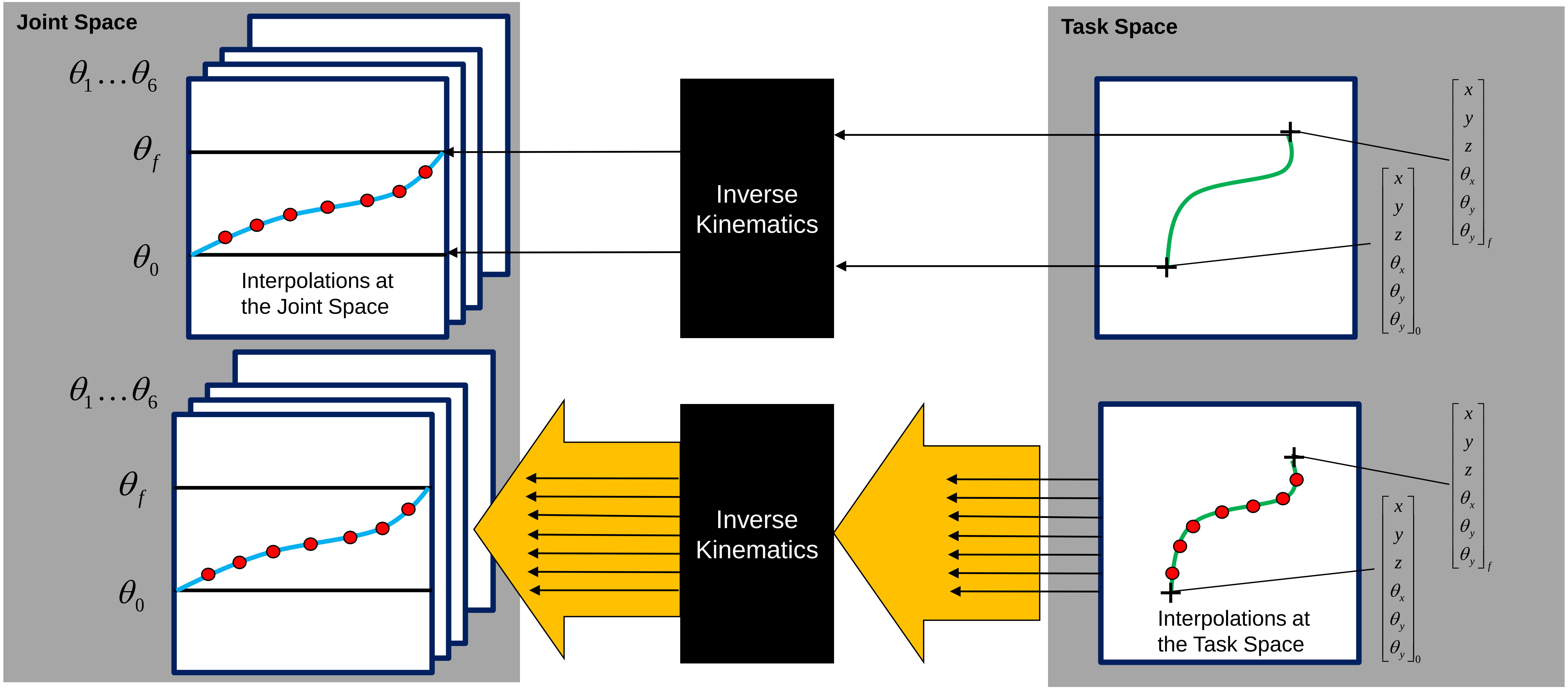
