Jacobian: Velocities and Static Forces 1/4
Kinematics Relations - Joint & Cartesian Spaces

• A robot is often used to manipulate object attached to its tip (end effector).

• The location of the robot tip may be specified using one of the following descriptions:

  • **Joint Space**

    \[ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix} \]

  • **Cartesian Space**

    \[ ^0_N T = \begin{bmatrix} ^0_N R & ^0 P_N \\ 0 & 1 \end{bmatrix} \]

    \[ X = \begin{bmatrix} ^0 P_N \\ ^0 r_N \end{bmatrix} \]

    Euler Angles
The robot kinematic equations relate the two description of the robot tip location.

\[ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix} \]

\[ X = FK(\theta) \]

\[ \theta = IK(X) \]

Tip Location in Joint Space

Tip Location in Cartesian Space
Kinematics Relations - Forward & Inverse

\[ \dot{\theta} = \frac{d}{dt} [\theta] = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix} \]

\[ \dot{X} = \frac{d}{dt} [X] = \begin{bmatrix} v_N \\ \omega_N \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \]

Tip Velocity in Joint Space

Tip velocity in Cartesian Space
The Jacobian is a multi dimensional form of the derivative.

Suppose that for example we have 6 functions, each of which is a function of 6 independent variables

\[
y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6)
\]
\[
y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6)
\]
\[
\vdots
\]
\[
y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)
\]

We may also use a vector notation to write these equations as

\[
Y = F(X)
\]
If we wish to calculate the differential of \( y_i \) as a function of the differential \( x_i \), we use the chain rule to get:

\[
\delta y_1 = \frac{\partial f_1}{\partial x_1} \delta x_1 + \frac{\partial f_1}{\partial x_2} \delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6} \delta x_6
\]

\[
\delta y_2 = \frac{\partial f_2}{\partial x_1} \delta x_1 + \frac{\partial f_2}{\partial x_2} \delta x_2 + \cdots + \frac{\partial f_2}{\partial x_6} \delta x_6
\]

\[
\vdots
\]

\[
\delta y_6 = \frac{\partial f_6}{\partial x_1} \delta x_1 + \frac{\partial f_6}{\partial x_2} \delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6} \delta x_6
\]

Which again might be written more simply using a vector notation as:

\[
\delta Y = \frac{\partial F}{\partial X} \delta X
\]
The 6x6 matrix of partial derivative is defined as the Jacobian matrix

\[ \delta Y = \frac{\partial F}{\partial X} \delta X = J(X) \delta X \]

- By dividing both sides by the differential time element, we can think of the Jacobian as mapping velocities in X to those in Y

\[ \dot{Y} = J(X) \dot{X} \]

- Note that the Jacobian is time varying linear transformation
In the field of robotics the Jacobian matrix describes the relationship between the joint angle rates \( \dot{\theta}_N \) and the translation and rotation velocities of the end effector \( \dot{x} \). This relationship is given by:

\[
\dot{x} = J(\theta)\dot{\theta}
\]

\[
\dot{\theta} = J(\theta)^{-1}\dot{x}
\]
This expression can be expanded to:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{z} \\
\omega_x \\
\omega_y \\
\omega_z
\end{pmatrix}
= 
J(\theta)
\begin{pmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\vdots \\
\dot{\theta}_N
\end{pmatrix}
\]

Where:
- \( \dot{x} \) is a 6x1 vector of the end effector linear and angular velocities
- \( J(\theta) \) is a 6xN Jacobian matrix
- \( \dot{\theta}_N \) is a Nx1 vector of the manipulator joint velocities
- \( N \) is the number of joints
In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\tau$) and the forces and moments ($\vec{F}$) at the robot end effector (Static Conditions). This relationship is given by:

$$\tau = J(\theta)^T \vec{F}$$
Jacobian Matrix - Introduction

• This expression can be expanded to:

\[
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\vdots \\
\tau_N
\end{bmatrix}
= 
J(\theta)^T
\begin{bmatrix}
F_x \\
F_y \\
F_z \\
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

• Where:
  – \( \tau \) is a 6x1 vector of the robot joint torques
  – \( J(\theta)^T \) is a 6xN Transposed Jacobian matrix
  – \( F \) is a Nx1 vector of the forces and moments at the robot end effector
  – \( N \) is the number of joints
Jacobian Matrix - Calculation Methods

Differentiation the Forward Kinematics Eqs.

