MAE 263D

HW3 Solutions (Refer to HW1, HW2 for forward and inverse kinematic solutions.)

1) Direct Differentiation

$${}^0P_{T,x} = x = L_1c_1 + L_2c_{12} \qquad \dot{x} = -L_1s_1\dot{\theta}_1 - L_2s_{12}(\dot{\theta}_1+\dot{\theta}_2)$$

$${}^0P_{T,y} = y = L_1s_1 + L_2s_{12} \qquad \dot{y} = L_1c_1\dot{\theta}_1 + L_2c_{12}(\dot{\theta}_1+\dot{\theta}_2)$$

$${}^0P_{T,\alpha} = \alpha = \theta_1 + \theta_2 \qquad \dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2$$

$$\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}-L_1s_1-L_2s_{12} & -L_2s_{12}\\ L_1c_1+L_2c_{12} & L_2c_{12}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = {}^0J\dot{\vec{\theta}}$$

$$\Rightarrow {}^0J = \begin{bmatrix}-L_1s_1-L_2s_{12} & -L_2s_{12}\\ L_1c_1+L_2c_{12} & L_2c_{12}\\ 1 & 1\end{bmatrix}$$

or full version ${}^0J = \begin{bmatrix}{}^0J_v\\ {}^0J_w\end{bmatrix} = \begin{bmatrix}-L_1s_1-L_2s_{12} & -L_2s_{12}\\ L_1c_1+L_2c_{12} & L_2c_{12}\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}$

Velocity Propagation

$${}^0w_0 = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad {}^1w_1 = {}^1_0R\,{}^0w_0 + \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} = \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}$$

$${}^2w_2 = {}^2_1R\,{}^1w_1 + \begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{\theta}_2\end{bmatrix} = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}$$

$${}^3w_3 = {}^3_2R\,{}^2w_2 + \begin{bmatrix}0\\0\\0\end{bmatrix} = I\,{}^2w_2 = \begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix}$$

$${}^1v_1 = {}^1_0R({}^0w_0 \times {}^0P_1 + {}^0v_0) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$${}^2v_2 = {}^2_1R({}^1w_1 \times {}^1P_2 + {}^1v_1) = \begin{bmatrix}c_2 & s_2 & 0\\ -s_2 & c_2 & 0\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix} \times \begin{bmatrix}L_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}L_1s_2\dot{\theta}_1\\ L_1c_2\dot{\theta}_1\\ 0\end{bmatrix}$$

$${}^3v_3 = {}^3_2R({}^2w_2 \times {}^2P_3 + {}^2v_2) = I\left(\begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix} \times \begin{bmatrix}L_2\\0\\0\end{bmatrix} + {}^2v_2\right) = \begin{bmatrix}s_2L_1\dot{\theta}_1\\ c_2L_1\dot{\theta}_1 + L_2(\dot{\theta}_1+\dot{\theta}_2)\\ 0\end{bmatrix}$$

$$\Rightarrow {}^3J = \begin{bmatrix}{}^3J_v\\ {}^3J_w\end{bmatrix} = \begin{bmatrix}s_2L_1 & 0\\ c_2L_1+L_2 & L_2\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ 1 & 1\end{bmatrix}$$

## Force Propagation

$$ {}^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} & I & \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x c_2 - f_y s_2 \\ f_x s_2 + f_y c_2 \\ 0 \end{bmatrix} $$

$$ {}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L_2 f_y \end{bmatrix} $$

$$ {}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ L_2 f_y \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} f_x c_2 - f_y s_2 \\ f_x s_2 + f_y c_2 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 \\ 0 \\ L_1 f_x s_2 + (L_1 c_2 + L_2) f_y \end{bmatrix} $$

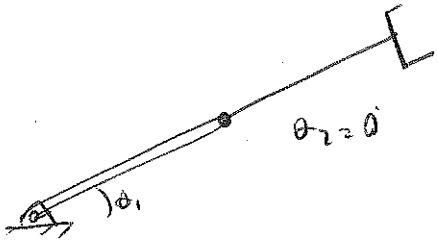
$$ \tau_i = {}^1n_i^T\,\hat{z}_i $$

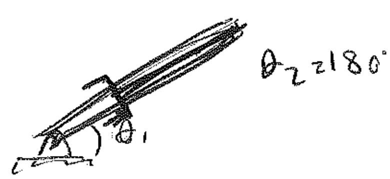
$$ \tau = \begin{bmatrix} L_1 s_1 & L_1 c_2 + L_2 & 0 & 0 & 0 & 1 \\ 0 & L_2 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix} $$

$$ \Rightarrow {}^3J = \begin{bmatrix} L_1 s_1 & 0 \\ L_1 c_2 + L_2 & L_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} $$

