Inverse Manipulator Kinematics

Review
Direct Versus Inverse Kinematics

**Direct (Forward) Kinematics**

*Given:* Joint angles and links geometry  
*Compute:* Position and orientation of the end effector relative to the base frame  

\[ f(\theta) = B_T^0 T = N_T^0 \]

**Inverse Kinematics**

*Given:* Position and orientation of the end effector relative to the base frame  
*Compute:* All possible sets of joint angles and links geometry which could be used to attain the given position and orientation of the end effector  

\[ \theta = f^{-1}(B_T^0 T) = f^{-1}(N_T^0) \]
Central Topic - Inverse Manipulator Kinematics - Examples

- **Geometric Solution - Concept**
  Decompose spatial geometry into several plane geometry

  **Examples** - RRR (3R) manipulators - Geometric Solution

- **Algebraic Solution - Concept**

  \[
  ^0_N T = ^0_1 T \ldots ^{N-1}_N T = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]

  Direct Kinematics Goal (Numeric values)

  **Examples** - PUMA 560 - Algebraic Solution
Solvability

- Existence of Solutions
- Multiple Solutions
- Method of solutions
  - Close form solution
    - Algebraic solution
    - Geometric solution
  - Numerical solutions
Solvability - Existence of Solution

• For a solution to exist, $^0_T N$ must be in the *workspace* of the manipulator.

• *Workspace* - Definitions

  – *Dexterous Workspace (DW)*: The subset of space in which the robot end effector can reach *all orientation*.

  – *Reachable Workspace (RW)*: The subset of space in which the robot end effector can reach in *at least 1 orientation*.

• The Dexterous Workspace is a subset of the Reachable Workspace

  \[ DW \subset RW \]
Solvability - Existence of Solution - Workspace - 2R
Example 1 - $L_1 = L_2$

Reachable Workspace

Dexterous Workspace
Solvability - Existence of Solution - Workspace - 2R

Example 2 - $L_1 \neq L_2$

Reachable Workspace

NO Dexterous Workspace

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Solvability - Existence of Solution - Workspace - 3R

Example 3 - \( L_1 = L_2 \)

Reachable Workspace & Dexterous Workspace

End Effector Rotation
Solvability - Multiple Solutions

- **Problem:** The fact that a manipulator has multiple solutions may cause problems because the system has to be able to choose one.

- **Solution:** Decision criteria
  - The closest (geometrically) - minimizing the amount that each joint is required to move
    - Note 1: input argument - present position of the manipulator
    - Note 2: Joint Weight - Moving small joints (wrist) instead of moving large joints (Shoulder & Elbow)
  - Obstacles exist in the workspace - avoiding collision

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Solvability - Multiple Solutions

• Multiple solutions are a common problem that can occur when solving inverse kinematics because the system has to be able to chose one

• The number of solutions depends on the number of joints in the manipulator but is also a function of the links parameters \( a_i, \alpha_i, d_i, \theta_i \)

• Example: The PUMA 560 can reach certain goals with 8 different (solutions) arm configurations
  – Four solutions are depicted
  – Four solutions are related to a “flipped” wrist

\[
\begin{align*}
\theta_4' &= \theta_4 + 180^\circ \\
\theta_5' &= -\theta_5 \\
\theta_6' &= \theta_6 + 180^\circ
\end{align*}
\]
Solvability - Multiple Solutions - Number of Solutions

- Task Definition - Position the end effector in a specific point in the plane (2D)

- No. of DOF = No. of DOF of the task

  Number of solution: 2
  (elbow up/down)

- No. of DOF > No. of DOF of the task

  Number of solution: \( \infty \)
  Self Motion - The robot can be moved without moving the end effector from the goal
Solvability - Methods of Solutions

• **Solution** (Inverse Kinematics)- A “solution” is the set of joint variables associated with an end effector’s desired position and orientation.

• **No general algorithms** that lead to the solution of inverse kinematic equations.

• **Solution Strategies**
  
  – **Closed form Solutions** - An analytic expression includes all solution sets.
    • **Algebraic Solution** - Trigonometric (Nonlinear) equations
    • **Geometric Solution** - Reduces the larger problem to a series of plane geometry problems.

  – **Numerical Solutions** - Iterative solutions will not be considered in this course.
Solvability

Robot - 6 DOF
Single Series Chain
Revolute & Prismatic Joints

Real-Time

Analytic Solution

Numeric Solution

Non Real-Time

Industrial Robots

Close Form Solution
Sufficient Condition
Three adjacent axes (rotary or prismatic) must intersect

Iterations

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Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 1-7
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 1

• Equation

\[ \sin \theta = a \quad a \in [-1,1] \]
\[ \cos \theta = b \quad b \in [-1,1] \]

• Solution (Unique)

\[ \theta = A \tan 2(a,b) \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 2

- Equation
  \[
  \sin \theta = a \quad a \in [-1, 1] \\
  \cos \theta = \pm \sqrt{1 - a^2} \\
  \cos \theta = b \quad b \in [-1, 1] \\
  \sin \theta = \pm \sqrt{1 - b^2}
  \]

- Solution
  \[
  \theta = A \tan 2(a, \pm \sqrt{1 - a^2}) \\
  \theta = A \tan 2(\pm \sqrt{1 - b^2}, b)
  \]
  - Two Solutions
  \[
  \theta_1 = \theta \\
  \theta_2 = 180 - \theta \\
  \theta_1 = \theta \\
  \theta_2 = -\theta
  \]
  - Singularity at the Boundary
  \[
  \text{When } \theta = \pm 90^\circ, \quad |a| = 1 \\
  \text{When } \theta = 0^\circ, 180^\circ, \quad |b| = 1
  \]
Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 3

• Equation

\[ a(\cos \theta) + b(\sin \theta) = 0 \]

\[ \frac{\sin \theta}{\cos \theta} = -\frac{a}{b} \]

• Solution
  – Two Solutions 180° apart

\[ \theta = A \tan 2(a, -b) \]

\[ \theta = A \tan 2(-a, b) \]

  – Singularity

\[ b = 0 \]
Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 4

- Equation

\[ a \cos \theta + b \sin \theta = c \]
\[ a, b, c \neq 0 \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 4
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 4

