}^{=0}}{\Delta t}\right) = {}^{A}\left({}^{A}V_{B\,ORG}\right) = {}^{A}V_{B\,ORG} = 0$$

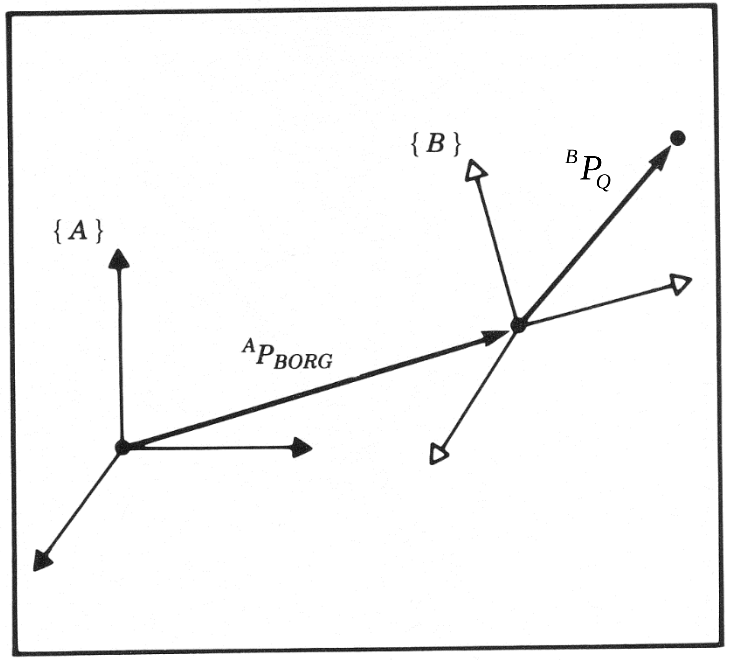
$$\frac{{}^{A}d}{dt}\left({}^{B}P_{Q}\right) \approx \lim_{\Delta t \to 0} {}^{A}\left(\frac{{}^{B}P_{Q}(t+\Delta t) - {}^{B}P_{Q}(t)}{\Delta t}\right) = {}^{A}\left({}^{B}V_{Q}\right)$$

$${}^{A}V_{Q} = {}^{A}_{B}R\,{}^{B}V_{Q}$$

Linear Velocity – Translation (No Rotation) – Problem 1&2 - Summary

- **Problem No. 1** – Change in a position of Point Q
- **Problem No. 2** – Translation of frame {B}

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}({}^{B}V_{Q}) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$



Linear Velocity – Translation – Simultaneous Derivation

$$^{A}P_{Q} = {^{A}P_{BORG}} + {^{B}P_{Q}}$$

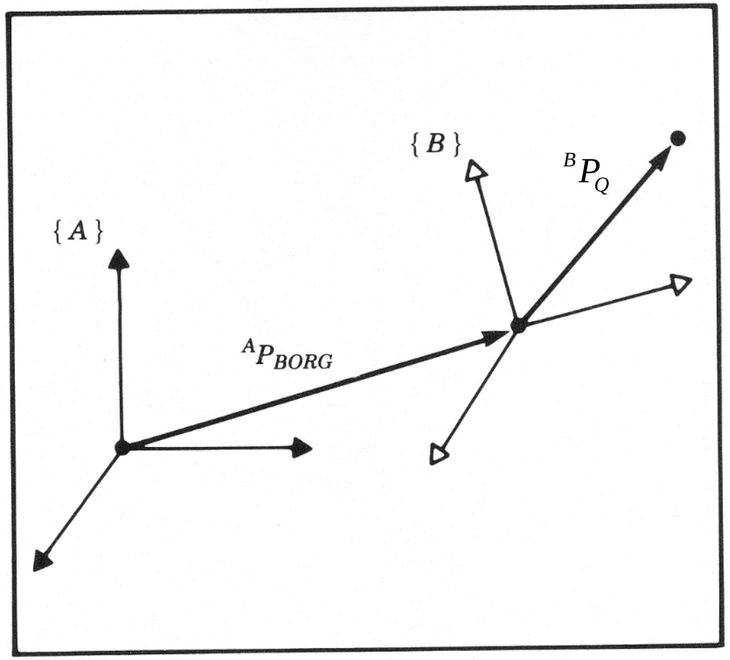
- Differentiate with respect to coordinate system {A}

$$\frac{^{A}d}{dt}\left(^{A}P_{Q}\right) = \frac{^{A}d}{dt}\left(^{A}P_{BORG}\right) + \frac{^{A}d}{dt}\left(^{B}P_{Q}\right)$$

$$^{A}\left(^{A}\dot{P}_{Q}\right) = {^{A}\left(^{A}\dot{P}_{BORG}\right)} + {^{A}\left(^{B}\dot{P}_{Q}\right)}$$

$$^{A}\left(^{A}V_{Q}\right) = {^{A}\left(^{A}V_{BORG}\right)} + {^{A}\left(^{B}V_{Q}\right)}$$

$$^{A}V_{Q} = {^{A}V_{BORG}} + {^{A}(^{B}V_{Q})} = {^{A}V_{BORG}} + {^{A}_{B}R}\,{^{B}V_{Q}}$$



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$$ {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Linear Velocities - Frames

- When describing the velocity (linear or angular) of an object, there are two important frames that are being used:

  – ***Represented Frame (Reference Frame)*** : e.g. {A}
  This is the frame used to **represent (express)** the object’s velocity.

  – ***Computed Frame***: e.g. {B}
  This is the frame in which the velocity is **measured** (differentiate the position).

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Linear Velocities - Frames

- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.
- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

Represented
(Reference Frame) -– Projected on

$$ {}^{Represented}\left({}^{Computed}V_{Q}\right) = \frac{{}^{Represented}d}{dt}\,{}^{Computed}P_{Q} $$

Computed
(Measured) - Differentiate with respect to

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Frame - Linear Velocity

- We can always remove the outer, leading superscript by explicitly including the rotation matrix which accomplishes the change in the reference frame

$$^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

- Note that in the general case $^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} \neq {}^{A}V_{Q}$ because ${}^{A}_{B}R$ may be time-verging ${}^{A}_{B}\dot{R} \neq 0$

- If the calculated velocity is written in terms of of the frame of differentiation the result could be indicated by a single leading superscript.

$$^{A}({}^{A}V_{Q}) = {}^{A}V_{Q}$$

- In a similar fashion when the angular velocity is expresses and measured as a vector

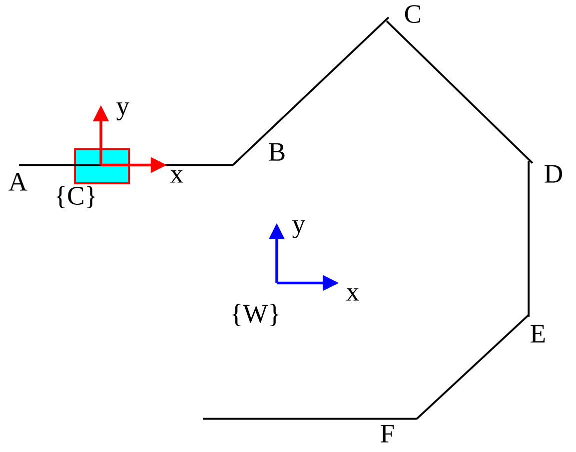
$$^{A}({}^{B}\Omega_{C}) = {}^{A}_{B}R\,{}^{B}\Omega_{C}$$