## Singularities

$$ \det({}^0J) = 0 \Rightarrow \begin{vmatrix} L_1 s_2 & 0 \\ L_2 + L_1 c_2 & L_2 \end{vmatrix} = L_1 L_2 s_2 = 0 \Rightarrow \text{Singularity } \theta_2 = 0^\circ,\ \theta_2 = 180^\circ $$



## 2) Direct Differentiation

$$\Rightarrow \dot{x} = -l_1 s_1(\dot\theta_1) - l_1 s_{12}(\dot\theta_1 + \dot\theta_2)$$

$$\dot{y} = l_1 s_1(\dot\theta_1) + L_2 c_{12}(\dot\theta_1 + \dot\theta_2)$$

$$\dot\alpha = \dot\theta_1 + \dot\theta_2 + \dot\theta_3$$

$$\Rightarrow {}^0J = \begin{bmatrix} {}^0J_v \\ {}^0J_w \end{bmatrix} = \begin{bmatrix} -L_1 s_1 - l_2 s_{12} & -l_2 s_{12} & 0 \\ l_2 c_{12} + L_1 c_1 & L_2 c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

## Velocity Propagation

$${}^0w_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^1w_1 = {}^1_0R\,{}^0w_0 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \qquad {}^2w_2 = {}^2_1R\,{}^1w_1 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix}$$

$${}^1w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \qquad {}^2w_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}$$

$${}^3w_3 = {}^3_2R\,{}^2w_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 + \dot\theta_3 \end{bmatrix} \Rightarrow {}^3w_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{bmatrix}$$

$${}^1v_1 = {}^1_0R\,({}^0w_0 \times {}^0P_1 + {}^0v_0) = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2v_2 = {}^2_1R\,({}^1w_1 \times {}^1P_2 + {}^1v_1) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} L_1 s_2 \dot\theta_1 \\ L_1 c_2 \dot\theta_1 \\ 0 \end{bmatrix}$$

$${}^3v_3 = {}^3_2R\,({}^2w_2 \times {}^2P_3 + {}^2v_2) = \begin{bmatrix} L_1 s_{23}\dot\theta_1 + L_2 s_3(\dot\theta_1 + \dot\theta_2) \\ L_1 c_{23}\dot\theta_1 + L_2 c_3(\dot\theta_1 + \dot\theta_2) \\ 0 \end{bmatrix}$$

$$\Rightarrow {}^3J = \begin{bmatrix} L_1 s_{23} + L_2 s_3 & L_2 s_3 & 0 \\ L_1 c_{23} + L_2 c_3 & L_2 c_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

# Force Propagation

$$ {}^{3}f_{3} = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^{2}f_{2} = {}^{2}_{3}R\,{}^{3}f_{3} = \begin{bmatrix} c_3 f_x - s_3 f_y \\ s_3 f_x - c_3 f_y \\ 0 \end{bmatrix} $$

$$ {}^{1}f_{1} = {}^{1}_{2}R\,{}^{2}f_{2} = \begin{bmatrix} c_{23} f_x - s_{23} f_y \\ s_{23} f_x + c_{23} f_y \end{bmatrix} $$

$$ {}^{3}n_{3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{2}n_{2} = {}^{2}_{3}R\,{}^{3}n_{3} + {}^{2}P_{3} \times {}^{2}f_{2} = \begin{bmatrix} 0 \\ 0 \\ l_2 s_3 f_x + l_2 c_3 f_y \end{bmatrix} $$

$$ {}^{1}n_{1} = {}^{1}_{2}R\,{}^{2}n_{2} + {}^{1}P_{2} \times {}^{1}f_{1} = \begin{bmatrix} 0 \\ 0 \\ l_2 s_3 f_x + l_1 s_{23} f_x + l_2 c_3 f_y + l_1 c_{23} f_y \end{bmatrix} $$

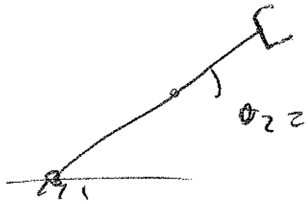
$$ \Rightarrow \tau = \begin{bmatrix} l_2 s_3 + l_1 s_{23} & l_2 c_3 + l_1 c_{23} & 0 & 0 & 0 & 1 \\ l_2 s_3 & l_2 c_3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix} $$

$$ \Rightarrow {}^{3}J = \begin{bmatrix} l_1 s_{23} + l_2 s_3 & l_2 s_3 & 0 \\ l_1 c_{23} + l_2 c_3 & l_2 c_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} $$