• Solution
  – Two Solutions
    \[ \theta = A \tan 2(\pm \sqrt{a^2 + b^2 - c^2}, c) + A \tan 2(b, a) \]

  – For a solution to exist
    \[ a^2 + b^2 - c^2 > 0 \]
  – No solution (outside of the workspace)
    \[ a^2 + b^2 - c^2 < 0 \]
  – One solution (singularity)
    \[ a^2 + b^2 - c^2 = 0 \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 5

- Equation
  \[
  \sin \theta \sin \phi = a \\
  \cos \theta \sin \phi = b
  \]

- Solution
  \[
  \theta = A \tan 2(a, b) \quad \text{if } \sin \phi \text{ is positive} \\
  \theta = A \tan 2(-a, -b) \quad \text{if } \sin \phi \text{ is negative}
  \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 6

• Equation

\[ \cos \theta + c \sin \theta = d \]
\[ \cos \theta + f \sin \theta = g \]

• Solution

\[ \theta = A \tan 2(af - ce, df - cg) \]

– For an exiting solution (the determinant must be positive)

\[ af - ce > 0 \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 7

- Transcendental equations are difficult to solve because they are a function of $c\theta, s\theta$

  $$ f(c\theta, s\theta) = k $$

- Making the following substitutions yields an expression in terms of a single veritable $u$, Using this substitutions, transcendental equation are converted into polynomial equation

  $$ u = \tan \frac{\theta}{2} $$

  $$ \cos \theta = \frac{1-u^2}{1+u^2} $$

  $$ \sin \theta = \frac{2u}{1+u^2} $$
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 7

• Transcendental equation

\[ ac\theta + bs\theta = c \]

• Substitute \( c\theta, s\theta \) with the following equations

\[ \cos \theta = \frac{1-u^2}{1+u^2} \]
\[ \sin \theta = \frac{2u}{1+u^2} \]

• yields

\[ a(1-u^2) + 2bu = c(1+u^2) \]
\[ (a+c)u^2 - 2bu + (c-a) = 0 \]
Inverse Kinematics - Generalized Algebraic (Analytical) Solutions – Case 7

- Which is solved by the quadratic formula to be

\[ u = \frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a + c} \]

\[ \theta = 2 \tan^{-1}\left(\frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a + c}\right) \]

- Note
  - If \( u \) is complex there is no real solution to the original transcendental equation
  - If \( a + c = 0 \) then \( \theta = 180^\circ \)
Inverse Manipulator Kinematics

Geometric Approach
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 1/

- **Given:**
  - **Manipulator Geometry**
  - **Goal Point Definition:** The position $x_d$, $y_d$, $z_d$ of the wrist in space

![Manipulator Diagram]

- **Problem:**
  What are the joint angles $(\theta_1, \theta_2, \theta_3)$ as a function of the goal (wrist position and orientation)
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 2/
inverse kinematics - 3d rrr (3r) - geometric solution - 3/

- The planar geometry - top view of the robot

\[
\begin{align*}
\theta_1 &= A \tan 2(y_d, x_d) \\
r_1 &= \sqrt{x_d^2 + y_d^2}
\end{align*}
\]
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 4/

\[ \{o3\} = \{S\} \]
\[ \{q\} = \{T\} \]
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 5/

- The planar geometry - side view of the robot:

\[
r_2 = \sqrt{r_1^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2}
\]

- where

\[
\hat{z} = z_d - (L_1 + L_2)
\]
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 6/

• By Apply the law of cosines we get

\[ r_2^2 = L_3^2 + L_4^2 - 2L_3L_4 \cos(180 + \theta_3) = L_3^2 + L_4^2 + 2L_3L_4 \cos(\theta_3) \]

• Rearranging gives

\[ c_3 = \frac{r_2^2 - (L_3^2 + L_4^2)}{2L_3L_4} \]

• and

\[ s_3 = \sqrt{1 - c_3^2} \]

• Solving for \( \theta_3 \) we get

\[ \theta_3 = A \tan(\pm \sqrt{1 - c_3^2}, c_3) \]

• Where \( c_3 \) is defined above in terms of known parameters \( L_3, L_4, x_d, y_d, \) and \( z_d \)
Finally we need to solve for $\theta_2$

\[ \theta_2 = \alpha + \beta \]

\[ \alpha = A \tan 2(\hat{z}, r_1) \]

where

\[ r_1 = \sqrt{x_d^2 + y_d^2} \]

\[ \hat{z} = z_d - (L_1 + L_2) \]
Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 9/

- Based on the law of cosines we can solve for $\beta$

$$L_4^2 = r_2^2 + L_3^2 - 2r_2L_3 \cos(\beta)$$

$$c_\beta = \frac{r_2^2 + L_3^2 - L_4^2}{2r_2L_3}$$

$$\beta = A \tan 2(\pm \sqrt{1 - c_\beta^2}, c_\beta)$$

$$\theta_2 = A \tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) + A \tan 2(\pm \sqrt{1 - c_\beta^2}, c_\beta)$$
Inverse Kinematics - 3D RRR (3R) -
Geometric Solution - 10/

Summary

\[ \theta_1 = A \tan 2(y_d, x_d) \]

\[ \theta_2 = A \tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) + \]

\[ A \tan 2(\pm \sqrt{1 - \left( \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2x_d^2 + 2y_d^2 + (z_d - (L_1 + L_2)L_3)^2} \right)^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2x_d^2 + 2y_d^2 + (z_d - (L_1 + L_2)L_3)^2}) \]

\[ \theta_3 = A \tan(\pm \sqrt{1 - \left( \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4} \right)^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4}) \]
Inverse Manipulator Kinematics

Analytical Approach
Central Topic - Inverse Manipulator Kinematics - Examples

• **Algebraic Solution - Concept**

\[
^0_T = ^0_T \cdots ^{N-1}_T = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Direct Kinematics            Goal (Numeric values)

**Examples** - PUMA 560 - Algebraic Solution
Solvability - PUMA 560

Given: PUMA 560 - 6 DOF, $^0_6 T$

Solve: $\theta_1 \cdots \theta_6$

\[
^0 T = ^1 T ^2 T ^3 T ^4 T ^5 T ^6 = \begin{bmatrix}
r_{11} & r_{12} & r_{13} & p_x \\
r_{21} & r_{22} & r_{23} & p_y \\
r_{31} & r_{32} & r_{33} & p_z \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Total Number of Equations: 12

