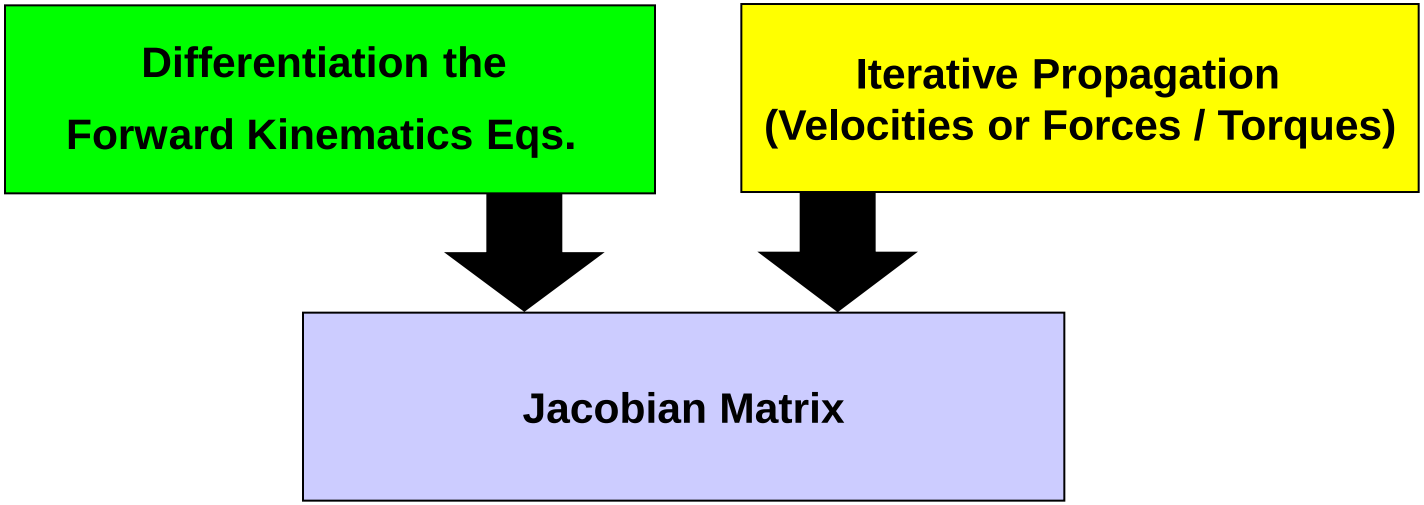
Linear and Angular Velocities 2/4

Jacobian Matrix - Calculation Methods



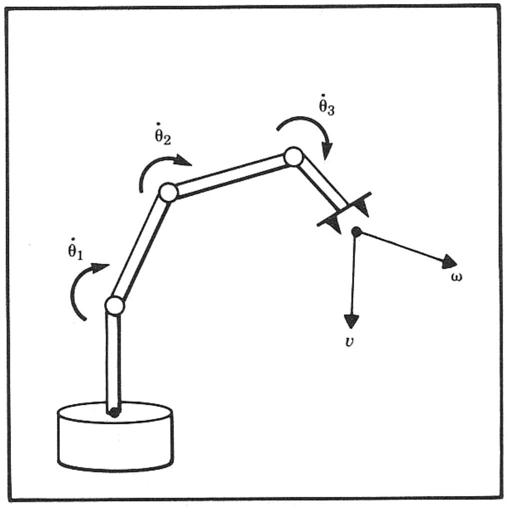
Jacobian Matrix - Introduction

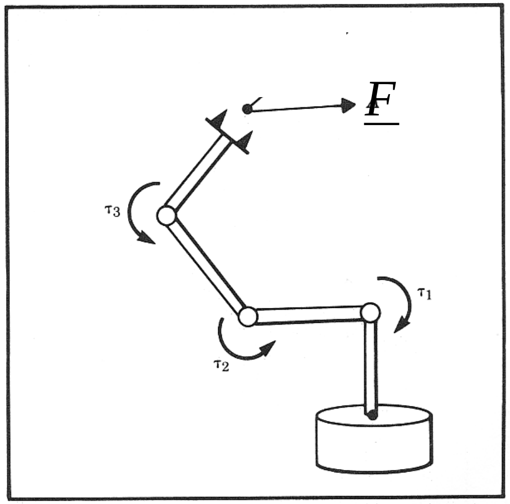
- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$





