Jacobian: Velocities and Static Forces 1/4

Kinematics Relations - Joint & Cartesian Spaces

- A robot is often used to manipulate object attached to its tip (end effector).

- The location of the robot tip may be specified using one of the following descriptions:

- ***Joint Space***

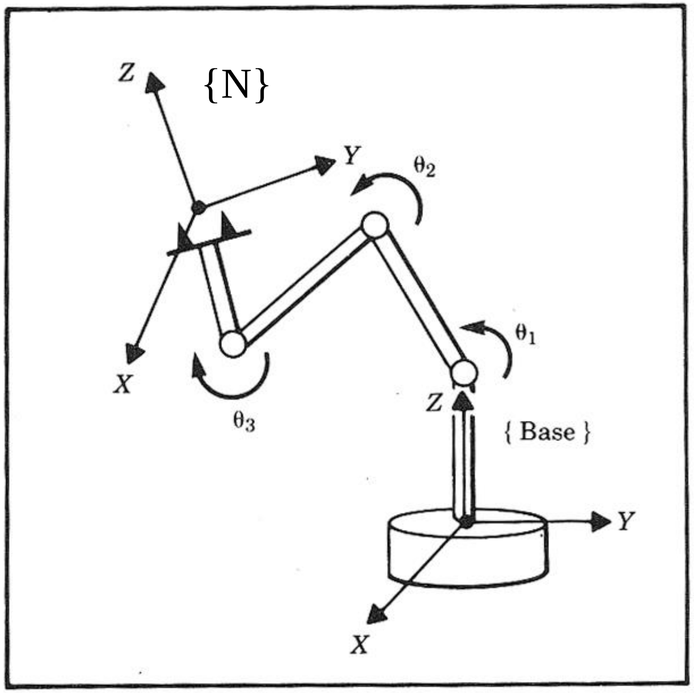
$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

- ***Cartesian Space***

$${}^{0}_{N}T = \begin{bmatrix} {}^{0}_{N}R & {}^{0}P_N \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} \left[{}^{0}P_N\right] \\ \left[{}^{0}r_N\right] \end{bmatrix}$$

Euler Angles



Kinematics Relations - Forward & Inverse

- The robot kinematic equations relate the two description of the robot tip location

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

$$X = FK(\theta)$$

$$\theta = IK(X)$$

$$X = \begin{bmatrix} \begin{bmatrix} {}^0P_N \end{bmatrix} \\ \begin{bmatrix} {}^0r_N \end{bmatrix} \end{bmatrix}$$

Tip Location in Joint Space

Tip Location in Cartesian Space

Kinematics Relations - Forward & Inverse

$$\dot{\theta} = \frac{d}{dt}[\theta] = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}$$

$$\dot{X} = \frac{d}{dt}[X] = \begin{bmatrix} [v_N] \\ [\omega_N] \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Tip Velocity in Joint Space

Tip velocity in Cartesian Space

Jacobian Matrix - Introduction

- **The Jacobian is a multi dimensional form of the derivative.**
- Suppose that for example we have 6 functions, each of which is a function of 6 independent variables

$$
\begin{aligned}
y_1 &= f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
y_2 &= f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\
&\vdots \\
y_6 &= f_6(x_1, x_2, x_3, x_4, x_5, x_6)
\end{aligned}
$$

- We may also use a vector notation to write these equations as

$$Y = F(X)$$

Jacobian Matrix - Introduction

- If we wish to calculate the differential of y_i as a function of the differential x_i we use the chain rule to get

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

- Which again might be written more simply using a vector notation as

$$\delta Y = \frac{\partial F}{\partial X}\delta X$$

Jacobian Matrix - Introduction

- The 6x6 matrix of partial derivative is defined as the Jacobian matrix

$$\delta Y = \frac{\partial F}{\partial X}\delta X = J(X)\delta X$$

- By dividing both sides by the differential time element, we can think of the Jacobian as mapping velocities in X to those in Y

