Manipulator Dynamics 4

Langrangian Formulation of Manipulator Dynamics 1/

1. Define a set of ***generalized coordinates*** for $\boldsymbol{i=1,2,3\ldots N}$.

These coordinates can be chosen arbitrarily as long as they provide a set of independent variables that map the system in a 1-to-1 manner. The usual variable set for serial manipulators is:

$$q_i = \begin{cases} \theta_i \text{ if revolute joint} \\ d_i \text{ if prismatic joint} \end{cases}$$

2. Define a set of ***generalized velocities*** $\dot{q}_i$ for $\boldsymbol{i=1,2,3\ldots N}$

3. Define a set of ***generalized forces (and moments)*** Q_i for $\boldsymbol{i=1,2,3\ldots N}$

The generalized forces must satisfy

$$Q_i \delta q_i = \delta W$$

where δq_i is a small change in the generalized coordinate and δW is the work done corresponding to that small change.

Langrangian Formulation of Manipulator Dynamics 2/

4. Write the equations describing the ***kinetic and potential energies*** as functions of the generalized coordinates as well as the resulting Lagrangian.

Let K denote the expression describing the kinetic energy. where $K = f\left(q_i, \dot{q}_i, t\right)$

$$k_i = \frac{1}{2} v_{c\,i}{}^T m_i v_{c\,i} + \frac{1}{2} {}^i\omega_i{}^{c\,i}I_i{}^i\omega_i$$

Let P denote the expression describing the potential energy. where $P = f\left(q_i, t\right)$

$$P = \sum mgh_i + \frac{1}{2} kx^2$$

Let L denote the Lagrangian given by:

$$L = K - P$$

Langrangian Formulation of Manipulator Dynamics 3/

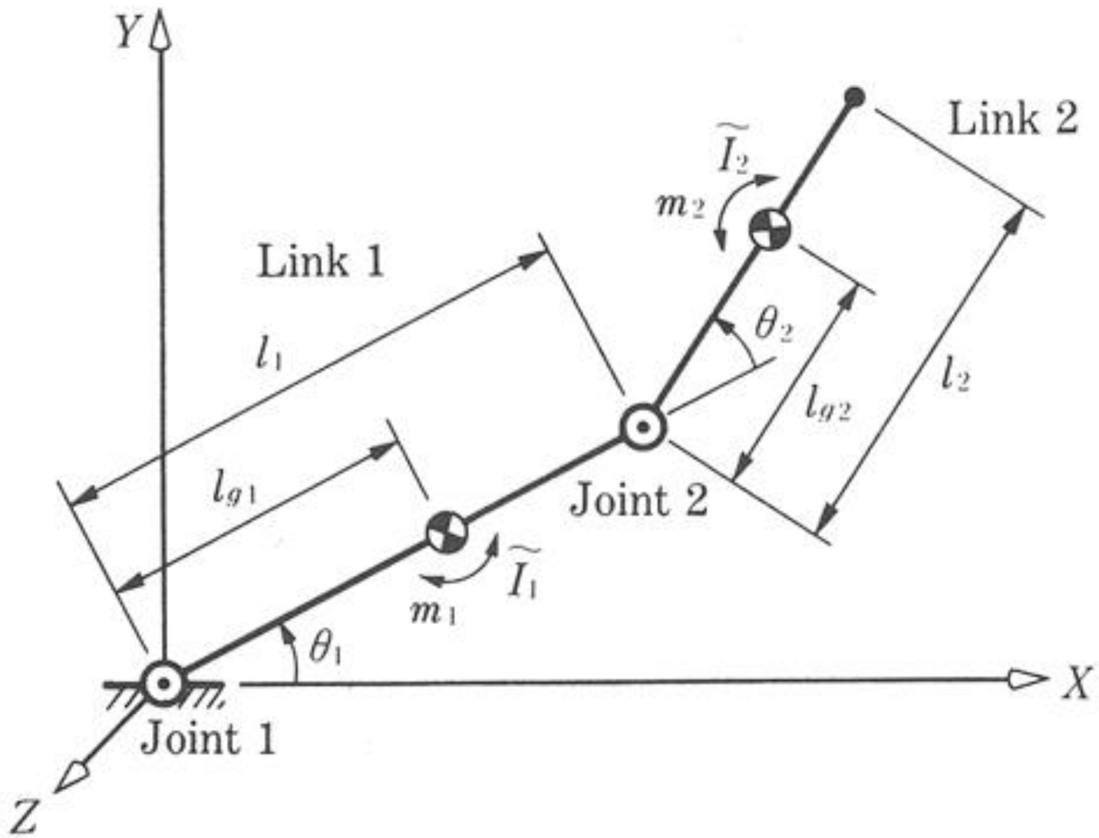
5. The equations of motion are given by

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

or, more practically, by

$$Q_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i}$$

Langrangian Formulation - 2R Robot Example



Langrangian Formulation - 2R Robot Example

Step 1: Let $q_1 = \theta_1$ and $q_2 = \theta_2$

Step 2: Let $\dot{q}_1 = \dot{\theta}_1$ and $\dot{q}_2 = \dot{\theta}_2$

Step 3: Let external forces/torques $Q_i = \tau_i$

Step 4:

- Kinetic Energy: $$k_i = \frac{1}{2} m_i v_{ci}{}^T v_{ci} + \frac{1}{2} {}^i\omega_i \, {}^{ci}I_i \, {}^i\omega_i$$

- For $i=1$

$$k_1 = \frac{1}{2} m_1 L_{g1}{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} I_1 \dot{\theta}_1{}^2$$

Langrangian Formulation - 2R Robot Example

- To find the velocity of the center of mass of link 2, first consider its position given by

$$ {}^{0}P_{g2} = \begin{bmatrix} L_1 c1 + L_{g2} c12 \\ L_1 s1 + L_{g2} s12 \end{bmatrix} $$

- The derivative squared gives

$$ v_{ci}^{\ T} v_{ci} = L_1^{\ 2}\theta_1^{\ 2} + L_{g2}^{\ \ 2}\left(\dot\theta_1 + \dot\theta_2\right)^2 + 2L_1 L_{g2} c2\left(\dot\theta_1^{\ 2} + \dot\theta_1\dot\theta_2\right) $$

- For *i=2*

$$ k_2 = \frac{1}{2} m_2 \left[L_1^{\ 2}\theta_1^{\ 2} + L_{g2}^{\ \ 2}\left(\dot\theta_1 + \dot\theta_2\right)^2 + 2L_1 L_{g2} c2\left(\dot\theta_1^{\ 2} + \dot\theta_1\dot\theta_2\right)\right] + \frac{1}{2} I_2 \left(\dot\theta_1 + \dot\theta_2\right)^2 $$

Langrangian Formulation - 2R Robot Example

- Potential Energy: $$p = \sum mgh_i$$

- For *i=1* $$p_1 = m_1 g L_{g1} s1$$

- For *i=2* $$p_2 = m_2 g\left(L_1 s1 + L_{g2} s12\right)$$

- Lagrangian: $$L = k_1 + k_2 - p_1 - p_2$$

Langrangian Formulation - 2R Robot Example

- **Step 5:** Solving

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$Q_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_1}\right) - \frac{\partial K}{\partial \theta_1} + \frac{\partial P}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_2}\right) - \frac{\partial K}{\partial \theta_2} + \frac{\partial P}{\partial \theta_2}$$

Langrangian Formulation - 2R Robot Example

$$\tau_1 = [m_1 L_{g1} + I_1 + m_2 (L_1{}^2 + L_{g2}{}^2 + 2 L_1 L_{g2} c_2) + I_2]\ddot{\theta}_1$$

$$+ [m_2 (L_{g2}{}^2 + L_1 L_{g2} c_2 + I_2]\ddot{\theta}_2$$

$$- m_2 L_1 L_{g2} s_2 (2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2{}^2)$$

$$+ m_1 g L_{g1} c_1 + m_2 g (L_1 c_1 + L_{g2} c_{12})$$

$$\tau_2 = [m_2 (L_{g2}{}^2 + L_1 L_{g2} c_2) + I_2]\ddot{\theta}_1$$

$$+ [m_2 L_{g2} + I_2]\ddot{\theta}_2$$

$$+ m_2 L_1 L_{g2} s_2 \dot{\theta}_1{}^2$$

$$+ m_2 g L_{g2} c_{12}$$

Gravity Effects - Langrangian Formulation

$$\tau_i = \frac{d}{dt}\left(\frac{\partial K(\theta,\dot{\theta})}{\partial \dot{\theta}_i}\right) - \frac{\partial K(\theta,\dot{\theta})}{\partial \theta_i} + \frac{\partial P(\theta)}{\partial \theta_i}$$

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

Manipulators - Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta) + F(\theta, \dot{\theta})$$