Independent Equations: 3 - Rotation Matrix

Independent Equations: 3 - Position Vector

Type of Equations: Non-linear

$r_{11} = c_1 [c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_{5} c_{6}] + s_1 (s_4 c_5 c_6 + c_4 s_6),$ 
$r_{21} = s_1 [c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_{5} c_{6}] - c_1 (s_4 c_5 c_6 + c_4 s_6),$ 
$r_{31} = -s_{23}(c_4 c_5 c_6 - s_4 s_6) - c_{23} s_{5} c_{6},$

$r_{12} = c_1 [c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_{5} s_{6}] + s_1 (c_4 c_6 - s_4 c_5 s_6),$ 
$r_{22} = s_1 [c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_{5} s_{6}] - c_1 (c_4 c_6 - s_4 c_5 s_6),$ 
$r_{32} = -s_{23}(-c_4 c_5 s_6 - s_4 c_6) + c_{23} s_{5} s_{6},$

$r_{13} = -c_1 (c_{23} c_4 s_5 + s_{23} c_5) - s_1 s_4 s_5,$ 
$r_{23} = -s_1 (c_{23} c_4 s_5 + s_{23} c_5) + c_1 s_4 s_5,$ 
$r_{33} = s_{23} c_4 s_5 - c_{23} c_5,$

$p_x = c_1 [a_2 c_2 + a_3 c_{23} - d_4 s_{23}] - d_3 s_1,$ 
$p_y = s_1 [a_2 c_2 + a_3 c_{23} - d_4 s_{23}] + d_3 c_1,$ 
$p_z = -a_3 s_{23} - a_2 s_2 - d_4 c_{23}.$
Central Topic - Inverse Manipulator Kinematics - Examples

• **Geometric Solution - Concept**
  Decompose spatial geometry into several plane geometry

  **Examples** - Planar RRR (3R) manipulators - Geometric Solution

• **Algebraic Solution - Concept**

  \[
  _N^0 T = _N^1 T \ldots _N^{N-1} T = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} & p_x \\
    r_{21} & r_{22} & r_{23} & p_y \\
    r_{31} & r_{32} & r_{33} & p_z \\
    0 & 0 & 0 & 1
  \end{bmatrix}
  \]

  Direct Kinematics Goal (Numeric values)

  **Examples** - PUMA 560 - Algebraic Solution
Inverse Kinematics - PUMA 560 - Algebraic Solution - 1/

• **Given:**
  
  – **Direct Kinematics:** The homogenous transformation from the base to the wrist $^{B}T_{W}$
  
  – **Goal Point Definition:** The position and orientation of the wrist in space
Inverse Kinematics - PUMA 560 - Algebraic Solution - 2/

- **Problem:**
  What are the joint angles \( (\theta_1 \cdots \theta_6) \) as a function of the wrist position and orientation (or when \( ^0\vec{T}_6 \) is given as numeric values)

\[
^0\vec{T}_6 = ^0\vec{T}(\theta_1) \cdot ^1\vec{T}(\theta_2) \cdot ^2\vec{T}(\theta_3) \cdot ^3\vec{T}(\theta_4) \cdot ^4\vec{T}(\theta_5) \cdot ^5\vec{T}(\theta_6) = \\
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Direct Kinematics

Goal
Inverse Kinematics - PUMA 560 - Algebraic Solution - 3/

- **Solution (General Technique):** Multiplying each side of the direct kinematics equation by an inverse transformation matrix for separating out variables in search of solvable equation.

- Put the dependence on $\theta_1$ on the left hand side of the equation by multiplying the direct kinematics eq. with $[^{0}\mathbf{T}(\theta_1)]^{-1}$ gives:

$$
[^{0}\mathbf{T}(\theta_1)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} =
[^{0}\mathbf{T}(\theta_1)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} \begin{bmatrix} 1T(\theta_2) & 2T(\theta_2) & 3T(\theta_3) & 4T(\theta_4) & 5T(\theta_5) & 6T(\theta_6) \end{bmatrix}
$$

$$
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} =
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} \begin{bmatrix} 1T(\theta_2) & 2T(\theta_2) & 3T(\theta_3) & 4T(\theta_4) & 5T(\theta_5) \end{bmatrix}
$$

$$
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} =
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} \begin{bmatrix} 1T(\theta_2) & 2T(\theta_2) & 3T(\theta_3) & 4T(\theta_4) \end{bmatrix}
$$

$$
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} =
[^{0}\mathbf{T}(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} \begin{bmatrix} 0T & 0T \end{bmatrix} \begin{bmatrix} 1T(\theta_2) & 2T(\theta_2) & 3T(\theta_3) & 4T(\theta_4) \end{bmatrix}
$$
Inverse Kinematics - PUMA 560 -
Algebraic Solution - 4/

- Put the dependence on $\theta_1$ on the left hand side of the equation by multiplying the direct kinematics eq. with $[1^T_1(\theta_1)]^{-1}$ gives

$$
[0^T_1(\theta_1)]^{-1} 0^T = [1^T_1(\theta_1)]^{-1} 0^T(\theta_1) \ 1^T_2(\theta_2) \ 2^T_3(\theta_3) \ 3^T_4(\theta_4) \ 4^T_5(\theta_5) \ 5^T_6(\theta_6)
$$

$$
0^T_1 = \begin{bmatrix}
c \theta_1 & -s \theta_1 & 0 & 0 \\
s \theta_1 & c \theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
1^T_0 = [0^T_1]^{-1} = \begin{bmatrix}
c \theta_1 & -s \theta_1 & 0 & 0 \\
s \theta_1 & c \theta_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

$$
[1^A_B T]^{-1} = [B^A T] = \begin{bmatrix}
A^B R^T & -A^B R^T A^P_{BORG} \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

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Inverse Kinematics - PUMA 560 - Algebraic Solution - 5/

\[
\begin{bmatrix}
  c_1 & s_1 & 0 & 0 \\
  -s_1 & c_1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix} = {^1}_6T
\]
Inverse Kinematics - PUMA 560 - Algebraic Solution - 6/
Solution for $\theta_1$

$$\begin{bmatrix}
  c_1 & s_1 & 0 & 0 \\
  -s_1 & c_1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} & p_x \\
  r_{21} & r_{22} & r_{23} & p_y \\
  r_{31} & r_{32} & r_{33} & p_z \\
  0 & 0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
  1 \\
  0 \\
  0 \\
  0
\end{bmatrix}$$