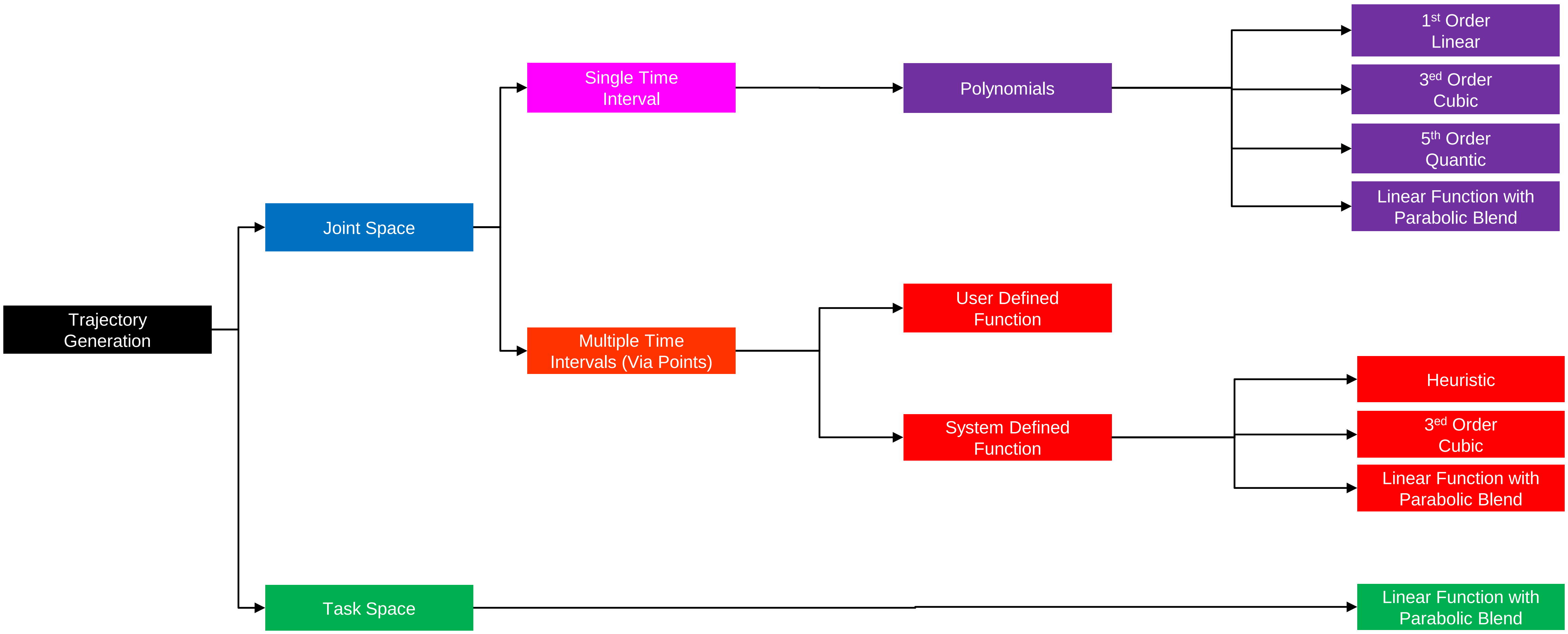
Trajectory Generation (2/2)

