

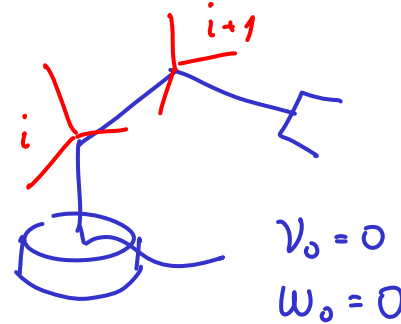
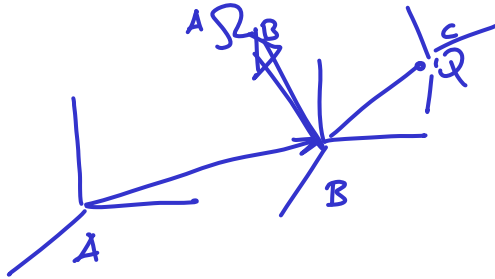


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## Velocity Propagation Between Robot Links 3/4



# Introduction – Velocity Propagation



$${}^A v_c = {}^A v_{BORG} + {}^A R^B v_c + {}^A \Omega_B \times {}^A R^B P_c \rightarrow {}^{i+1} v_{i+1} = {}^{i+1} R^i ({}^i \omega_i \times {}^i P_{i+1} + {}^i v_i) + \begin{bmatrix} 0 \\ 0 \\ d_{i+1} \end{bmatrix}$$

$${}^A \Omega_c = {}^A \Omega_B + {}^A R^B \Omega_c \rightarrow {}^{i+1} \omega_{i+1} = {}^{i+1} R^i \omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$



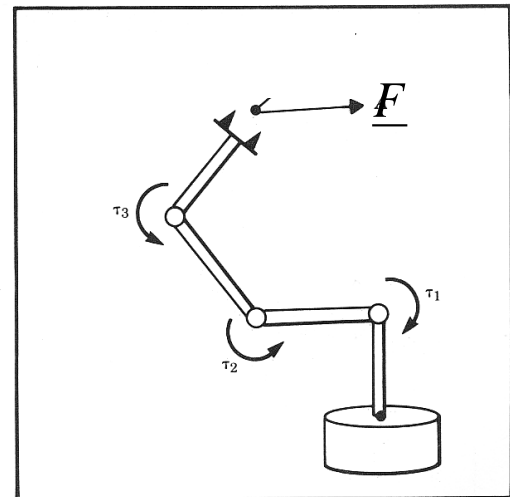
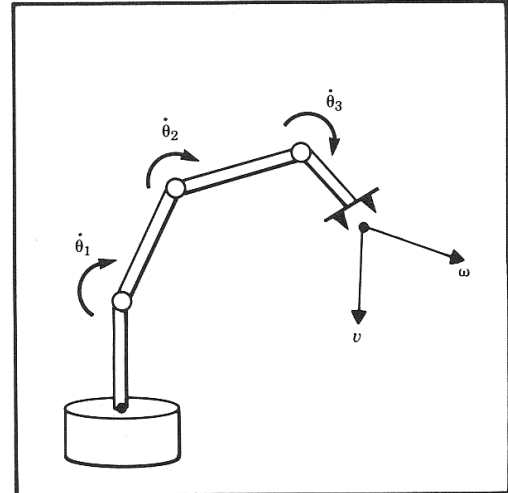
## Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ( $\dot{\underline{\theta}}_N$ ) and the translation and rotation velocities of the end effector ( $\dot{\underline{x}}$ ). This relationship is given by:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = J(\underline{\theta}) \dot{\underline{\theta}}$$

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ( $\underline{\tau}$ ) and the forces and moments ( $\underline{F}$ ) at the robot end effector (**Static Conditions**). This relationship is given by:

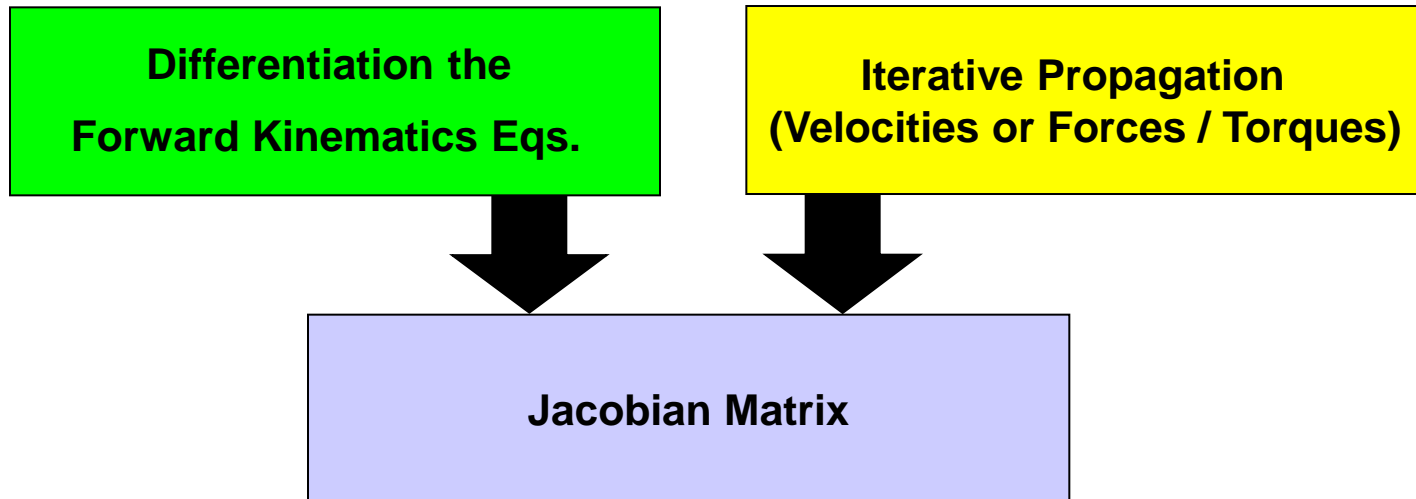
$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$





## Jacobian Matrix - Calculation Methods

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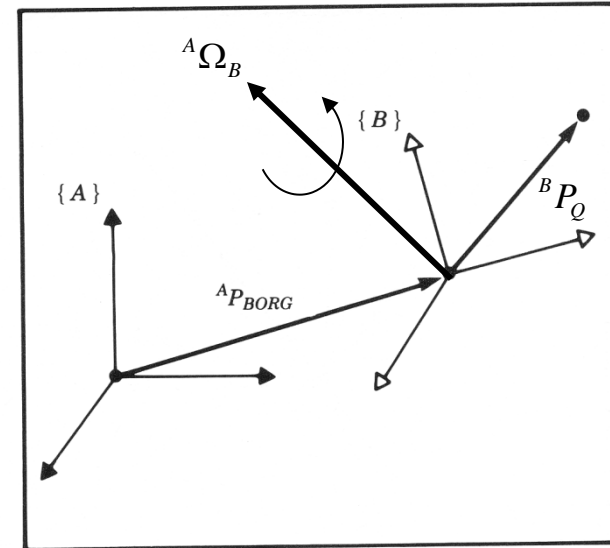
## Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$${}^A V_Q = {}^A V_{BORG} + {}_B^A R^B V_Q + {}^A \Omega_B \times {}_B^A R^B P_Q$$

- Matrix Form

$${}^A V_Q = {}^A V_{BORG} + {}_B^A R^B V_Q + {}_B^A \dot{R}_\Omega \left( {}_B^A R^B P_Q \right)$$



- Angular Velocity

- Vector Form

$${}^A \Omega_C = {}^A \Omega_B + {}_B^A R^B \Omega_C$$



- Matrix Form

$${}^A {}_C \dot{R}_\Omega = {}_B^A \dot{R}_\Omega + {}_B^A R^B \dot{R}_\Omega {}_B^A R^T$$