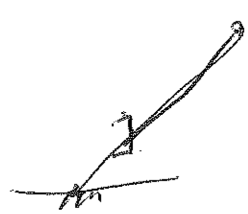
## Singularities

$$ \Rightarrow \det(J) = 0 \Rightarrow \begin{vmatrix} l_1 s_{23} + l_2 s_3 & l_2 s_3 \\ l_1 c_{23} + l_2 c_3 & l_2 c_3 \end{vmatrix} = 0 \Rightarrow \underline{\theta_2 = 0^\circ, 180^\circ} $$

$\Rightarrow$



$\theta_2 = 0$



$\theta_2 = 180^\circ$

## 3) Direct Differentiation

$$\Rightarrow \dot{x} = -L_3 c_1 s_{23}(\dot\theta_2 + \dot\theta_3) - L_3 c_{23}\dot\theta_1 s_1 - L_2 c_1 s_2 \dot\theta_2 - L_2 c_2 s_1 \dot\theta_1$$

$$\dot{y} = -L_3 s_1 s_{23}(\dot\theta_2 + \dot\theta_3) + L_3 c_{23} c_1 \dot\theta_1 + L_2 c_2 c_1 \dot\theta_1 - L_2 s_1 s_2 \dot\theta_2$$

$$\dot{z} = -L_3 c_{23}(\dot\theta_2 + \dot\theta_3) - L_2 c_2 \dot\theta_2$$

$$^0_4T = \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & -s_1 & L_3 c_1 c_{23} + L_2 c_1 c_2 \\ s_1 c_{23} & -s_1 s_{23} & c_1 & L_3 s_1 c_{23} + L_2 c_2 s_1 \\ -s_{23} & -c_{23} & 0 & -L_3 s_{23} - L_2 s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow {}^0J = \begin{bmatrix} -s_1(L_3 c_{23} + L_2 c_2) & -c_1(L_3 s_{23} + L_2 s_2) & -L_3 s_{23} c_1 \\ c_1(L_3 c_{23} + L_2 c_2) & -s_1(L_3 s_{23} + L_2 s_2) & -L_3 s_{23} s_1 \\ 0 & -L_3 c_{23} - L_2 c_2 & -L_3 c_{23} \\ 0 & -s_1 & -s_1 \\ 0 & c_1 & c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

## Velocity Propagation

$$^1w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \qquad {}^2w_2 = {}^2_1R\, {}^1w_1 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} -s_2\dot\theta_1 \\ -c_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

$$^3w_3 = {}^3_2R\, {}^2w_2 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_3 \end{bmatrix} = \begin{bmatrix} -s_{23}\dot\theta_1 \\ -c_{23}\dot\theta_1 \\ \dot\theta_2 + \dot\theta_3 \end{bmatrix}$$

$$^4w_4 = {}^4_3R\, {}^3w_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = {}^3w_3$$

$$^1v_1 = {}^1_0R\left({}^0w_0 \times {}^0P_1 + {}^0v_0\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^2v_2 = {}^2_1R\left({}^1w_1 \times {}^1P_2 + {}^1v_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$^3v_3 = {}^3_2R\left({}^2w_2 \times {}^2P_3 + {}^2v_2\right) = \begin{bmatrix} L_2 s_3 \dot\theta_2 \\ L_2 c_3 \dot\theta_2 \\ L_2 c_2 \dot\theta_1 \end{bmatrix}$$

$$^4v_4 = {}^4_3R\left({}^3w_3 \times {}^3P_4 + {}^3v_3\right) = \begin{bmatrix} L_2 s_3 \dot\theta_2 \\ L_3\dot\theta_2 + L_2 c_3 \dot\theta_2 + L_3\dot\theta_3 \\ L_3 c_{23}\dot\theta_1 + L_2 c_2 \dot\theta_1 \end{bmatrix}$$

$$\Rightarrow {}^4J = \begin{bmatrix} 0 & L_2S_3 & 0 \\ 0 & L_3+L_2C_3 & L_3 \\ L_3C_{23}+L_2C_2 & 0 & 0 \\ -S_{23} & 0 & 0 \\ -C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

## Force Propagation

$${}^4f_4 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad {}^3f_3 = {}^3_4R\,{}^4f_4 = {}^4f_4$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} C_3f_x - S_3f_y \\ S_3f_x + C_3f_y \\ f_z \end{bmatrix}$$

$${}^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} C_{23}f_x - S_{23}f_y \\ f_z \\ -f_xS_{23} - C_{23}f_y \end{bmatrix}$$

$${}^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3 = \begin{bmatrix} 0 \\ -L_3f_z \\ L_3f_y \end{bmatrix}$$

$${}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} L_3f_zS_3 \\ -C_3L_3f_z - L_2f_z \\ L_3f_y + L_2S_3f_x + L_2C_3f_y \end{bmatrix}$$