$$\dot{Y} = J(X)\dot{X}$$

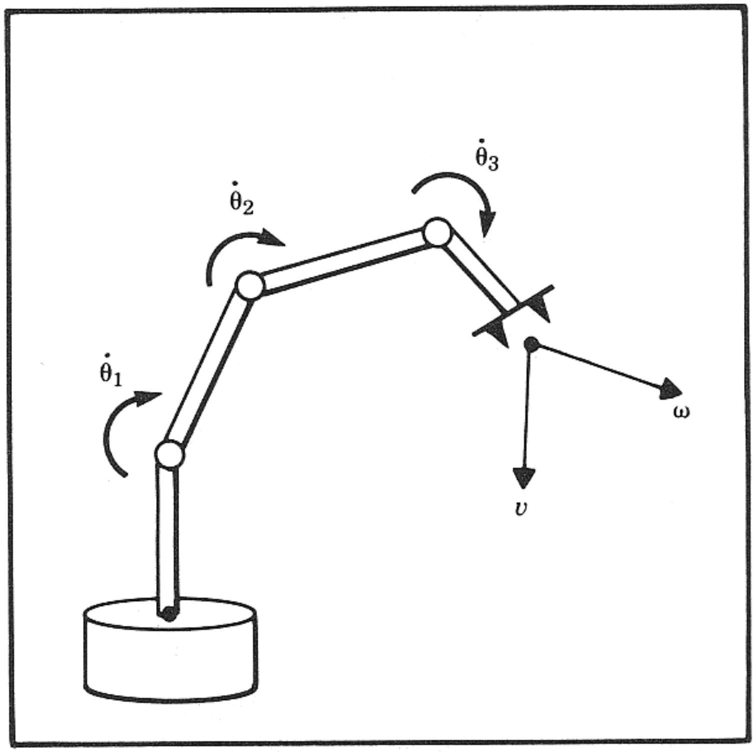
- Note that the Jacobian is time varying linear transformation

Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\dot{\underline{\theta}}_N$) and the translation and rotation velocities of the end effector ($\dot{\underline{x}}$). This relationship is given by:

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}}$$

$$\dot{\underline{\theta}} = J(\underline{\theta})^{-1}\dot{\underline{x}}$$



Jacobian Matrix - Introduction

- This expression can be expanded to:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & & \\ & J(\underline{\theta}) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \cdots \\ \dot{\theta}_N \end{bmatrix}$$

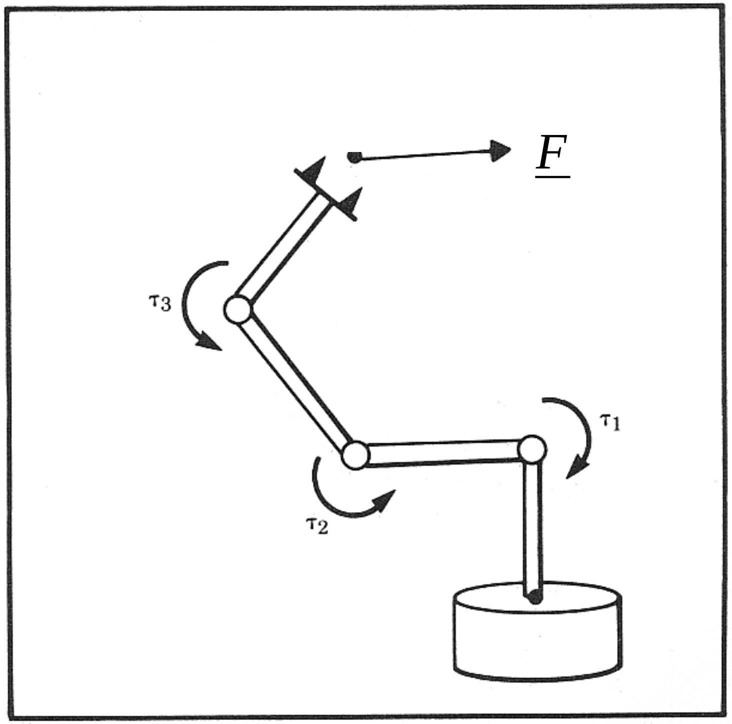
6x1 **6xN** **Nx1**

- Where:
  - $\underline{\dot{x}}$ is a 6x1 vector of the end effector linear and angular velocities
  - $J(\underline{\theta})$ is a 6xN Jacobian matrix
  - $\underline{\dot{\theta}}_N$ is a Nx1 vector of the manipulator joint velocities
  - N is the number of joints