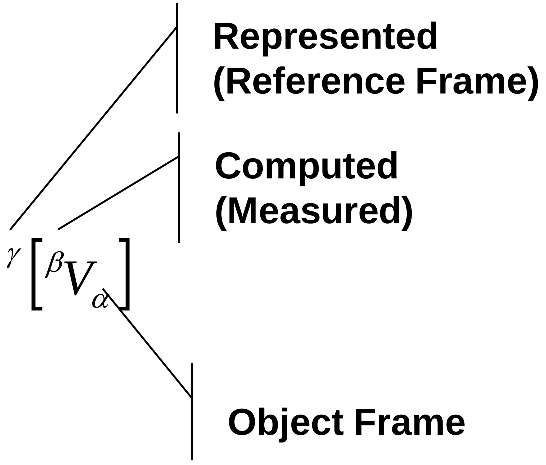
$$ {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Frames - Linear Velocity - Example

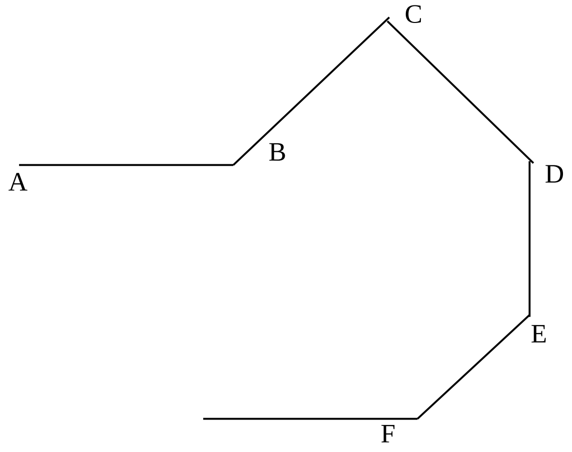
- ***Given:*** The driver of the car maintains a ***speed*** of 100 km/h (as shown to the driver by the car's speedometer).

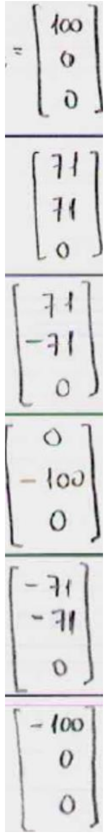
- ***Problem:*** Express the velocities ${}^{C}\left[{}^{C}V_C\right]$ ${}^{W}\left[{}^{W}V_C\right]$ ${}^{W}\left[{}^{C}V_C\right]$ ${}^{C}\left[{}^{W}V_C\right]$ in each section of the road A, B, C, D, E, F where {C} - Car frame, and {W} - World frame





Frames - Linear Velocity - Example





$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Frames - Linear Velocity - Example

$${}^{A}_{B}R = Rot(\hat{z},\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rot(\hat{z},+45^{o}) = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

$$Rot(\hat{z},-45^{o}) = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}$$

$$Rot(\hat{z},+90^{o}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Rot(\hat{z},-90^{o}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Frames - Linear Velocity - Example

$${}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q$$

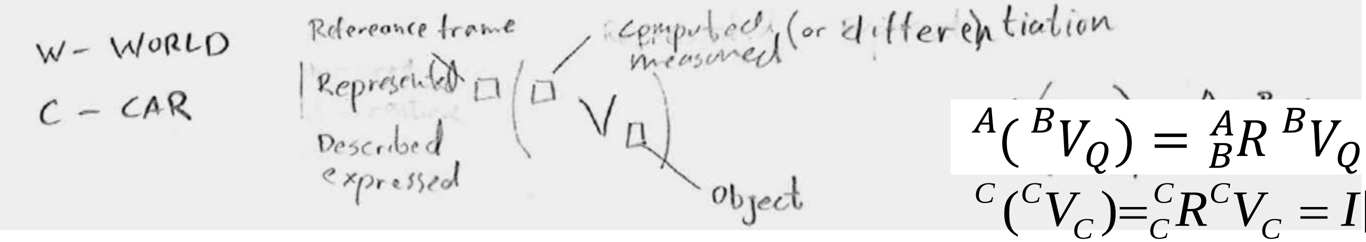
- ${}^{A}_{B}\dot{R} = 0$ is not time-varying (in this example)

$${}^{C}({}^{C}V_C) = {}^{C}_{C}R\,{}^{C}V_C = I[0] = [0]$$

$${}^{W}({}^{W}V_C) = {}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C$$

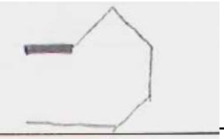
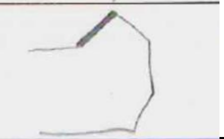
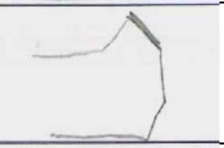
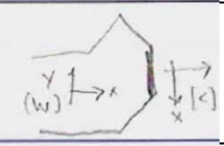
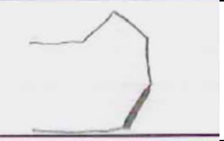
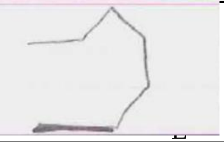
$${}^{W}({}^{C}V_C) = {}^{W}_{C}R\,{}^{C}V_C = {}^{W}_{C}R[0] = [0]$$

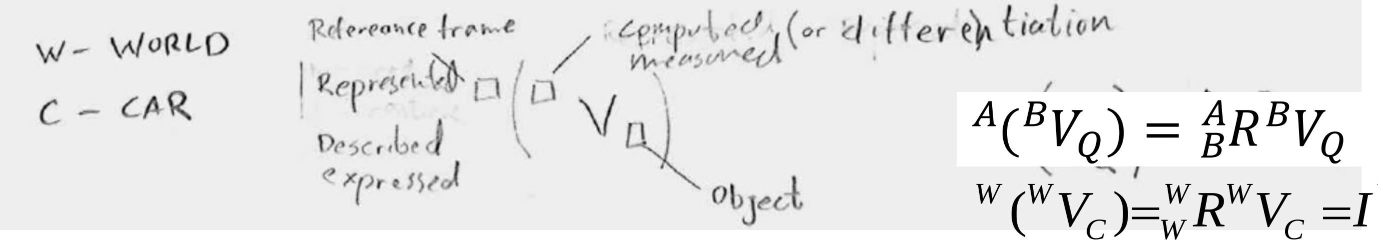
$${}^{C}({}^{W}V_C) = {}^{C}_{W}R\,{}^{W}V_C$$



$${}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q$$

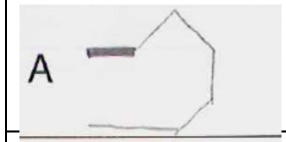
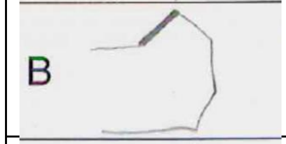
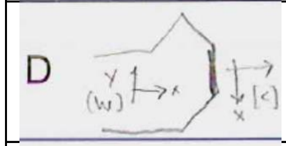
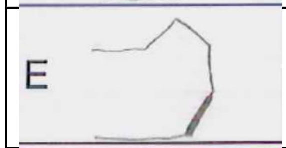
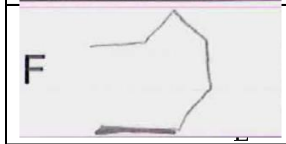
$${}^{C}({}^{C}V_C) = {}^{C}_{C}R\,{}^{C}V_C = I[0] = [0]$$