$${}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} L_3S_{23}f_z + L_2S_2f_z \\ L_2S_3f_x + L_2C_3f_y + L_3f_y \\ L_3C_{23}f_z + L_2C_2f_z \end{bmatrix}$$

$$\Rightarrow \tau = \begin{bmatrix} 0 & 0 & L_2C_2+L_3C_{23} & -S_{23} & -C_{23} & 0 \\ L_2S_3 & L_2C_3+L_3 & 0 & 0 & 0 & 1 \\ 0 & L_3 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

$$\Rightarrow {}^4J = \begin{bmatrix} 0 & L_2S_3 & 0 \\ 0 & L_2C_3+L_3 & L_3 \\ L_2C_2+L_3C_{23} & 0 & 0 \\ -S_{23} & 0 & 0 \\ -C_{23} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

## Singularities

$\Rightarrow \det({}^{0}J) = 0 \Rightarrow \begin{vmatrix} 0 & L_2 S_3 & 0 \\ 0 & L_2 C_3 + L_3 & L_3 \\ L_2 C_2 + L_3 C_{23} & 0 & 0 \end{vmatrix} = 0$

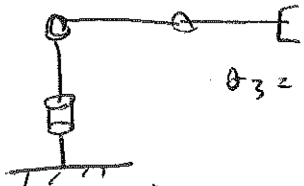
$\Rightarrow L_2^2 L_3 S_3 C_2 + L_3^2 L_2 S_3 C_{23} = 0$

$S_3 (L_2 C_2 + L_3 C_{23}) = 0$

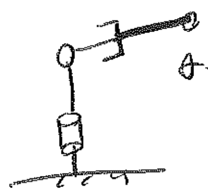
$\Rightarrow$ Singularity when $\theta_3 = 0°$

$\theta_3 = 180°$

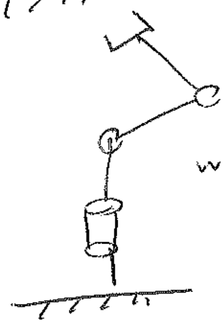
and when $L_2 C_2 + L_3 C_{23} = 0$



$\theta_3 = 0°$



$\theta_3 = 180°$



when $\underline{L_2 C_2 + L_3 C_{23} = 0}$

## 4) Direct Differentiation

$$
{}^{0}_{3}T = \begin{bmatrix} c_1 c_2 & -s_1 & -c_1 s_2 & -d_3 c_1 s_2 + L_2 s_1 \\ c_2 s_1 & c_1 & -s_1 s_2 & -d_3 s_1 s_2 + L_2 c_1 \\ s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$\dot{x} = L_2 c_1 \dot{\theta}_1 + d_3 s_1 s_2 \dot{\theta}_1 - d_3 c_1 c_2 \dot{\theta}_2 - c_1 s_2 \dot{d}_3$$

$$\dot{y} = L_2 s_1 \dot{\theta}_1 - d_3 c_1 s_2 \dot{\theta}_1 - d_3 c_2 s_1 \dot{\theta}_2 - s_1 s_2 \dot{d}_3$$

$$\hat{z} = -d_3 s_2 \dot{\theta}_2 + \dot{d}_3 c_2$$

$$
\Rightarrow {}^{0}J = \begin{bmatrix} L_2 c_1 + d_3 s_1 s_2 & -d_3 c_1 c_2 & -c_1 s_2 \\ L_2 s_1 - d_3 c_1 s_2 & -d_3 c_2 s_1 & -s_1 s_2 \\ 0 & -d_3 s_2 & c_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

## Velocity Propagation

$$
{}^{1}w_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad {}^{2}w_2 = {}^{2}_{1}R\, {}^{1}w_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} s_2 \dot{\theta}_1 \\ c_2 \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}
$$

$$
{}^{3}w_3 = {}^{3}_{2}R\, {}^{2}w_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s_2 \dot{\theta}_1 \\ -\dot{\theta}_2 \\ c_2 \dot{\theta}_1 \end{bmatrix}
$$

$$
{}^{i}v_i = {}^{i}_{i-1}R \left( {}^{i-1}w_{i-1} \times {}^{i-1}P_i + {}^{i-1}v_{i-1} \right)
$$

$$
\Rightarrow {}^{1}v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{2}v_2 = \begin{bmatrix} L_2 c_2 \dot{\theta}_1 \\ -L_2 s_2 \dot{\theta}_1 \\ 0 \end{bmatrix} \qquad {}^{3}v_3 = \begin{bmatrix} L_2 c_2 \dot{\theta}_1 - d_3 \dot{\theta}_2 \\ -d_3 s_2 \dot{\theta}_1 \\ -L_2 s_2 \dot{\theta}_1 + \dot{d}_3 \end{bmatrix}
$$

$$
\Rightarrow {}^{3}J = \begin{bmatrix} L_2 c_2 & -d_3 & 0 \\ -d_3 s_2 & 0 & 0 \\ -L_2 s_2 & 0 & 1 \\ s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}
$$