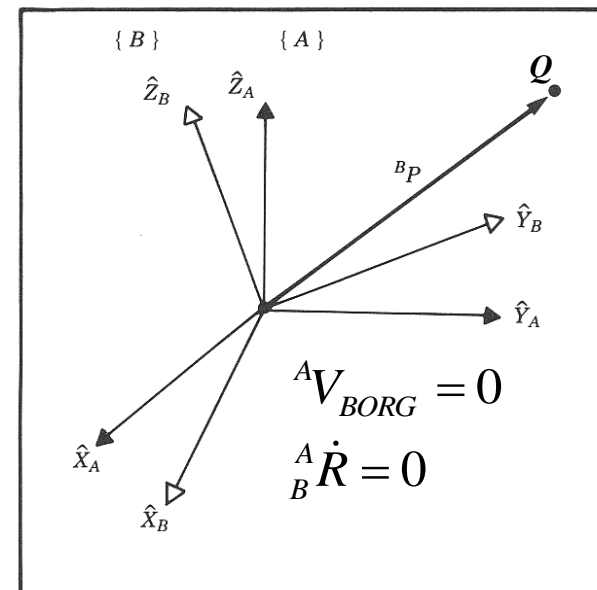
## Frame - Velocity

- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.
- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

$${}^A({}^B V_Q) = \frac{{}^A d}{{}^A dt} {}^B P_Q$$

**Represented  
(Reference Frame)**

**Computed  
(Measured)**





## Position Propagation

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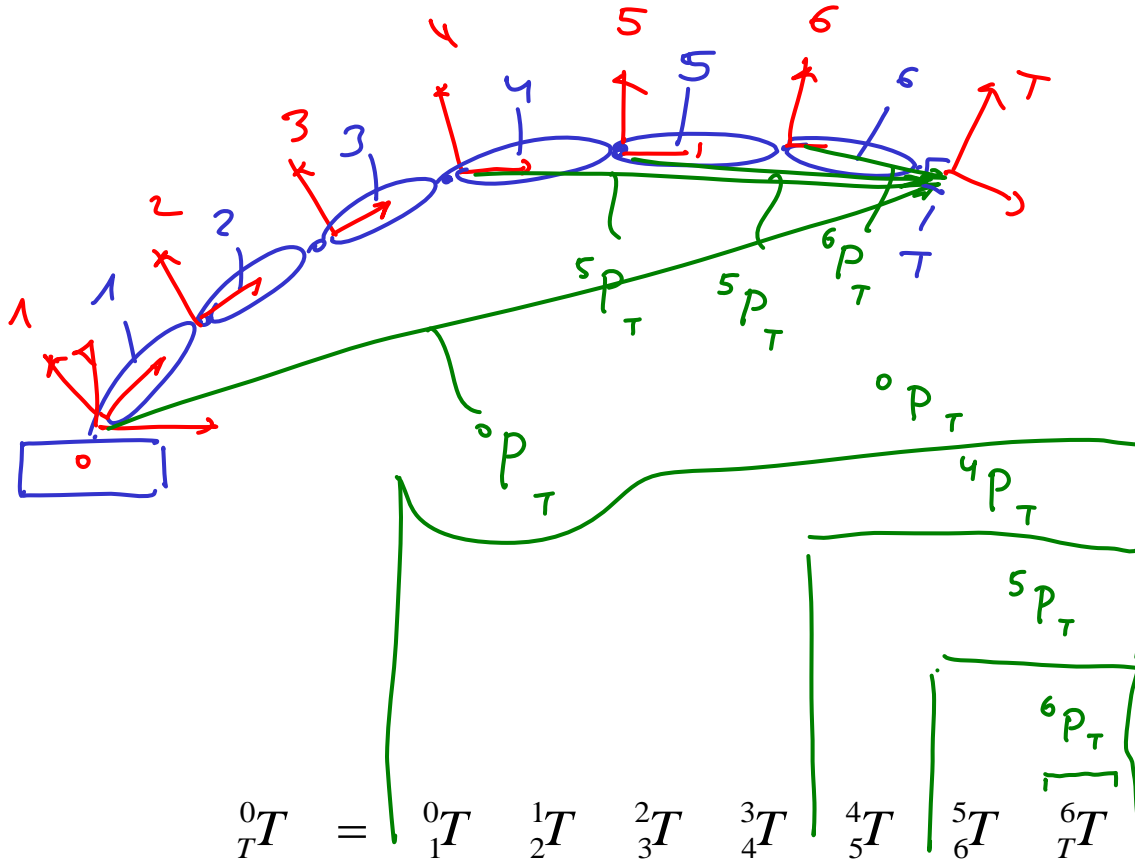
- The homogeneous transform matrix provides a complete description of the linear and angular position relationship between adjacent links.
- These descriptions may be combined together to describe the position of a link relative to the robot base frame {0}.

$${}^oT_i = {}^oT_1 {}^1T_2 \cdots {}^{i-1}T_i$$

- A similar description of the linear and angular velocities between adjacent links as well as the base frame would also be useful.



# Position Propagation







## Motion of the Link of a Robot

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- In considering the motion of a robot link we will always use link frame  $\{0\}$  as the reference frame (Computed AND Represented). However any frame can be used as the reference (represented) frame including the link's own frame ( $i$ )

Where:  $v_i$  - is the linear velocity of the origin of link frame ( $i$ ) with respect to frame  $\{0\}$  (Computed AND Represented)

$\omega_i$  - is the angular velocity of the origin of link frame ( $i$ ) with respect to frame  $\{0\}$  (Computed AND Represented)

- Expressing the velocity of a frame  $\{i\}$  (associated with link  $i$ ) relative to the robot base (frame  $\{0\}$ ) using our previous notation is defined as follows:

$$v_i \equiv {}^0V_i = [{}^0V_i]$$

$$\omega_i \equiv {}^0\Omega_i = [{}^0\Omega_i]$$

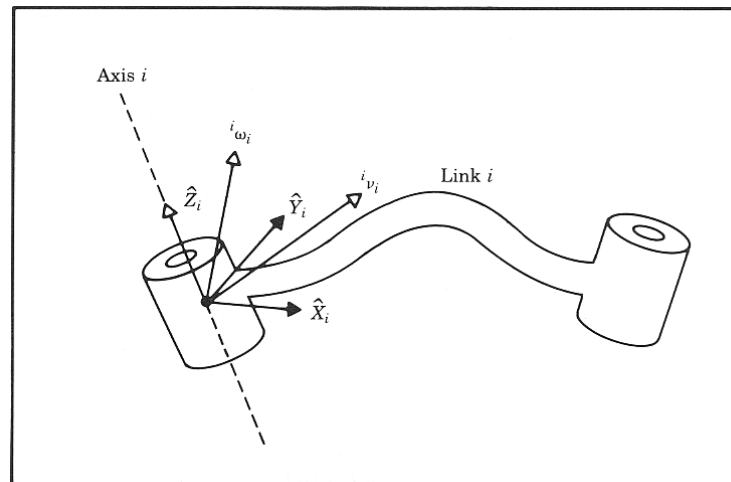


## Velocities - Frame & Notation

- The velocities **differentiate (computed)** relative to the base frame  $\{0\}$  are often **represented** relative to other frames  $\{k\}$ . The following notation is used for this conditions

$${}^k v_i \equiv {}^k [{}^0 V_i] = {}^k R [{}^0 V_i] = {}^k R \cdot v_i$$

$${}^k \omega_i \equiv {}^k [{}^0 \Omega_i] = {}^k R [{}^0 \Omega_i] = {}^k R \cdot \omega_i$$