Velocity Propagation – Link / Joint Abstraction

Velocity Propagation – Intuitive Explanation

Velocity Propagation – Intuitive Explanation

Central Topic - Simultaneous Linear and Rotational Velocity

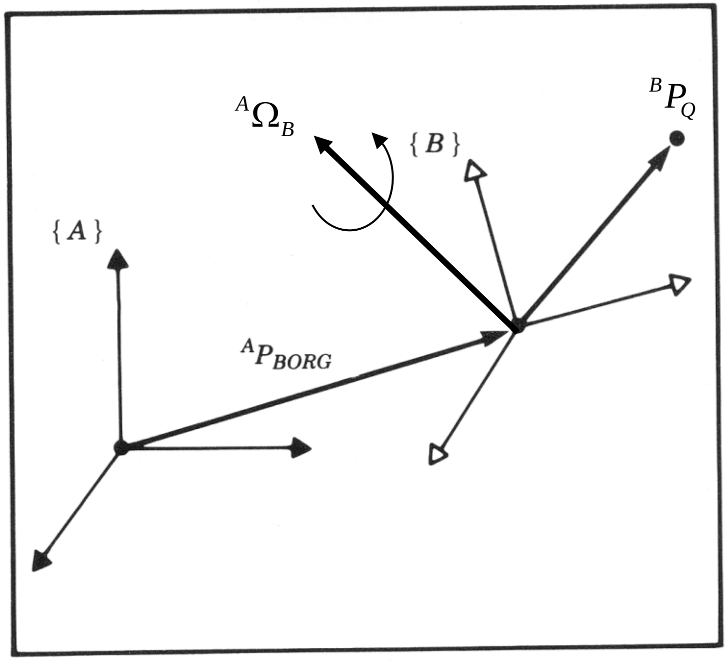
$$ {}^{A}V_{Q} = f\left({}^{B}P_{Q},\ \ {}^{B}V_{Q},\ \ {}^{A}V_{BORG},\ \ {}^{A}\Omega_{B},\ \ {}^{A}_{B}R\right) $$

- Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form

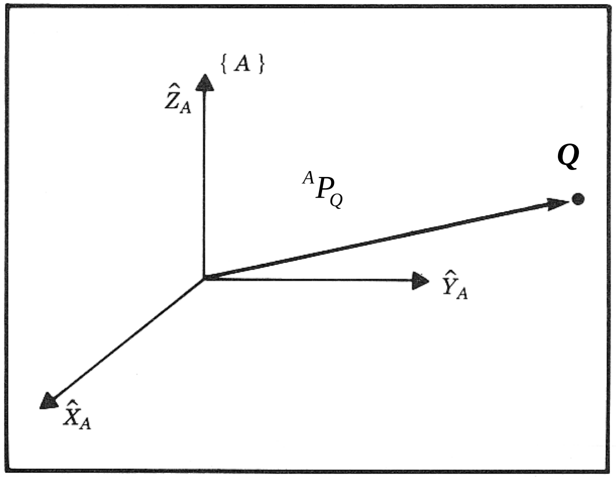
$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Definitions - Linear Velocity

- ***Linear velocity*** - The instantaneous rate of change in linear position of a point relative to some frame.



$$ {}^{A}V_Q = \frac{d}{dt}\,{}^{A}P_Q \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_Q(t+\Delta t) - {}^{A}P_Q(t)}{\Delta t} \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_Q(t) - {}^{A}P_Q(t-\Delta t)}{\Delta t} $$

$$ {}^AV_Q = \boxed{{}^AV_{BORG}} + {}^A_BR\,\boxed{{}^BV_Q} + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = \boxed{{}^AV_{BORG}} + {}^A_BR\,\boxed{{}^BV_Q} + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right) $$

Definitions - Linear Velocity

- The position of point Q in frame {A} is represented by the ***linear position vector***

$$ {}^AP_Q = \begin{bmatrix} {}^AP_{Qx} \\ {}^AP_{Qy} \\ {}^AP_{Qz} \end{bmatrix} $$

- The velocity of a point Q relative to frame {A} is represented by the ***linear velocity vector***

$$ {}^AV_Q = \frac{{}^Ad}{dt}\begin{bmatrix} {}^AP_{Qx} \\ {}^AP_{Qy} \\ {}^AP_{Qz} \end{bmatrix} = \begin{bmatrix} {}^A\dot{P}_{Qx} \\ {}^A\dot{P}_{Qy} \\ {}^A\dot{P}_{Qz} \end{bmatrix} $$

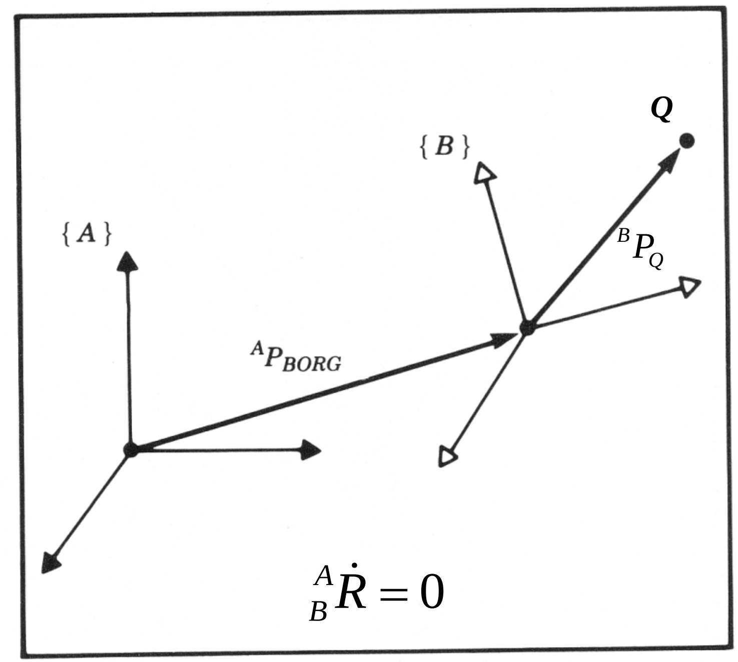
$${}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right)$$