Equating the (2,4) elements from both sides of the equation we have

$$-s_1 p_x + c_1 p_y = d_3$$

To solve the equation of this form we make the trigonometric substitution

$$p_x = \rho \cos \phi$$
$$p_y = \rho \sin \phi$$
Inverse Kinematics - PUMA 560 - Algebraic Solution - 8/

\[ \rho = \sqrt{p_x^2 + p_y^2} \]

\[ \phi = A \tan 2(p_x, p_y) \]

- Substituting \( p_x, p_y \) with \( \rho, \phi \) we obtain

\[ c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho} \]

- Using the difference of angles formula

\[ \sin(\phi - \theta_1) = \frac{d_3}{\rho} \]
Based on

\[ \sin^2(\phi - \theta_1) + \cos^2(\phi - \theta_1) = 1 \]

\[ \cos(\phi - \theta_1) = \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \]

and so

\[ \phi - \theta_1 = A \tan 2 \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right) \]

The solution for \( \theta_1 \) may be written

\[ \theta_1 = A \tan 2(p_y, p_x) - A \tan 2 \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right) \]

Note: we have found two possible solutions for \( \theta_1 \) corresponding to the +/- sign
Inverse Kinematics - PUMA 560 -
Algebraic Solution - 10/

- Solution for $\theta_3$
- Equating the $(1,4)$ element and $(3,4)$ element

\[
\begin{bmatrix}
c_1 & s_1 & 0 & 0 \\
-s_1 & c_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
r_{11} & r_{12} & r_{13} & p_x \\
r_{21} & r_{22} & r_{23} & p_y \\
r_{31} & r_{32} & r_{33} & p_z \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
= ^{1}_{0}T
\]

- We obtain

\[
c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2
\]
\[
-p_z = a_3 c_{23} + d_4 s_{23} + a_2 c_2
\]
If we square the following equations and add the resulting equations:

\[-s_1 p_x + c_1 p_y = d_3\]
\[c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2\]
\[-p_z = a_3 s_{23} + d_4 c_{23} + a_2 s_2\]
Inverse Kinematics - PUMA 560 - Algebraic Solution - 11/ (Continue)
we obtain
\[ a_3 c_3 - d_4 s_3 = K \]

where
\[
K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - a_4^2}{2a_2}
\]

Note that the dependence on \( \theta_1 \) has been removed. Moreover, the eq. for \( \theta_3 \) is of the same form as the eq. for \( \theta_1 \) and so may be solved by the same kind of trigonometric substitution to yield a solution for \( \theta_3 \).
Inverse Kinematics - PUMA 560 - Algebraic Solution - 13/

\[ \theta_3 = A \tan 2(a_3, d_4) - A \tan 2\left(K, \pm \sqrt{a_3^2 + d_4^2 - K^2}\right) \]

- Note that the +/- sign leads to two different solutions for \( \theta_3 \)
Inverse Kinematics - PUMA 560 - Algebraic Solution - 14/

• Solution for $\theta_2$

$$[0_T(\theta_1, \theta_2, \theta_3)]^{-1} 0_T = [0_T(\theta_1)]^{-1} 0_T(\theta_1) \begin{bmatrix} 1 & 2 & \cdots & 6 \end{bmatrix}$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 & r_{11} & r_{12} & r_{13} & p_x \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 & r_{21} & r_{22} & r_{23} & p_y \\ -s_1 & c_1 & 0 & -d_3 & r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & -c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• Equating the (1,4) element and (2,4) element we obtain

$$c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3$$

$$-c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4$$

• These equations may be solved simultaneously for $s_{23}$ and $c_{23}$ resulting in
Inverse Kinematics - PUMA 560 - Algebraic Solution - 14/ (Continue)
Inverse Kinematics - PUMA 560 - Algebraic Solution - 15/

\[ s_{23} = \frac{(-a_3 - a_2 c_2) p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2} \]

\[ c_{23} = \frac{(a_2 s_3 - d_4) p_z - (-a_3 - a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2} \]

- Since the denominator are equal and positive, we solve for the sum of \( \theta_2 \) and \( \theta_3 \) as

\[ \theta_{23} = A \tan 2[(-a_3 - a_2 c_2) p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4), \]

\[ (a_2 s_3 - d_4) p_z - (-a_3 - a_2 c_3)(c_1 p_x + s_1 p_y)]] \]

- The equation computes four values of \( \theta_{23} \) according to the four possible combination of solutions for \( \theta_1 \) and \( \theta_3 \).
Inverse Kinematics - PUMA 560 - Algebraic Solution - 16/

• Then, four possible solutions for $\theta_2$ are computed as

$$\theta_2 = \theta_{23} - \theta_3$$

• Solution for $\theta_4$
• Equating the $(1,3)$ and the $(3,3)$ elements

$$\begin{bmatrix}
    c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\
    -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\
    -s_1 & c_1 & 0 & -d_3 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    r_1 \\
    r_{21} \\
    r_{31} \\
    0
\end{bmatrix}
\begin{bmatrix}
    r_{12} \\
    r_{22} \\
    r_{32} \\
    0
\end{bmatrix}
\begin{bmatrix}
    r_{13} \\
    r_{23} \\
    r_{33} \\
    0
\end{bmatrix}
\begin{bmatrix}
    p_x \\
    p_y \\
    p_z \\
    1
\end{bmatrix}
= 
\begin{bmatrix}
    c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 \\
    s_5c_6 & -s_5s_6 & -c_5 \\
    -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_3 \\
    d_4 \\
    0 \\
    1
\end{bmatrix}
$$

• we get

$$r_{13}c_1c_{23} + r_{23}s_1c_{23} - s_{23}r_{33} = -c_4s_5$$

$$-r_{13}s_1 + r_{23}c_1 = s_4s_5$$

$$c_4 = \frac{-r_{13}c_1c_{23} - r_{23}s_1c_{23} + s_{23}r_{33}}{s_5}$$

$$s_4 = \frac{-r_{13}s_1 + r_{23}c_1}{s_5}$$
As long as $s_5 \neq 0$ we can solve for $\theta_4$

$$\theta_4 = A \tan 2(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_23 - r_{23}s_1c_23 + s_23r_{33})$$