Task Space Versus Joint Space - Interpolations



Trajectory Generation – Roadmap Diagram

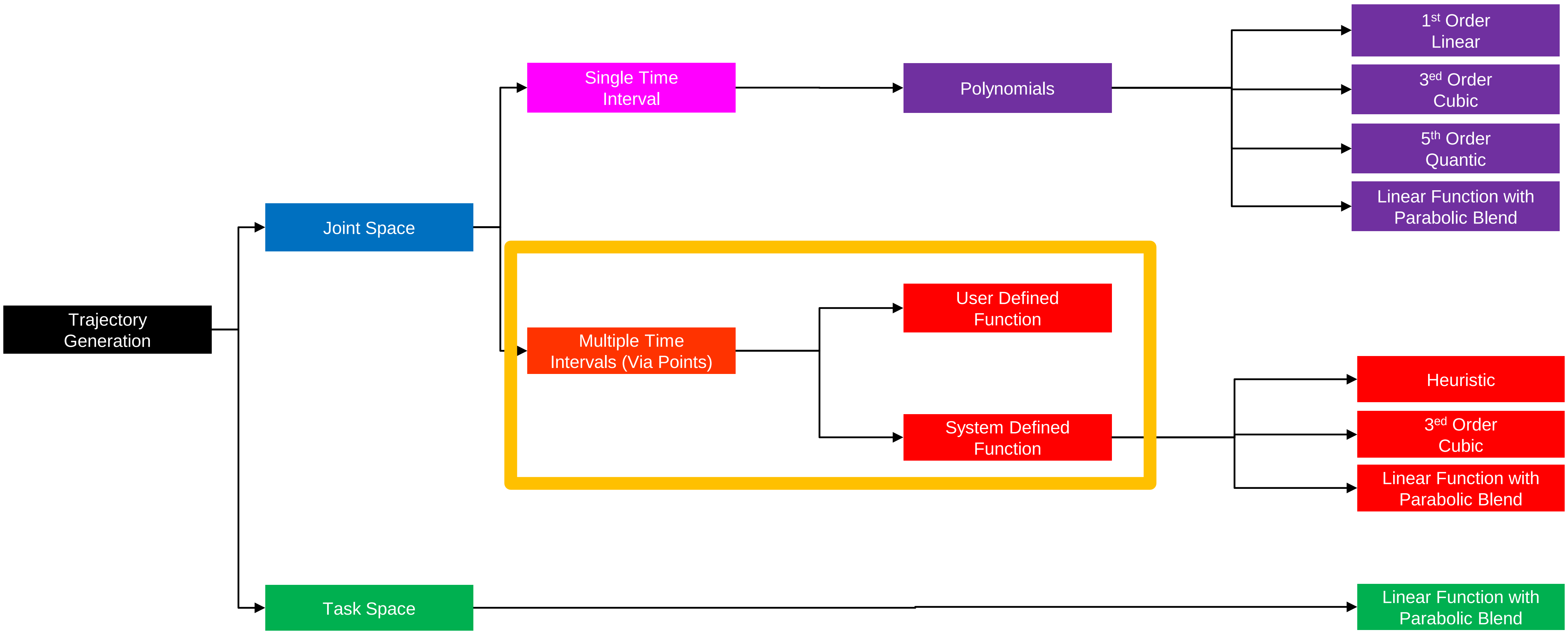


Joint Space Schemes

Multiple Time Intervals

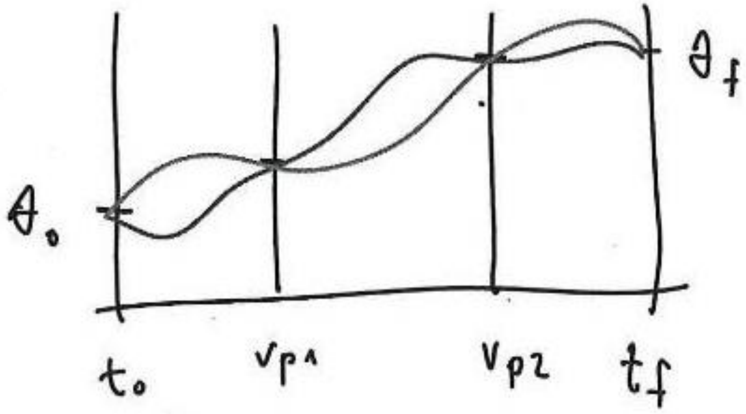
Via Point

Trajectory Generation – Roadmap Diagram



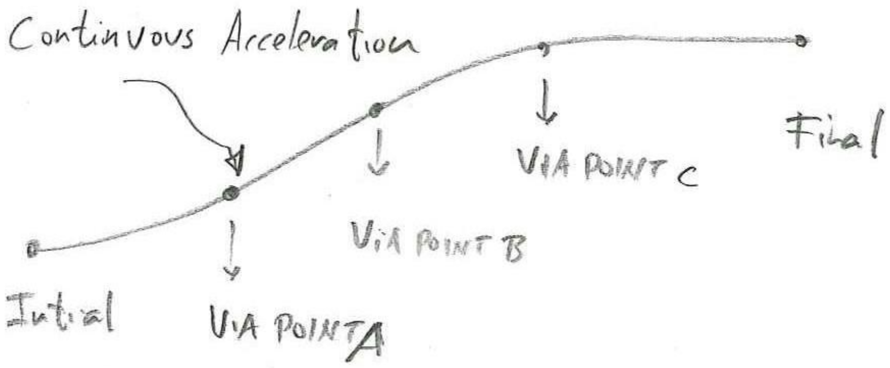
Joint Space Schemes – Multiple Time Intervals – Via Points

- Define multiple functions for each joint such that
  - The value at t_0 is the **initial position** of the joint and whose value at t_f is the **desire goal position** of the joint $\theta(t)$ at the end of the time interval
  - In between the beginning/ending points the user define via points that each joint must pass trough.
  - Note that all the joints reach the via point at the same time to guarantee a specific position and orientation of the end effector
- There are many smooth functions that may be used to interpolate the joint value.



Joint Space Schemes – Multiple Time Intervals – Via Points – Velocity Definition

- Specify the desired velocities at each segment:
  1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.
  2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) automatically using a suitable heuristic method.
  3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.