Jacobian Matrix - Introduction

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$



Jacobian Matrix - Introduction

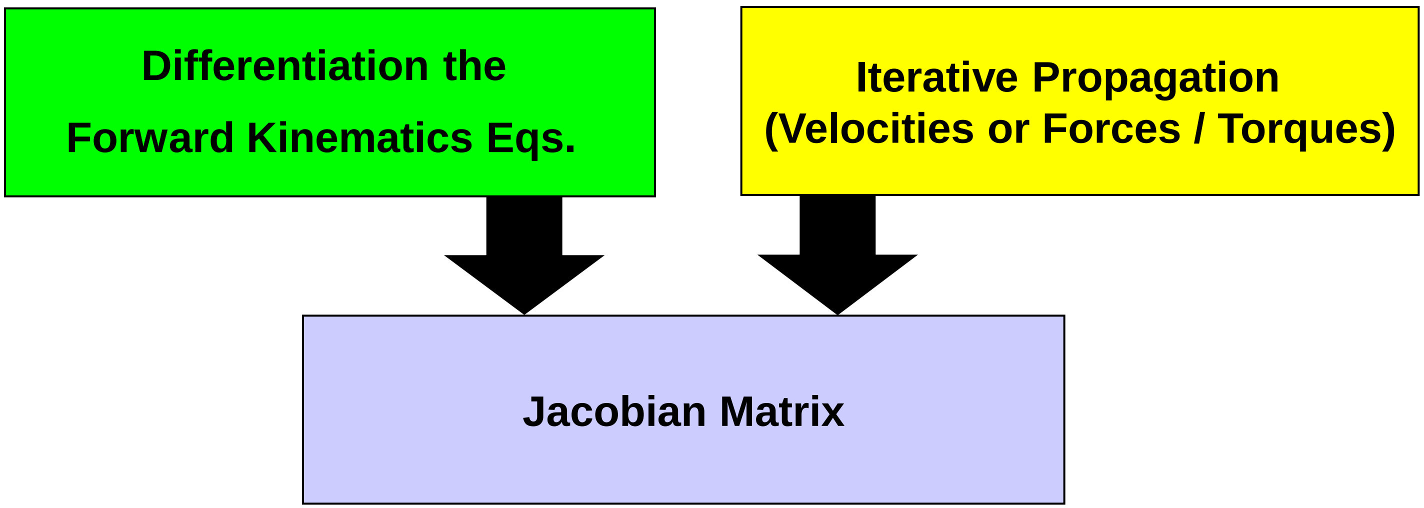
- This expression can be expanded to:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_N \end{bmatrix} = \begin{bmatrix} \\ \\ J(\underline{\theta}) \\ \\ \\ \end{bmatrix}^T \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

6x1 **6xN** **Nx1**

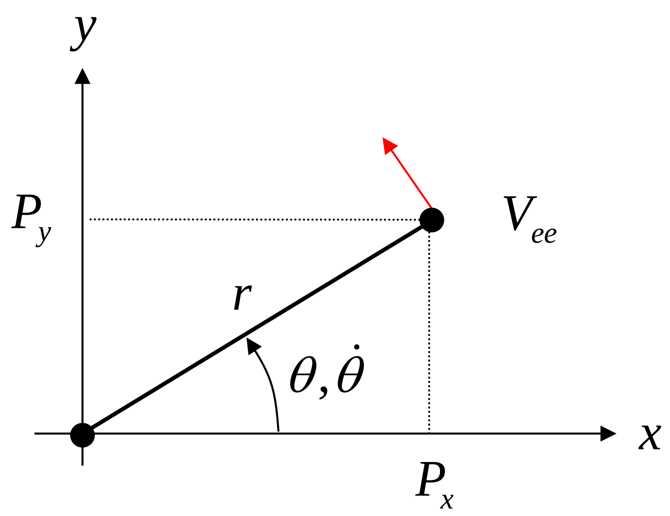
- Where:
  - $\underline{\tau}$ is a 6x1 vector of the robot joint torques
  - $J(\underline{\theta})^T$ is a 6xN Transposed Jacobian matrix
  - $\underline{F}$ is a Nx1 vector of the forces and moments at the robot end effector
  - N is the number of joints

Jacobian Matrix - Calculation Methods



Jacobian Matrix by Differentiation - 1R - 1/4

- Consider a simple planar 1R robot



- The end effector position is given by

$$^{0}P_x = x = r\cos\theta$$

$$^{0}P_y = y = r\sin\theta$$

Jacobian Matrix by Differentiation - 1R - 2/4

- The velocity of the end effector is defined by

$$
{}^{0}V_x = {}^{0}\dot{P}_x = \dot{x} = -\dot{\theta}\, r \sin\theta = -\omega\, r \sin\theta
$$

$$
{}^{0}V_y = {}^{0}\dot{P}_y = \dot{y} = \dot{\theta}\, r \cos\theta = \omega\, r \cos\theta
$$

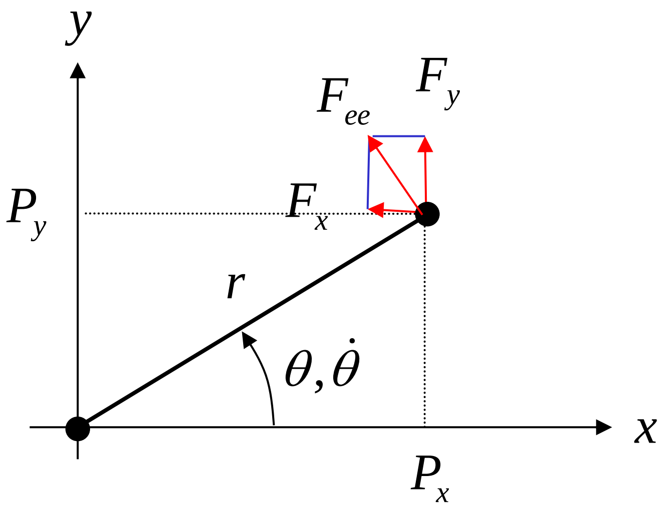
- Expressed in matrix form we have

$$
\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}
$$

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix}
$$

2x1 2x1 1x1

Jacobian Matrix by Differentiation - 1R - 3/4



- The moment about the joint generated by the force acting on the end effector is given by

$$\tau = -rF_x \sin\theta + rF_y \cos\theta$$

Jacobian Matrix by Differentiation - 1R - 4/4

- Expressed in matrix form we have

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

$$[\tau] = [-r\sin\theta \quad r\cos\theta]\begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