Linear Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The orientation of frame {A} with respect to frame {B} is not changing as a function of time $\,{}^A_B\dot{R} = 0$

- ***Problem:*** describe the motion of of the vector ${}^BP_Q$ relative to frame {A}

- ***Solution:*** Frame {B} is located relative to frame {A} by a position vector ${}^AP_{BORG}$ and the rotation matrix ${}^A_BR$ (assume that the orientation is not changing in time ${}^A_B\dot{R} = 0$) expressing both components of the velocity in terms of frame {A} gives



$${}^AV_Q = {}^AV_{BORG} + {}^A\left({}^BV_Q\right) = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q$$

Linear Velocity – Translation

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Linear Velocity – Translation

$${}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Linear & Angular Velocities - Frames

- When describing the velocity (linear or angular) of an object, there are two important frames that are being used:

  - ***Represented Frame (Reference Frame)*** :

    This is the frame used to **represent (express)** the object's velocity.

  - ***Computed Frame***

    This is the frame in which the velocity is **measured** (differentiate the position).

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frame - Velocity

- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.

- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

Represented (Reference Frame)

$$ {}^{A}({}^{B}V_{Q}) = \frac{{}^{A}d}{dt}\,{}^{B}P_{Q} $$

Computed (Measured)

$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Frame - Linear Velocity

- We can always remove the outer, leading superscript by explicitly including the rotation matrix which accomplishes the change in the reference frame

$${}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

- Note that in the general case ${}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} \neq {}^{A}V_{Q}$ because ${}^{A}_{B}R$ may be time-verging ${}^{A}_{B}\dot{R} \neq 0$

- If the calculated velocity is written in terms of of the frame of differentiation the result could be indicated by a single leading superscript.

$${}^{A}({}^{A}V_{Q}) = {}^{A}V_{Q}$$

- In a similar fashion when the angular velocity is expresses and measured as a vector

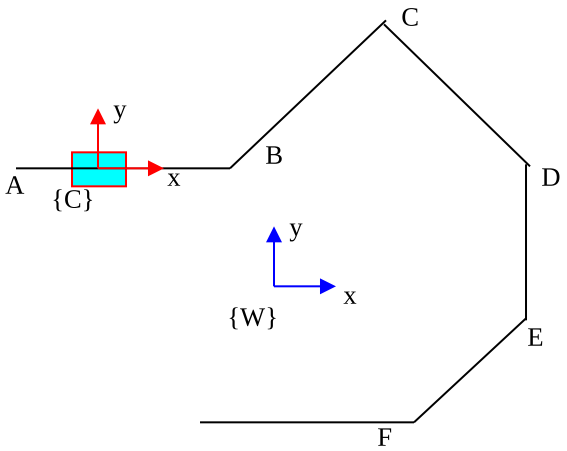
$${}^{A}({}^{B}\Omega_{C}) = {}^{A}_{B}R\,{}^{B}\Omega_{C}$$

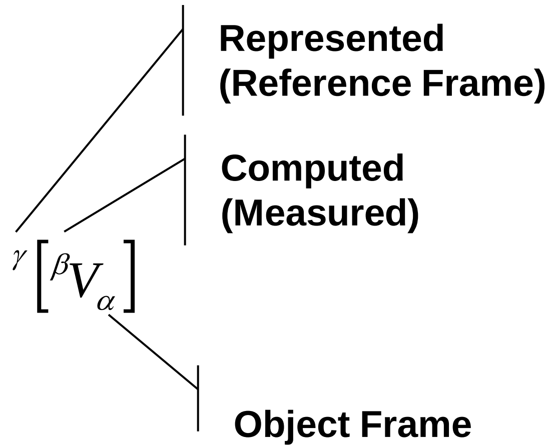
$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

- ***Given:*** The driver of the car maintains a ***speed*** of 100 km/h (as shown to the driver by the car's speedometer).

- ***Problem:*** Express the velocities ${}^{C}\left[{}^{C}V_{C}\right]$ ${}^{W}\left[{}^{W}V_{C}\right]$ ${}^{W}\left[{}^{C}V_{C}\right]$ ${}^{C}\left[{}^{W}V_{C}\right]$ in each section of the road A, B, C, D, E, F where {C} - Car frame, and {W} - World frame





$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

$$ {}^{A}_{B}R = Rot(\hat{z},\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ Rot(\hat{z},+45^{o}) = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},-45^{o}) = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},+90^{o}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Frames - Linear Velocity - Example

$$ {}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

- $ {}^{A}_{B}\dot{R} = 0 $ is not time-varying (in this example)

$$ {}^{C}({}^{C}V_C) = {}^{C}_{C}R\,{}^{C}V_C = I[0] = [0] $$

$$ {}^{W}({}^{W}V_C) = {}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C $$

$$ {}^{W}({}^{C}V_C) = {}^{W}_{C}R\,{}^{C}V_C = {}^{W}_{C}R[0] = [0] $$

$$ {}^{C}({}^{W}V_C) = {}^{C}_{W}R\,{}^{W}V_C $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