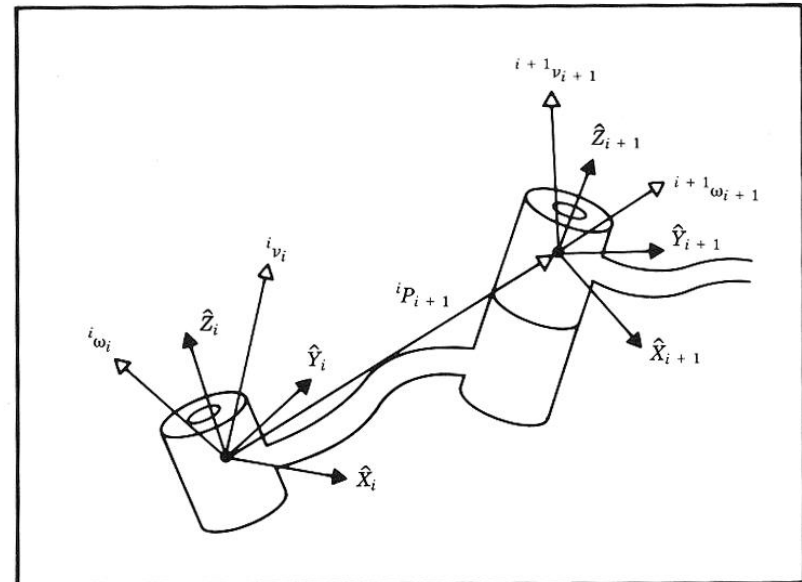




## Velocity Propagation

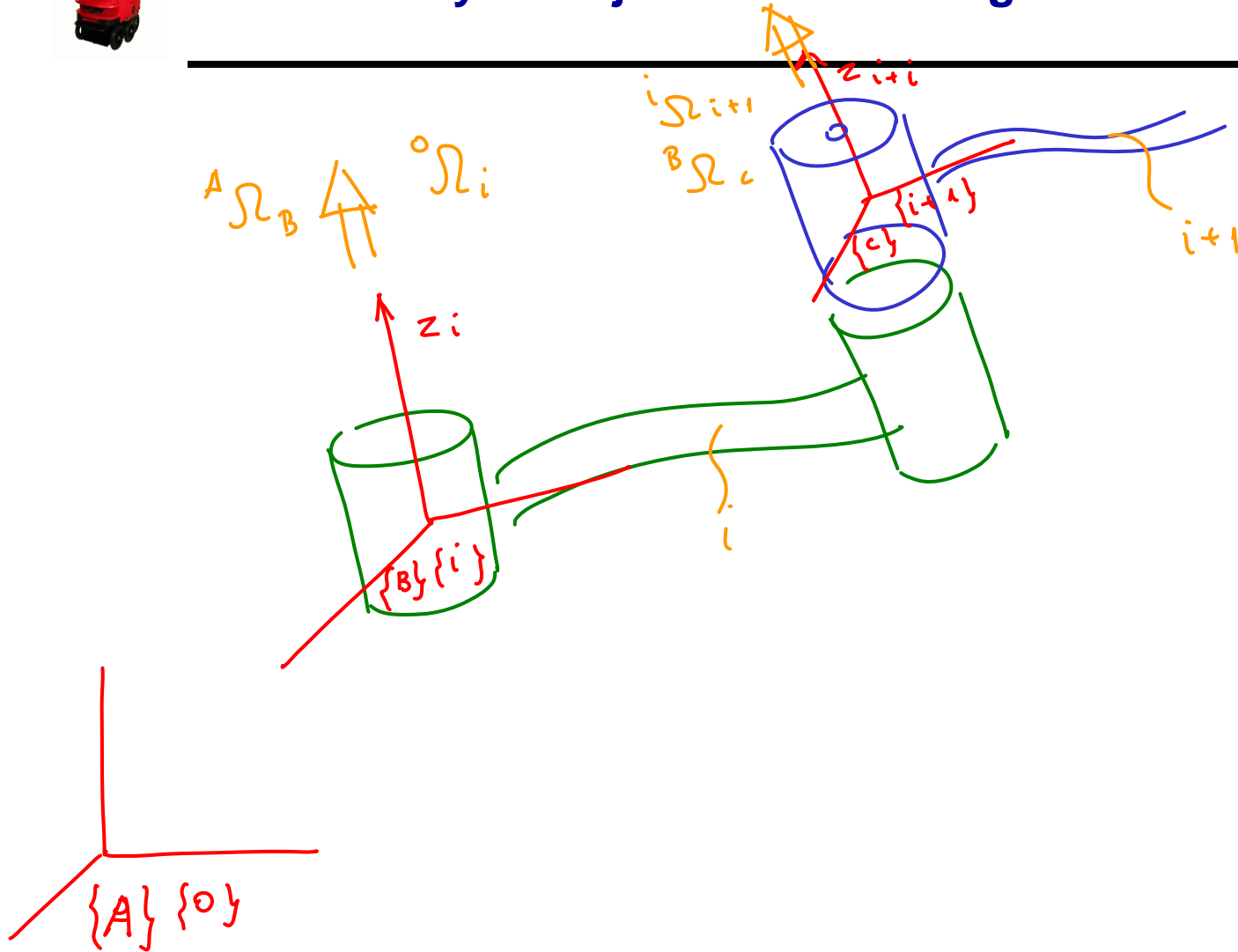
- **Given:** A manipulator - A chain of rigid bodies each one capable of moving relative to its neighbor
- **Problem:** Calculate the linear and angular velocities of the link of a robot
- **Solution (Concept):** Due to the robot structure (serial mechanism) we can **compute the velocities** of each link in order **starting from the base**.

The velocity of link  $i+1$  will be that of link  $i$ , plus whatever new velocity components were added by joint  $i+1$





# Velocity of Adjacent Links - Angular Velocity 0/5





## Velocity of Adjacent Links - Angular Velocity 1/5

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- From the relationship developed previously

$$\longrightarrow {}^A\Omega_C = {}^A\Omega_B + {}^A R^B \Omega_C$$

- we can re-assign link names to calculate the velocity of any link  $i$  relative to the base frame  $\{0\}$

$$\left\{ \begin{array}{l} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{array} \right.$$

$${}^0\Omega_{i+1} = {}^0\Omega_i + {}^0R^i \Omega_{i+1}$$

- By pre-multiplying both sides of the equation by  ${}^{i+1}_0R$ , we can convert the frame of reference for the base  $\{0\}$  to frame  $\{i+1\}$



## Velocity of Adjacent Links - Angular Velocity 2/5

$${}^{i+1}_0 R^0 \Omega_{i+1} = {}^{i+1}_0 R^0 \Omega_i + {}^{i+1}_0 R^i R^i \Omega_{i+1}$$
$${}^{i+1} \omega_{i+1} = {}^{i+1} \omega_i + {}^{i+1} R^i \Omega_{i+1}$$

The diagram shows the derivation of the angular velocity equation. The top equation is annotated with orange brackets and arrows. The bottom equation is enclosed in a box with orange circles highlighting the terms that correspond to the terms in the top equation.

- Using the recently defined notation, we have

- ${}^{i+1} \omega_{i+1}$  - Angular velocity of frame  $\{i+1\}$  measured relative to the robot base, and expressed in frame  $\{i+1\}$  - **Recall the car example**  ${}^c [{}^w V_c] = {}^c v_c$
- ${}^{i+1} \omega_i$  - Angular velocity of frame  $\{i\}$  measured relative to the robot base, and expressed in frame  $\{i+1\}$
- ${}^{i+1} R^i \Omega_{i+1}$  - Angular velocity of frame  $\{i+1\}$  measured relative to frame  $\{i\}$  and expressed in frame  $\{i+1\}$



## Velocity of Adjacent Links - Angular Velocity 3/5

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$${}^{i+1}\omega_{i+1} = \boxed{{}^{i+1}\omega_i} + {}^{i+1}R^i \Omega_{i+1}$$

- Angular velocity of frame  $\{i\}$  measured relative to the robot base, **expressed in frame  $\{i+1\}$**

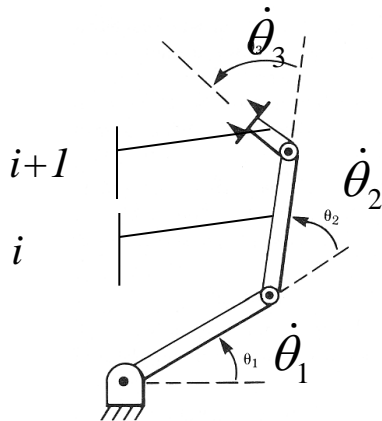
$${}^{i+1}\omega_i = {}^{i+1}R^i \omega_i$$
$${}^{i+1}R \begin{bmatrix} {}^iR \\ \Omega_i \end{bmatrix}$$



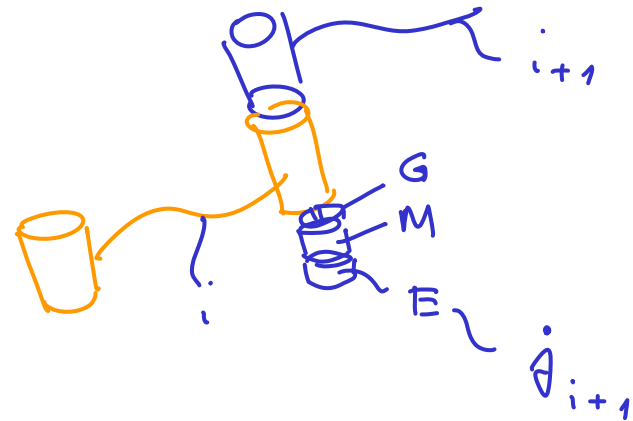
## Velocity of Adjacent Links - Angular Velocity 4/5

$${}^{i+1}\omega_{i+1} = {}^{i+1}\omega_i + \boxed{{}^{i+1}R^i \Omega_{i+1}}$$

- Angular velocity of frame  $\{i+1\}$  measured (differentiate) in frame  $\{i\}$  and represented (expressed) in frame  $\{i+1\}$
- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (**angular velocity**) or prismatic joint (Linear velocity).
- Based on the frame attachment convention in which we assign the Z axis pointing along the  $i+1$  joint axis such that the two are coincide (rotations of a link is preformed only along its Z- axis) we can rewrite this term as follows:



$${}^{i+1}R^i \Omega_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$







## Velocity of Adjacent Links - Angular Velocity 5/5

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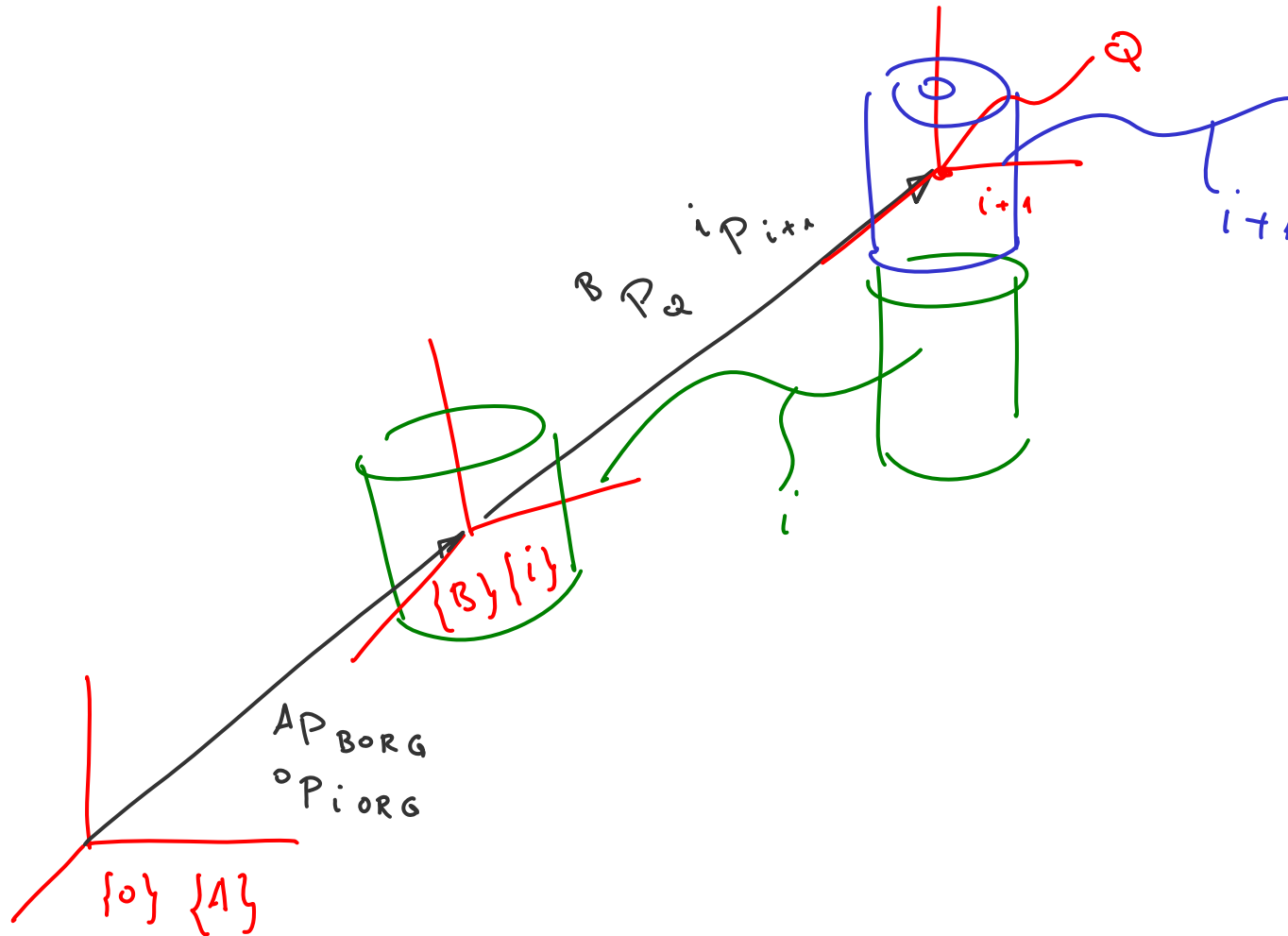
- The result is a **recursive equation** that shows the angular velocity of one link in terms of the angular velocity of the previous link plus the relative motion of the two links.

$${}^{i+1}\omega_{i+1} = {}^{i+1}R^i \omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

- Since the term  ${}^{i+1}\omega_{i+1}$  depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.



# Velocity of Adjacent Links - Linear Velocity 0/6





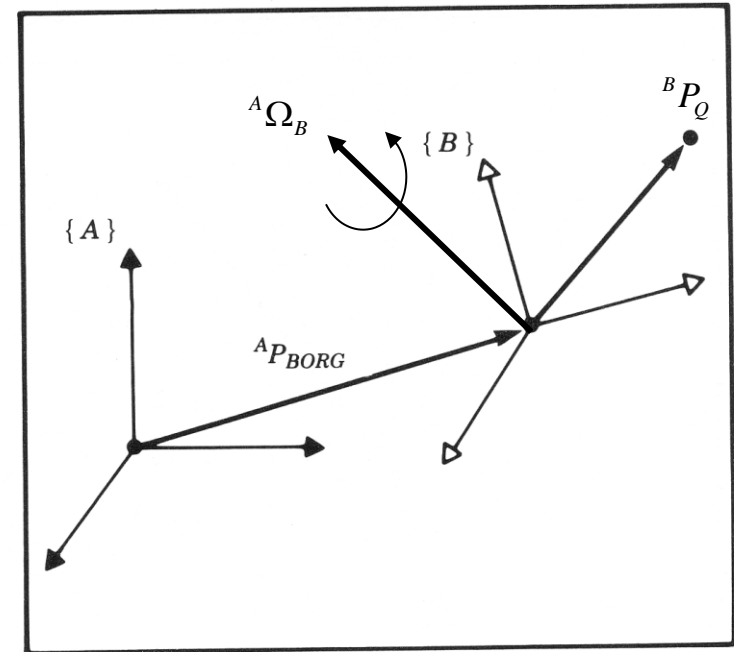
## Velocity of Adjacent Links - Linear Velocity 1/6

- Simultaneous Linear and Rotational Velocity
- The derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame
- Vector Form

$${}^A V_Q = {}^A V_{BORG} + {}_B^A R^B V_Q + {}^A \Omega_B \times {}_B^A R^B P_Q$$

- Matrix Form

$${}^A V_Q = {}^A V_{BORG} + {}_B^A R^B V_Q + {}_B^A \dot{R}_\Omega \left( {}_B^A R^B P_Q \right)$$





## Velocity of Adjacent Links - Linear Velocity 2/6

- From the relationship developed previously (matrix form)

$$\longrightarrow {}^A V_Q = {}^A V_{BORG} + {}^A R^B V_Q + {}^A \dot{R}_\Omega \left( {}^A R^B P_Q \right)$$

- we re-assign link frames for adjacent links ( $i$  and  $i+1$ ) with the velocity computed relative to the robot base frame  $\{0\}$

$$\left\{ \begin{array}{l} A \rightarrow 0 \\ B \rightarrow i \\ C \rightarrow i+1 \end{array} \right.$$

$\xrightarrow{i+1} V_{i+1}$

$$\longrightarrow \underbrace{{}^0 V_{i+1}} = \underbrace{{}^0 \dot{R}_\Omega} \left( {}^0 R^i P_{i+1} \right) + {}^0 V_i + {}^0 R^i V_{i+1}$$

- By pre-multiplying both sides of the equation by  ${}^0 R^{i+1}$ , we can convert the frame of reference for the left side to frame  $\{i+1\}$



## Velocity of Adjacent Links - Linear Velocity 3/6

$$\begin{matrix} \downarrow & & \downarrow & & \downarrow & & \downarrow \\ {}^{i+1}_0 R^0 V_{i+1} = & {}^{i+1}_0 R_i^0 \dot{R}_\Omega & ({}^0 R^i P_{i+1}) + & {}^{i+1}_0 R^0 V_i + & \boxed{{}^{i+1}_0 R_i^0 R^i V_{i+1}} \end{matrix}$$

- Which simplifies to

$${}^{i+1}_0 R^0 V_{i+1} = \boxed{{}^{i+1}_0 R_i^0 \dot{R}_\Omega ({}^0 R^i P_{i+1}) + {}^{i+1}_0 R^0 V_i} + \boxed{{}^{i+1}_i R^i V_{i+1}}$$

- Factoring out  ${}^{i+1}_i R$  from the left side of the first two terms

$${}^{i+1}_0 R^0 V_{i+1} = \boxed{{}^{i+1}_i R ({}^i R_i^0 \dot{R}_\Omega {}^0 R^i P_{i+1} + {}^i R^0 V_i)} + {}^{i+1}_i R^i V_{i+1}$$



## Velocity of Adjacent Links - Linear Velocity 4/6

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$${}^{i+1}R^0V_{i+1} = {}^{i+1}R \left( {}^iR^0\dot{R}_{\Omega i} {}^0R^iP_{i+1} + {}^iR^0V_i \right) + \boxed{{}^{i+1}R^iV_{i+1}}$$

${}^{i+1}R^iV_{i+1}$  - Linear velocity of frame  $\{i+1\}$  measured relative to frame  $\{i\}$  and expressed in frame  $\{i+1\}$

- Assuming that a joint has only 1 DOF. The joint configuration can be either revolute joint (angular velocity) or prismatic joint (Linear velocity).
- Based on the frame attachment convention in which we assign the Z axis pointing along the  $i+1$  joint axis such that the two are coincide (**translation of a link is preformed only along its Z- axis**) we can rewrite this term as follows:

$${}^{i+1}R^iV_{i+1} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$



## Velocity of Adjacent Links - Linear Velocity 5/6

$${}_{0}^{i+1}R^0V_{i+1} = {}_{i}^{i+1}R \left( {}_{0}^{i}R_i^0\dot{R}_{\Omega i}^0R^i P_{i+1} + {}_{0}^{i}R^0V_i \right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

$${}_{0}^{i}R_i^0\dot{R}_{\Omega i}^0R = {}_{0}^{i}R_i^0\dot{R}_{\Omega 0}^iR^T = {}_{0}^{i}R^0\Omega_i = {}_{0}^{i}R\omega_i = {}^i\omega_i$$

Multiply by Matrix

Definition

$${}_{0}^{i+1}R^0V_{i+1} = {}^{i+1}v_{i+1}$$

Definition

$${}_{0}^{i}R^0V_i = {}^i v_i$$

Definition



## Velocity of Adjacent Links - Linear Velocity 6/6

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- The result is a **recursive equation** that shows the linear velocity of one link in terms of the previous link plus the relative motion of the two links.

$$\longrightarrow \quad {}^{i+1}v_{i+1} = {}^{i+1}R \left( {}^i\omega_i \times {}^iP_{i+1} + {}^i v_i \right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

- Since the term  ${}^{i+1}v_{i+1}$  depends on all previous links through this recursion, the angular velocity is said to propagate from the base to subsequent links.