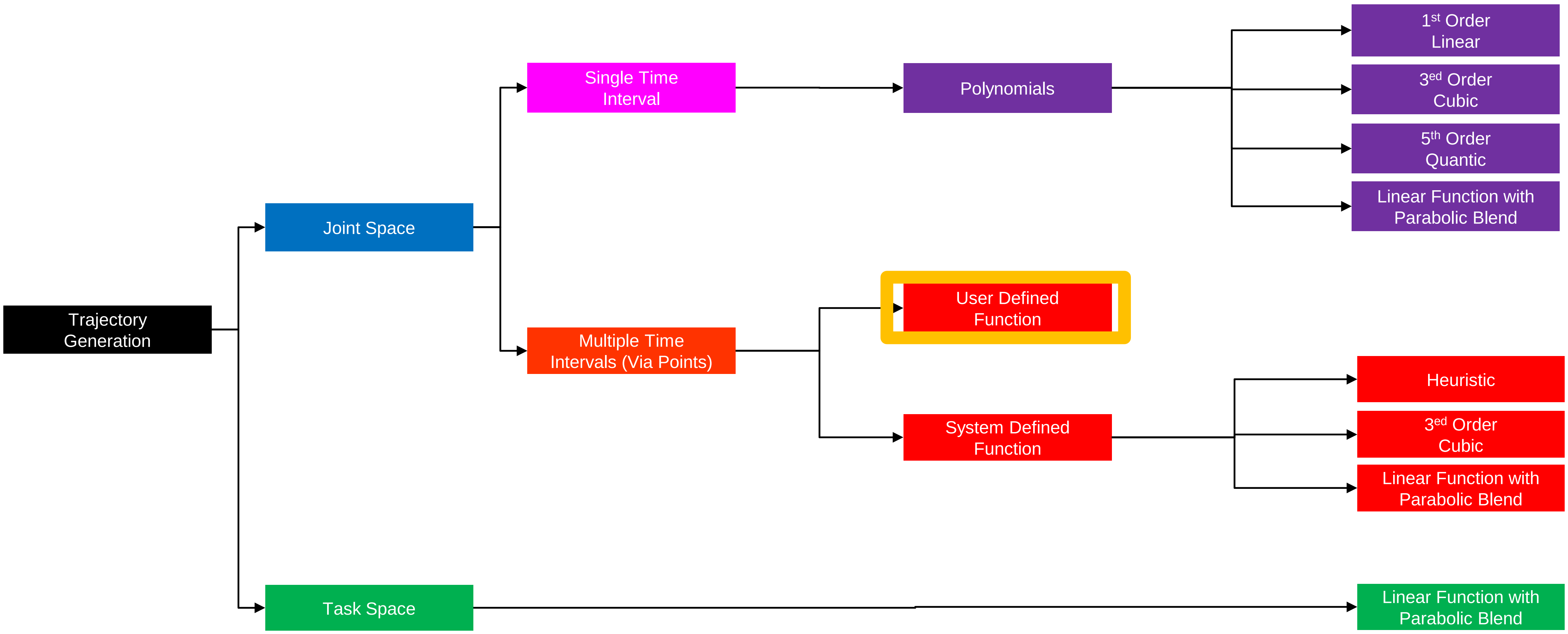
Joint Space Schemes

Multiple Time Intervals

Via Point

User Defined Function

Trajectory Generation – Roadmap Diagram



Joint Space Schemes – Multiple Time Intervals – Via Points – Velocity Definition

1. **User Definition** – Desired Cartesian linear and angular velocity of the tool frame at each via point.

– **Mapping** (Cartesian Space to Joint Space) - Cartesian velocities at the via point are "mapped" to desired joint rates by using the inverse Jacobian

$$\dot{\theta}_f = J^{-1}\dot{X}_f$$

– **Singularity** - If the manipulator is at a singular point at a particular via point then the user is not free to choose an arbitrarily velocity at this point.

