



Manipulator Dynamics 1



Forward Dynamics

Problem

Given: Joint torques and links geometry, mass, inertia, friction, joint torques

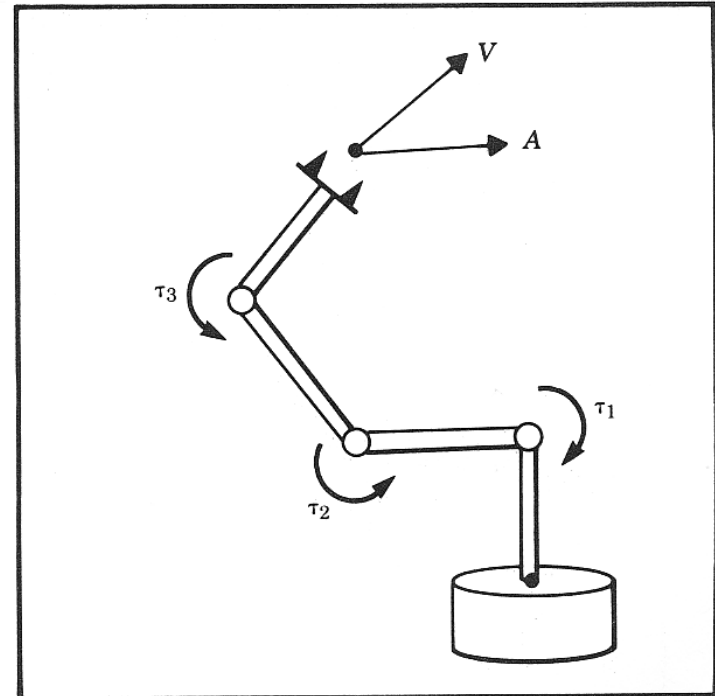
Compute: Angular acceleration of the links (solve differential equations)

Solution

solve a set of differential equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \mathbf{G}(\boldsymbol{\Theta}) + \mathbf{F}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}})$$





Inverse Dynamics

Problem

Given: Angular acceleration, velocity and angles of the links in addition to the links geometry, mass, inertia, friction

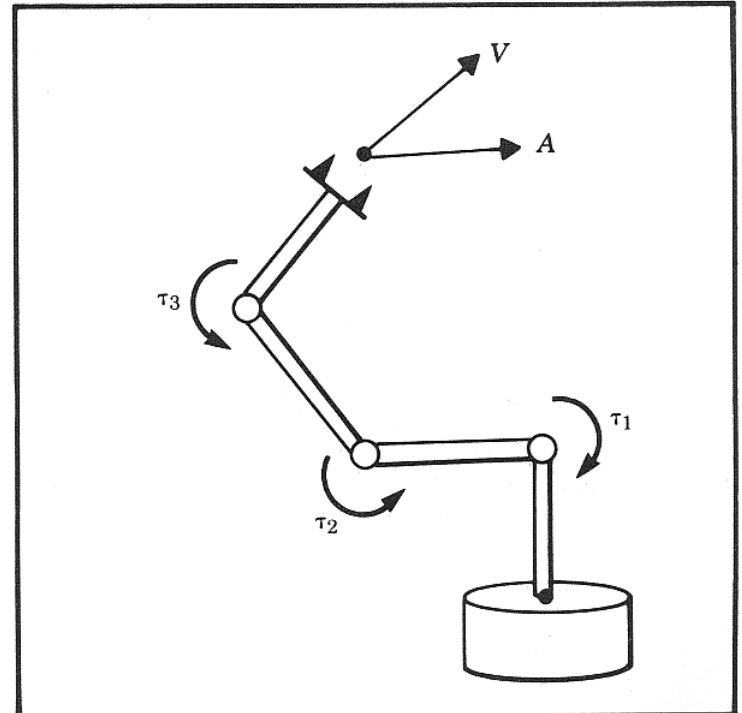
Compute: Joint torques

Solution

Solve a set of algebraic equations

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = \mathbf{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \mathbf{G}(\boldsymbol{\Theta}) + \mathbf{F}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}})$$