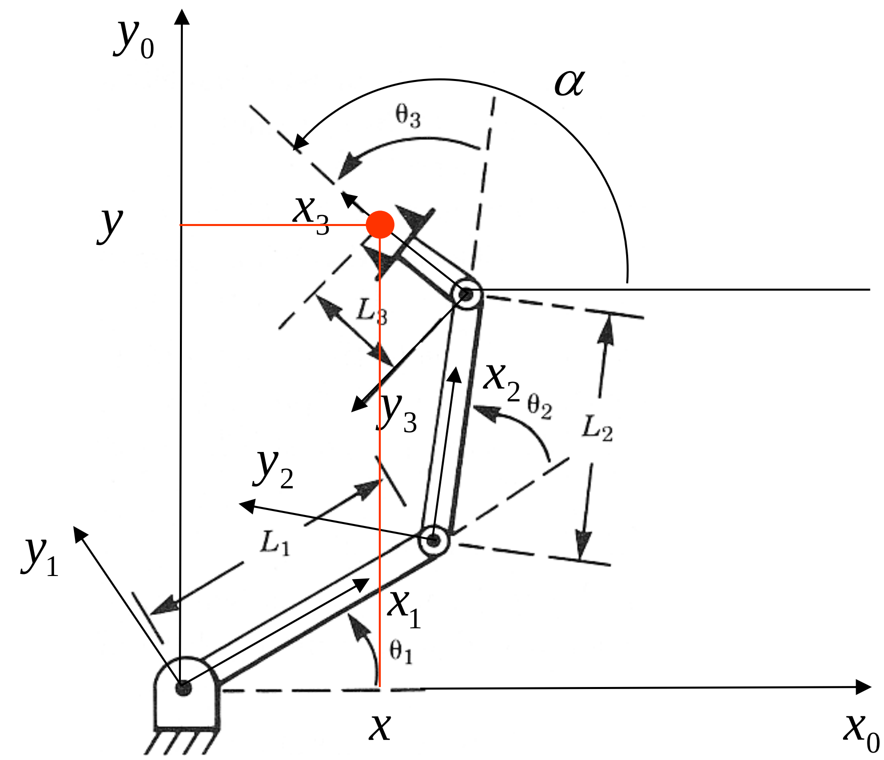
1x1 **1x2** **2x1**

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix}[\dot{\theta}]$$

Jacobian Matrix by Differanciation - 3R - 1/4

- Consider the following 3 DOF Planar manipulator



Jacobian Matrix by Differanciation - 3R - 2/4

- **Problem:** Compute the Jacobian matrix that describes the relationship

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} \qquad \underline{\tau} = J(\underline{\theta})^T \underline{F}$$

- **Solution:**

- The end effector position and orientation is defined in the base frame by

$$\underline{x} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix}$$

Jacobian Matrix by Differanciation - 3R - 3/4

- The forward kinematics gives us relationship of the end effector to the joint angles:

$$^{0}P_{3org,x} = x = L_1 c_1 + L_2 c_{12} + L_3 c_{123}$$

$$^{0}P_{3org,y} = y = L_1 s_1 + L_2 s_{12} + L_3 s_{123}$$

$$^{0}P_{3org,\alpha} = \alpha = \theta_1 + \theta_2 + \theta_3$$

- Differentiating the three expressions gives

$$\dot{x} = -L_1 s_1 \dot{\theta}_1 - L_2 s_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) - L_3 s_{123}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)$$

$$= -\left(L_1 s_1 + L_2 s_{12} + L_3 s_{123}\right)\dot{\theta}_1 - \left(L_2 s_{12} + L_3 s_{123}\right)\dot{\theta}_2 - \left(L_3 s_{123}\right)\dot{\theta}_3$$

$$\dot{y} = L_1 c_1 \dot{\theta}_1 + L_2 c_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + L_3 c_{123}\left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right)$$

$$= \left(L_1 c_1 + L_2 c_{12} + L_3 c_{123}\right)\dot{\theta}_1 + \left(L_2 c_{12} + L_3 c_{123}\right)\dot{\theta}_2 + \left(L_3 c_{123}\right)\dot{\theta}_3$$

$$\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

Jacobian Matrix by Differanciation - 3R - 4/4

- Using a matrix form we get

$$\underline{\dot{x}} = {}^{0}J\left(\underline{\theta}\right)\underline{\dot{\theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees a freedom. Before tackling these problems, consider this brief review of linear algebra.