Road Section	**Velocity**			
	${}^{C}\left[{}^{C}V_{C}\right]$	${}^{W}\left[{}^{W}V_{C}\right]$	${}^{W}\left[{}^{C}V_{C}\right]$	${}^{C}\left[{}^{W}V_{C}\right]$
A				
B				
C				
D				
E				
F				

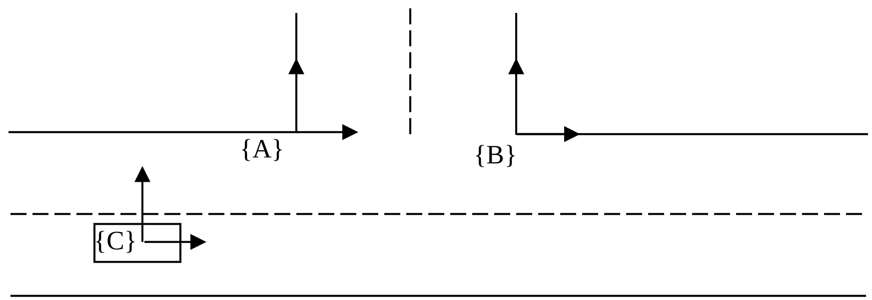
$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,\boxed{^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,\boxed{^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Linear Velocity - Free Vector

- Linear velocity vectors are insensitive to shifts in origin.

- Consider the following example:



- The velocity of the object in {C} relative to both {A} and {B} is the same, that is

$$^{A}V_{C} = {}^{B}V_{C}$$

- As long as {A} and {B} remain fixed relative to each other (translational but not rotational), then the velocity vector remains unchanged (that is, a ***free vector***).

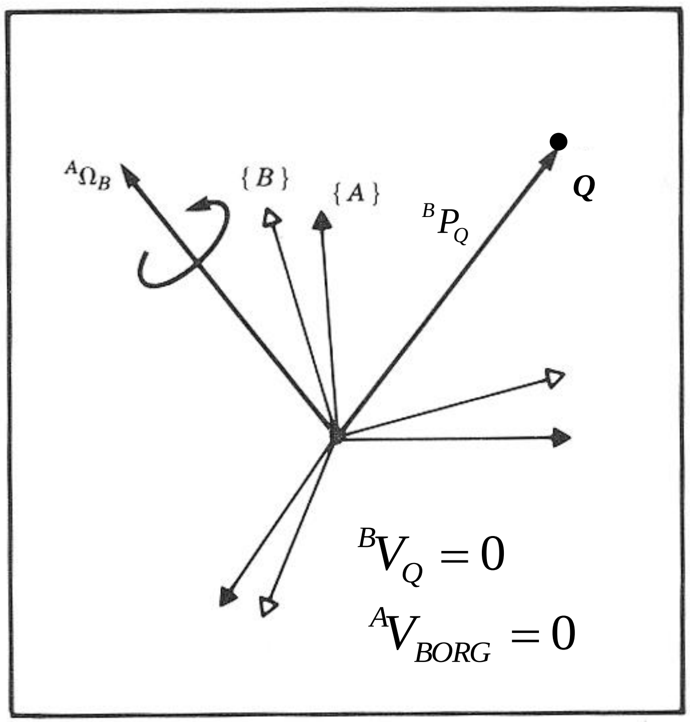
$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The vector ${}^{B}P_{Q}$ is constant as view from frame {B} ${}^{B}V_{Q} = 0$

- ***Problem:*** describe the velocity of the vector ${}^{B}P_{Q}$ representing the the point **Q** relative to frame {A}

- ***Solution:*** Even though the vector ${}^{B}P_{Q}$ is constant as view from frame {B} it is clear that point **Q** will have a velocity as seen from frame {A} due to the rotational velocity ${}^{A}\Omega_{B}$



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \boxed{{}^{A}\Omega_{B}} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach

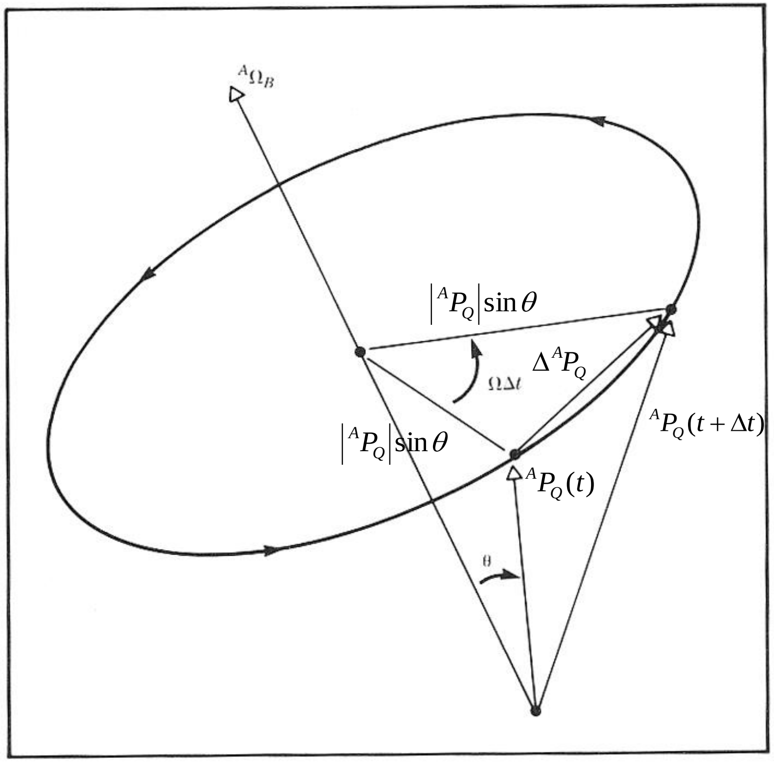
- The figure shows to instants of time as the vector ${}^{A}P_Q$ rotates around ${}^{A}\Omega_B$ This is what an observer in frame {A} would observe.

- The Magnitude of the differential change is

$$\left|\Delta {}^{A}P_Q\right| = \left(\left|{}^{A}\Omega_B\right|\Delta t\right)\left(\left|{}^{A}P_Q\right|\sin\theta\right)$$

- Using a vector cross product we get

$$\frac{\Delta {}^{A}P_Q}{\Delta t} = {}^{A}V_Q = {}^{A}\Omega_B \times {}^{A}P_Q$$



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\, {}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\, {}^{B}P_{Q} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\, {}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega} \left({}^{A}_{B}R\, {}^{B}P_{Q} \right) $$

Angular Velocity - Rigid Body - Intuitive Approach

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}_{B}^{A}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + {}_{B}^{A}\dot{R}_{\Omega}\left({}_{B}^{A}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

- Rotation in 2D

Angular Velocity - Rigid Body - Intuitive Approach

- In the general case, the vector Q may also be changing with respect to the frame {B}. Adding this component we get.

$${}^{A}V_Q = {}^{A}\left({}^{B}V_Q\right) + {}^{A}\Omega_B \times {}^{A}P_Q$$

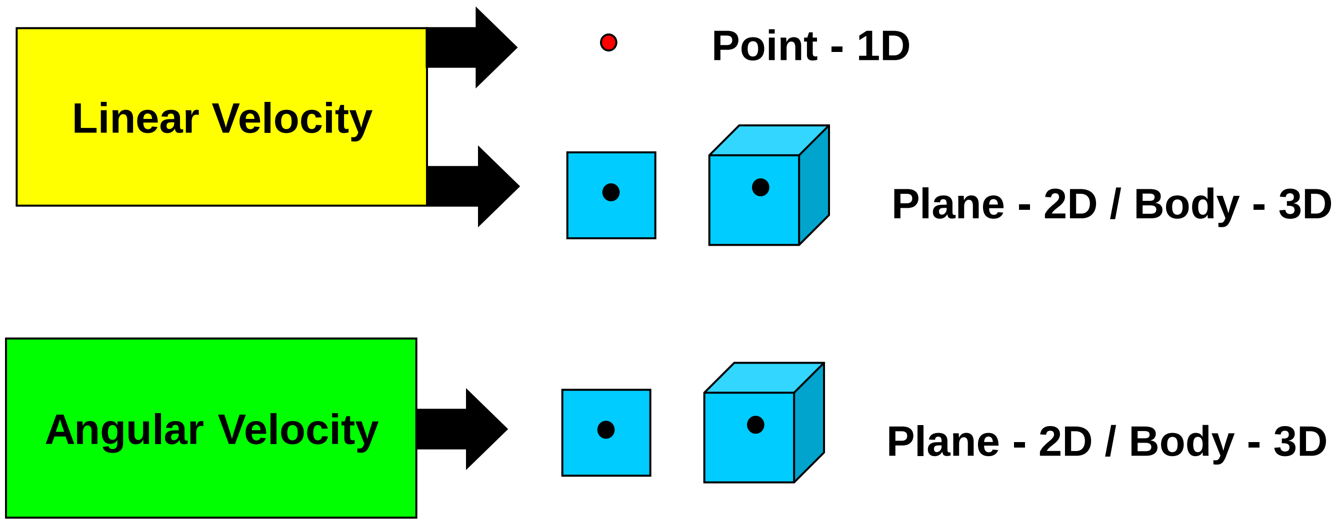
- Using the rotation matrix to remove the dual-superscript, and since the description of ${}^{A}P_Q$ at any instance is ${}^{A}_{B}R\,{}^{B}P_Q$ we get

$${}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\;{}^{B}P_Q$$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Definitions - Angular Velocity