Dynamics - Newton-Euler Equations

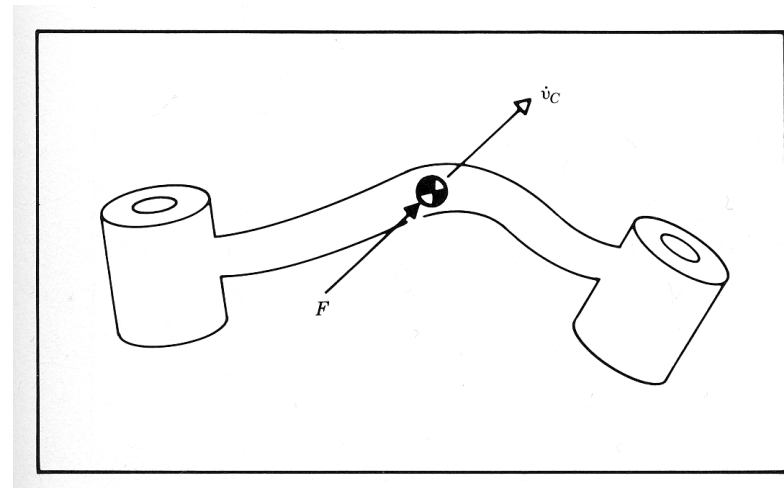
Newton Equation

- For a rigid body (like a robot link) whose center of mass is experiencing an acceleration, the force acting at the center of mass that caused the acceleration is given by Newton's equation

$$F = \frac{d(mv_c)}{dt}$$

- For our robot manipulators, whose link masses are constant, this equation simplifies to

$$F = m\dot{v}_c$$





Dynamics - Newton-Euler Equations

Eular Equation

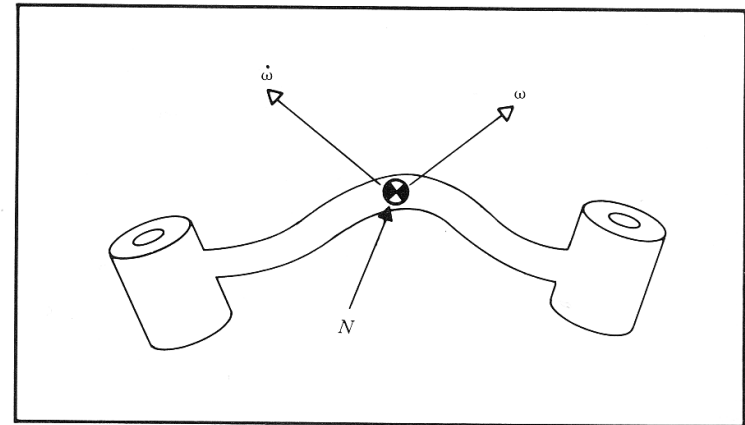
- For a rotating rigid body, the moment that causes an angular acceleration is given by Euler's equation

$$N = \frac{d({}^c I \omega)}{dt}$$

- For our robot manipulators, whose **link moment of inertia is constant**, this equation simplifies to

$$N = {}^c I \dot{\omega} + \omega \times {}^c I \omega$$

- The second term on the right will be non-zero when the link's angular velocity vector is not coincident with the link's principle axis of inertia.





Iterative Newton-Euler Equations Approach Overview and Solution Procedure



Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

\dot{v}_c – The linear acceleration of the center of mass

$\dot{\omega}$ – The angular acceleration

${}^c I$ – The Inertia tensor (moment of inertia)

F - The sum of all the forces applied on the center of mass

N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^c I \dot{\omega} + \omega \times {}^c I \omega$$



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Expression of Force & Torque With Respect to the CM



Linear Momentum



Linear Momentum



Linear Momentum



Angular Momentum



Angular Momentum



Angular Momentum



Iterative Newton Euler Eq. Solution Procedure



Dynamics - Newton-Euler Equations

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Propagation of Acceleration – Angular Vector Approach