Singularity - The Concept

- ***Motivation:*** We would like the hand of a robot (end effecror) to move with a certain velocity vector in Cartesian space. Using linear transformation relating the joint velocity to the Cartesian velocity we could calculate the necessary joint rates at each instance along the path.

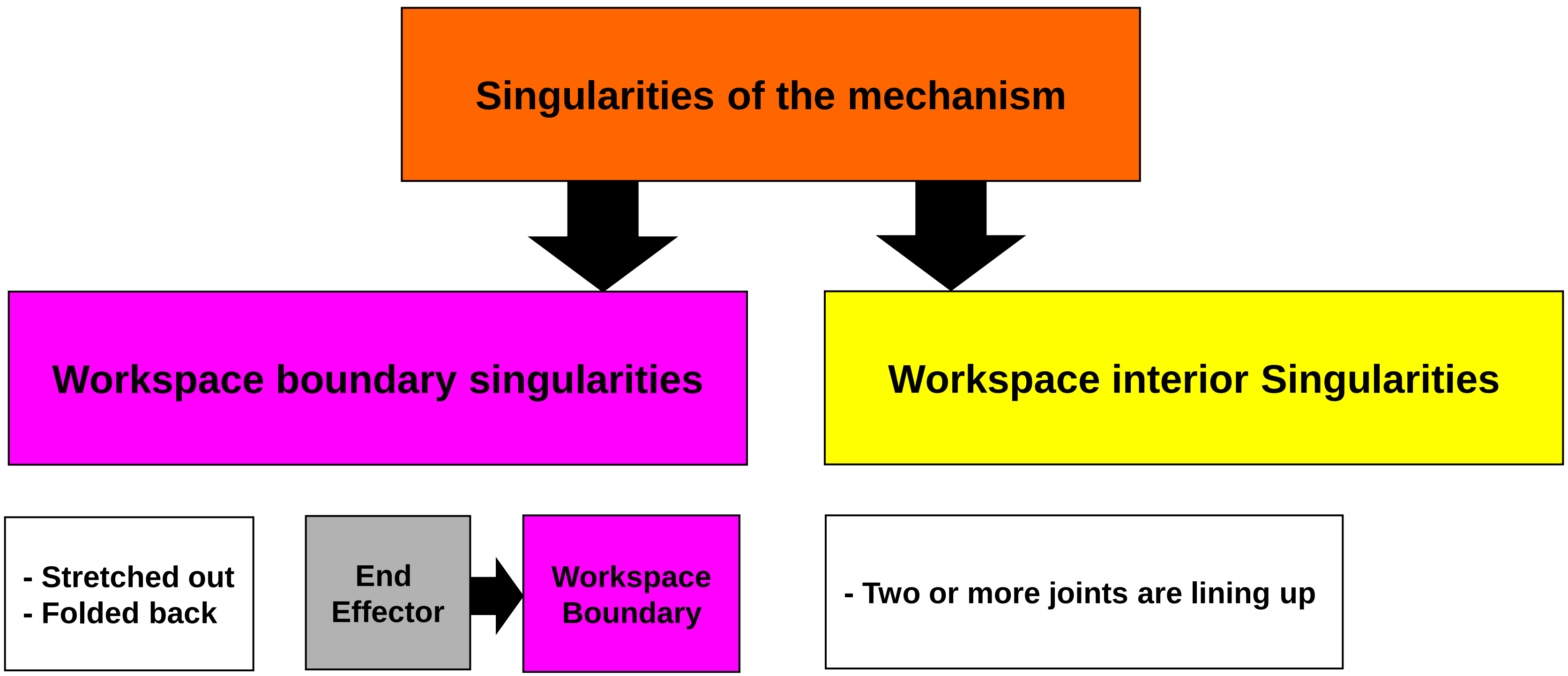
$$\underline{\dot{\theta}} = J\left(\underline{\theta}\right)^{-1} \underline{\dot{x}}$$

- ***Given:*** a linear transformation relating the joint velocity to the Cartesian velocity (usually the end effector)