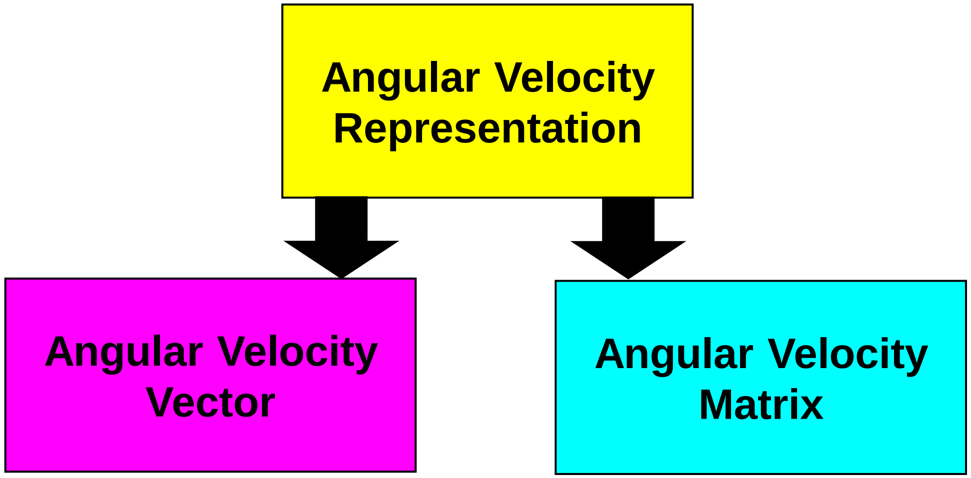
- ***Angular Velocity:*** The instantaneous rate of change in the orientation of one frame relative to another.



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \boxed{{}^{A}\Omega_{B}} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + \boxed{{}^{A}_{B}\dot{R}_{\Omega}} \left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Angular Velocity

- Just as there are many ways to represent orientation (Euler Angles, Roll-Pitch-Yaw Angles, Rotation Matrices, etc.) there are also many ways to represent the rate of change in orientation.



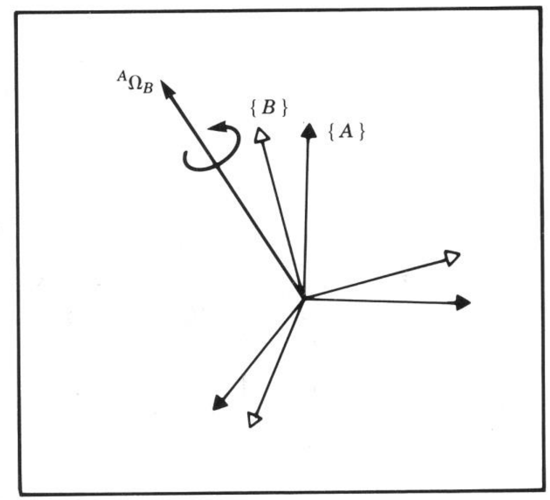
- The angular velocity vector is convenient to use because it has an easy to grasp physical meaning. However, the matrix form is useful when performing algebraic manipulations.

$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity - Vector

- ***Angular Velocity Vector:*** A vector whose direction is the instantaneous axis of rotation of one frame relative to another and whose magnitude is the rate of rotation about that axis.

$${}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$



- The angular velocity vector ${}^{A}\Omega_B$ describes the instantaneous change of rotation of frame {B} relative to frame {A}

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Angular Velocity - Matrix

- The rotation matrix (${}^{A}_{B}R$) defines the orientation of frame {B} relative to frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are the unit vectors of {B} represented in {A}.

$$ {}^{A}_{B}R = {}^{A}\begin{bmatrix} \left[{}^{B}P_{x}\right] & \left[{}^{B}P_{y}\right] & \left[{}^{B}P_{z}\right] \end{bmatrix} $$

- If we look at the derivative of the rotation matrix, the columns will be the velocity of each unit vector of {B} relative to {A}:

$$ {}^{A}_{B}\dot{R} = \frac{d}{dt}\left[{}^{A}_{B}R\right] = {}^{A}\begin{bmatrix} \left[{}^{B}V_{x}\right] & \left[{}^{B}V_{y}\right] & \left[{}^{B}V_{z}\right] \end{bmatrix} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Angular Velocity - Matrix

- The relationship between the rotation matrix ${}^{A}_{B}R$ and the derivative of the rotation matrix ${}^{A}_{B}\dot{R}$ can be expressed as follows:

$$ {}^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R $$

$$ {}^{A}\begin{bmatrix} \left[{}^{B}V_{x}\right] & \left[{}^{B}V_{y}\right] & \left[{}^{B}V_{z}\right] \end{bmatrix} = {}^{A}_{B}\dot{R}_{\Omega}\;{}^{A}\begin{bmatrix} \left[{}^{B}P_{x}\right] & \left[{}^{B}P_{y}\right] & \left[{}^{B}P_{z}\right] \end{bmatrix} $$