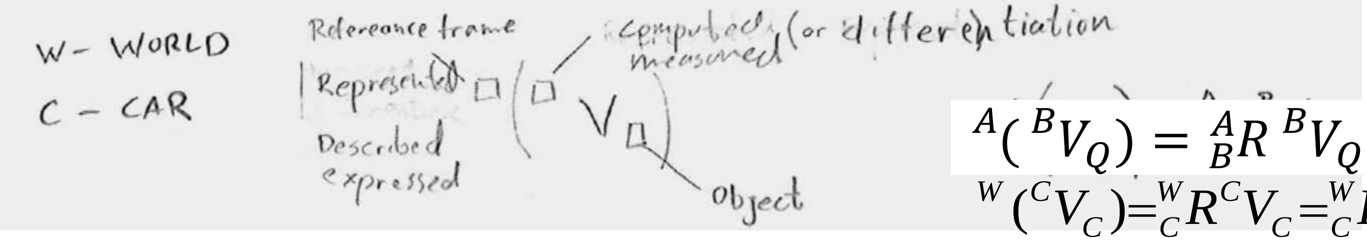
Road Section	Velocity			
	${}^{C}[{}^{C}V_C]$	${}^{W}[{}^{W}V_C]$	${}^{W}[{}^{C}V_C]$	${}^{C}[{}^{W}V_C]$
A 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
B 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
C 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
D 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
E 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
F 	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$			



$$^{A}(^{B}V_Q) = {}^{A}_{B}R\,^{B}V_Q$$

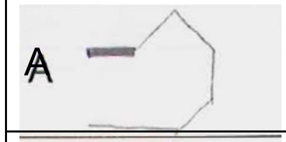
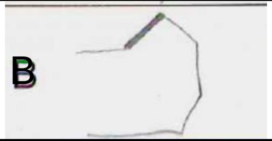
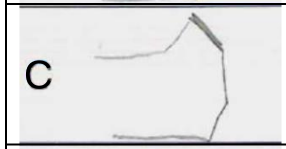
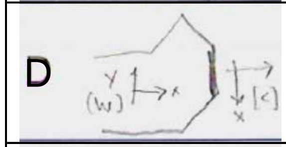
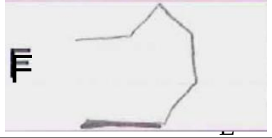
$$^{W}(^{W}V_C) = {}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C$$

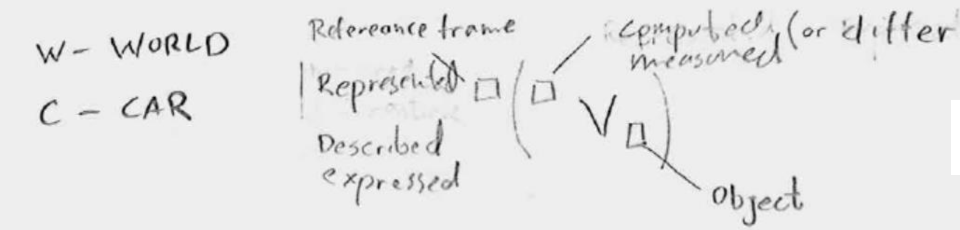
Road Section	Velocity			
	$^{C}[^{C}V_C]$	$^{W}[^{W}V_C]$	$^{W}[^{C}V_C]$	$^{C}[^{W}V_C]$
 A	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}100\\0\\0\end{bmatrix}$		
 B	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}71\\71\\0\end{bmatrix}$		
 C	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}71\\-71\\0\end{bmatrix}$		
 D	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}0\\-100\\0\end{bmatrix}$		
 E	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}-71\\-71\\0\end{bmatrix}$		
 F	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}-100\\0\\0\end{bmatrix}$		



$$^{A}(^{B}V_Q) = {}^{A}_{B}R\,^{B}V_Q$$

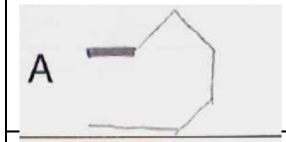
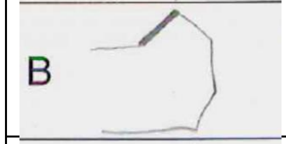
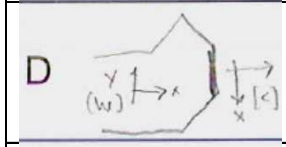
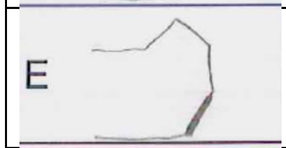
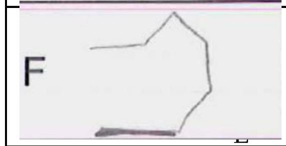
$$^{W}(^{C}V_C) = {}^{W}_{C}R\,^{C}V_C = {}^{W}_{C}R[0] = [0]$$

Road Section	Velocity			
	$^{C}[^{C}V_C]$	$^{W}[^{W}V_C]$	$^{W}[^{C}V_C]$	$^{C}[^{W}V_C]$
A 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
B 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}71\\71\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
C 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}71\\-71\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
D 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}0\\-100\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
E 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}-71\\-71\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
F 	${}^{C}_{C}R\,^{C}V_C = I\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,^{W}V_C = I\,^{W}V_C = \begin{bmatrix}-100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	



$$ {}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

$$ {}^{C}({}^{W}V_C) = {}^{C}_{W}R\,{}^{W}V_C $$

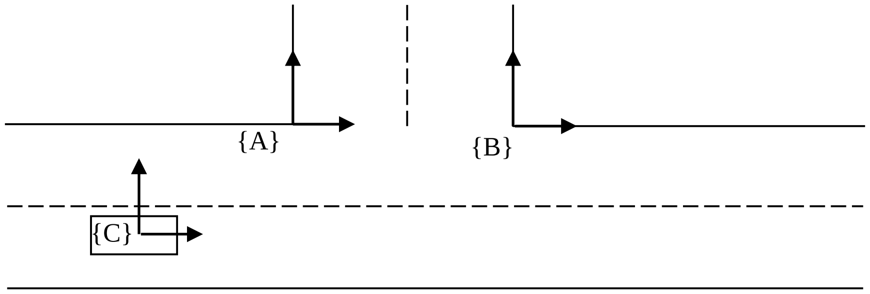
Road Section	Velocity			
	${}^{C}\left[{}^{C}V_C\right]$	${}^{W}\left[{}^{W}V_C\right]$	${}^{W}\left[{}^{C}V_C\right]$	${}^{C}\left[{}^{W}V_C\right]$
 A	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},0) = I\begin{bmatrix}100\\0\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
 B	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}71\\71\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},45)\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}.707 & -.707 & 0\\.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
 C	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}71\\-71\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},-45)\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}.707 & .707 & 0\\-.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
 D	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}0\\-100\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},90)\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
 E	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}-71\\-71\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
 F	${}^{C}_{C}R\,{}^{C}V_C = I\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C = \begin{bmatrix}-100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_C = \begin{bmatrix}0\\0\\0\end{bmatrix}$	

$${}^AV_Q = {}^AV_{BORG} + {}^A_BR\,\boxed{{}^BV_Q} + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,\boxed{{}^BV_Q} + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right)$$

Linear Velocity - Free Vector

- Linear velocity vectors are insensitive to shifts in origin.