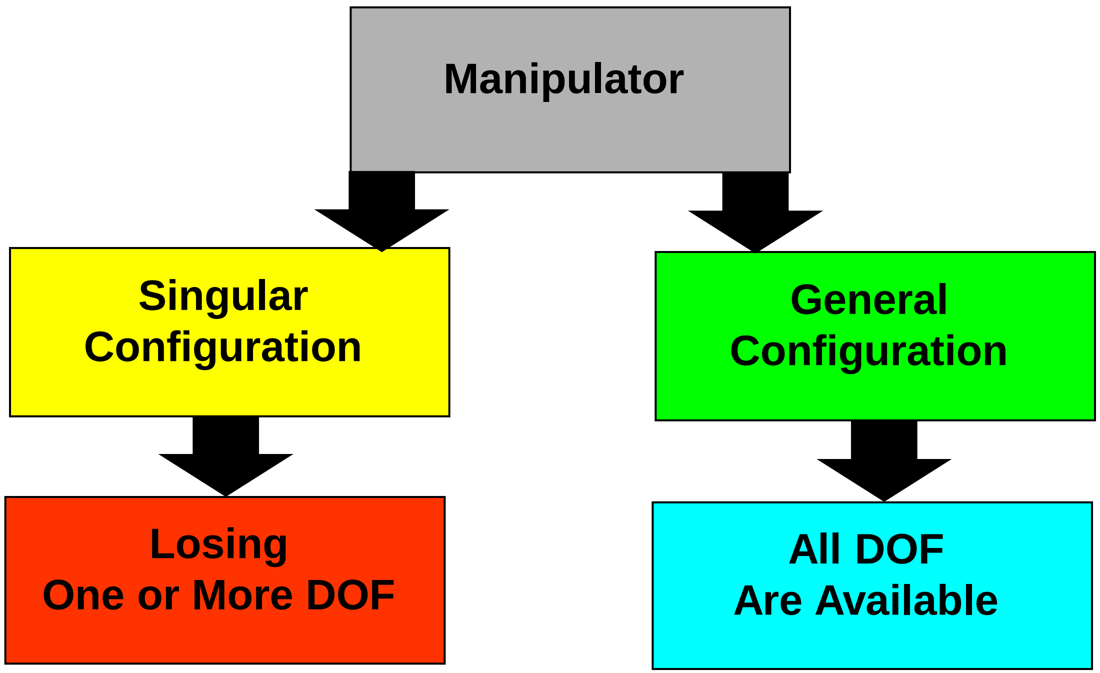
- ***Question:*** Is the Jacobian matrix invertable? (Or) Is it nonsingular?
  Is the Jacobian invertable for all values of θ ?
  If not, where is it not invertable?

Singularity - The Concept

- ***Answer (Conceptual):*** Most manipulator have values of θ where the Jacobian becomes singular . Such locations are called ***singularities of the mechanism*** or ***singularities*** for short



Singularity - The Concept



- Losing one or more DOF means that there is a some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot (end effector) no matter which joint rate are selected

Singularity – Physical Interpretation - Examples

Brief Linear Algebra Review - 1/

- Inverse of Matrix A exists ***if and only if*** the determinant of A is non-zero.

A^{-1} Exists ***if and only if***

$$Det(A) = |A| \neq 0$$

- If the determinant of A is equal to zero, then the matrix A is a singular matrix

$$Det(A) = |A| = 0$$

A Singular

Brief Linear Algebra Review - 2/

- The rank of the matrix A is the size of the largest squared Matrix S for which

$$Det(S) \neq 0$$

- Example 1 - $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $A = S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $|A| = |S| = 3$ $Rank(A) = 2$

- Example 2 - $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $S = [1]$ $|S| = 1$ $Rank(A) = 1$

Brief Linear Algebra Review - 3/

- If two rows or columns of matrix A are equal or related by a constant, then

$$Det(A) = 0$$

- Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & -3 \\ 10 & -6 & -5 \end{bmatrix}$$

$$\det(A) = |A| = 2\begin{vmatrix} -3 & -3 \\ -6 & -5 \end{vmatrix} - 0\begin{vmatrix} 6 & -3 \\ 10 & -5 \end{vmatrix} - 1\begin{vmatrix} 6 & -3 \\ 10 & -6 \end{vmatrix} = 6 + 0 - 6 = 0$$

Brief Linear Algebra Review - 4/

- ***Eigenvalues***

$$AX = \lambda X$$

$$(A - \lambda I)X = 0$$

- Eigenvalues are the roots of the polynomial

$$Det(A - \lambda I)$$

- If $X \neq 0$ each solution to the characteristic equation λ (Eigenvalue) has a corresponding Eigenvector

Brief Linear Algebra Review - 4/

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(A - \lambda I)X = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$Det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

$$\lambda_1 = -1$$

$$\lambda_2 = 3$$

Brief Linear Algebra Review - 4/

$$\lambda_1 = -1$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 3$$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Brief Linear Algebra Review - 5/