– **Difficult**

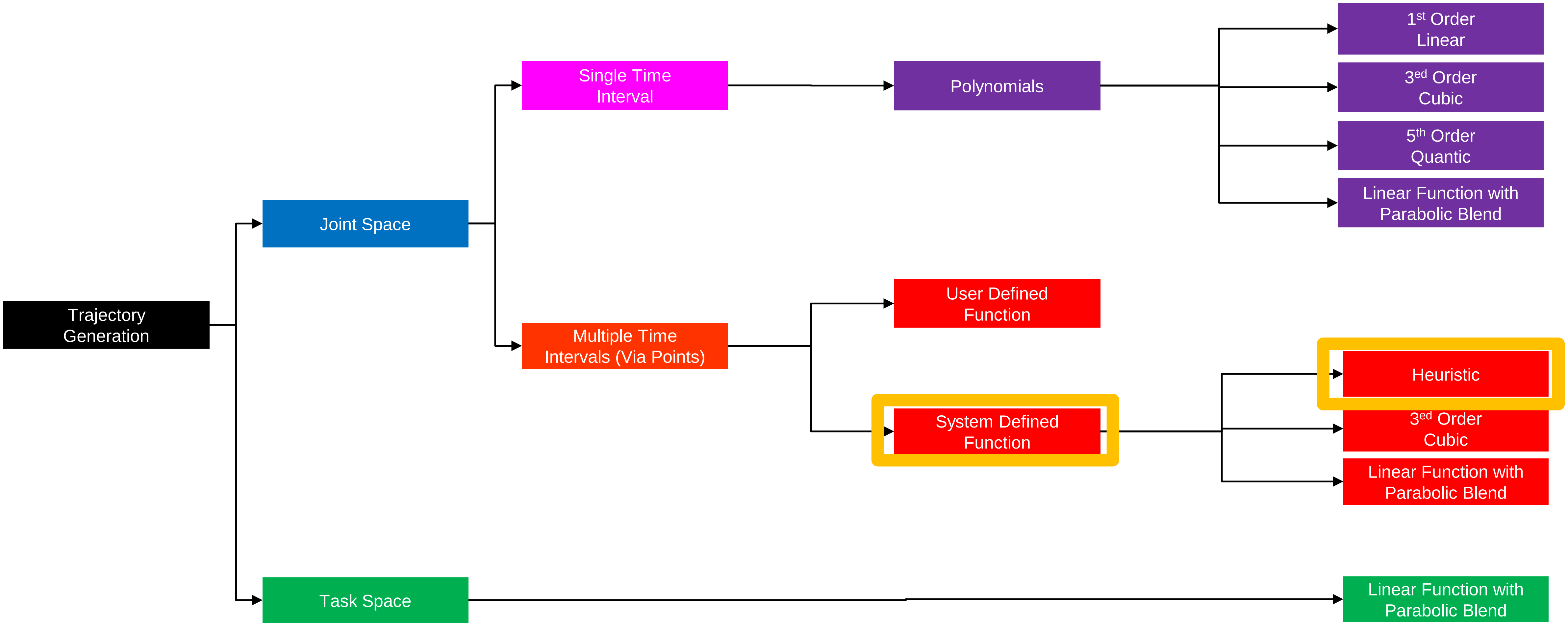
Joint Space Schemes

Multiple Time Intervals

Via Point

System Defined Function – Heuristic

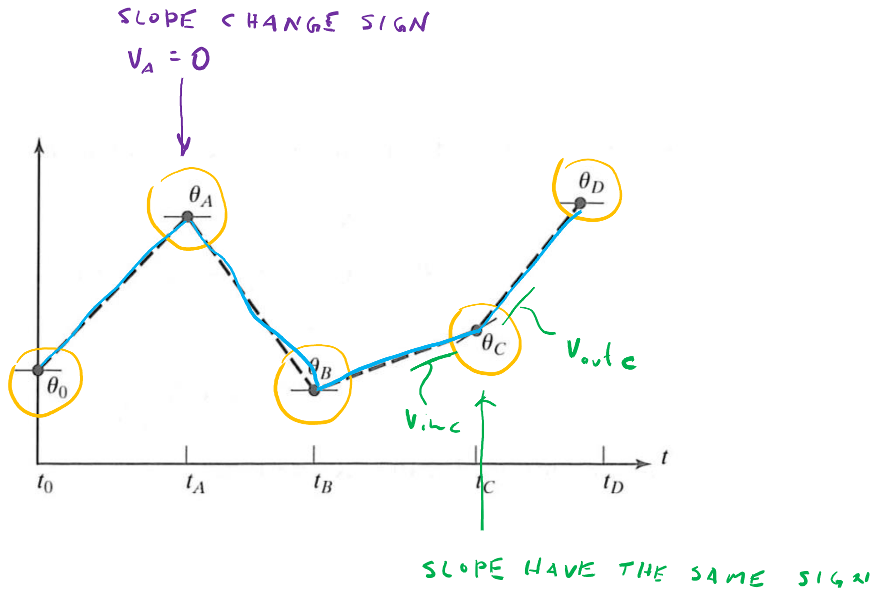
Trajectory Generation – Roadmap Diagram



2. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) using a suitable heuristic method given a trajectory .