## Force Propagation

$$^3f_3 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$$^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} f_x \\ f_z \\ -f_y \end{bmatrix}$$

$$^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} c_2 f_x - f_z s_2 \\ f_y \\ s_2 f_x + c_2 f_z \end{bmatrix}$$

$$^3n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} -d_3 f_y \\ 0 \\ -d_3 f_x \end{bmatrix}$$

$$^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} -d_3 c_2 f_y - l_2 s_2 f_x - l_2 c_2 f_z \\ d_3 f_x \\ -d_3 s_2 f_y + l_2 c_2 f_x - l_2 s_2 f_z \end{bmatrix}$$

$$\tau = \begin{pmatrix} l_2 c_2 & -d_3 s_2 & -l_2 s_2 & s_2 & 0 & c_2 \\ -d_3 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{pmatrix}$$

$$\Rightarrow {}^3J = \begin{bmatrix} l_2 c_2 & -d_3 & 0 \\ -d_3 s_2 & 0 & 0 \\ -l_2 s_2 & 1 & 1 \\ s_2 & 0 & 0 \\ 0 & -1 & 0 \\ c_2 & 0 & 0 \end{bmatrix}$$

### Singularity

$\Rightarrow \det(J) = 0 \Rightarrow d_3(-d_3 s_2) = 0 \Rightarrow$ Singularity when $\theta_2 = 0°$

$\theta_2 = 180°$

## 5) Direct Differentiation

$$ {}^{0}_{3}T = \begin{bmatrix} -s_1 & 0 & c_1 & d_3 c_1 \\ c_1 & 0 & s_1 & d_3 s_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \dot{x} = -d_3 s_1 \dot{\theta}_1 + c_1 \dot{d}_3 $$
$$ \dot{y} = d_3 c_1 \dot{\theta}_1 + s_1 \dot{d}_3 $$
$$ \dot{z} = \dot{d}_2 $$

$$ \Rightarrow {}^{0}J = \begin{bmatrix} -d_3 s_1 & 0 & c_1 \\ d_3 c_1 & 0 & s_1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} $$

## Velocity Propagation

$$ {}^{0}w_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^{1}w_1 = {}^{1}_{0}R\,{}^{0}w_0 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^{2}w_2 = {}^{2}_{1}R\,{}^{1}w_1 \cancel{+ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} $$

$$ {}^{3}w_3 = {}^{3}_{2}R\,{}^{2}w_2 = \begin{bmatrix} 0 \\ \dot{\theta}_1 \\ 0 \end{bmatrix} $$

$$ {}^{i}v_i = {}^{i}_{i-1}R\left({}^{i-1}w_{i-1} \times {}^{i-1}P_i + {}^{i-1}v_{i-1}\right) $$

$$ \Rightarrow {}^{1}v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^{2}v_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} \qquad {}^{3}v_3 = \begin{bmatrix} d_3 \dot{\theta}_1 \\ \dot{d}_2 \\ \dot{d}_3 \end{bmatrix} $$

$$ \Rightarrow {}^{3}J = \begin{bmatrix} d_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} $$

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# Force Propagation

$${}^3f_3 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 = \begin{pmatrix} f_x \\ -f_z \\ f_y \end{pmatrix}$$

$${}^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} f_z \\ f_x \\ f_y \end{bmatrix}$$

$${}^3n_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2n_2 = {}^2P_3 \times {}^2f_2 + {}^2_3R\,{}^3n_3 = \begin{pmatrix} -d_3f_y \\ 0 \\ d_3f_x \end{pmatrix}$$

$${}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} -d_2f_x \\ -d_3f_y + d_2f_z \\ -d_3f_x \end{bmatrix}$$

$$\Rightarrow \tau = \begin{pmatrix} -d_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

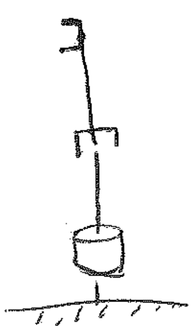
$${}^3J = \begin{pmatrix} -d_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

## Singularity

$$\Rightarrow \det({}^0J) = 0 \Rightarrow \begin{vmatrix} d_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 \Rightarrow d_3 = 0$$