- Any singular matrix ($Det(A) = 0$) has at least one Eigenvalue equal to zero

Brief Linear Algebra Review - 6/

- If A is non-singular ($Det(A) \neq 0$), and λ is an eigenvalue of A with corresponding to eigenvector X, then

$$A^{-1}X = \lambda^{-1}X$$

Brief Linear Algebra Review - 7/

- If the n x n matrix A is of full rank (that is, ***Rank (A) = n***), then the only solution to

$$AX = 0$$

is the trivial one

$$X = 0$$

- If A is of less than full rank (that is ***Rank (A) < n***), then there are n-r linearly independent (orthogonal) solutions

$$x_j \qquad 0 \le j \le n - r$$

for which

$$Ax_j = 0$$

Brief Linear Algebra Review - 8/

- If A is square, then A and A^T have the same eigenvalues

Properties of the Jacobian - Velocity Mapping and Singularities

- ***Example:*** Planar 3R

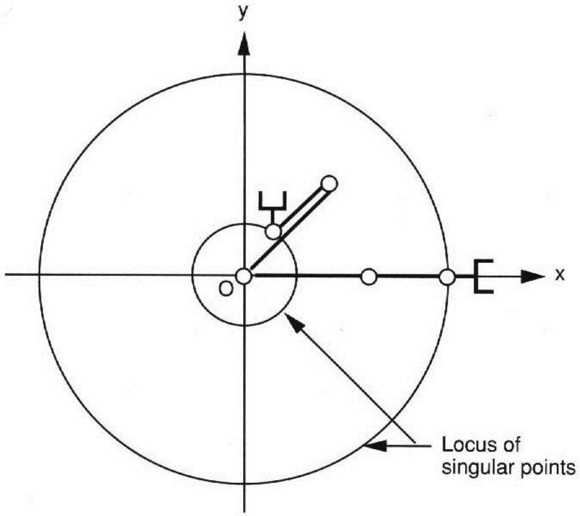
$$\det(J(\theta)) = \begin{vmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{vmatrix} = L_1 L_2 s_2$$

$$\det(J(\theta)) = L_1 L_2 s_2 = 0$$

- Note that $\det(J(\theta))$ is not a function of θ_1, θ_3

Properties of the Jacobian - Velocity Mapping and Singularities

$$\text{singular configuration} \begin{cases} \theta_2 = 0 & \text{Stretched Out} \\ \theta_2 = \pi & \text{Fold Back} \end{cases}$$



- The manipulator loses 1 DEF. The end effector can only move along the tangent direction of the arm. Motion along the radial direction is not possible.

Properties of the Jacobian - Force Mapping and Singularities

- The relationship between joint torque and end effector force and moments is given by:

$$\underline{\tau} = J\left(\underline{\theta}\right)^T \underline{F}$$

- The rank of $J\left(\theta\right)^T$ is equals the rank of $J\left(\theta\right)$.

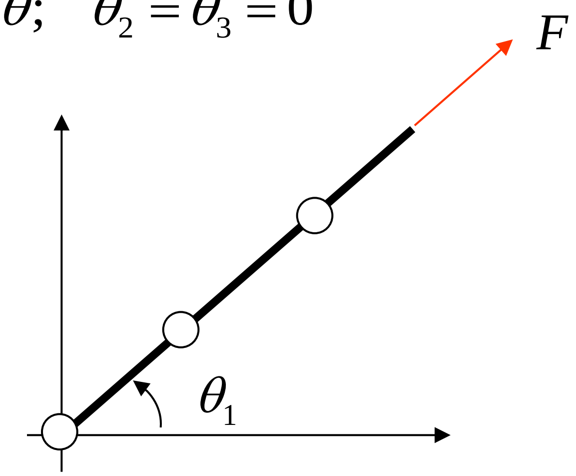
- At a singular configuration there exists a non trivial force $\underline{F}$ such that

$$J\left(\underline{\theta}\right)^T \underline{F} = 0$$

- In other words, a finite force can be applied to the end effector that produces no torque at the robot's joints. In the singular configuration, the manipulator can "lock up."

Properties of the Jacobian - Force Mapping and Singularities

- ***Example:*** Planar 3R $\theta_1 = \theta; \quad \theta_2 = \theta_3 = 0$



- In this case the force acting on the end effector (relative to the {0} frame) is given by

$$^{0}F = \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix}$$

Properties of the Jacobian - Force Mapping and Singularities

$$\underline{{}^0\tau} = {}^0J(\underline{\theta})^T \, \underline{{}^0F} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & L_1 c_1 + L_2 c_{12} + L_3 c_{123} & 1 \\ -L_2 s_{12} - L_3 s_{123} & L_2 c_{12} + L_3 c_{123} & 1 \\ -L_3 s_{123} & L_3 c_{123} & 1 \end{bmatrix} \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix}$$