Iterative Propagation (Velocities or Forces / Torques)

Jacobian Matrix
Consider a simple planar 1R robot

The end effector position is given by

$$
^0 P_x = x = r \cos \theta \\
^0 P_y = y = r \sin \theta
$$
The velocity of the end effector is defined by

\[ 0V_x = 0\dot{P}_x = \dot{x} = -\dot{\theta} r \sin \theta = -\omega r \sin \theta \]
\[ 0V_y = 0\dot{P}_y = \dot{y} = \dot{\theta} r \cos \theta = \omega r \cos \theta \]

Expressed in matrix form we have

\[ \dot{x} = J(\theta)\dot{\theta} \]

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r \sin \theta \\ r \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix} \]
The moment about the joint generated by the force acting on the end effector is given by

\[ \tau = -r F_x \sin \theta + r F_y \cos \theta \]
Jacobian Matrix by Differentiation - 1R - 4/4

- Expressed in matrix form we have

\[ \tau = J(\theta)^T F \]

\[ \begin{bmatrix} \tau \end{bmatrix} = \begin{bmatrix} -r \sin \theta & r \cos \theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix} \]

\[ \begin{bmatrix} 1 \times 1 \\ 1 \times 2 \\ 2 \times 1 \]

\[ \dot{x} = J(\theta) \dot{\theta} \]

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r \sin \theta \\ r \cos \theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix} \]
Consider the following 3 DOF Planar manipulator.
• **Problem:** Compute the Jacobian matrix that describes the relationship

\[ \dot{x} = J(\theta)\dot{\theta} \quad \tau = J(\theta)^T F \]

• **Solution:**

• The end effector position and orientation is defined in the base frame by

\[
\begin{bmatrix}
    x \\
    y \\
    \alpha
\end{bmatrix}
\]
The forward kinematics gives us relationship of the end effector to the joint angles:

\[
0P_{3 \text{org}, x} = x = L_1 c_1 + L_2 c_{12} + L_3 c_{123}
\]

\[
0P_{3 \text{org}, y} = y = L_1 s_1 + L_2 s_{12} + L_3 s_{123}
\]

\[
0P_{3 \text{org}, \alpha} = \alpha = \theta_1 + \theta_2 + \theta_3
\]

Differentiating the three expressions gives

\[
\dot{x} = -L_1 s_1 \dot{\theta}_1 - L_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) - L_3 s_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)
\]

\[
\dot{y} = L_1 c_1 \dot{\theta}_1 + L_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) + L_3 c_{123}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)
\]

\[
\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3
\]
Jacobian Matrix by Differanciation - 3R - 4/4

• Using a matrix form we get

\[ \dot{x} = ^0J(\theta)\dot{\theta} \]

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{\alpha}
\end{bmatrix}
= 
\begin{bmatrix}
-L_1s_1 - L_2s_{12} - L_3s_{123} & -L_2s_{12} - L_3s_{123} & -L_3s_{123} \\
L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123} \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\dot{\theta}_3
\end{bmatrix}
\]

• The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees of freedom. Before tackling these problems, consider this brief review of linear algebra.
Singularity - The Concept

• **Motivation:** We would like the hand of a robot (end effector) to move with a certain velocity vector in Cartesian space. Using linear transformation relating the joint velocity to the Cartesian velocity we could calculate the necessary joint rates at each instance along the path.

\[ \dot{\theta} = J(\theta)^{-1} \dot{x} \]

• **Given:** a linear transformation relating the joint velocity to the Cartesian velocity (usually the end effector)

• **Question:** Is the Jacobian matrix invertable? (Or) Is it nonsingular? Is the Jacobian invertable for all values of \( \theta \)? If not, where is it not invertable?
**Answer (Conceptual):** Most manipulators have values of $\theta$ where the Jacobian becomes singular. Such locations are called **singularities of the mechanism** or **singularities** for short.