$\Rightarrow$ Singularity when $d_3 = 0$

Note: This case is only a singularity for position not orientation



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6) Direct Differentiation

$$\dot{x} = 0 \qquad \dot{y} = 0 \qquad \dot{z} = 0$$

$$^3_6T = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_6s_4 - c_4c_5 & -c_4s_5 & 0 \\ c_4s_6 + c_5c_6s_4 & c_4c_6 - c_5s_4s_6 & -s_4s_5 & 0 \\ c_6s_5 & -s_5 s_4 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0\theta_x = s_4\theta_5 - c_4s_5\theta_6$$

$$^0\theta_y = -c_4\theta_5 - s_4s_5\theta_6 \quad \Rightarrow \quad ^0J_w = \begin{bmatrix} 0 & s_4 & -c_4s_5 \\ 0 & -c_4 & -s_4s_5 \\ 1 & 0 & c_5 \end{bmatrix}$$

$$^0\theta_z = \theta_4 + c_5\theta_6$$

$$\Rightarrow \ ^0J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & s_4 & -c_4s_5 \\ 0 & -c_4 & -s_4s_5 \\ 1 & 0 & c_5 \end{bmatrix}$$

Velocity Propagation

$$^4\omega_4 = \cancel{^4_3R\,^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix}} = \begin{bmatrix} c_4 & s_4 & 0 \\ -s_4 & c_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix}$$

$$^4_3R\,^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix}$$

$$^5\omega_5 = {}^5_4R\,^4\omega_4 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_5 \end{bmatrix} = \begin{bmatrix} s_5\dot{\theta}_4 \\ c_5\dot{\theta}_4 \\ \dot{\theta}_5 \end{bmatrix}$$

$$^6\omega_6 = {}^6_5R\,^5\omega_5 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_6 \end{bmatrix} = \begin{bmatrix} c_6s_5\dot{\theta}_4 - s_6\dot{\theta}_5 \\ -s_5s_6\dot{\theta}_4 - c_6\dot{\theta}_5 \\ c_5\dot{\theta}_4 + \dot{\theta}_6 \end{bmatrix}$$

$$^iv_i = {}^i_{i-1}R\left(^{i-1}\omega_{i-1} \times {}^{i-1}P_i + {}^{i-1}v_{i-1}\right)$$

$$\Rightarrow \ ^4v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ^5v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad ^6v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \ ^6J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_5c_6 & -s_6 & 0 \\ -s_5s_6 & -c_6 & 0 \\ c_5 & 0 & 1 \end{bmatrix}$$

# Force Propagation

$$ {}^6f_6 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad {}^6n_6 = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} $$

$$ {}^5f_5 = {}^5_6R\,{}^6f_6 = \begin{bmatrix} c_6 f_x - s_6 f_y \\ f_z \\ -s_6 f_x - c_6 f_y \end{bmatrix} $$

$$ {}^4f_4 = {}^4_5R\,{}^5f_5 = \begin{bmatrix} c_5 c_6 f_x - c_5 s_6 f_y - s_5 f_z \\ s_6 f_x + c_6 f_y \\ c_6 s_5 f_x - s_5 s_6 f_y + c_5 f_z \end{bmatrix} $$

$$ {}^5n_5 = {}^5_6R\,{}^6n_6 + {}^5P_6 \times {}^5f_5 = \begin{bmatrix} c_6 n_x - s_6 n_y \\ n_z \\ -s_6 n_x - c_6 n_y \end{bmatrix} $$ → only $n_x, n_y, n_z$ terms

$$ {}^4n_4 = {}^4_5R\,{}^5n_5 + {}^4P_5 \times {}^4f_4 = \begin{bmatrix} c_5 c_6 n_x - c_5 s_6 n_y - s_5 n_z \\ s_6 n_x + c_6 n_y \\ s_5 c_6 n_x - s_6 s_5 n_y + c_5 n_z \end{bmatrix} $$

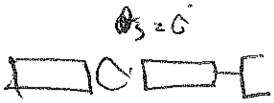
$$ \Rightarrow \quad \tau = \begin{bmatrix} 0 & 0 & 0 & s_5 c_6 & -s_5 s_6 & c_5 \\ 0 & 0 & 0 & -s_6 & -c_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix} $$

$$ \Rightarrow \quad {}^3J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_5 c_6 & -s_6 & 0 \\ -s_5 s_6 & -c_6 & 0 \\ c_5 & 0 & 1 \end{bmatrix} $$

# Singularity

$$ \det({}^6J) = 0 \Rightarrow \begin{vmatrix} c_6 s_5 & -s_6 & 0 \\ -s_5 s_6 & -c_6 & 0 \\ c_5 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{matrix} \theta_5 = 0 \\ \theta_5 = 180^\circ \end{matrix} $$