When $\theta_5 = 0$ the manipulator is in a **singular configuration** in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot. In this case all that can be solved for is the sum or difference of $\theta_4$ and $\theta_6$. This situation is detected by checking whether both arguments of $A\tan 2$ defining $\theta_4$ are near zero. If so $\theta_4$ is chosen arbitrary (usually chosen to be equal to the present value of joint 4), and $\theta_6$ is computed later.
Inverse Kinematics - PUMA 560 - Algebraic Solution - 18/

- Solution for $\theta_5$

\[ [4T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = [4T(\theta_1, \theta_2, \theta_3, \theta_4)]^{-1} \begin{bmatrix} 0 T(\theta_1) \\ 1 T(\theta_2) \\ 2 T(\theta_3) \\ 3 T(\theta_4) \\ 4 T(\theta_5) \\ 5 T(\theta_6) \end{bmatrix} \]

\[
\begin{bmatrix}
    c_1 c_23c_4 + s_1 s_4 & s_1 c_23c_4 - c_1 s_4 & -s_23c_4 & -a_2c_3c_4 + d_3s_4 - a_3c_4 \\
    -c_1 c_23s_4 + s_1 c_4 & -s_1 c_23s_4 - c_1 c_4 & s_23s_4 & a_2c_3s_4 + d_3c_4 - a_3s_4 \\
    -c_1 s_23 & -s_1 s_23 & c_23 & a_2 s_3 - d_4 \\
    0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
    r_{11} \\
    r_{21} \\
    r_{31} \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    r_{12} \\
    r_{22} \\
    r_{32} \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    r_{13} \\
    r_{23} \\
    r_{33} \\
    0 \\
\end{bmatrix}
\begin{bmatrix}
    p_x \\
    p_y \\
    p_z \\
    1 \\
\end{bmatrix}
\begin{bmatrix}
    c_5 c_6 \\
    s_6 \\
    s_5 c_6 \\
    c_5 \\
\end{bmatrix}
\begin{bmatrix}
    -c_5 s_6 \\
    c_6 \\
    -s_5 s_6 \\
    c_5 \\
\end{bmatrix}
\begin{bmatrix}
    -s_5 \\
    0 \\
    0 \\
\end{bmatrix}
\]

- Equating the (1,3) and the (3,3) elements we get

\[ r_{13}(c_1 c_23c_4 + s_1 s_4) + r_{23}(s_1 c_23c_4 - c_1 s_4) - r_{33}(s_23c_4) = -s_5 \]

\[ r_{13}(-c_1 s_23) + r_{23}(-s_1 s_23) + r_{33}(-c_23) = c_5 \]

- We can solve for $\theta_5$

\[ \theta_5 = A \tan(2(s_5, c_5)) \]
Inverse Kinematics - PUMA 560 - Algebraic Solution - 19/

- Solution for $\theta_6$

\[
[0^T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} 0^T = [0^T(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)]^{-1} 0^T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \end{bmatrix}
\]

- Equating the (3,1) and the (1,1) elements we get

\[
\begin{align*}
& r_{11}(c_1 s_{23} c_5 - s_1 s_{23} s_5) + r_{21}(c_{23}) + r_{31}(+c_1 s_{23} s_5 + s_1 s_{23} c_5) = s_6 \\
& r_{11}(c_1 c_{23} c_4 c_5 + s_1 s_4 c_5 - s_1 c_{23} c_4 s_5 + c_1 s_4 s_5) + r_{21}(c_{23} c_4) + r_{31}(c_1 c_{23} c_4 s_5 + s_1 s_4 s_5 + s_1 c_{23} c_4 c_5 - c_1 s_4 c_5) = c_6
\end{align*}
\]

- We can solve for $\theta_6$

\[
\theta_6 = A \tan 2(s_6, c_6)
\]
Inverse Kinematics - PUMA 560 - Algebraic Solution - 21/

- Summary - Number of Solutions

- Four solution

\[
\theta_1 = A \tan 2(p_y, p_x) - A \tan 2 \left( \frac{d_3}{\rho}, \pm \sqrt{1 - \frac{d_3^2}{\rho^2}} \right)
\]

\[
\theta_3 = A \tan 2(a_3, d_4) - A \tan 2 \left( K, \pm \sqrt{a_3^2 + d_4^2 - K^2} \right)
\]

- For each of the four solutions the wrist can be flipped

\[
\theta'_4 = \theta_4 + 180^\circ
\]

\[
\theta'_5 = -\theta_5
\]

\[
\theta'_6 = \theta_6 + 180^\circ
\]
• After all eight solutions have been computed, some or all of them may have to be discarded because of joint limit violations.

• Of the remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.
Pieper’s Solution - Three consecutive Axes Intersect

- *High Level Introduction to the Solution*
Pieper’s Solution - Three consecutive Axes Intersect

- High Level Introduction to the Solution
Pieper’s Solution - Three consecutive Axes Intersect

- When the last three axes of a 6 DOF robot intersect, the origins of link frame \{4\}, \{5\}, and \{6\} are all located at the point of intersection. This point is given in the base coordinate system as

\[
0 P_{4_{org}} = 0 T_1^1 T_2^2 T_3^3 P_{4_{org}}
\]

- From the general forward kinematics method for determining homogeneous transforms using DH parameters, we know:

\[
\begin{bmatrix}
  c \theta_i & -s \theta_i & 0 \\
  s \theta_i c \alpha_{i-1} & c \theta_i c \alpha_{i-1} & -s \alpha_{i-1} \\
  s \theta_i s \alpha_{i-1} & c \theta_i s \alpha_{i-1} & c \alpha_{i-1} \\
  0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
i^{-1} R \\
  a_{i-1} \\
  -s \alpha_{i-1} d_i \\
  c \alpha_{i-1} d_i \\
\end{bmatrix}
\]

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Pieper’s Solution - Three consecutive Axes Intersect