**Workspace boundary singularities**
- Stretched out
- Folded back

**Workspace interior Singularities**
- Two or more joints are lining up
Losing one or more DOF means that there is a some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot (end effector) no matter which joint rate are selected.
Inverse of Matrix A exists \textit{if and only if} the determinant of A is non-zero.

\[ A^{-1} \text{ Exists } \textit{if and only if} \]

\[ Det(A) = |A| \neq 0 \]

If the determinant of A is equal to zero, then the matrix A is a singular matrix

\[ Det(A) = |A| = 0 \]

A Singular
The rank of the matrix $A$ is the size of the largest squared matrix $S$ for which

$$\text{Det}(S) \neq 0$$

- Example 1 -

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad A = S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad |A| = |S| = 3 \quad \text{Rank}(A) = 2$$

- Example 2 -

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 1 \end{bmatrix} \quad |S| = 1 \quad \text{Rank}(A) = 1$$
• If two rows or columns of matrix $A$ are equal or related by a constant, then

$$Det(A) = 0$$

• Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & -3 \\ 10 & -6 & -5 \end{bmatrix}$$

$$\det(A) = |A| = 2 \begin{vmatrix} -3 & -3 \\ -6 & -5 \end{vmatrix} - 0 \begin{vmatrix} 6 & -3 \\ 10 & -5 \end{vmatrix} - 1 \begin{vmatrix} 6 & -3 \\ 10 & -6 \end{vmatrix} = 6 + 0 - 6 = 0$$
• **Eigenvalues**

\[ AX = \lambda X \]

\[ (A - \lambda I)X = 0 \]

• Eigenvalues are the roots of the polynomial

\[ Det(A - \lambda I) \]

• If \( X \neq 0 \) each solution to the characteristic equation \( \lambda \) (Eigenvalue) has a corresponding Eigenvector
Brief Linear Algebra Review - 4/

\[ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \]

\[(A - \lambda I)X = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0\]

\[Det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0\]

\[\lambda_1 = -1\]
\[\lambda_2 = 3\]
Brief Linear Algebra Review - 4/

\[ \lambda_1 = -1 \]

\[
\begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = 0
\]

\[ X = \begin{bmatrix}
-1 \\
1
\end{bmatrix} \]

\[ \lambda_2 = 3 \]

\[
\begin{bmatrix}
-2 & 2 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} = 0
\]

\[ X = \begin{bmatrix}
1 \\
1
\end{bmatrix} \]
Any singular matrix \( \text{Det}(A) = 0 \) has at least one Eigenvalue equal to zero.
• If $A$ is non-singular ($Det(A) \neq 0$), and $\lambda$ is an eigenvalue of $A$ with corresponding to eigenvector $X$, then

$$A^{-1}X = \lambda^{-1}X$$
Brief Linear Algebra Review - 7/

• If the $n \times n$ matrix $A$ is of full rank (that is, $\text{Rank} (A) = n$), then the only solution to

$$AX = 0$$

is the trivial one

$$X = 0$$

• If $A$ is of less than full rank (that is $\text{Rank} (A) < n$), then there are $n-r$ linearly independent (orthogonal) solutions

$$x_j \quad 0 \leq j \leq n - r$$

for which

$$Ax_j = 0$$
• If $A$ is square, then $A$ and $A^T$ have the same eigenvalues
Properties of the Jacobian - Velocity Mapping and Singularities

- **Example:** Planar 3R

\[
\begin{vmatrix}
-L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\
L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\
1 & 1 & 1
\end{vmatrix} = L_1 L_2 s_2
\]

\[
\det(J(\theta)) = L_1 L_2 s_2 = 0
\]

- Note that \(\det(J(\theta))\) is not a function of \(\theta_1, \theta_3\)
Properties of the Jacobian - Velocity Mapping and Singularities

A singular configuration is given by:

\[
\begin{align*}
\theta_2 &= 0 & \text{Stretched Out} \\
\theta_2 &= \pi & \text{Fold Back}
\end{align*}
\]

- The manipulator loses 1 DEF. The end effector can only move along the tangent direction of the arm. Motion along the radial direction is not possible.
Properties of the Jacobian -
Force Mapping and Singularities

• The relationship between joint torque and end effector force and moments is given by:

\[ \tau = J(\theta)^T F \]

• The rank of \( J(\theta)^T \) is equals the rank of \( J(\theta) \).