Propagation of Acceleration – Angular Vector Approach



Propagation of Acceleration – Angular Vector Approach



Propagation of Acceleration – Angular Vector Approach



Propagation of Acceleration – Angular Matrix Approach

- To derive a general formula for the angular acceleration, we will differentiate the angular velocity

$${}^{i+1}\omega_{i+1} = {}^{i+1}R^i \omega_i + \begin{bmatrix} 0 & 0 & \dot{\theta}_{i+1} \end{bmatrix}^T$$

- Applying the chain rule, we find:

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R^i \dot{\omega}_i + \boxed{{}^{i+1}\dot{R}^i} \omega_i + \begin{bmatrix} 0 & 0 & \ddot{\theta}_{i+1} \end{bmatrix}^T$$

- Recall that

$${}^A_B \dot{R} = \frac{d}{dt} \begin{bmatrix} A \\ B \end{bmatrix} R = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} {}^A_B R = \boxed{{}^A_B \dot{R}_\Omega {}^A_B R}$$

- Substitution of this result yields



Propagation of Acceleration - Angular

Matrix form (Revolute Joint)

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R^i \dot{\omega}_i + {}^{i+1}\dot{R}_\Omega {}^i R^i \omega_i + \begin{bmatrix} 0 & 0 & \ddot{\theta}_{i+1} \end{bmatrix}^T$$

- Converting from matrix to vector form gives the angular acceleration vector

Vector form (Revolute Joint)

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R^i \dot{\omega}_i + {}^{i+1}R^i \omega_i \times \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{bmatrix}$$

- If joint $i+1$ is prismatic, the angular terms are zero ($\dot{\theta}_{i+1} = \ddot{\theta}_{i+1} = 0$) and the above equation simplifies to:

Matrix form (Prismatic Joint)

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}R^i \dot{\omega}_i$$



Dynamics - Newton-Euler Equations

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$$N = {}^c I \dot{\omega} + \omega \times {}^c I \omega$$



Propagation of Acceleration - Linear Simultaneous Linear and Rotational Velocity

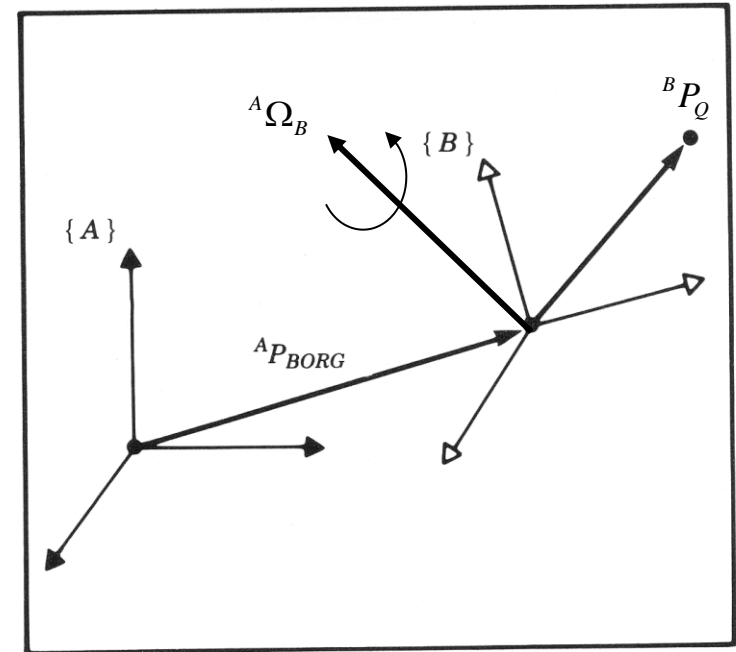
$${}^A V_Q = f({}^B P_Q, {}^B V_Q, {}^A V_{BORG}, {}^A \Omega_B, {}^A R_B)$$

- Vector Form

$${}^A V_Q = {}^A V_{BORG} + {}^A R_B {}^B V_Q + {}^A \Omega_B \times {}^A R_B P_Q$$

- Matrix Form

$${}^A V_Q = {}^A V_{BORG} + {}^A R_B {}^B V_Q + {}^A \dot{R}_B \left({}^A R_B P_Q \right)$$