- Consider the following example:



- The velocity of the object in {C} relative to both {A} and {B} is the same, that is

$${}^AV_C = {}^BV_C$$

- As long as {A} and {B} remain fixed relative to each other (translational but not rotational), then the velocity vector remains unchanged (that is, a ***free vector***).

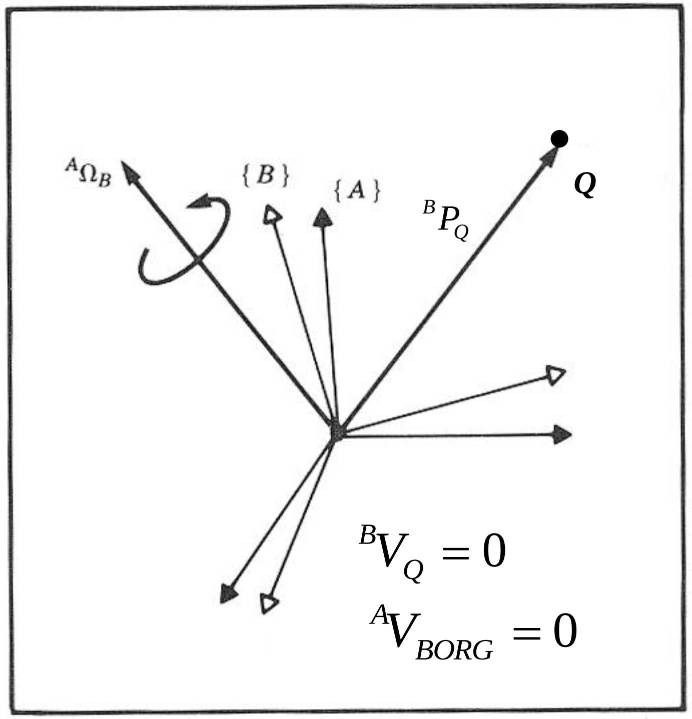
Angular Velocity - Rigid Body - Intuitive Approach

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The vector ${}^{B}P_{Q}$ is constant as view from frame {B} ${}^{B}V_{Q} = 0$
- ***Problem:*** describe the velocity of the vector ${}^{B}P_{Q}$ representing the the point **Q** relative to frame {A}
- ***Solution:*** Even though the vector ${}^{B}P_{Q}$ is constant as view from frame {B} it is clear that point **Q** will have a velocity as seen from frame {A} due to the rotational velocity ${}^{A}\Omega_{B}$



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}_{B}^{A}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}_{B}^{A}\dot{R}_{\Omega}\left({}_{B}^{A}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

- Pure 3D Rotation - The length of the vector Q does not change its length in frame B

$$ {}^{A}P_{Q} = CONST $$

$$ {}^{B}V_{Q} = 0 $$

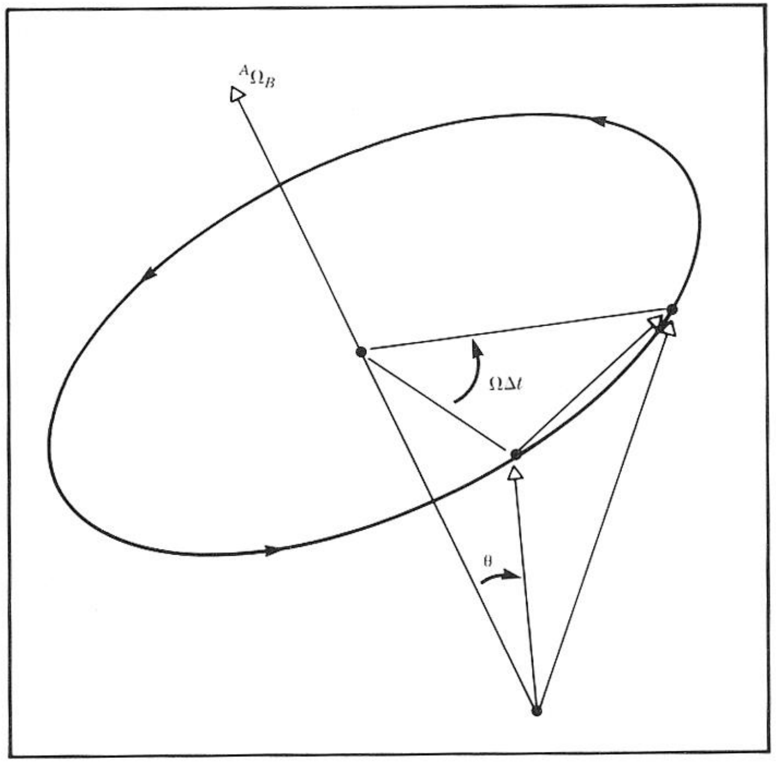
$$ \Delta {}^{A}P_{Q} = \left(\left|{}^{A}P_{Q}\right| sin\theta\right)\left({}^{A}\Omega_{Q}\Delta t\right) $$

$$ {}^{A}V_{Q} = {}^{A}\Omega_{Q} \times {}^{A}P_{Q} $$

- In general the vector ${}^{A}P_{Q}$ can change with respect to frame {B}

$$ {}^{A}V_{Q} = {}^{A}\left({}^{B}\mathrm{V}_{Q}\right) + {}^{A}\Omega_{B} \times {}^{A}\mathrm{P}_{Q} $$