Heuristic method

- – Consider a path defined by via points
- – Connect the via points with straight lines
- – If the slope change sign
  - – Set the velocity at the via point to be zero
- – If the slope have the same sign
  - – Calculate the average between the to velocities at the via point.



$$V_c = \frac{V_{in\,c} + V_{out\,c}}{2}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Velocity Definition

3. **System Definition** – The system automatically chooses the velocities (Cartesian or angular) to cause the acceleration at the via point to be continuous.

– **Spline** - Enforcing the velocity and the acceleration to be continuous at the via point

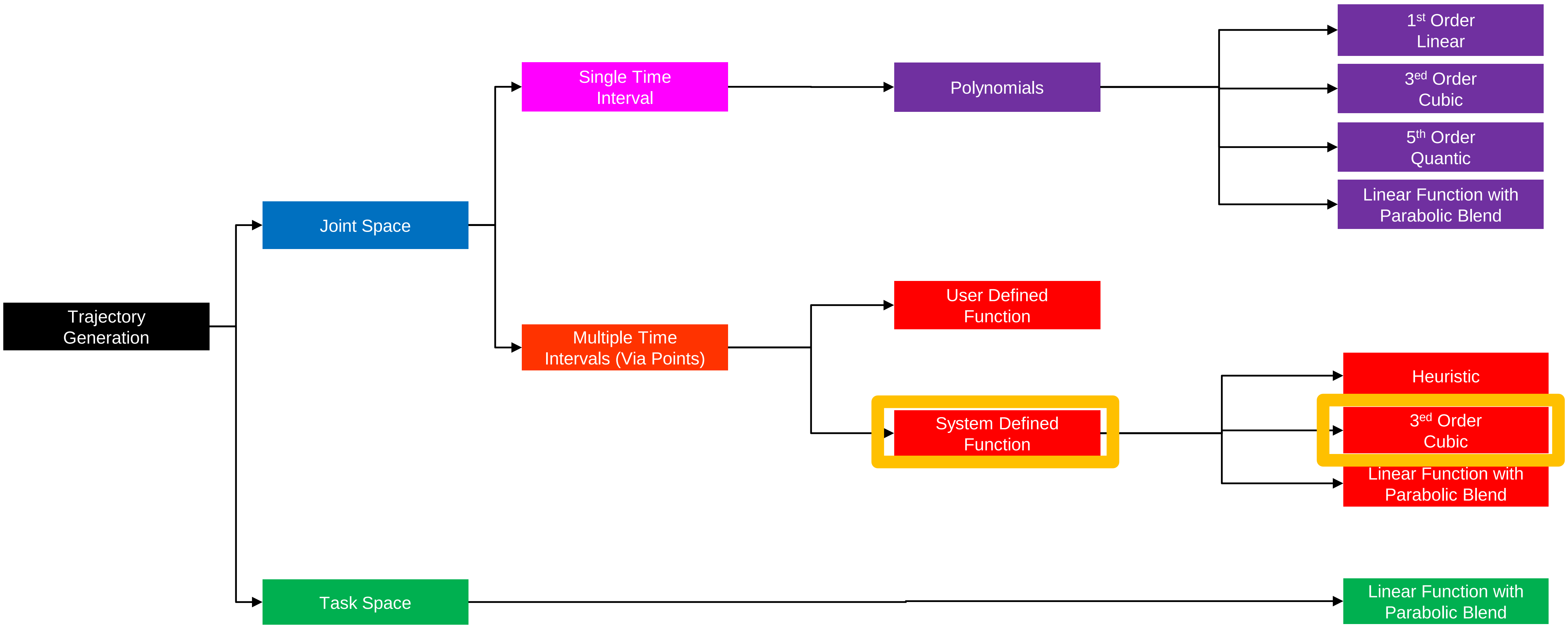
Joint Space Schemes

Multiple Time Intervals

Via Point

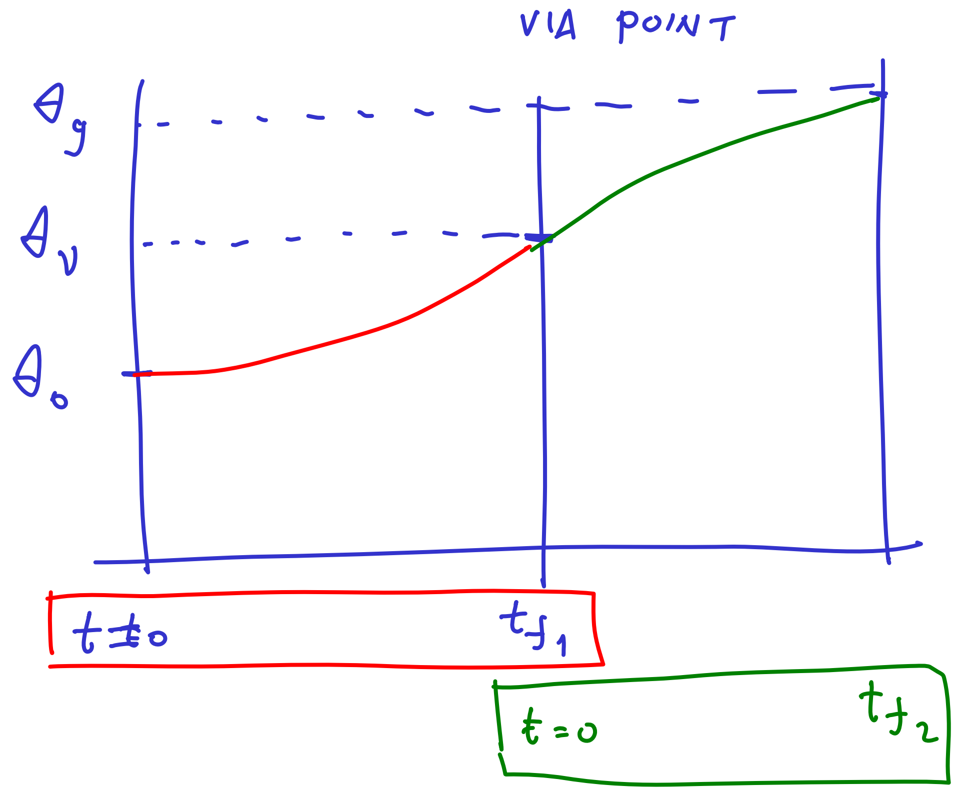
System Defined Function – Cubic Polynomials

Trajectory Generation – Roadmap Diagram



Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- Solve for the coefficients of two cubic functions that are connected in a two segment spline with a continuous acceleration at the intermediate via point.



$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$

$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t + 3a_{13}t^2$$