Propagation of Acceleration – Linear Matrix Approach

- To derive a general formula for the linear acceleration, we will differentiate the linear velocity. However, instead of differentiating the **recursive equation** like we did for the angular acceleration derivation, we'll begin at a slightly earlier step. Recall the three-part expression:

$${}^A V_Q = {}^A \dot{R}_B \left({}^B R^B P_{Q\text{org}} \right) + {}^A V_{B\text{org}} + {}^B R^B V_Q$$

- Re-assigning the link frames ($A \rightarrow 0$ $B \rightarrow i$ $Q \rightarrow i+1$), we find

$${}^0 V_{i+1} = {}^0 \dot{R}_{\Omega i} \left({}^i R^i P_{i+1} \right) + {}^0 V_i + {}^i R^i V_{i+1}$$

- Differentiating using the chain rule gives:

$${}^0 \dot{V}_{i+1} = {}^0 \ddot{R}_{\Omega i} {}^i R^i P_{i+1} + {}^0 \dot{R}_{\Omega i} \left({}^0 \dot{R}_{\Omega i} {}^i R^i P_{i+1} + {}^0 \dot{R}_{\Omega i} {}^i R^i \dot{P}_{i+1} \right) + {}^0 \dot{V}_i + {}^0 \dot{R}_{\Omega i} {}^i R^i V_{i+1} + {}^i R^i \dot{V}_{i+1}$$



Propagation of Acceleration - Linear

$${}^0\dot{V}_{i+1} = {}^0\ddot{R}_{\Omega i} {}^0R^i P_{i+1} + {}^0\dot{R}_{\Omega i} {}^0\dot{R}_{\Omega i} {}^0R^i P_{i+1} + \boxed{{}^0\dot{R}_{\Omega i} {}^0R^i \dot{P}_{i+1}} + {}^0\dot{V}_i + \boxed{{}^0\dot{R}_{\Omega i} {}^0R^i V_{i+1}} + {}^0R^i \dot{V}_{i+1}$$

- Combining the two like terms, we find:

$${}^0\dot{V}_{i+1} = {}^0\ddot{R}_{\Omega i} {}^0R^i P_{i+1} + {}^0\dot{R}_{\Omega i} {}^0\dot{R}_{\Omega i} {}^0R^i P_{i+1} + {}^0\dot{V}_i + \boxed{2{}^0\dot{R}_{\Omega i} {}^0R^i V_{i+1}} + {}^0R^i \dot{V}_{i+1}$$

- Pre-multiplying both sides of the equation by $\boxed{{}^{i+1}_0 R}$ gives:

$$\boxed{{}^{i+1}_0 R} {}^0\dot{V}_{i+1} = \boxed{{}^{i+1}_0 R} {}^0\ddot{R}_{\Omega i} {}^0R^i P_{i+1} + \boxed{{}^{i+1}_0 R} {}^0\dot{R}_{\Omega i} {}^0\dot{R}_{\Omega i} {}^0R^i P_{i+1} + \boxed{{}^{i+1}_0 R} {}^0\dot{V}_i + 2\boxed{{}^{i+1}_0 R} {}^0\dot{R}_{\Omega i} {}^0R^i V_{i+1} + \boxed{{}^{i+1}_0 R} {}^0R^i \dot{V}_{i+1}$$

- Expanding terms gives

$${}^{i+1}_0 R {}^0\dot{V}_{i+1} = \boxed{{}^{i+1}_i R} \boxed{{}^i_0 R} {}^0\ddot{R}_{\Omega i} {}^0R^i P_{i+1} + \boxed{{}^{i+1}_i R} \boxed{{}^i_0 R} {}^0\dot{R}_{\Omega i} {}^0\dot{R}_{\Omega i} {}^0R^i P_{i+1} + {}^{i+1}_0 R {}^0\dot{V}_i + 2\boxed{{}^{i+1}_i R} \boxed{{}^i_0 R} {}^0\dot{R}_{\Omega i} {}^0R^i V_{i+1} + {}^{i+1}_0 R {}^0R^i \dot{V}_{i+1}$$