$\theta_5 = 0^\circ$



$\theta_5 = 180^\circ$

## 7) Direct Differentiation

$$\dot{x} = (-L_2 S_{12} - L_1 S_1)\dot{\theta}_1 - L_2 S_{12}\dot{\theta}_2$$
$$\dot{y} = (L_2 C_{12} + L_1 C_1)\dot{\theta}_1 + L_2 C_{12}\dot{\theta}_2$$
$$\dot{z} = -\dot{d}_4$$
$$\dot{\theta}_x = 0$$
$$\dot{\theta}_y = 0$$
$$\dot{\theta}_z = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

$$\Rightarrow {}^0J = \begin{pmatrix} -L_2 S_{12} - L_1 S_1 & -L_2 S_{12} & 0 & 0 \\ L_2 C_{12} + L_1 C_1 & L_2 C_{12} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

## Velocity Propagation

$${}^0w_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^1w_1 = {}^1_0R\,{}^0w_0 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

$${}^2w_2 = {}^2_1R\,{}^1w_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$${}^3w_3 = {}^3_2R\,{}^2w_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} = {}^4w_4$$

$${}^iv_i = {}^i_{i-1}R\left({}^{i-1}w_{i-1} \times {}^{i-1}P_i + {}^{i-1}v_{i-1}\right)$$

$$\Rightarrow {}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^2v_2 = \begin{pmatrix} L_1 S_2 \dot{\theta}_1 \\ L_1 C_2 \dot{\theta}_1 \\ 0 \end{pmatrix} \quad {}^3v_3 = \begin{bmatrix} (L_1 S_{23} + L_2 S_3)\dot{\theta}_1 + L_2 S_3 \dot{\theta}_2 \\ (L_1 C_{23} + L_2 C_3)\dot{\theta}_1 + L_2 C_3 \dot{\theta}_2 \\ 0 \end{bmatrix}$$

$${}^4v_4 = \begin{bmatrix} (L_1 S_{23} + L_2 S_3)\dot{\theta}_1 + L_2 S_3 \dot{\theta}_2 \\ (L_1 C_{23} + L_2 C_3)\dot{\theta}_1 + L_2 C_3 \dot{\theta}_2 \\ -\dot{d}_4 \end{bmatrix}$$

$$\Rightarrow {}^4J = \begin{bmatrix} L_1 S_{23} + L_2 S_3 & L_2 S_3 & 0 & 0 \\ L_1 C_{23} + L_2 C_3 & L_2 C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

## Force Propagation

$${}^4f_4 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \qquad {}^4n_4 = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}$$

$${}^3f_3 = {}^3_4R\,{}^4f_4 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} C_3 f_x - S_3 f_y \\ S_3 f_x + C_3 f_y \\ f_z \end{bmatrix}$$

$${}^1f_1 = {}^1_2R\,{}^2f_2 = \begin{bmatrix} C_{23} f_x - S_{23} f_y \\ S_{23} f_x + C_{23} f_y \\ f_z \end{bmatrix}$$

$${}^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3 = \begin{bmatrix} -d_4 f_y \\ d_4 f_x \\ 0 \end{bmatrix}$$

$${}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2 = \begin{bmatrix} -S_3 d_4 f_x - C_3 d_4 f_y \\ C_3 d_4 f_x - S_3 d_4 f_y \\ L_2 S_3 f_x + L_2 C_3 f_y \end{bmatrix}$$

$${}^1n_1 = {}^1_2R\,{}^2n_2 + {}^1P_2 \times {}^1f_1 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} {}^2n_2 + \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \times {}^1f_1$$

$$\Rightarrow {}^1n_{1z} = L_2 S_3 f_x + L_2 C_3 f_y + L_1 C_{23} f_y + L_1 S_{23} f_x$$

$$\Rightarrow \quad \tau = \begin{pmatrix} l_2 s_3 + l_1 s_{23} & l_2 c_3 + l_1 c_{23} & 0 & 0 & 0 & 1 \\ l_2 s_3 & l_2 c_3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \\ n_x \\ n_y \\ n_z \end{pmatrix}$$

$$\Rightarrow \quad {}^4J = \begin{pmatrix} l_2 s_3 + l_1 s_{23} & l_2 c_3 + l_1 c_{23} & 0 & 0 \\ l_2 s_3 & l_2 c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

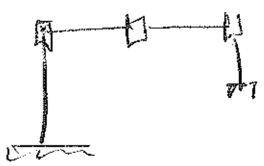
Singularity

$\Rightarrow \det({}^0J) = 0 \Rightarrow S_2 = 0 \Rightarrow \theta_2 = 0°$

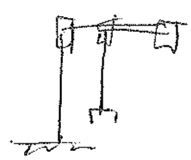
$\theta_2 = 180°$

Assuming Lengths are nonzero

$\theta_2 = 0°$



$\theta_2 = 180°$



8) Direct Differentiation

$$\dot{x} = (-L_3 S_{23} - L_2 S_2)\dot{\theta}_2 - L_3 S_{23}\dot{\theta}_3$$
$$\dot{y} = (L_3 C_{23} - L_2 C_2)\dot{\theta}_2 + L_3 C_{23}\dot{\theta}_3$$
$$\dot{z} = \dot{d}_1$$
$$\dot{\theta}_x = 0$$
$$\dot{\theta}_y = 0$$
$$\dot{\theta}_z = \theta_4 + \theta_2 + \theta_3$$