Manipulators – Non Linear Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

Manipulators – Non Linear Control Problem

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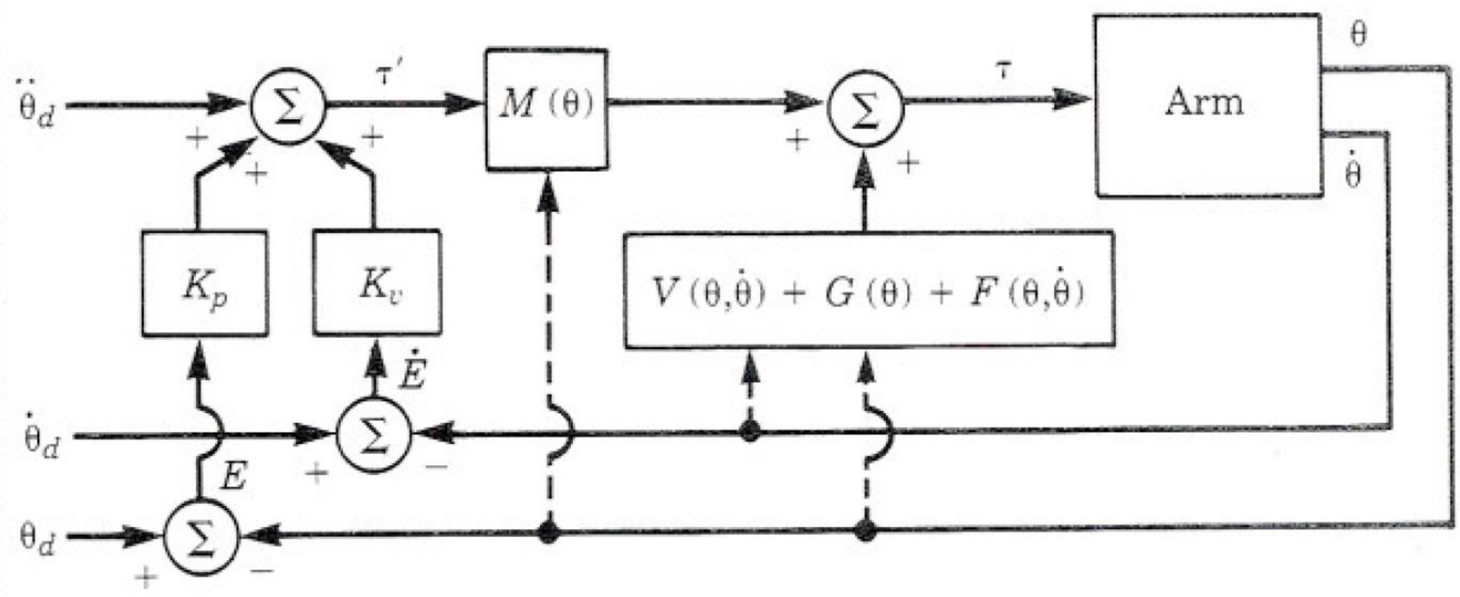

Equation of Motion – Non Rigid Body Effects

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

- Viscous Friction $\tau_{friction} = v\dot{\theta}$
- Coulomb Friction $\tau_{friction} = c\,\mathrm{sgn}(\dot{\theta})$
- Model of Friction $\tau_{friction} = v\dot{\theta} + c\,\mathrm{sgn}(\dot{\theta}) = f(\theta,\dot{\theta})$

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Dynamic Equations - State Space Equation

- It is often convenient to express the dynamic equations of a manipulator in a single equation

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

where

$M(\theta)$ - Mass matrix (includes inertia terms) - ***nxn Matrix***

$V(\theta,\dot{\theta})$ - Centrifugal (square of joint velocity) and Coriolis (product of two different joint velocities) terms - ***nx1 Vector***

$G(\theta)$ - gravitational terms - ***nx1 Vector***.

Dynamic Equations - Configuration Space Equation

- By rewriting the velocity dependent term $V(\theta,\dot{\theta})$ in a different form, we can write the dynamic equations as

$$\tau = M(\theta)\ddot{\theta} + B(\theta)\left[\dot{\theta}\,\dot{\theta}\right] + C(\theta)\left[\dot{\theta}^2\right] + G(\theta)$$

where

$B(\theta)$ - Centrifugal coefficients(square of joint velocity)

$C(\theta)$ - Coriolis coefficients (product of two different joint velocities)

- This form can be useful for applications using force control. Each of the matrices is a function of manipulator configuration only (that is, joint position) and can be updated at a rate depending on the magnitude of joint changes.

Dynamic Equations - Cartesian State Space Equation

- It can sometimes be desirable to have a relationship between the end effector's Cartesian accelerations and the joint torques. Beginning from the Configuration Space equation

$$\tau = M(\theta)\ddot{\theta} + V(\theta, \dot{\theta}) + G(\theta)$$

- we can substitute the joint moments using our definition of the Jacobian matrix:

$$\tau = J^{T}(\theta)F \qquad F = J^{-T}(\theta)\tau$$

$$\dot{x} = J(\theta)\dot{\theta}$$

- By differentiation, we find

$$\ddot{x} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta}$$

Dynamic Equations - Cartesian State Space Equation

- Solving for joint acceleration gives

$$\ddot{\theta} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{\theta}$$

- Substitution yields

$$F = J^{-T}\tau = J^{-T}M(\theta)J^{-1}\ddot{x} - J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta)$$

$$\boxed{F = M_x(\theta)\ddot{x} + V_x(\theta,\dot{\theta}) + G_x(\theta)}$$

Where

$$M_x(\theta) = J^{-T}M(\theta)J^{-1}$$

$$V_x(\theta,\dot{\theta}) = J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta})$$

$$G_x(\theta) = J^{-T}G(\theta)$$

- This equation relates the forces and moments at the end effector to the Cartesian accelerations of the end effector and the manipulator joint positions and velocities.