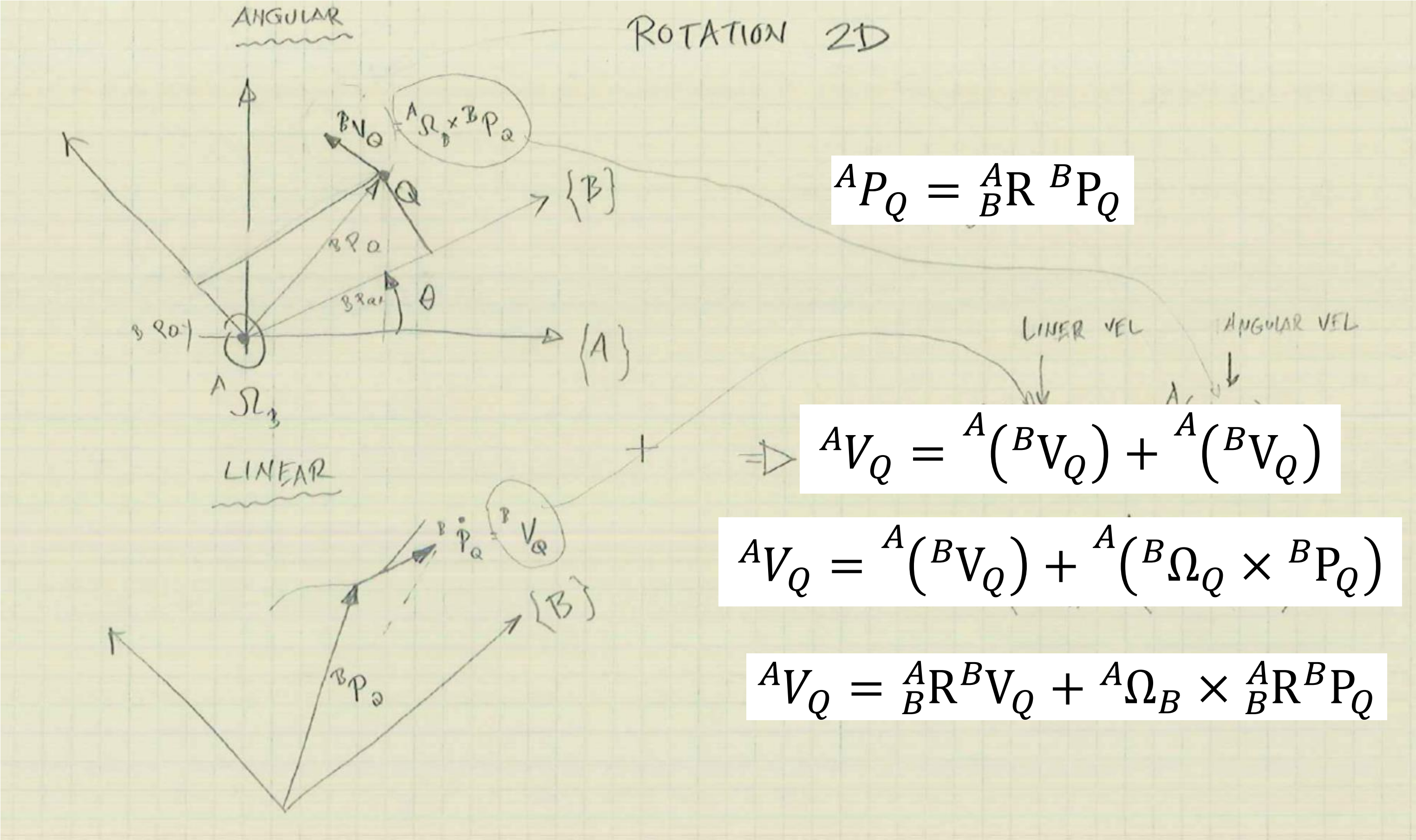
$$ {}^{A}V_{Q} = {}_{B}^{A}\mathrm{R}\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}_{B}^{A}\mathrm{R}\,{}^{B}\mathrm{P}_{Q} $$



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

- Rotation in 2D



$$ {}^{A}P_{Q} = {}^{A}_{B}R\,{}^{B}P_{Q} $$

$$ {}^{A}V_{Q} = {}^{A}({}^{B}V_{Q}) + {}^{A}({}^{B}V_{Q}) $$

$$ {}^{A}V_{Q} = {}^{A}({}^{B}V_{Q}) + {}^{A}({}^{B}\Omega_{Q} \times {}^{B}P_{Q}) $$

$$ {}^{A}V_{Q} = {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

- In the general case, the vector Q may also be changing with respect to the frame {B}. Adding this component we get.

$$ {}^{A}V_Q = {}^{A}\left({}^{B}V_Q\right) + {}^{A}\Omega_B \times {}^{A}P_Q $$

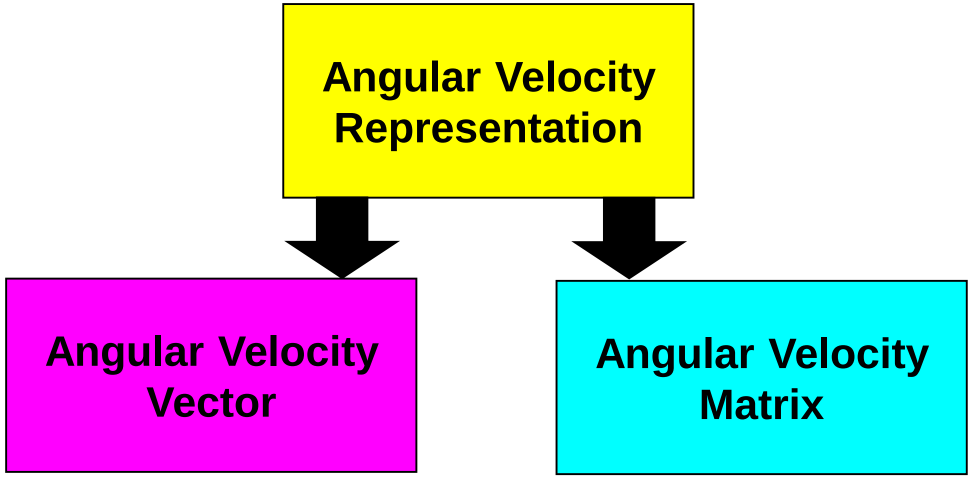
- Using the rotation matrix to remove the dual-superscript,

$$ {}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\;{}^{B}P_Q $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \boxed{{}^{A}\Omega_{B}} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \boxed{{}^{A}_{B}\dot{R}_{\Omega}} \left({}^{A}_{B}R\,{}^{B}P_{Q} \right) $$

Definitions - Angular Velocity

- Just as there are many ways to represent orientation (Euler Angles, Roll-Pitch-Yaw Angles, Rotation Matrices, etc.), there are also many ways to represent the rate of change in orientation.



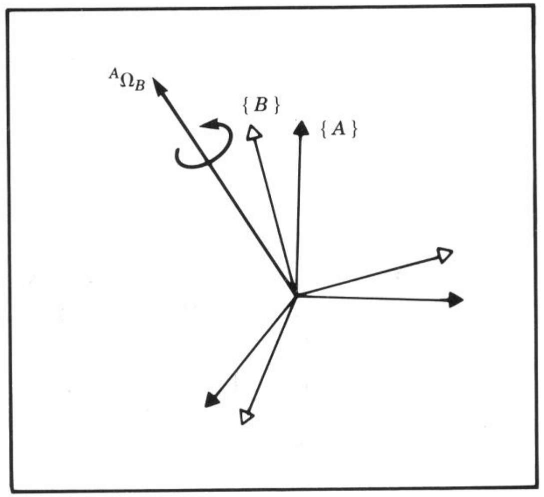
- The angular velocity vector is convenient to use because it has an easy to grasp physical meaning. However, the matrix form is useful when performing algebraic manipulations.

$${}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right)$$

Definitions - Angular Velocity - Vector

- ***Angular Velocity Vector:*** A vector whose direction is the instantaneous axis of rotation of one frame relative to another and whose magnitude is the rate of rotation about that axis.

$${}^A\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$



- The angular velocity vector ${}^A\Omega_B$ describes the instantaneous change of rotation of frame {B} relative to frame {A}

$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}_{B}^{A}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}_{B}^{A}\dot{R}_{\Omega}\left({}_{B}^{A}R\,{}^{B}P_{Q}\right)$$