- where ${}^{A}_{B}\dot{R}_{\Omega}$ is defined as the ***angular velocity matrix***

$$ {}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_{z} & \Omega_{y} \\ \Omega_{z} & 0 & -\Omega_{x} \\ -\Omega_{y} & \Omega_{x} & 0 \end{bmatrix} \qquad {}^{A}\Omega_{B} \equiv \begin{bmatrix} \Omega_{x} \\ \Omega_{y} \\ \Omega_{z} \end{bmatrix} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + \boxed{{}^{A}\Omega_{B}} \times {}_{B}^{A}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}_{B}^{A}R\,{}^{B}V_{Q} + \boxed{{}_{B}^{A}\dot{R}_{\Omega}} \left({}_{B}^{A}R\,{}^{B}P_{Q} \right) $$

Angular Velocity - Matrix & Vector Forms

	Matrix Form	Vector Form
Definition	${}_{B}^{A}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	${}^{A}\Omega_{B} \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$
Multiply by Constant	$k \left[{}_{B}^{A}\dot{R}_{\Omega} \right]$	$k \left[{}^{A}\Omega_{B} \right]$
Multiply by Vector	$\left[{}_{B}^{A}\dot{R}_{\Omega} \right] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$	${}^{A}\Omega_{B} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
Multiply by Matrix	$\left[{}_{t}^{s}R \right] \left[{}_{B}^{A}\dot{R}_{\Omega} \right] \left[{}_{t}^{s}R \right]^{T}$	$\left[{}_{t}^{s}R \right] \left[{}^{A}\Omega_{B} \right]$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Simultaneous Linear and Rotational Velocity - Vector Versus Matrix Representation

Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

$$ \Omega \times P = \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ P_x & P_y & P_z \end{vmatrix} = i\,(\Omega_y P_z - \Omega_z P_y) - j\,(\Omega_x P_z - \Omega_z P_x) + k\,(\Omega_x P_y - \Omega_y P_x) $$

$$ \dot{R}_{\Omega} P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} -\Omega_z P_y + \Omega_y P_z \\ \Omega_z P_x - \Omega_x P_z \\ -\Omega_y P_x + \Omega_x P_y \end{bmatrix} $$

Simultaneous Linear and Rotational Velocity

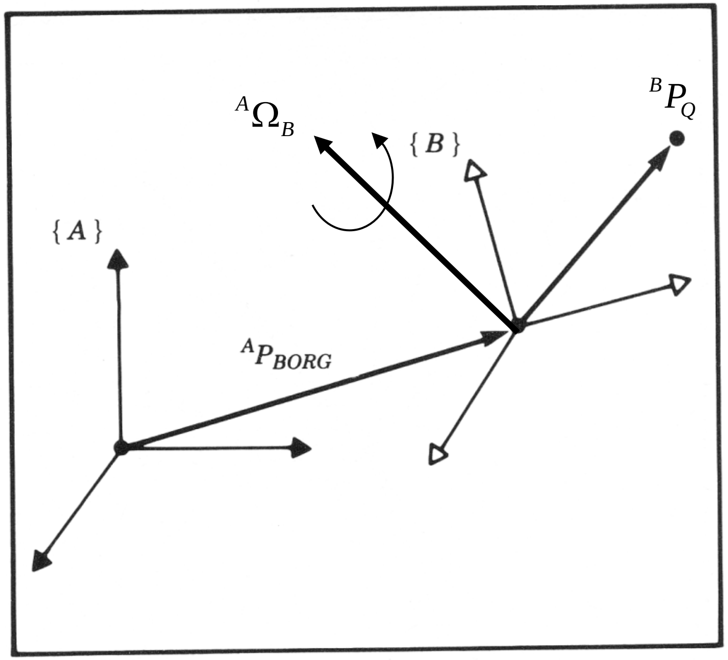
- The final results for the derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame

- Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

- Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



Changing Frame of Representation - Linear Velocity

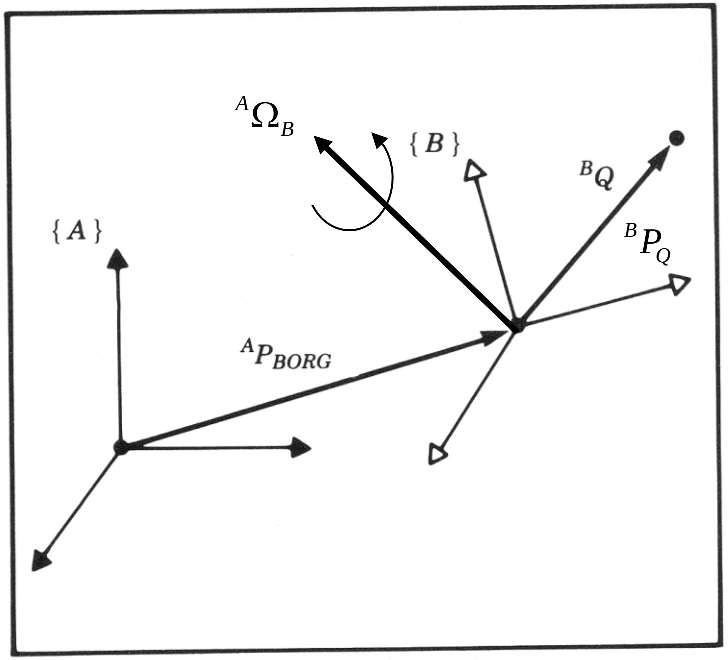
- We have already used the homogeneous transform matrix to compute the location of position vectors in other frames:

$$ {}^{A}P_{Q} = {}^{A}_{B}T\, {}^{B}P_{Q} $$

- To compute the relationship between velocity vectors in different frames, we will take the derivative:

$$ \frac{d}{dt}\left[{}^{A}P_{Q}\right] = \frac{d}{dt}\left[{}^{A}_{B}T\, {}^{B}P_{Q}\right] $$

$$ {}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\, {}^{B}P_{Q} + {}^{A}_{B}T\, {}^{B}\dot{P}_{Q} $$



$${}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q}$$

Changing Frame of Representation - Linear Velocity

- Recall that

$${}^{A}_{B}T = \begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- so that the derivative is

$${}^{A}_{B}\dot{T} = \frac{d}{dt}\begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}\right] & & & \left[{}^{A}\dot{P}_{B\,org}\right] \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix} == \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\right] & & & \left[{}^{A}V_{B\,org}\right] \\ & & & \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^{A}\dot{P}_Q = {}^{A}_{B}\dot{T}\,{}^{B}P_Q + {}^{A}_{B}T\,{}^{B}\dot{P}_Q$$

Changing Frame of Representation - Linear Velocity

$${}^{A}_{B}\dot{T} = \begin{bmatrix} \multicolumn{3}{c}{\left[{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\right]} & \left[{}^{A}V_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Substitute the previous results into the original equation ${}^{A}\dot{P}_Q = {}^{A}_{B}\dot{T}\,{}^{B}P_Q + {}^{A}_{B}T\,{}^{B}\dot{P}_Q$ we get

$$\begin{bmatrix} \left[{}^{A}V_Q\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\cdot{}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_Q\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\,org}\right] \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_Q\right] \\ 0 \end{bmatrix}$$

- This expression is equivalent to the following three-part expression:

$${}^{A}V_Q = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_Q$$

Changing Frame of Representation - Linear Velocity

$${}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

- Converting from matrix to vector form yields

$${}^{A}V_{Q} = {}^{A}\Omega_{B} \times \left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

Changing Frame of Representation - Angular Velocity

- We use rotation matrices to represent angular position so that we can compute the angular position of {C} in {A} if we know the angular position of {C} in {B} and {B} in {A} by

$$ {}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R $$

- To derive the relationship describing how angular velocity propagates between frames, we will take the derivative

$$ {}^{A}_{C}\dot{R} = {}^{A}_{B}\dot{R}\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R} $$

- Substituting the angular velocity matrixes

$$ {}^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R \qquad {}^{B}_{C}\dot{R} = {}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R \qquad {}^{A}_{C}\dot{R} = {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R $$

- we find

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

Changing Frame of Representation - Angular Velocity

- Post-multiplying both sides by ${}^A_C R^T$,which for rotation matrices, is equivalent to ${}^A_C R^{-1}$

$$ {}^A_C\dot{R}_\Omega \, {}^A_C R \, {}^A_C R^T = {}^A_B\dot{R}_\Omega \, {}^A_C R \, {}^A_C R^T + {}^A_B R \, {}^B_C\dot{R}_\Omega \, {}^B_C R \, {}^A_C R^T $$

$$ \boxed{ {}^A_C\dot{R}_\Omega = {}^A_B\dot{R}_\Omega + {}^A_B R \, {}^B_C\dot{R}_\Omega \, {}^A_B R^T } $$

- The above equation provides the relationship for changing the frame of representation of angular velocity matrices.

- The vector form is given by

$$ \boxed{ {}^A\Omega_C = {}^A\Omega_B + {}^A_B R \, {}^B\Omega_C } $$

- To summarize, the angular velocities of frames may be added as long as they are expressed in the same frame.

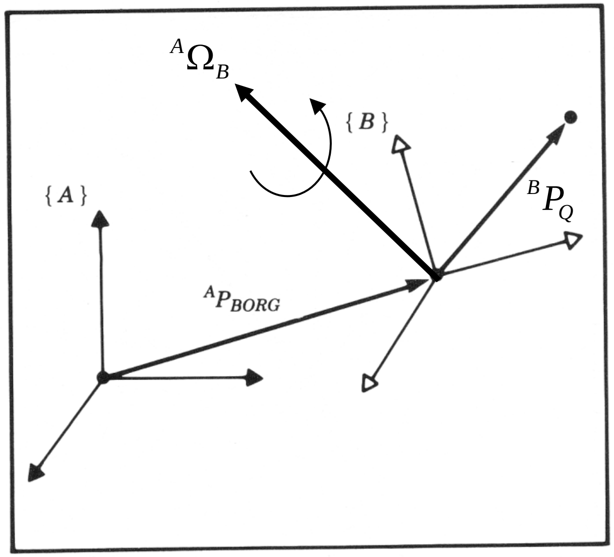
Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

  - Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



- Angular Velocity
  - Vector Form

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

  - Matrix Form

$$ {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$