$$\dot{\theta}(t) = a_{21} + 2a_{22}t + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$

$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- The joint angle velocity and acceleration for each segment (8 unknowns)

$$\theta(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$

$$\theta(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$

$$\dot{\theta}(t) = a_{11} + 2a_{12}t \ \ + 3a_{13}t^2$$

$$\dot{\theta}(t) = a_{21} + 2a_{22}t \ \ + 3a_{23}t^2$$

$$\ddot{\theta}(t) = 2a_{12} + 6a_{13}t$$

$$\ddot{\theta}(t) = 2a_{22} + 6a_{23}t$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- Position at the beginning and end of each segment

- Segment 1

$$\theta_0(t=0) = a_{10}$$ eq 1

$$\theta_{via}(t=t_{f_1}) = a_{10} + a_{11}t_{f_1} + a_{12}t_{f_1}^{\ 2} + a_{13}t_{f_1}^{\ 3}$$ eq 2

- Segment 2

$$\theta_{via}(t=0) = a_{20}$$ eq 3

$$\theta_g(t=t_{f_2}) = a_{20} + a_{21}t_{f_2} + a_{22}t_{f_2}^{\ 2} + a_{23}t_{f_2}^{\ 3}$$ eq 4

Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- Velocity at the beginning of the interval

$$\dot{\theta}(t=0)=a_{11} \quad \text{eq 5}$$

- Velocity at the end of the interval

$$\dot{\theta}(t=t_{f_2})=a_{21}+2a_{22}t_{f_2}+3a_{23}t_{f_2}^{\ 2} \quad \text{eq 6}$$