$$\Rightarrow {}^0J = \begin{bmatrix} 0 & -L_2S_2 - L_3S_{23} & -L_3S_{23} & 0 \\ 0 & L_2C_2 + L_3C_{23} & L_3C_{23} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Velocity Propagation

$${}^1w_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad {}^2w_2 = {}^2_1R\,{}^1w_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^3w_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} + {}^3_2R\,{}^2w_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$${}^4w_4 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_4 \end{bmatrix} + {}^4_3R\,{}^3w_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 \end{bmatrix} = {}^5w_5$$

$${}^iv_i = {}^i_{i-1}R\left({}^{i-1}w_{i-1} \times {}^{i-1}P_i + {}^{i-1}v_{i-1}\right)$$

$$\Rightarrow {}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \underset{\text{prismatic}}{\begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix} \qquad {}^2v_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix} \qquad {}^3v_3 = \begin{bmatrix} L_2S_3\dot{\theta}_2 \\ L_2C_3\dot{\theta}_2 \\ \dot{d}_1 \end{bmatrix}$$

$$\Rightarrow \; {}^4v_4 = \begin{bmatrix} l_2 s_{34}\dot\theta_2 + l_3 s_4 \dot\theta_2 + l_3 s_4 \dot\theta_3 \\ l_2 c_{34}\dot\theta_2 + l_3 c_4 \dot\theta_2 + l_3 c_4 \dot\theta_3 \\ \dot d_1 \end{bmatrix} = {}^5v_5$$

$$ {}^5J = \begin{bmatrix} 0 & l_2 s_{34} + l_3 s_4 & l_3 s_4 & 0 \\ 0 & l_2 c_{34} + l_3 c_4 & l_3 c_4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

## Force Propagation

$${}^5f_5 = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = {}^4f_4$$

$${}^3f_3 = {}^3_4R\,{}^4f_4 = \begin{bmatrix} c_4 f_x - s_4 f_y \\ s_4 f_x + c_4 f_y \\ f_z \end{bmatrix}$$

$${}^2f_2 = {}^2_3R\,{}^3f_3 = \begin{bmatrix} c_{34} f_x - s_{34} f_y \\ s_{34} f_x + c_{34} f_y \\ f_z \end{bmatrix} = {}^1f_1$$

$${}^4n_4 = {}^4_5R\,{}^5n_5 + {}^4P_5 \times {}^4f_4 = \begin{bmatrix} l_4 f_y \\ -l_4 f_x \\ 0 \end{bmatrix}$$

$${}^3n_3 = {}^3_4R\,{}^4n_4 + {}^3P_4 \times {}^3f_3 = \begin{bmatrix} +s_4 l_4 f_x + c_4 l_4 f_y \\ -c_4 l_4 f_x + l_4 s_4 f_y - l_3 f_z \\ l_3 s_4 f_x + l_3 c_4 f_y \end{bmatrix}$$

$${}^2n_2 = {}^2_3R\,{}^3n_3 + {}^2P_3 \times {}^2f_2 \Rightarrow {}^2n_{2_z} = l_2 s_{34} f_x + l_3 s_4 f_x + l_2 c_{34} f_y + l_3 c_4 f_y$$

$$\Rightarrow \quad \tau = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ l_3 s_4 + l_2 s_{34} & l_3 c_4 + l_2 c_{34} & 0 & 0 & 0 & 1 \\ l_3 s_4 & l_3 c_4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

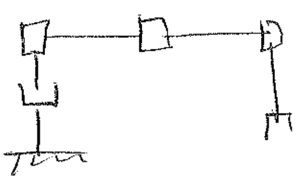
$$\Rightarrow {}^{s}J = \begin{bmatrix} 0 & L_2 s_{34} + L_3 s_4 & L_3 s_4 & 0 \\ 0 & L_2 c_{34} + L_3 c_4 & L_3 c_4 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

## Singularity

$$\Rightarrow \det({}^{o}J) = 0 \Rightarrow \begin{vmatrix} L_3 s_4 + L_2 s_{34} & L_3 s_4 \\ L_2 c_4 + L_2 c_{34} & L_3 c_4 \end{vmatrix} = 0 \Rightarrow s_3 = 0 \Rightarrow \theta_3 = 0, \quad \theta_3 = 180^\circ$$

Assuming lengths are non-zero.

$\theta^3 = 0$



$\theta^3 = 180^\circ$