Definitions - Angular Velocity - Matrix

- ***Angular Velocity Matrix:***

$$\left[{}_{B}^{A}\dot{R}_{\Omega}\right]\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{matrix} -\Omega_z y + \Omega_y z \\ \Omega_z x - \Omega_x z \\ -\Omega_y \mathrm{x} + \Omega_x y \end{matrix}$$

$${}^{A}\Omega_{B} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} i & j & w \\ \Omega_x & \Omega_y & \Omega_z \\ x & y & z \end{vmatrix} = \begin{matrix} \Omega_y z - \Omega_z y \\ -\Omega_x z + \Omega_z x \\ \Omega_{\mathrm{x}} \mathrm{y} - \Omega_y x \end{matrix}$$

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity - Matrix

- The rotation matrix (${}^{A}_{B}R$) defines the orientation of frame {B} relative to frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are the unit vectors of {B} represented in {A}.

$${}^{A}_{B}R = {}^{A}\begin{bmatrix} \left[{}^{B}P_x\right] & \left[{}^{B}P_y\right] & \left[{}^{B}P_z\right] \end{bmatrix}$$

- If we look at the derivative of the rotation matrix, the columns will be the velocity of each unit vector of {B} relative to {A}:

$${}^{A}_{B}\dot{R} = \frac{d}{dt}\left[{}^{A}_{B}R\right] = {}^{A}\begin{bmatrix} \left[{}^{B}V_x\right] & \left[{}^{B}V_y\right] & \left[{}^{B}V_z\right] \end{bmatrix}$$

$$ {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A_B\dot{R}_\Omega \left({}^A_BR\,{}^BP_Q \right) $$

Definitions - Angular Velocity - Matrix

- The relationship between the rotation matrix ${}^A_BR$ and the derivative of the rotation matrix ${}^A_B\dot{R}$ can be expressed as follows:

$$ {}^A_B\dot{R} = {}^A_B\dot{R}_\Omega \, {}^A_BR $$

$$ {}^A\begin{bmatrix} \left[{}^BV_x\right] & \left[{}^BV_y\right] & \left[{}^BV_z\right] \end{bmatrix} = {}^A_B\dot{R}_\Omega \, {}^A\begin{bmatrix} \left[{}^BP_x\right] & \left[{}^BP_y\right] & \left[{}^BP_z\right] \end{bmatrix} $$

- where ${}^A_B\dot{R}_\Omega$ is defined as the ***angular velocity matrix***

$$ {}^A_B\dot{R}_\Omega \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \qquad {}^A\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Matrix & Vector Forms

	Matrix Form	Vector Form
Definition	${}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	${}^{A}\Omega_{B} \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$
Multiply by Constant	$k\ \left[{}^{A}_{B}\dot{R}_{\Omega}\right]$	$k\ \left[{}^{A}\Omega_{B}\right]$
Multiply by Vector	$\left[{}^{A}_{B}\dot{R}_{\Omega}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	${}^{A}\Omega_{B} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \omega \times r$
Multiply by Matrix	$\left[\ {}^{s}_{t}R\ \right] \left[\ {}^{A}_{B}\dot{R}_{\Omega}\ \right] \left[\ {}^{s}_{t}R\ \right]^{T}$	$\left[\ {}^{s}_{t}R\ \right] \left[\ {}^{A}\Omega_{B}\ \right]$

Simultaneous Linear and Rotational Velocity

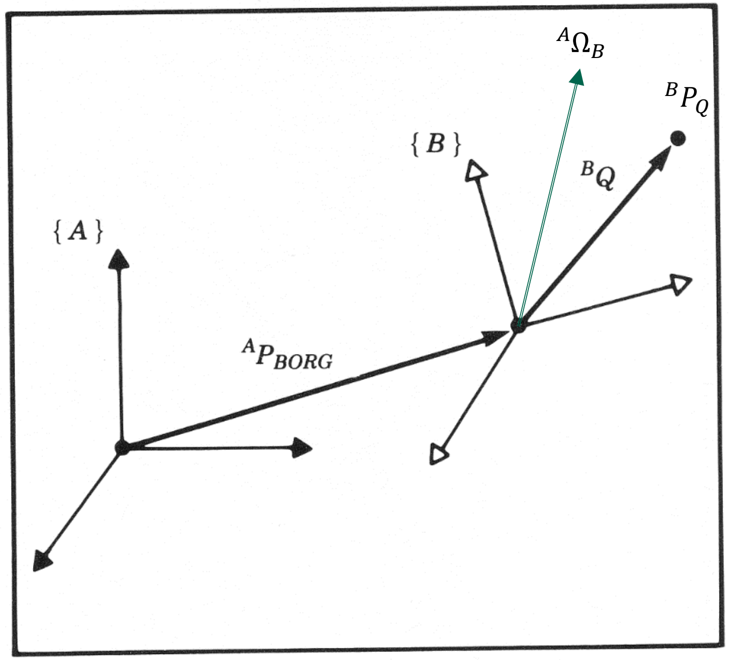
- The final results for the derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame

- Vector Form

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

- Matrix Form

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$



Velocity – Derivation Method No. 3

Homogeneous Transformation Form

Changing Frame of Representation - Linear Velocity

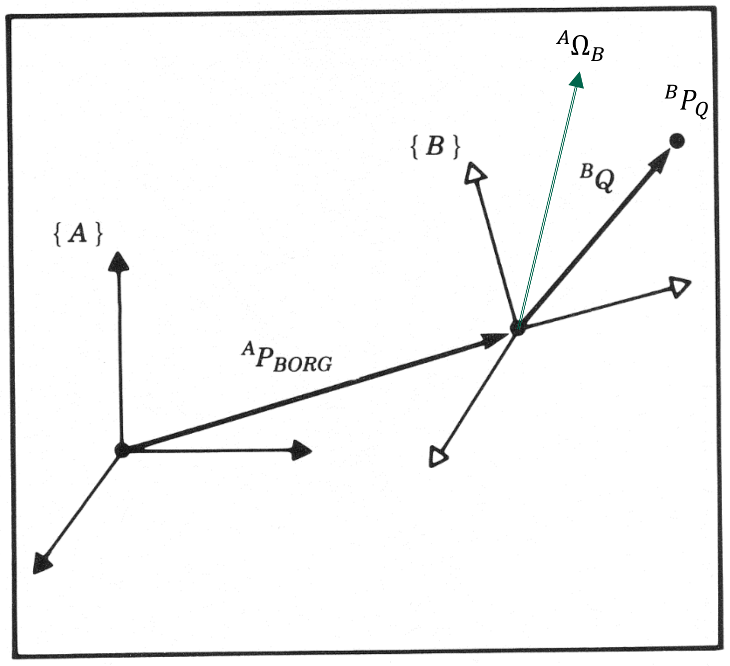
- We have already used the homogeneous transform matrix to compute the location of position vectors in other frames:

$$^{A}P_{Q} = {}^{A}_{B}T\,{}^{B}P_{Q}$$

- To compute the relationship between velocity vectors in different frames, we will take the derivative:

$$\frac{d}{dt}\left[{}^{A}P_{Q}\right] = \frac{d}{dt}\left[{}^{A}_{B}T\,{}^{B}P_{Q}\right]$$

$$^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q}$$



$${}^A\dot{P}_Q = {}^A_B\dot{T}\,{}^BP_Q + {}^A_BT\,{}^B\dot{P}_Q$$

Changing Frame of Representation - Linear Velocity

- Recall that

$${}^A_BT = \begin{bmatrix} & [{}^A_BR] & & [{}^AP_{B\ org}] \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- so that the derivative is

$${}^A_B\dot{T} = \frac{d}{dt}\begin{bmatrix} & [{}^A_BR] & & [{}^AP_{B\ org}] \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & [{}^A_B\dot{R}] & & [{}^A\dot{P}_{B\ org}] \\ 0 & 0 & 0 & 0 \end{bmatrix} == \begin{bmatrix} & [{}^A_B\dot{R}_\Omega\,{}^A_BR] & & [{}^AV_{B\ org}] \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$ {}^A\dot{P}_Q = {}^A_B\dot{T}\,{}^B P_Q + {}^A_B T\,{}^B\dot{P}_Q $$