# Velocity of Adjacent Links - Summary

- Angular Velocity

0 - Prismatic Joint

$${}^{i+1}\omega_{i+1} = {}^{i+1}R^i \omega_i + \begin{matrix} \text{0} \\ \text{0} \\ \dot{\theta}_{i+1} \end{matrix}$$

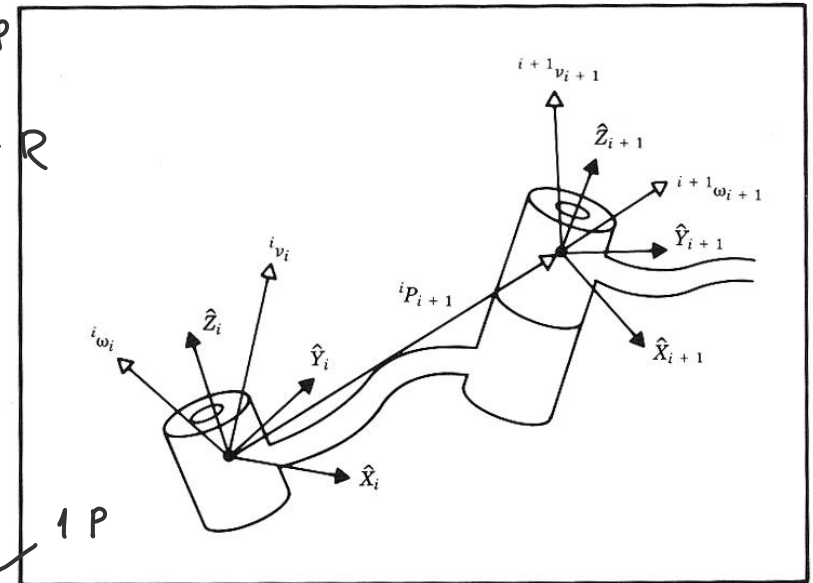
0-P  
R

- Linear Velocity

0 - Revolute Joint

$${}^{i+1}v_{i+1} = {}^{i+1}R^i (\omega_i \times {}^iP_{i+1} + v_i) + \begin{matrix} \text{0} \\ \text{0} \\ \dot{d}_{i+1} \end{matrix}$$

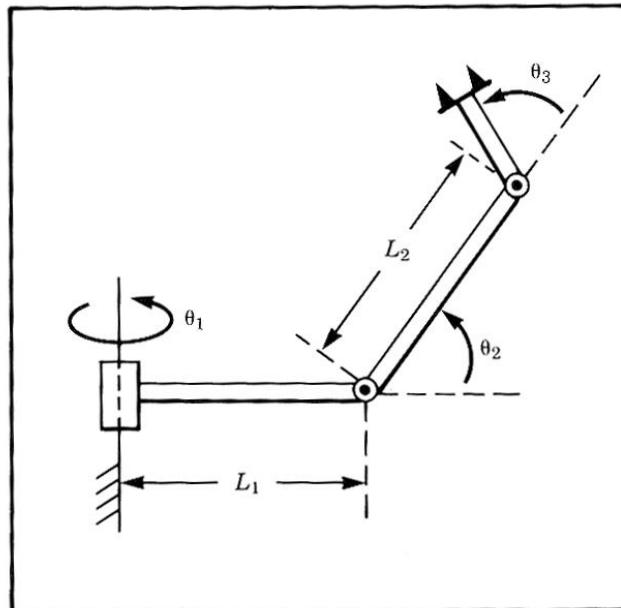
1-P  
R





## Angular and Linear Velocities - 3R Robot - Example

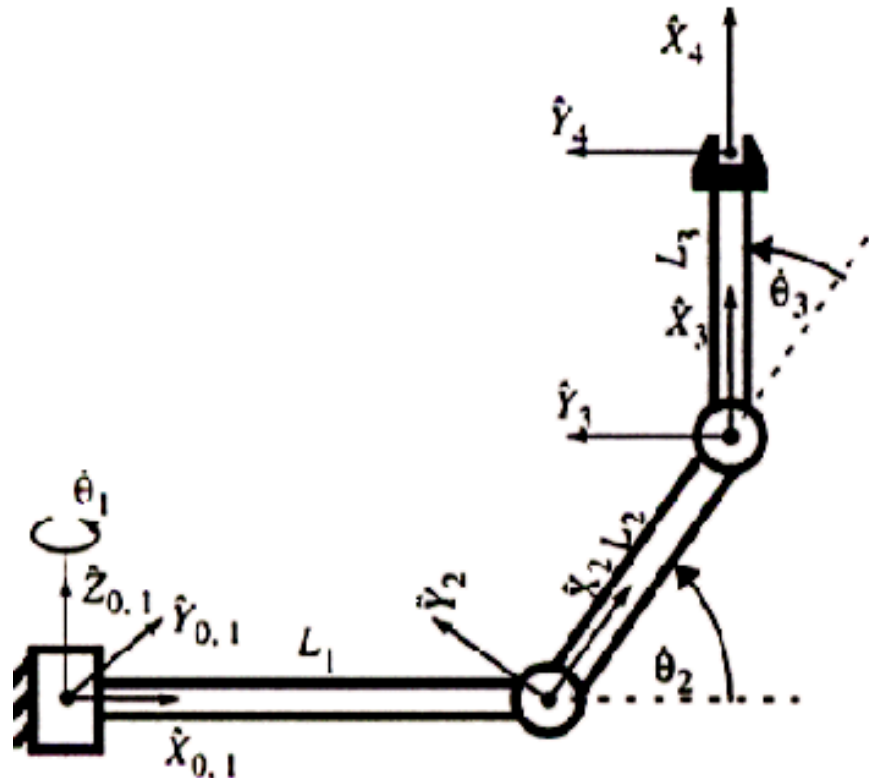
- For the manipulator shown in the figure, compute the angular and linear velocity of the “tool” frame relative to the base frame expressed in the “tool” frame (that is, calculate  ${}^4\omega_4$  and  ${}^4v_4$  ).





## Angular and Linear Velocities - 3R Robot - Example

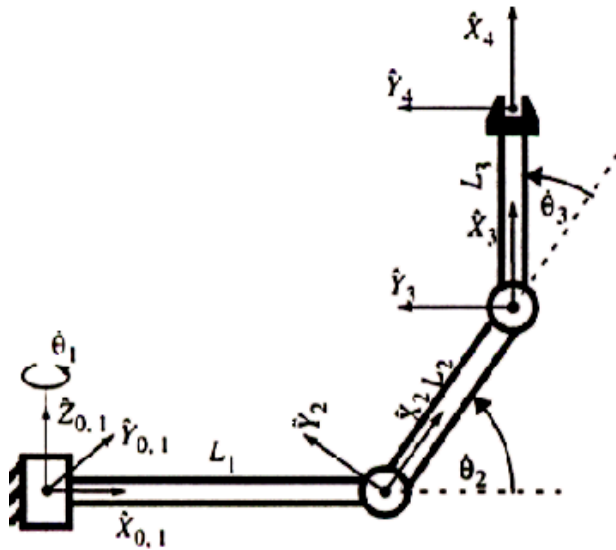
- Frame attachment





## Angular and Linear Velocities - 3R Robot - Example

- DH Parameters



$i$	$\alpha_{i-1}$	$a_{i-1}$	$d_i$	$\theta_i$
1	0	0	0	$\theta_1$
2	90	L1	0	$\theta_2$
3	0	L2	0	$\theta_3$
4	0	L3	0	0



## Angular and Linear Velocities - 3R Robot - Example

- From the DH parameter table, we can specify the homogeneous transform matrix for each adjacent link pair:

$${}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$${}^0T_1 = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$${}^2T_3 = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$${}^1T_2 = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$${}^3T_4 = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



## Angular and Linear Velocities - 3R Robot - Example

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- Compute the angular velocity of the end effector frame relative to the base frame expressed at the end effector frame.

$${}^{i+1}\omega_{i+1} = {}^{i+1}R^i \omega_i + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

- For  $i=0$

$${}^1\omega_1 = {}^1R^0 \omega_0 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$



## Angular and Linear Velocities - 3R Robot - Example

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- For  $i=1$   ${}^2\omega_2 = {}^2R^1\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$
- For  $i=2$   ${}^3\omega_3 = {}^3R^2\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$
- For  $i=3$   ${}^4\omega_4 = {}^4R^3\omega_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$
- Note  ${}^3\omega_3 = {}^4\omega_4$