Propagation of Acceleration - Linear

$${}^{i+1}_0 R^0 \dot{V}_{i+1} = {}^{i+1}_i R \left[{}^i_0 R^0 \ddot{R}_{\Omega i} {}^0 R^i P_{i+1} + {}^{i+1}_i R \left[{}^i_0 R^0 \dot{R}_{\Omega i} {}^0 \dot{R}_{\Omega i} {}^i P_{i+1} + {}^{i+1}_0 R^0 \dot{V}_i + 2 {}^{i+1}_i R \left[{}^i_0 R^0 \dot{R}_{\Omega i} {}^0 R^i V_{i+1} + {}^{i+1}_0 R^0 R^i \dot{V}_{i+1} \right] \right] \right]$$

- Simplifying the previous equation using (Note: ${}^s R_B^A \dot{R}_{\Omega t} {}^s R^T = {}^s R^0 A \Omega_B$)

$${}^i_0 R^0 \ddot{R}_{\Omega i} {}^0 R^i = {}^i_0 R^0 \ddot{R}_{\Omega 0} {}^i R^T = {}^i_0 R^0 \dot{\Omega}_i = {}^i_0 R \dot{\omega}_i = {}^i \dot{\omega}_i$$

$${}^i_0 R^0 \dot{R}_{\Omega i} {}^0 \dot{R}_{\Omega i} {}^i R = {}^i \omega_i \times {}^i \omega_i$$

$${}^{i+1}_0 R^0 \dot{V}_i = {}^{i+1}_i R \left[{}^i_0 R^0 \dot{V}_i = {}^{i+1}_i R^i \dot{v}_i \right]$$

$${}^i_0 R^0 \dot{R}_{\Omega i} {}^0 R = {}^i_0 R^0 \omega_i = {}^i \omega_i$$

- we have

$${}^{i+1} \dot{v}_{i+1} = {}^{i+1}_i R \left[{}^i \dot{\omega}_i \times {}^i P_{i+1} + {}^i \omega_i \times {}^i \omega_i \times {}^i P_{i+1} + {}^i \dot{v}_i \right] + 2 {}^i \omega_i \times {}^{i+1} v_{i+1} + {}^{i+1} \dot{v}_{i+1}$$



Propagation of Acceleration - Linear

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R \left[{}^i\dot{\omega}_i \times {}^i P_{i+1} + {}^i\omega_i \times {}^i\omega_i \times {}^i P_{i+1} + {}^i\dot{v}_i \right] + 2{}^{i+1}\omega_{i+1} \times {}^{i+1}v_{i+1} + {}^{i+1}\dot{v}_{i+1}$$

- This equation can be written equivalently as:

General form

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R \left[{}^i\dot{\omega}_i \times {}^i P_{i+1} + {}^i\omega_i \times {}^i\omega_i \times {}^i P_{i+1} + {}^i\dot{v}_i \right] + 2{}^{i+1}\omega_{i+1} \times \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_{i+1} \end{bmatrix}$$

- If joint $i+1$ is revolute joint, the linear terms are zero and the above equation simplifies to:

Revolute Joint

$${}^{i+1}\dot{v}_{i+1} = {}^{i+1}R \left[{}^i\dot{\omega}_i \times {}^i P_{i+1} + {}^i\omega_i \times {}^i\omega_i \times {}^i P_{i+1} + {}^i\dot{v}_i \right]$$



Propagation of Acceleration - Linear

- From the general equation, we can also get the solution for the **acceleration of the center of mass** for link i . Appropriate frame substitution and elimination of prismatic terms gives we find:

$${}^i \dot{v}_{ci} = {}^i \dot{\omega}_i \times {}^i P_{ci} + {}^i \omega_i \times ({}^i \omega_i \times {}^i P_{ci}) + {}^i \dot{v}_i$$

- Frame $\{C_i\}$ attached to each link with its origin located at the Center of mass of the link, and with the same orientation as the link frame $\{i\}$



Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

\dot{v}_c - The linear acceleration of the center of mass

$\dot{\omega}$ - The angular acceleration

${}^c I$ - The Inertia tensor (moment of inertia)

F - The sum of all the forces applied on the center of mass

N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^c I \dot{\omega} + \omega \times {}^c I \omega$$