- For \( i = 4 \)

\[
{^3T_4} = \begin{bmatrix}
  c \theta_4 & -s \theta_4 & 0 & a_3 \\
  s \theta_4 c \alpha_3 & c \theta_4 c \alpha_3 & -s \alpha_3 & -s \alpha_3 d_4 \\
  s \theta_4 s \alpha_3 & c \theta_4 s \alpha_3 & c \alpha_3 & c \alpha_3 d_4 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

- Using the fourth column and substituting for \( \frac{3}{4}P_{4org} \) we find

\[
{^0P_{4org}} = {^1T_2}{^2T_3}{^3P_{4org}} = {^1T_2}{^2T_3}
\begin{bmatrix}
a_3 \\
-s \alpha_3 d_4 \\
c \alpha_3 d_4 \\
1
\end{bmatrix}
\]

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• where

$$
\begin{bmatrix}
    f_1(\theta_3) \\
    f_2(\theta_3) \\
    f_3(\theta_3) \\
    1
\end{bmatrix}
= \frac{2}{3}T
\begin{bmatrix}
    a_3 \\
    -s\alpha_3d_4 \\
    c\alpha_3d_4 \\
    1
\end{bmatrix}
$$

$$
\begin{bmatrix}
    f_1(\theta_3) \\
    f_2(\theta_3) \\
    f_3(\theta_3) \\
    1
\end{bmatrix}
= \begin{bmatrix}
    c\theta_3 & -s\theta_3 & 0 & a_2 \\
    s\theta_3c\alpha_2 & c\theta_3c\alpha_2 & -s\alpha_2 & -s\alpha_2d_3 \\
    s\theta_3s\alpha_2 & c\theta_3s\alpha_2 & c\alpha_2 & c\alpha_2d_3 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_3 \\
    -s\alpha_3d_4 \\
    c\alpha_3d_4 \\
    1
\end{bmatrix}
$$

$$
\begin{align*}
    f_1(\theta_3) &= a_3c_3 + d_4s\alpha_3s_3 + a_2 \\
    f_2(\theta_3) &= a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2 \\
    f_3(\theta_3) &= a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2
\end{align*}
$$
Pieper’s Solution - Three consecutive Axes Intersect

- Repeating the same process again

\[
^0 P_{\text{org}} = ^0 T_2 T_3 P_{\text{org}} = ^0 T_2 \begin{bmatrix}
    f_1(\theta_3) \\
    f_2(\theta_3) \\
    f_3(\theta_3) \\
    1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    g_1(\theta_2) \\
    g_2(\theta_2) \\
    g_3(\theta_2) \\
    1
\end{bmatrix}
= ^1 T_2 \begin{bmatrix}
    f_1(\theta_3) \\
    f_2(\theta_3) \\
    f_3(\theta_3) \\
    1
\end{bmatrix}
\]

\[
\begin{bmatrix}
    g_1(\theta_2) \\
    g_2(\theta_2) \\
    g_3(\theta_2) \\
    1
\end{bmatrix} = \begin{bmatrix}
    c\theta_2 & -s\theta_2 & 0 & a_1 \\
    s\theta_2 c\alpha_1 & -s\theta_2 c\alpha_2 & -c\alpha_1 & -s\alpha_1 d_2 \\
    s\theta_2 s\alpha_1 & c\theta_2 s\alpha_1 & c\alpha_1 & c\alpha_1 d_2 \\
    0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    f_1(\theta_3) \\
    f_2(\theta_3) \\
    f_3(\theta_3) \\
    1
\end{bmatrix}
\]
Pieper’s Solution - Three consecutive Axes Intersect

\[
g_1(\theta_2) = c_2 f_1 + s_2 f_2 + a_1
\]
\[
g_2(\theta_2) = s_2 s \alpha_1 f_1 + c_2 c \alpha_1 f_2 - s \alpha_1 f_3 - d_2 s \alpha_1
\]
\[
g_3(\theta_2) = s_2 s \alpha_1 f_1 + c_2 s \alpha_1 f_2 + c \alpha_1 f_3 + d_2 c \alpha_1
\]

- Repeating the same process for the last time

\[
0 P_{4\text{org}} = \begin{bmatrix}
0 & T^1 & T^2 & T^3 & P_{4\text{org}} = 0 T
\end{bmatrix}
\]

\[
0 P_{4\text{org}} = \begin{bmatrix}
1 & g_1(\theta_2) \\
g_2(\theta_2) \\
1 & g_3(\theta_2)
\end{bmatrix}
\]

\[
0 P_{4\text{org}} = \begin{bmatrix}
c \theta_1 & -s \theta_1 & 0 & a_0 & g_1(\theta_2) \\
s \theta_1 c \alpha_0 & c \theta_1 c \alpha_0 & -s \alpha_0 & -s \alpha_0 d_1 & g_2(\theta_2) \\
s \theta_1 s \alpha_0 & c \theta_1 s \alpha_0 & c \alpha_0 & c \alpha_0 d_1 & g_3(\theta_2)
\end{bmatrix}
\]

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- **Frame \{0\}** - The frame attached to the base of the robot or link 0 called frame \{0\}. This frame does not move and for the problem of arm kinematics can be considered as the *reference frame*.

- Assign \{0\} to match \{1\} when the first joint veritable is zero

\[
0 P_{4_{org}} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
0 P_{4_{org}} = \begin{bmatrix}
c g_1 - s g_2 \\
s g_1 + c g_2 \\
g_3 \\
1
\end{bmatrix}
\]

\[
\begin{bmatrix}
c \theta & -s \theta & 0 & a_0 \\
s \theta c & c \theta c & -s \alpha & -s \alpha d_1 \\s \theta s & c \theta s & c \alpha & c \alpha d_1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
g_1(\theta_2) \\
g_2(\theta_2) \\
g_3(\theta_2)
\end{bmatrix}
\]
Pieper’s Solution - Three consecutive Axes Intersect

• Through algebraic manipulation of these equations, we can solve for the desired joint angles ( $\theta_1, \theta_2, \theta_3$ ).