## Angular and Linear Velocities - 3R Robot - Example

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- Compute the linear velocity of the end effector frame relative to the base frame expressed at the end effector frame.
- 
- Note that the term involving the prismatic joint has been dropped from the equation (it is equal to zero).

$${}^{i+1}v_{i+1} = {}^{i+1}R \left( \omega \times {}^i P_{i+1} + {}^i v_i \right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

↗ 0





## Angular and Linear Velocities - 3R Robot - Example

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- For  $i=0$

$${}^1v_1 = {}^1R^0 \left\{ \omega_0 \times {}^0P_1 + {}^0v_0 \right\} = \begin{bmatrix} c1 & s1 & 0 \\ -s1 & c1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- For  $i=1$

$${}^2v_2 = {}^2R^1 \left\{ \omega_1 \times {}^1P_2 + {}^1v_1 \right\} = \begin{bmatrix} c2 & 0 & s2 \\ -s2 & 0 & c2 \\ 0 & -1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} L1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \dot{\theta}_1 \end{bmatrix}$$



## Angular and Linear Velocities - 3R Robot - Example

- For  $i=3$

$$\begin{aligned} {}^3v_3 = {}^3R\left\{{}^2\omega_2 \times {}^2P_3 + {}^2v_2\right\} &= \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} L2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -L1\dot{\theta}_1 \end{bmatrix} \right\} \\ &= \begin{bmatrix} c3 & s3 & 0 \\ -s3 & c3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ L2\dot{\theta}_1 \\ -L2c2\dot{\theta}_1 - L1\dot{\theta}_1 \end{bmatrix} \right\} = \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix} \end{aligned}$$



## Angular and Linear Velocities - 3R Robot - Example

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- For  $i=4$

$${}^4v_4 = {}^4R^3 \left\{ \omega_3 \times {}^3P_4 + {}^3v_3 \right\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} \times \begin{bmatrix} L3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L2s3\dot{\theta}_2 \\ L2c3\dot{\theta}_2 \\ (-L1 - L2c2)\dot{\theta}_1 \end{bmatrix} \right\}$$
$$= \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \end{bmatrix}$$



## Angular and Linear Velocities - 3R Robot - Example

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- Note that the linear and angular velocities ( ${}^4\omega_4, {}^4v_4$ ) of the end effector where differentiate (measured) in frame  $\{0\}$  however represented (expressed) in frame  $\{4\}$
- In the car example: Observer sitting in the “Car”  ${}^C[{}^wV_C]$   
Observer sitting in the “World”  ${}^w[{}^wV_C]$

$${}^k v_i \equiv {}^k [{}^0V_i] = {}^k R [{}^0V_i] = {}^k R \cdot v_i$$

$${}^k \omega_i \equiv {}^k [{}^0\Omega_i] = {}^k R [{}^0\Omega_i] = {}^k R \cdot \omega_i$$

Solve for  $v_4$  and  $\omega_4$  by multiply both side of the questions from the left by  ${}^4R^{-1}$

$${}^4v_4 = {}^4R \cdot v_4$$

$${}^4\omega_4 = {}^4R \cdot \omega_4$$



## Angular and Linear Velocities - 3R Robot - Example

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- Multiply both sides of the equation by the inverse transformation matrix, we finally get the linear and angular velocities expressed and measured in the stationary frame {0}

$$v_4 = {}^4R^{-1} \cdot {}^4v_4 = {}^4R^T \cdot {}^4v_4 = {}^0R \cdot {}^4v_4$$

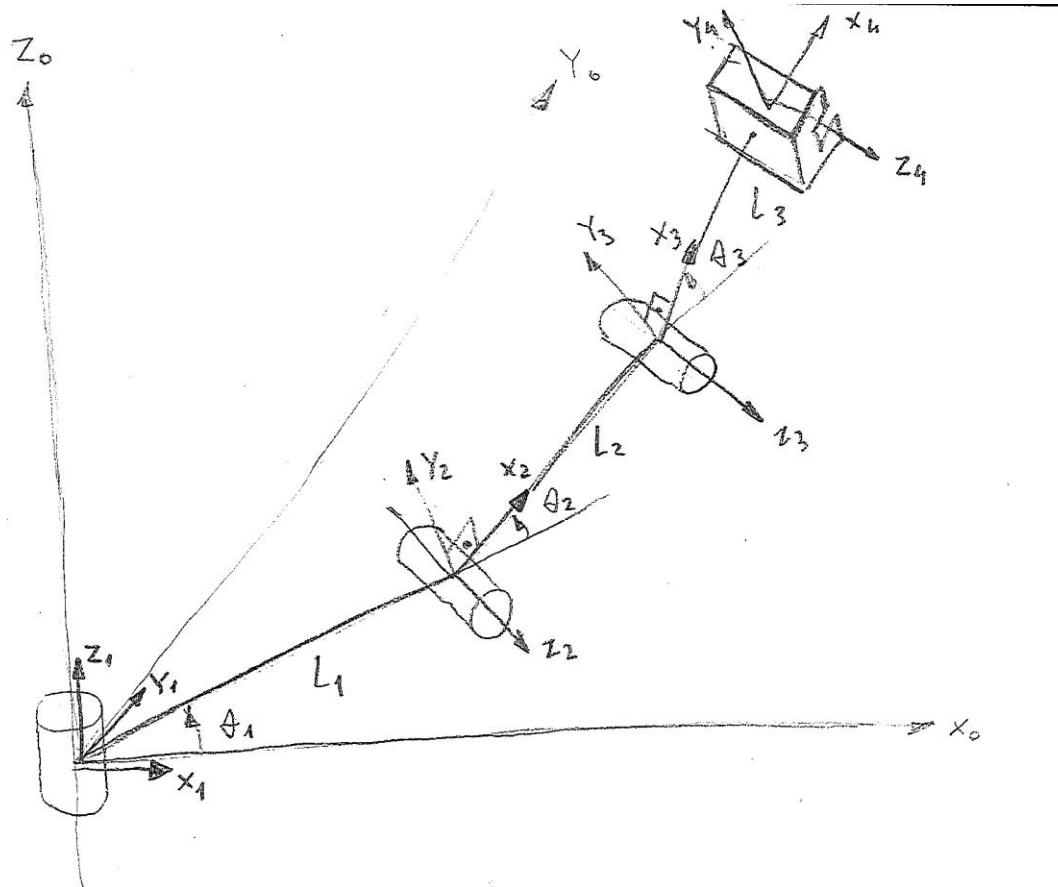
$$\omega_4 = {}^4R^{-1} \cdot \omega_4 = {}^4R^T \cdot \omega_4 = {}^0R \cdot \omega_4$$

$${}^0T_4 = {}^0T_1 {}^1T_2 {}^2T_3 {}^3T_4$$



# Angular and Linear Velocities - 3R Robot - Example

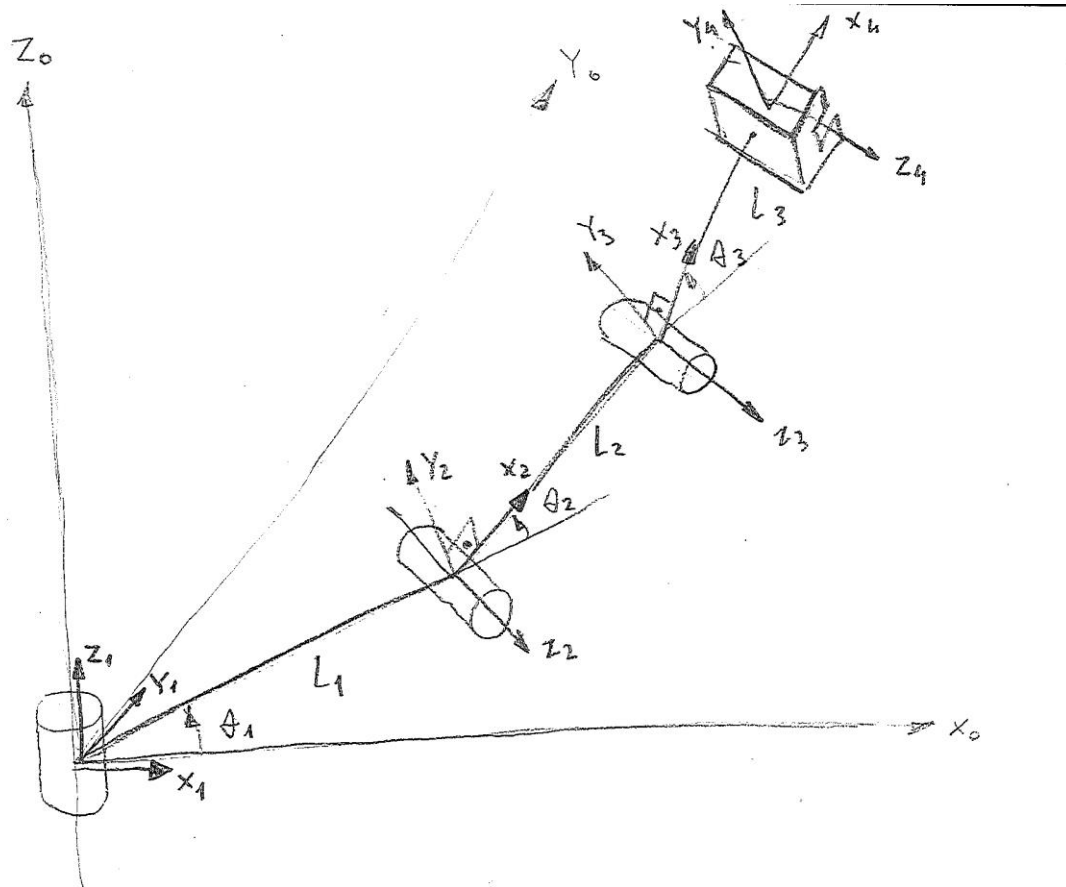
$${}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$