Iterative Newton-Euler Equations

- The Newton and Euler equations are re-written for the forces and moments at each link:

$${}^i F_i = m_i {}^i \dot{v}_{c_i}$$

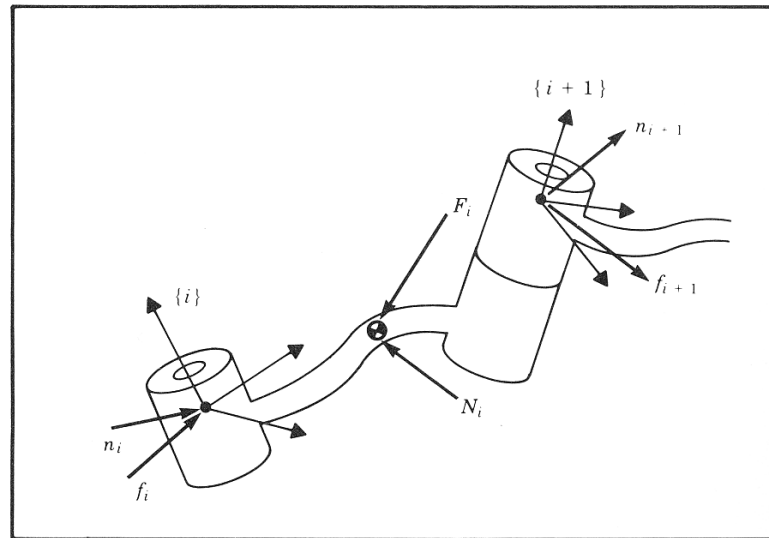
$${}^i N_i = {}^{c_i} I_i {}^i \dot{\omega}_i + {}^i \omega_i \times {}^{c_i} I_i {}^i \omega_i$$

- Where $\{C_i\}$ is a frame who has its origin at the link's center of mass and has the same orientation as the link frame $\{i\}$.



Sum of Forces and Moment on a Link

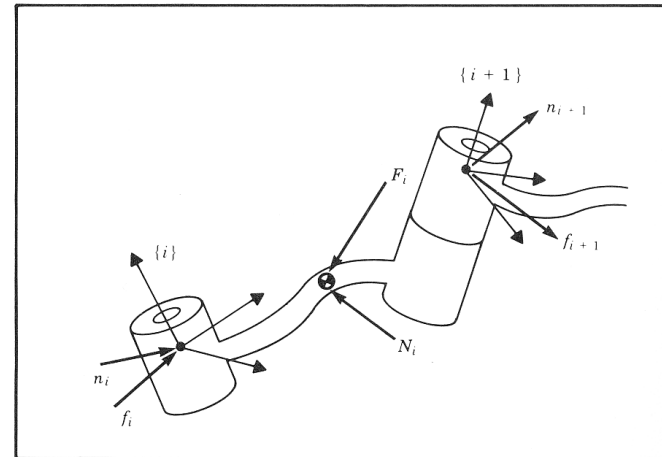
- In addition to calculating the **forces and torques arising from link accelerations**, we also need to account for how they **affect the neighboring links** as well as the end effectors interactions with the environment.



- Balancing the forces shown in the above figure, we can find the total force and torque on each link.