• The first step is to square the magnitude of the distance from the frame {0} origin to frame {4} origin.

\[
r^2 = \left(0 P_{4orgx}\right)^2 + \left(0 P_{4orgy}\right)^2 + \left(0 P_{4orgz}\right)^2 = g_1^2 + g_2^2 + g_3^2
\]

• Using the previously define function for $g_i$ we have

\[
r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + a_1\left(c_2 f_1 - s_2 f_2\right)
\]
Applying a substitution of temporary variables, we can write the magnitude squared term along with the z-component of the \{0\} frame origin to the \{4\} frame origin distance.

\[ r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + a_1(c_2f_1 - s_2f_2) \]

\[ Z = 0P_{4orgz} = g_3 = s_2s\alpha_1f_1 + c_2s\alpha_1f_2 + c\alpha_1f_3 + d_2c\alpha_1 \]

- These equations are useful because dependence on \( \theta_1 \) has been eliminated, and dependence on \( \theta_2 \) takes a simple form.
Pieper’s Solution - Three consecutive Axes Intersect

- Consider 3 cases while solving for $\theta_3$:

- **Case 1** - $a_1 = 0$

  \[ r^2 = (k_1c_2 + k_2s_2)2a_1 + k_3 \]

  \[ Z = (k_1s_2 - k_2c_2)s\alpha_1 + k_4 \]

  \[ r^2 = k_3 \]

  \[ k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 \]

  \[ f_1(\theta_3) = a_3c_3 + d_4s\alpha_3s_3 + a_2 \]

  \[ f_2(\theta_3) = a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2 \]

  \[ f_3(\theta_3) = a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2 \]

- Solution Methodology - Reduction to Polynomial $\Rightarrow$ Quadratic Equation

  \[ u = \tan\left(\frac{\theta}{2}\right) \]

  \[ \cos \theta = \frac{1-u^2}{1+u^2} \]

  \[ \sin \theta = \frac{2u}{1+u^2} \]
Pieper’s Solution - Three consecutive Axes Intersect

- **Case 2** - \( s\alpha_1 = 0 \)

\[
r^2 = (k_1 c_2 + k_2 s_2)2a_1 + k_3 \quad Z = k_4
\]
\[
Z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 \quad k_4 = f_3 c\alpha_1 + d_2 c\alpha_1
\]
\[
f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2
\]

- **Solution Methodology** - Reduction to Polynomial \( \Rightarrow \) Quadratic Equation

\[
u = \tan \frac{\theta}{2} \quad \cos \theta = \frac{1-u^2}{1+u^2} \quad \sin \theta = \frac{2u}{1+u^2}
\]
Pieper’s Solution - Three consecutive Axes Intersect

- **Case 3 (General case)**: We can find $\theta_3$ through the following algebraic manipulation:

$$r^2 = (k_1 c_2 + k_2 s_2)2a_1 + k_3$$

$$Z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4$$

$$\frac{r^2 - k_3}{2a_1} = (k_1 c_2 + k_2 s_2)$$

$$\frac{Z - k_4}{s\alpha_1} = (k_1 s_2 - k_2 c_2)$$

- squaring both sides, we find

$$\left(\frac{r^2 - k_3}{2a_1}\right)^2 = (k_1 c_2 + k_2 s_2)^2 = k_1^2 c_2^2 + k_2^2 s_2^2 + 2k_1 k_2 c_2 s_2$$

$$\left(\frac{Z - k_4}{s\alpha_1}\right)^2 = (k_1 s_2 - k_2 c_2)^2 = k_1^2 s_2^2 + k_2^2 c_2^2 - 2k_1 k_2 c_2 s_2$$
Pieper’s Solution - Three consecutive Axes Intersect

- Adding these two equations together and simplifying using the trigonometry identity (Reduction to Polynomial), we find a fourth order equation for $\theta_3$

$$\left( \frac{r^2 - k_3}{2a_1} \right)^2 + \left( \frac{Z - k_4}{s\alpha_1} \right)^2 = k_1^2 + k_2^2$$

\begin{align*}
k_1 &= f_1 \\
k_2 &= -f_2 \\
k_3 &= f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 \\
k_4 &= f_3c\alpha_1 + d_2c\alpha_1 \\
f_1(\theta_3) &= a_3c_3 + d_4s\alpha_3s_3 + a_2 \\
f_2(\theta_3) &= a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2 \\
f_3(\theta_3) &= a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2
\end{align*}
Pieper’s Solution - Three consecutive Axes Intersect

- With $\theta_3$ solved, substitute into $r^2, Z$ to find $\theta_2$

\[
r^2 = (k_1c_2 + k_2s_2)2a_1 + k_3
\]
\[
Z = (k_1s_2 - k_2c_2)s\alpha_1 + k_4
\]

\[
k_1 = f_1
\]
\[
k_2 = -f_2
\]
\[
k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3
\]
\[
k_4 = f_3 c\alpha_1 + d_2 c\alpha_1
\]
\[
f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2
\]
\[
f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2
\]
\[
f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2
\]
Pieper’s Solution - Three consecutive Axes Intersect

- With $\theta_2, \theta_3$ solved, substitute into $^0P_{4org}$ to find

$$^0P_{4org} = \begin{bmatrix} c_1g_1 - s_1g_2 \\ s_1g_1 + c_1g_2 \\ g_3 \\ 1 \end{bmatrix}$$

$$^0P_{4orgx} = c_1g_1 - s_1g_2$$

$$^0P_{4orgy} = s_1g_1 + c_1g_2$$

$$g_1(\theta_2) = c_2f_1 + s_2f_2 + a_1$$

$$g_2(\theta_2) = s_2\alpha_1f_1 + c_2\alpha_1f_2 - s\alpha_1f_3 - d_2s\alpha_1$$

$$g_3(\theta_2) = s_2\alpha_1f_1 + c_2\alpha_1f_2 + c\alpha_1f_3 + d_2c\alpha_1$$

$$f_1(\theta_3) = a_3c_3 + d_4s\alpha_3s_3 + a_2$$

$$f_2(\theta_3) = a_3c\alpha_2s_3 - d_4s\alpha_3c\alpha_2c_3 - d_4s\alpha_2c\alpha_3 - d_3s\alpha_2$$

$$f_3(\theta_3) = a_3s\alpha_2s_3 - d_4s\alpha_3s\alpha_2c_3 + d_4c\alpha_2c\alpha_3 + d_3c\alpha_2$$

- Solve for $\theta_1$ using the reduction to polynomial method
Pieper’s Solution - Three consecutive Axes Intersect