## Angular and Linear Velocities - 3R Robot - Example

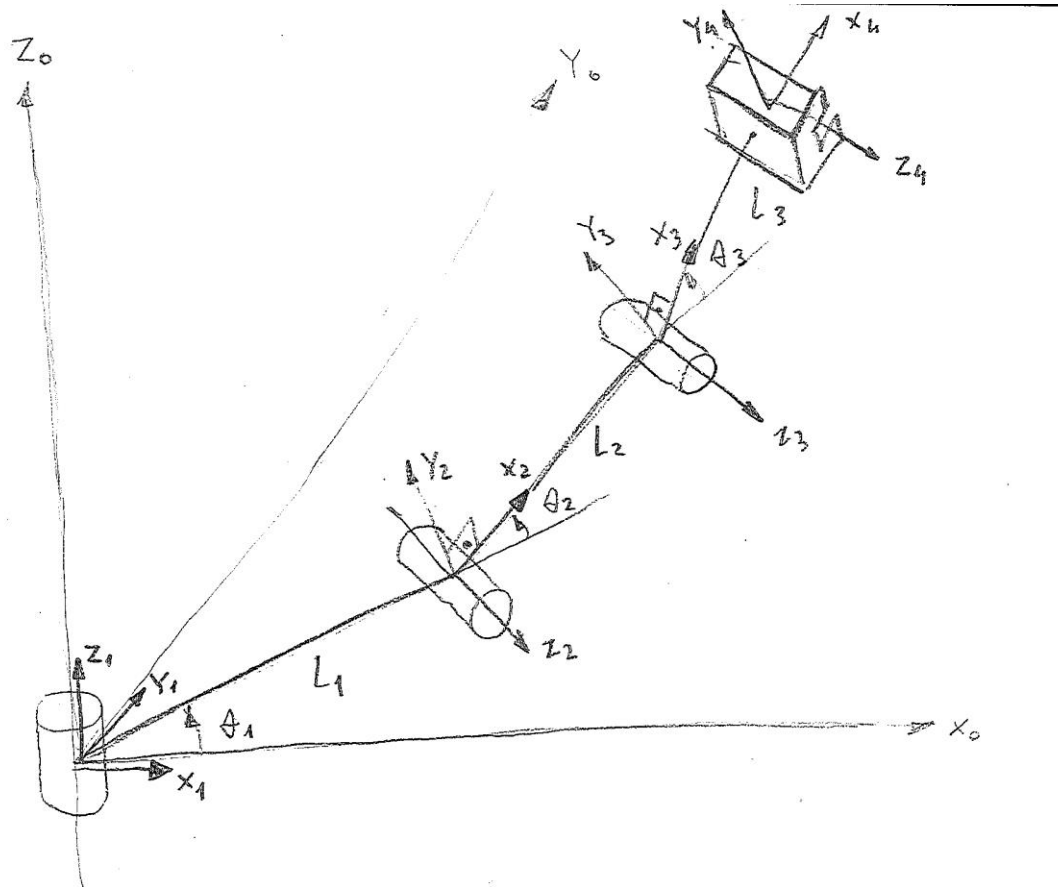
$${}^2\omega_2 = \begin{bmatrix} s2\dot{\theta}_1 \\ c2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$





## Angular and Linear Velocities - 3R Robot - Example

$${}^3\omega_3 = \begin{bmatrix} s_{23}\dot{\theta}_1 \\ c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