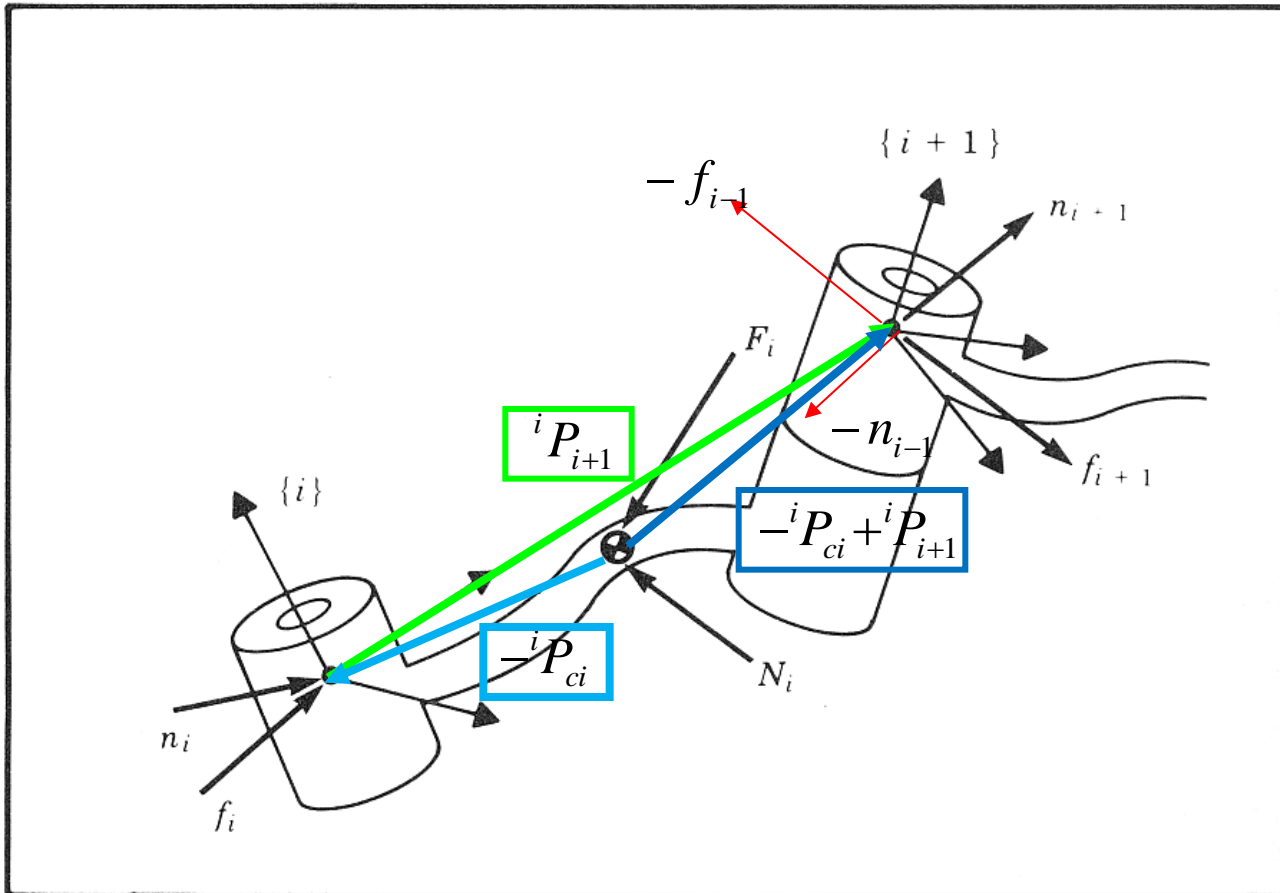


Sum of Forces and Moment on a Link





Sum of Forces and Moment on a Link





Sum of Forces and Moment on a Link

$${}^i F_i = m \dot{v}_{ci}$$

$${}^i N_i = {}^{ci} I \dot{\omega}_i + \omega_i \times {}^c I \omega_i$$

$${}^i F_i = {}^i f_i - {}^i f_{i+1} = {}^i f_i - {}_{i+1}^i R {}^{i+1} f_{i+1}$$

$${}^i N_i = {}^i n_i - {}^i n_{i+1} + ({}^i P_{ci}) \times {}^i f_i - ({}^i P_{ci} + {}^i P_{i+1}) \times {}^i f_{i+1}$$

$${}^i N_i = {}^i n_i - {}_{i+1}^i R {}^{i+1} n_{i+1} - \boxed{{}^i P_{ci} \times {}^i f_i + {}^i P_{ci} \times {}_{i+1}^i R {}^{i+1} f_{i+1}} - {}^i P_{i+1} \times {}_{i+1}^i R {}^{i+1} f_{i+1}$$

$$\boxed{{}^i P_{ci} \times ({}^i f_i - {}_{i+1}^i R {}^{i+1} f_{i+1})}$$

$$\boxed{{}^i P_{ci} \times {}^i F_i}$$

$${}^i N_i = {}^i n_i - {}_{i+1}^i R {}^{i+1} n_{i+1} - \boxed{{}^i P_{ci} \times {}^i F_i} - {}^i P_{i+1} \times {}_{i+1}^i R {}^{i+1} f_{i+1}$$