Changing Frame of Representation - Linear Velocity

$$ {}^A_B\dot{T} = \begin{bmatrix} \multicolumn{3}{c}{\left[{}^A_B\dot{R}_\Omega\,{}^A_B R\right]} & \left[{}^A V_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

- Substitute the previous results into the original equation ${}^A\dot{P}_Q = {}^A_B\dot{T}\,{}^B P_Q + {}^A_B T\,{}^B\dot{P}_Q$ we get

$$ \begin{bmatrix} \left[{}^A V_Q\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^A_B\dot{R}_\Omega \cdot {}^A_B R\right] & \left[{}^A V_{B\,org}\right] \\ 0\;0\;0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^B P_Q\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^A_B R\right] & \left[{}^A P_{B\,org}\right] \\ 0\;0\;0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^B V_Q\right] \\ 0 \end{bmatrix} $$

- This expression is equivalent to the following three-part expression:

$$ {}^A V_Q = {}^A_B\dot{R}_\Omega\left({}^A_B R\,{}^B P_Q\right) + {}^A V_{B\,org} + {}^A_B R\,{}^B V_Q $$

Changing Frame of Representation - Linear Velocity

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\, {}^{B}P_{Q} \right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\, {}^{B}V_{Q} $$

- Converting from matrix to vector form yields

$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times \left({}^{A}_{B}R\, {}^{B}P_{Q} \right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\, {}^{B}V_{Q} $$

Time Derivative of the Rotation Matrix

$$A A^{-1} = I$$

$$R R^{-1} = I$$

$$R^{-1} = R^T$$

$$R R^T = I$$

Rotation around the x axis by θ

$$R_x(\theta) R_x^T(\theta) = I$$

- Take the derivative using the chain rule

$$\left[\frac{d}{d\theta} R_x(\theta)\right] R_x^T(\theta) + R_x(\theta)\left[\frac{d}{d\theta} R^T(\theta)\right] = 0$$

- using the rule $(AB)^T = B^T A^T$

$$\left[\frac{d}{d\theta} R_x(\theta)\right] R_x^T(\theta) + \left(\left[\frac{d}{d\theta} R_x(\theta)\right] R_x(\theta)\right)^T = 0$$

- Introdusing S as the skew symmetric matrix as

$$S = \frac{d}{d\theta} R_x(\theta)\, R_x^T(\theta)$$

Time Derivative of the Rotation Matrix

- Rewriting the equation

$$S + S^T = 0$$

- S is also called the anti symmetric matric

$$S^T = -S$$

- S is always singular therefore

$$\det S = 0$$

Time Derivative of the Rotation Matrix

- Any matrix can be written as the sum of symetric and skew symmetric matrix

- In 3 dimention

$$S(v) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \qquad v = [x, y, z]$$

0 along the diagonal

Time Derivative of the Rotation Matrix

- An alternative way to express the vector cross product

$$a \times b = S(a)\, b$$

↳ $S(a)$: scew symmetric matrix made of the vector a

Time Derivative of the Rotation Matrix

Example – Rotation around x

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$S = \frac{d}{d\theta} R_x(\theta) \; R_x^T(\theta)$$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Time Derivative of the Rotation Matrix

- Recall the definition of $S(\nu)$:

$$S(\nu) = \begin{bmatrix} 0 & -\nu_z & \nu_y \\ \nu_z & 0 & -\nu_x \\ -\nu_y & \nu_x & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S([1,0,0])$$

- the derivative of the rotation around x by θ is equal to

$$\frac{d}{d\theta} R_x(\theta) = S(1,0,0)\, R_x(\theta)$$

Time Derivative of the Rotation Matrix

In a similar fashion

$$\begin{cases} \frac{d}{d\theta} R_x(\theta) = S([1,0,0])\, R_x(\theta) \\ \frac{d}{d\theta} R_y(\theta) = S([0,1,0])\, R_y(\theta) \\ \frac{d}{d\theta} R_z(\theta) = S([0,0,1])\, R_z(\theta) \end{cases}$$

Time Derivative of the Rotation Matrix

- Rotation around the vector L by angle θ

$$\frac{d}{d\theta} R_L(\theta) = S(L)\, R_L(\theta)$$

- Multiply both sides of the expression by $\left(\frac{d\theta}{dt}\right)$

$$\frac{d\theta}{dt}\frac{d}{d\theta} R_L(\theta) = \frac{d\theta}{dt} S(L)\, R_L(\theta)$$

$$\dot{R}_L(\theta) = S(\omega)\, R_L(\theta)$$

$\omega = \dot{\theta} L$

Scenarios 2
Angular Velocity
Changing the Frame of Representation

Sensation 2 – Two Consecutive Rotation

- Given - Three frames i.e frame {A} and frame {B} frame {C} all sharing the same origin. Frame {A} is stationary and Frames {B} and {C} rotate

- Two actions take place simultaneously

  – Frame B rotates with respect to frame A along an axis defined by the vector ${}^{A}\Omega_{B}$
  – Frame C rotates with respect to frame B along an axis defined by the vector ${}^{B}\Omega_{C}$

- **Challenge** – Express the rotation of frame {C} with respect to frame {A} or alternatively express the vector ${}^{A}\Omega_{C}$

Angular Velocity – Changing the Frame of Representation – Scenario No.2

- Angular Velocity Representation in Various Frames
  - Vector Form $${}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$
  - Matrix Form $${}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}$$

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} \qquad {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$

Changing Frame of Representation - Angular Velocity

- We use rotation matrices to represent angular position so that we can compute the angular position of {C} in {A} if we know the angular position of {C} in {B} and {B} in {A} by

$$ {}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R $$

- To derive the relationship describing how angular velocity propagates between frames, we will take the derivative

$$ {}^{A}_{C}\dot{R} = {}^{A}_{B}\dot{R}\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R} $$

- Substituting the angular velocity matrixes

$$ {}^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R \qquad {}^{B}_{C}\dot{R} = {}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R \qquad {}^{A}_{C}\dot{R} = {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R $$

- we find

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

Changing Frame of Representation - Angular Velocity

- Post-multiplying both sides by ${}^{A}_{C}R^{T}$,which for rotation matrices, is equivalent to ${}^{A}_{C}R^{-1}$

$${}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R\,{}^{A}_{C}R^{T} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{C}R\,{}^{A}_{C}R^{T} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R\,{}^{A}_{C}R^{T}$$

$${}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}$$

- The above equation provides the relationship for changing the frame of representation of angular velocity matrices.