• At a singular configuration there exists a non trivial force \( F \) such that

\[ J(\theta)^T F = 0 \]

• In other words, a finite force can be applied to the end effector that produces no torque at the robot’s joints. In the singular configuration, the manipulator can “lock up.”
Properties of the Jacobian - Force Mapping and Singularities

- **Example:** Planar 3R \( \theta_1 = \theta; \quad \theta_2 = \theta_3 = 0 \)

- In this case the force acting on the end effector (relative to the \{0\} frame) is given by

\[
0F = \begin{bmatrix}
FC_1 \\
FS_1 \\
0
\end{bmatrix}
\]
Properties of the Jacobian - Force Mapping and Singularities

\[ \mathbf{\tau} = \mathbf{J}(\theta)^T \mathbf{F} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & L_1 c_1 + L_2 c_{12} + L_3 c_{123} & 1 \\ -L_2 s_{12} - L_3 s_{123} & L_2 c_{12} + L_3 c_{123} & 1 \\ -L_3 s_{123} & L_3 c_{123} & 1 \end{bmatrix} \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix} \]

- For \( \theta_1 = \theta; \ \theta_2 = \theta_3 = 0 \) we get

\[ \mathbf{\tau} = \mathbf{J}(\theta)^T \mathbf{F} = \begin{bmatrix} -L_1 s_1 - L_2 s_1 - L_3 s_1 & L_1 c_1 + L_2 c_1 + L_3 c_1 & 1 \\ -L_2 s_1 - L_3 s_1 & L_2 c_1 + L_3 c_1 & 1 \\ -L_3 s_1 & L_3 c_1 & 1 \end{bmatrix} \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} -Fs_1 c_1 (L_1 + L_2 + L_3) + Fs_1 c_1 (L_1 + L_2 + L_3) \\ -Fs_1 c_1 (L_2 + L_3) + Fs_1 c_1 (L_2 + L_3) \\ -Fs_1 c_1 (L_3) + Fs_1 c_1 (L_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
Properties of the Jacobian -
Force Mapping and Singularities

• This situation is an old and famous one in mechanical engineering.

• For example, in the steam locomotive, “top dead center” refers to the following condition

- The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.
Properties of the Jacobian - Velocity Mapping and Singularities

• We have shown the relationship between joint space velocity and end effector velocity, given by

\[ \dot{x} = J(\theta)\dot{\theta} \]

• It is interesting to determine the inverse of this relationship, namely

\[ \dot{\theta} = J(\theta)^{-1} \dot{x} \]
Properties of the Jacobian - Velocity Mapping and Singularities

- Consider the square 6x6 case for \( J(\theta) \).

- If \( \text{rank} < 6 \) (\( Det(J(\theta)) = 0 \)), then there is no solution to the inverse equation (see Brief Linear Algebra Review - 1,7).

\[
\text{Rank}(J(\theta)) < 6
\]

\[
\dot{\theta} = J(\theta)^{-1} \dot{x}
\]

- However, if the rank = 5, then there is at least one non-trivial solution to the forward equation (see Brief Linear Algebra Review - 7). That is, for

\[
\dot{x} = J(\theta)\dot{\theta} = 0
\]
Properties of the Jacobian - Velocity Mapping and Singularities

• The solution is a direction \( \theta \) in the in joint velocity space for which joint motion produces no end effector motion.

• We call any joint configuration \( \theta = Q \) for which

\[
\text{Rank}(J(\theta)) < 6
\]

a **singular configuration**.
Properties of the Jacobian - Velocity Mapping and Singularities

- For certain directions of end effector motion, \( \dot{x}_i \quad 1 \leq i \leq 6 \)

\[
\dot{x} = J(\theta) \dot{\theta} = \lambda_i(\theta) \omega_i,
\]

where:
- \( \lambda_i \) are the eigenvalues of \( J(\theta) \)
- \( \omega_i \) are the eigenvectors of \( J(\theta) \)

- If \( J(\theta) \) is fully ranked (see Brief Linear Algebra Review - 6/), we have

\[
\omega_i = J(\theta)^{-1} \dot{x} = \lambda_i(\theta)^{-1} \dot{x}
\]
Properties of the Jacobian - Velocity Mapping and Singularities

• As the joint approach a singular configuration \( \boldsymbol{\theta} = \boldsymbol{Q} \) there is at least one eigenvalue for which \( \lambda_i \rightarrow 0 \). This results in

\[
\begin{align*}
\omega_i &= \frac{\dot{x}}{\lambda_i(\theta)} \rightarrow \frac{\dot{x}}{0} \rightarrow \infty
\end{align*}
\]

• In other word, as the joints approach the singular configuration, the end effector motion in a particular task direction \( \dot{x}_j \) causes the joint velocities to approach infinity. However, there are task velocities that can have solutions.

• If \( J(\theta) \) loses rank by only one, then there are \( n-1 \) eigenvectors in the task velocity space \( (\dot{x}_j) \) for which solutions do exist. However, there can be multiple solutions.