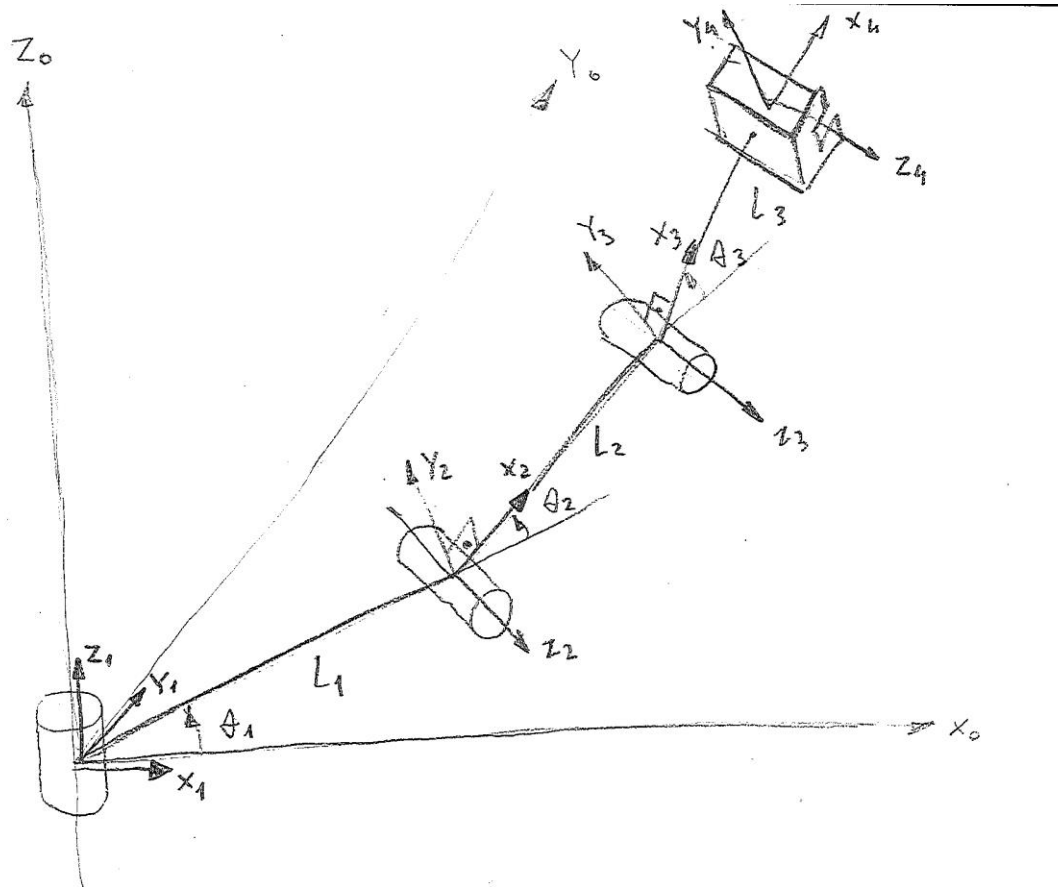






# Angular and Linear Velocities - 3R Robot - Example

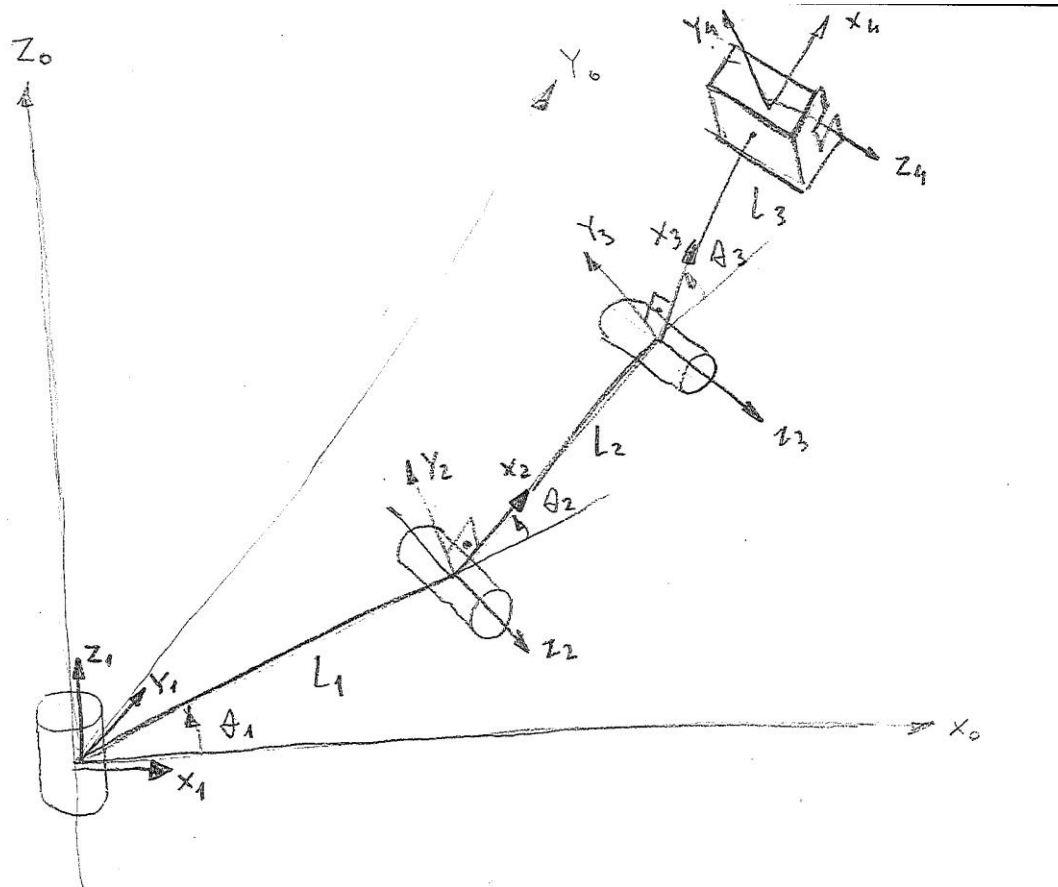
$${}^3\omega_3 = {}^4\omega_4$$





## Angular and Linear Velocities - 3R Robot - Example

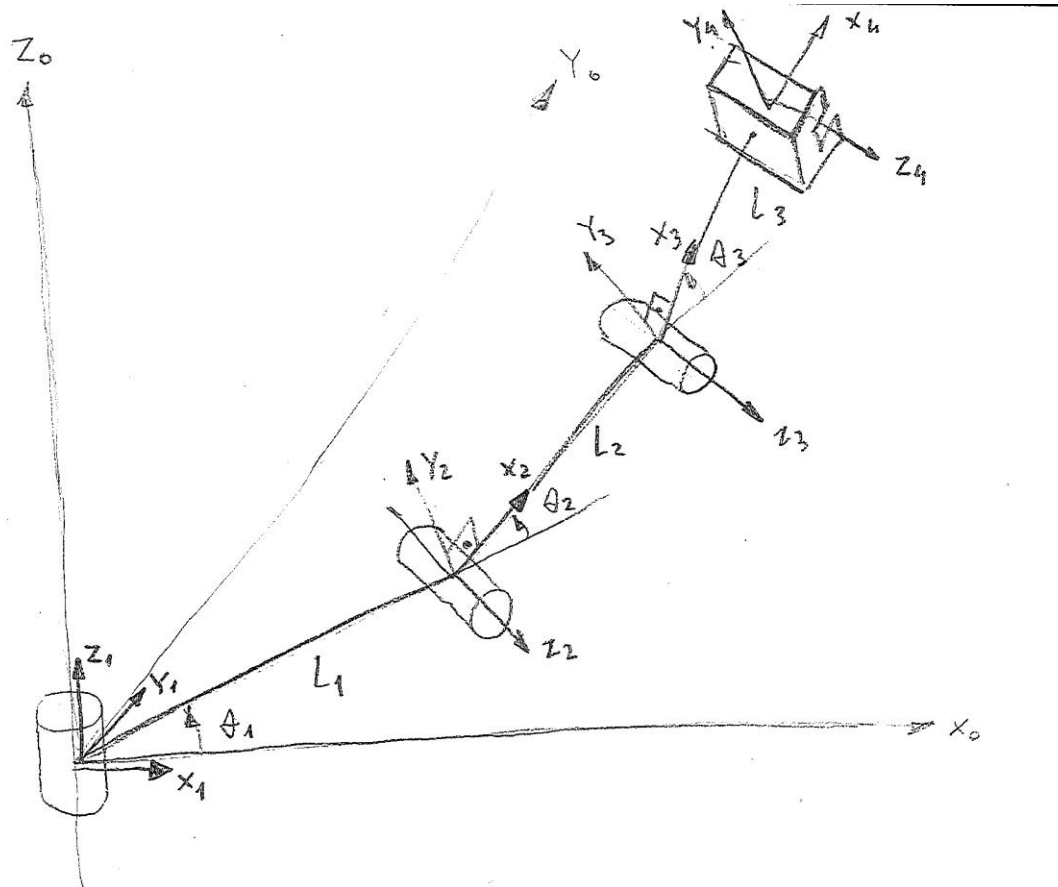
$${}^1v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$





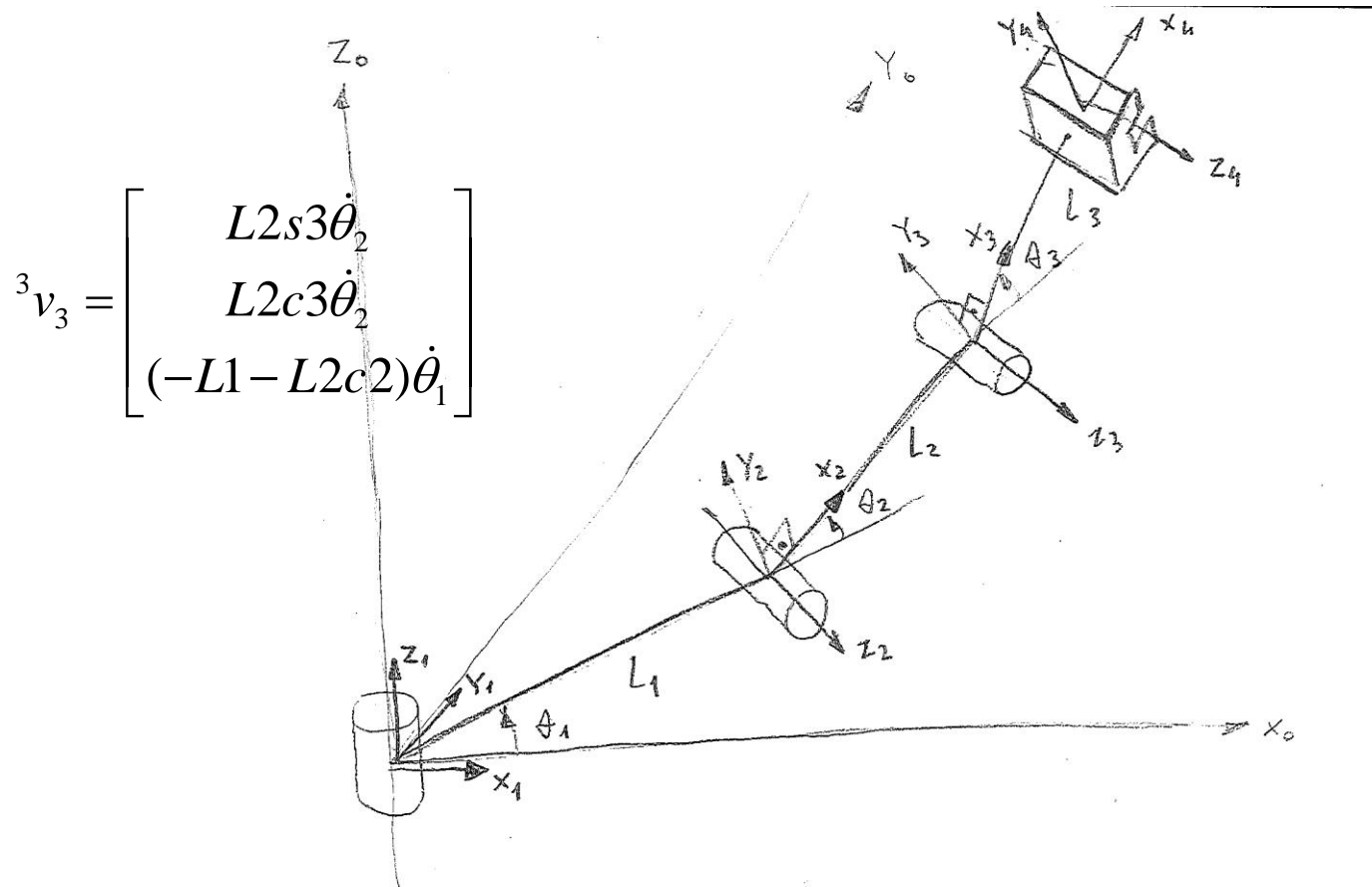
## Angular and Linear Velocities - 3R Robot - Example

$${}^2v_2 = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix}$$





# Angular and Linear Velocities - 3R Robot - Example





## Angular and Linear Velocities - 3R Robot - Example

$${}^4v_4 = \begin{bmatrix} L_2 s_3 \dot{\theta}_2 \\ (L_2 c_3 + L_3) \dot{\theta}_2 + L_3 \dot{\theta}_3 \\ -(L_1 + L_2 c_2 + L_3 c_{23}) \dot{\theta}_1 \end{bmatrix}$$