- The vector form is given by

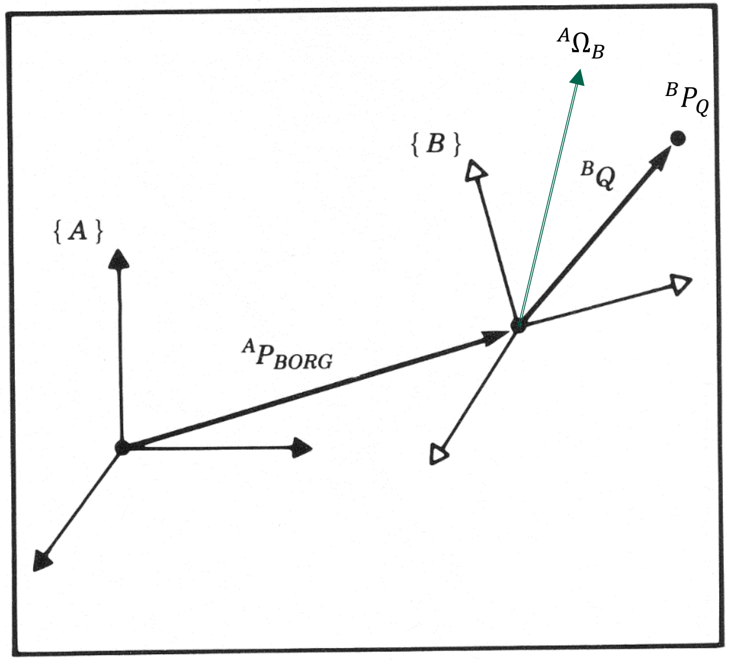
$${}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$

- To summarize, the angular velocities of frames may be added as long as they are expressed in the same frame.

Summary

Simultaneous Linear and Rotational Velocity - Scenario No.1

$$^{A}V_Q = f(^{B}P_Q,\ ^{B}V_Q,\ ^{A}V_{BORG},\ ^{A}\Omega_B,\ ^{A}_{B}R)$$



- Vector Form (Method No. 1)

$$^{A}V_Q = {^{A}V_{BORG}} + {^{A}_{B}R}\,{^{B}V_Q} + {^{A}\Omega_B} \times {^{A}_{B}R}\,{^{B}P_Q}$$

- Matrix Form (Method No. 2)

$$^{A}V_Q = {^{A}V_{BORG}} + {^{A}_{B}R}\,{^{B}V_Q} + {^{A}_{B}\dot{R}_\Omega}\left({^{A}_{B}R}\,{^{B}P_Q}\right)$$

- Matrix Formulation – Homogeneous Transformation Form – Method No. 3

$$\begin{bmatrix} [^{A}V_Q] \\ 0 \end{bmatrix} = \begin{bmatrix} [{^{A}_{B}\dot{R}_\Omega} \cdot {^{A}_{B}R}] & [^{A}V_{B\ org}] \\ 0\ 0\ 0 & 0 \end{bmatrix} \begin{bmatrix} [^{B}P_Q\ ] \\ 1 \end{bmatrix} + \begin{bmatrix} [^{A}_{B}R] & [^{A}P_{B\ org}] \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} [^{B}V_Q] \\ 0 \end{bmatrix}$$

Angular Velocity – Changing the Frame of Representation – Scenario No.2

- Angular Velocity Representation in Various Frames
  - Vector Form $${}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$
  - Matrix Form $${}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}$$

Linear Algebra - Review

Brief Linear Algebra Review - 1/

- Inverse of Matrix A exists ***if and only if*** the determinant of A is non-zero.

A^{-1} Exists ***if and only if***

$$Det(A) = |A| \neq 0$$

- If the determinant of A is equal to zero, then the matrix A is a singular matrix

$$Det(A) = |A| = 0$$

A Singular

Brief Linear Algebra Review - 2/

- The rank of the matrix A is the size of the largest squared Matrix S for which

$$Det(S) \neq 0$$

- Example 1 -

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad A = S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad |A| = |S| = 3 \quad Rank(A) = 2$$

- Example 2 -

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad S = [1] \quad |S| = 1 \quad Rank(A) = 1$$

Brief Linear Algebra Review - 3/

- If two rows or columns of matrix A are equal or related by a constant, then

$$Det(A) = 0$$

- Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & -3 \\ 10 & -6 & -5 \end{bmatrix}$$

$$\det(A) = |A| = 2\begin{vmatrix} -3 & -3 \\ -6 & -5 \end{vmatrix} - 0\begin{vmatrix} 6 & -3 \\ 10 & -5 \end{vmatrix} - 1\begin{vmatrix} 6 & -3 \\ 10 & -6 \end{vmatrix} = 6 + 0 - 6 = 0$$

Brief Linear Algebra Review - 4/

- ***Eigenvalues***

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

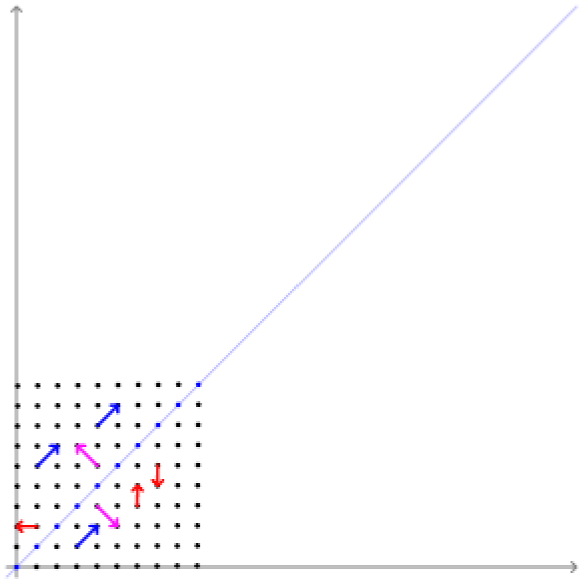
- Eigenvalues are the roots of the polynomial

$$Det(A - \lambda I)$$

- If $X \neq 0$ each solution to the characteristic equation λ (Eigenvalue) has a corresponding Eigenvector

Brief Linear Algebra Review - 4/

- Wikipedai - https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors



Brief Linear Algebra Review - 4/

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A - \lambda I)X = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$Det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

Brief Linear Algebra Review - 4/

$$\lambda_1 = 1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Brief Linear Algebra Review - 5/

- Any singular matrix ($Det(A) = 0$) has at least one Eigenvalue equal to zero

Brief Linear Algebra Review - 6/

- If A is non-singular ($Det(A) \neq 0$), and λ is an eigenvalue of A with corresponding to eigenvector X, then

$$A^{-1}X = \lambda^{-1}X$$

Brief Linear Algebra Review - 7/

- If the $n \times n$ matrix A is of full rank (that is, $\mathbf{Rank\ (A) = n}$), then the only solution to

$$AX = 0$$

is the trivial one

$$X = 0$$

- If A is of less than full rank (that is $\mathbf{Rank\ (A) < n}$), then there are $n\text{-}r$ linearly independent (orthogonal) solutions

for which x_j $\quad 0 \le j \le n - r$

$$Ax_j = 0$$

Brief Linear Algebra Review - 8/

- If A is square, then A and A^T have the same eigenvalues