- To complete our solution we need to solve for $\theta_4, \theta_5, \theta_6$

- Since the last three axes intersect these joint angle affect the orientation of only the last link. We can compute them based only upon the rotation portion of the specified goal $^0_6 R$

$$^0_6 R = ^4_6 R \bigg|_{\theta_4 = 0} ^4 R$$

$$^4_6 R \bigg|_{\theta_4 = 0} = ^0_4 R^{-1} \bigg|_{\theta_4 = 0} ^0_6 R$$

- The orientation of link frame $\{4\}$ relative to the base frame $\{0\}$ when $\theta_4 = 0$

- $\theta_4, \theta_5, \theta_6$ are the Euler angles applied to $^0_6 R \bigg|_{\theta_4 = 0}$
Central Topic - Inverse Manipulator Kinematics - Examples

- **Algebraic Solution** (closed form) - Piepers Method (Continue) - Last three consecutive axes intersect at one point

- Consider a 3 DOF (wrist) non-planar robot whose axes all intersect at a point.
Mapping - Rotated Frames - Z-Y-Z Euler Angles

Start with frame \{4\}.

- Rotate frame \{4\} about \(\hat{Z}_4\) by an angle \(\alpha\)
- Rotate frame \{4\} about \(\hat{Y}_4\) by an angle \(\beta\)
- Rotate frame \{4\} about \(\hat{Z}_4\) by an angle \(\gamma\)

**Note** - Each rotation is performed about an axis of the moving reference frame \{4\}, rather than a fixed reference.
Mapping - Rotated Frames - Z-Y-Z Euler Angles
Mapping - Rotated Frames - Z-Y-Z Euler Angles

\[ R_{Z'Y'Z'}(\alpha, \beta, \gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ R_{Z'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} \]

\[ ^A_B R_{X'Y'Z'}(\alpha, \beta, \gamma) = ^4_6 R_{\theta=0} \]
Three consecutive Axes Intersect - wrist

\[
\begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix} =
\begin{bmatrix}
    c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\
    s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\
    -s\beta c\gamma & s\beta s\gamma & c\beta
\end{bmatrix}
\]

Goal

Direct Kinematics
Three consecutive Axes Intersect - wrist

- Solve for $\beta$ using element $r_{31}, r_{32}, r_{33}$

\[
\begin{align*}
    r_{31} &= -s\beta c\alpha \\
    r_{32} &= s\beta s\alpha \\
    r_{33} &= c\beta
\end{align*}
\]

\[
\begin{align*}
    r_{31}^2 + r_{32}^2 &= s\beta^2(c\alpha^2 + s\alpha^2) \\
    r_{33} &= c\beta
\end{align*}
\]

\[
s\beta = \pm \sqrt{r_{31}^2 + r_{32}^2}
\]

- Using the Atan2 function, we find

\[
\beta = \text{Atan2}\left(\pm \sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)
\]
Three consecutive Axes Intersect - wrist

- Solve for $\alpha$ using elements $r_{23}, r_{13}$

\[
\begin{align*}
r_{13} &= c\alpha s\beta \\
r_{23} &= s\alpha s\beta
\end{align*}
\]

\[
\alpha = \text{Atan2}(r_{23} / s\beta, r_{13} / s\beta)
\]
Three consecutive Axes Intersect - wrist

- Solve for $\gamma$ using elements $r_{32}, r_{31}$

\[
r_{32} = s\beta s\gamma \\
r_{31} = -s\beta c\gamma
\]

\[
\gamma = \text{Atan2}(r_{32} / s\beta, -r_{31} / s\beta)
\]
Three consecutive Axes Intersect - wrist

- Note: Two answers exist for angle $\beta$ which will result in two answers each for angles $\alpha$ and $\gamma$.

\[
\beta = \text{Atan2}
\left(\pm \sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)
\]

\[
\alpha = \text{Atan2}(r_{33} / s\beta, r_{13} / s\beta)
\]

\[
\gamma = \text{Atan2}(r_{32} / s\beta, -r_{31} / s\beta)
\]

- If $\beta = 0^\circ, \beta = 180^\circ \Rightarrow s\beta = 0$ the solution degenerates.
Three consecutive Axes Intersect - wrist

\[
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
= \begin{bmatrix}
  c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\
  s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\
  -s\beta c\gamma & s\beta s\gamma & c\beta
\end{bmatrix}
\]

\[\beta = 0^\circ\]

\[
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
= \begin{bmatrix}
  c\alpha c\gamma - s\alpha s\gamma & -c\alpha s\gamma - s\alpha c\gamma & 0 \\
  s\alpha c\gamma + c\alpha s\gamma & -s\alpha s\gamma + c\alpha c\gamma & 0 \\
  0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
  c(\alpha + \gamma) & -s(\alpha + \gamma) & 0 \\
  s(\alpha + \gamma) & c(\alpha + \gamma) & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

- We are left with \((\gamma + \alpha)\) for every case. This means we can’t solve for either, just their sum.
Three consecutive Axes Intersect - wrist

- One possible convention is to choose $\alpha = 0^\circ$
- The solution can be calculated to be

$$\beta = 0$$
$$\alpha = 0$$
$$\gamma = \text{Atan2}(-r_{12}, r_{11}) = \text{Atan2}(s\gamma, c\gamma)$$

$$\beta = 180$$
$$\alpha = 0$$
$$\gamma = \text{Atan2}(r_{12}, -r_{11}) = \text{Atan2}(s\gamma, c\gamma)$$
For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector’s orientation never changes.
Gimbal Lock

Normal situation
The three gimbals are independent

Gimbal lock:
Two out of the three gimbals are in the same plane, one degree of freedom is lost

http://youtu.be/zc8b2Jo7mno
Gimbal Lock – Robotics

• In robotics, gimbal lock is commonly referred to as "wrist flip", due to the use of a "triple-roll wrist" in robotic arms, where three axes of the wrist, controlling yaw, pitch, and roll, all pass through a common point.

• An example of a wrist flip, also called a wrist singularity, is when the path through which the robot is traveling causes the first and third axes of the robot's wrist to line up. The second wrist axis then attempts to spin 180° in zero time to maintain the orientation of the end effector. The result of a singularity can be quite dramatic and can have adverse effects on the robot arm, the end effector, and the process.

• The importance of non-singularities in robotics has led the American National Standard for Industrial Robots and Robot Systems — Safety Requirements to define it as "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities".

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