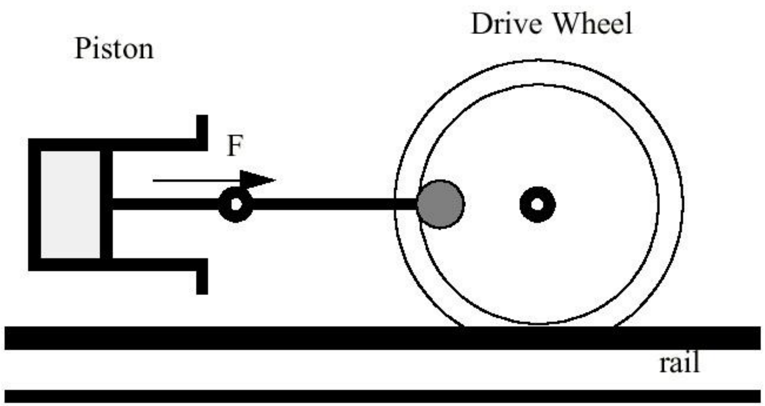
- For $\theta_1 = \theta; \quad \theta_2 = \theta_3 = 0$ we get

$$\underline{{}^0\tau} = {}^0J(\underline{\theta})^T \, \underline{{}^0F} = \begin{bmatrix} -L_1 s_1 - L_2 s_1 - L_3 s_1 & L_1 c_1 + L_2 c_1 + L_3 c_1 & 1 \\ -L_2 s_1 - L_3 s_1 & L_2 c_1 + L_3 c_1 & 1 \\ -L_3 s_1 & L_3 c_1 & 1 \end{bmatrix} \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -Fs_1c_1(L_1 + L_2 + L_3) + Fs_1c_1(L_1 + L_2 + L_3) \\ -Fs_1c_1(L_2 + L_3) + Fs_1c_1(L_2 + L_3) \\ -Fs_1c_1(L_3) + Fs_1c_1(L_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Properties of the Jacobian - Force Mapping and Singularities

- This situation is an old and famous one in mechanical engineering.
- For example, in the steam locomotive, "top dead center" refers to the following condition



- The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.

Properties of the Jacobian - Velocity Mapping and Singularities

- We have shown the relationship between joint space velocity and end effector velocity, given by

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

- It is interesting to determine the inverse of this relationship, namely

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

- Consider the square 6x6 case for $J(\underline{\theta})$.

- If rank < 6 ($Det(J(\underline{\theta}))=0$) , then there is no solution to the inverse equation (see Brief Linear Algebra Review - 1,7).

$$Rank(J(\underline{\theta}))<6$$

$$\dot{\underline{\theta}} = J(\underline{\theta})^{-1}\dot{\underline{x}}$$

- However, if the rank = 5, then there is at least one non-trivial solution to the forward equation (see Brief Linear Algebra Review - 7). That is, for

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}} = 0$$

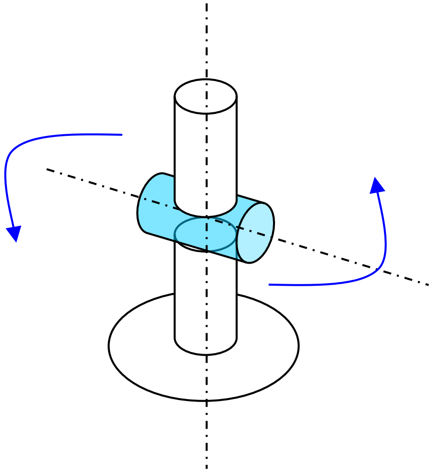
Properties of the Jacobian - Velocity Mapping and Singularities

- The solution is a direction $\left(\underline{\theta}\right)$ in the in joint velocity space for which joint motion produces no end effector motion.

- We call any joint configuration $\underline{\theta} = Q$ for which

$$Rank\left(J\left(\underline{\theta}\right)\right) < 6$$

a ***singular configuration***.



Properties of the Jacobian - Velocity Mapping and Singularities

- For certain directions of end effector motion , $\dot{\underline{x}}_i \quad 1 \le i \le 6$

$$\dot{\underline{x}} = J(\theta)\dot{\underline{\theta}} = \lambda_i(\underline{\theta})\underline{\omega}_i$$

where:

- λ_i are the eigenvalues of $J(\theta)$
- $\underline{\omega}_i$ are the eigenvectors of $J(\theta)$

- If $J(\theta)$ is fully ranked (see Brief Linear Algebra Review - 6/), we have

$$\underline{\omega}_i = J(\theta)^{-1}\dot{\underline{x}} = \lambda_i(\underline{\theta})^{-1}\dot{\underline{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

- As the joint approach a singular configuration $\underline{\theta} = Q$ there is at least one eigenvalue for which $\lambda_i \to 0$. This results in

$$\underline{\omega}_i = \frac{\dot{\underline{x}}}{\lambda_i(\underline{\theta})} \to \frac{\dot{\underline{x}}}{0} \to \infty$$

- In other word, as the joints approach the singular configuration, the end effector motion in a particular task direction $\dot{\underline{x}}_j$ causes the joint velocities to approach infinity. However, there are task velocities that can have solutions.

- If $J(\underline{\theta})$ loses rank by only one, then there are n-1 eigenvectors in the task velocity space ($\dot{\underline{x}}_j$) for which solutions do exist. However, there can be multiple solutions.