Sum of Forces and Moment on a Link

- Rearranging the force / torque equations so that they appear as iterative relationship from higher number neighbor to lower number neighbor. The total force and torque on each link.

$${}^i f_i = {}^i R^{i+1} f_{i+1} + {}^i F_i$$
$${}^i n_i = {}^i N_i + {}^i R^{i+1} n_{i+1} + {}^i P_{ci} \times {}^i F_i + {}^i P_{i+1} \times {}^i R^{i+1} f_{i+1}$$

- Compare with the same equation for the static conditions

$${}^i f_i = {}^i R^{i+1} f_{i+1}$$
$${}^i n_i = {}^i R^{i+1} n_{i+1} + {}^i P_{i+1} \times {}^i f_{i+1}$$



Sum of Forces and Moment on a Link

- The joint torque is simply the component of torque that projects onto the joint axis (Z axis by definition)

$$\tau_i = {}^i n_i \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Sum of Forces and Moment on a Link

- For a robot moving in free space

$${}^{N+1}f_{N+1} = 0$$

$${}^{N+1}n_{N+1} = 0$$

- If the robot is contacting the environment, the forces/ torques due to this contact may be included in the equations

$${}^{N+1}f_{N+1} \neq 0$$

$${}^{N+1}n_{N+1} \neq 0$$



Iterative Newton-Euler Equations - Solution Procedure

- **Step 1** - Calculate the link velocities and accelerations iteratively from the robot's base to the end effector
- **Step 2** - Write the Newton and Euler equations for each link.

Outward iterations: $i : 0 \rightarrow 5$

$${}^{i+1}\omega_{i+1} = {}^i R^{i+1} {}^i \omega_i + \dot{\theta}_{i+1} {}^{i+1} \hat{Z}_{i+1},$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^i R^{i+1} {}^i \dot{\omega}_i + {}^i R^{i+1} {}^i \omega_i \times \dot{\theta}_{i+1} {}^{i+1} \hat{Z}_{i+1} + \ddot{\theta}_{i+1} {}^{i+1} \hat{Z}_{i+1},$$

$${}^{i+1}\dot{v}_{i+1} = {}^i R^{i+1} ({}^i \dot{\omega}_i \times {}^i P_{i+1} + {}^i \omega_i \times ({}^i \omega_i \times {}^i P_{i+1}) + {}^i \dot{v}_i),$$

$$\begin{aligned} {}^{i+1}\dot{v}_{C_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ &\quad + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}) + {}^{i+1}\dot{v}_{i+1}, \end{aligned}$$

$${}^{i+1}F_{i+1} = m_{i+1} {}^{i+1}\dot{v}_{C_{i+1}},$$

$${}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1} {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1} {}^{i+1}\omega_{i+1}.$$



Iterative Newton-Euler Equations - Solution Procedure

- **Step 3** - Use the forces and torques generated by interacting with the environment (that is, tools, work stations, parts etc.) in calculating the joint torques from the end effector to the robot's base.

Inward iterations: $i : 6 \rightarrow 1$

$${}^i f_i = {}^i_{i+1} R^{i+1} f_{i+1} + {}^i F_i,$$

$$\begin{aligned} {}^i n_i = & {}^i N_i + {}^i_{i+1} R^{i+1} n_{i+1} + {}^i P_{C_i} \times {}^i F_i \\ & + {}^i P_{i+1} \times {}^i_{i+1} R^{i+1} f_{i+1}, \end{aligned}$$

$$\tau_i = {}^i n_i^T {}^i \hat{Z}_i.$$



Iterative Newton-Euler Equations - Solution Procedure

- **Error Checking** - Check the units of each term in the resulting equations
- **Gravity Effect** - The effect of gravity can be included by setting ${}^0\dot{v}_0 = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.



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