- Velocity at the mid point between the intervals

$$\dot{\theta}\left[Function1(t=t_{f_1})\right]=\dot{\theta}\left[Function2(t=0)\right]$$

$$a_{11}+2a_{12}t_{f_1}+3a_{13}t_{f_1}^{\ 2}=a_{21} \quad \text{eq 7}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- Acceleration at the mid point between the intervals

$$\ddot{\theta}\left[Function1(t = t_{f_1})\right] = \ddot{\theta}\left[Function2(t = 0)\right]$$

$$2a_{12} + 6a_{13}t_{f_1} = 2a_{22} \qquad \text{eq. 8}$$

- Solve 8 equations with 8 unknown

Joint Space Schemes – Multiple Time Intervals – Via Points – Cubic Polynomials

- Solution for the 8 equations

$$a_{10} = \theta_0$$

$$a_{11} = 0$$

$$a_{12} = \frac{12\theta_v - 3\theta_g - 9\theta_0}{4t_f^2}$$

$$a_{13} = \frac{-8\theta_v + 3\theta_g + 5\theta_0}{4t_f^3}$$

$$a_{10} = \theta_v$$

$$a_{21} = \frac{3\theta_g - 3\theta_0}{4t_f}$$

$$a_{22} = \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2}$$

$$a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}$$

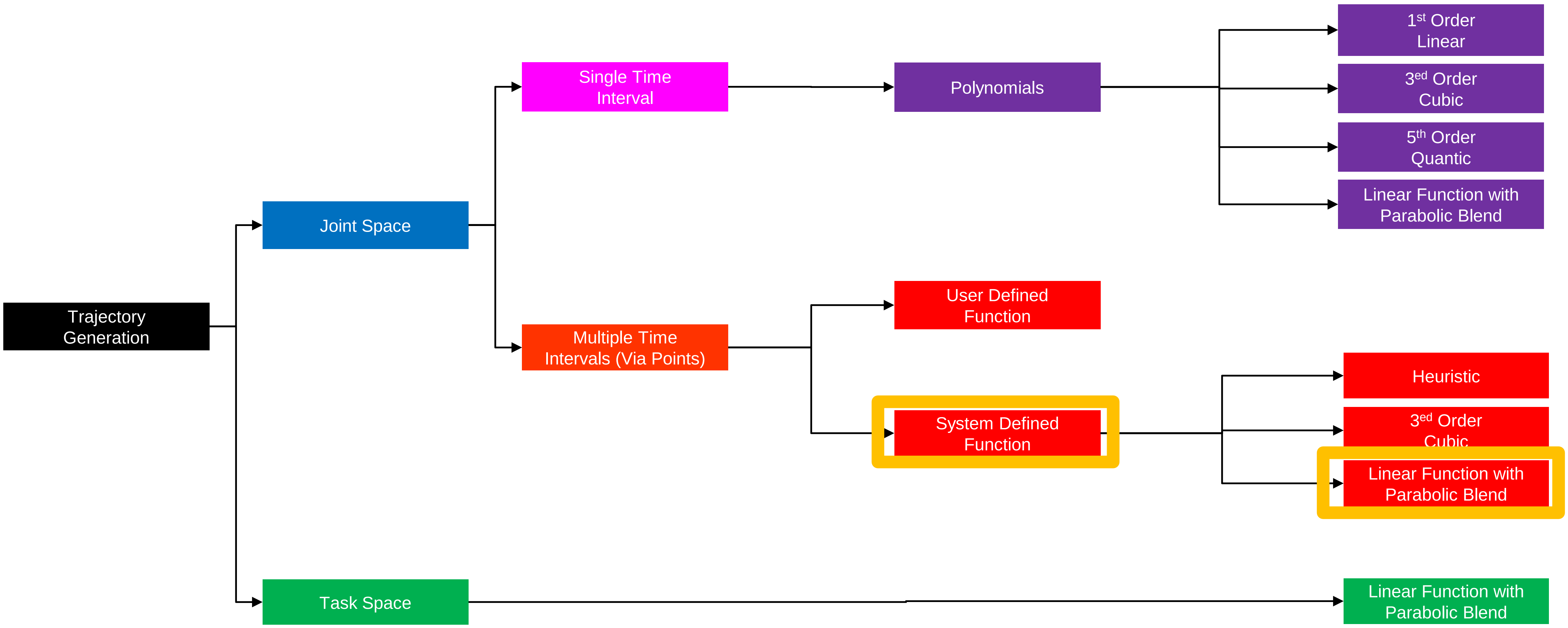
Joint Space Schemes

Multiple Time Intervals

Via Point

System Defined Function – Linear Function With Parabolic Blend

Trajectory Generation – Roadmap Diagram

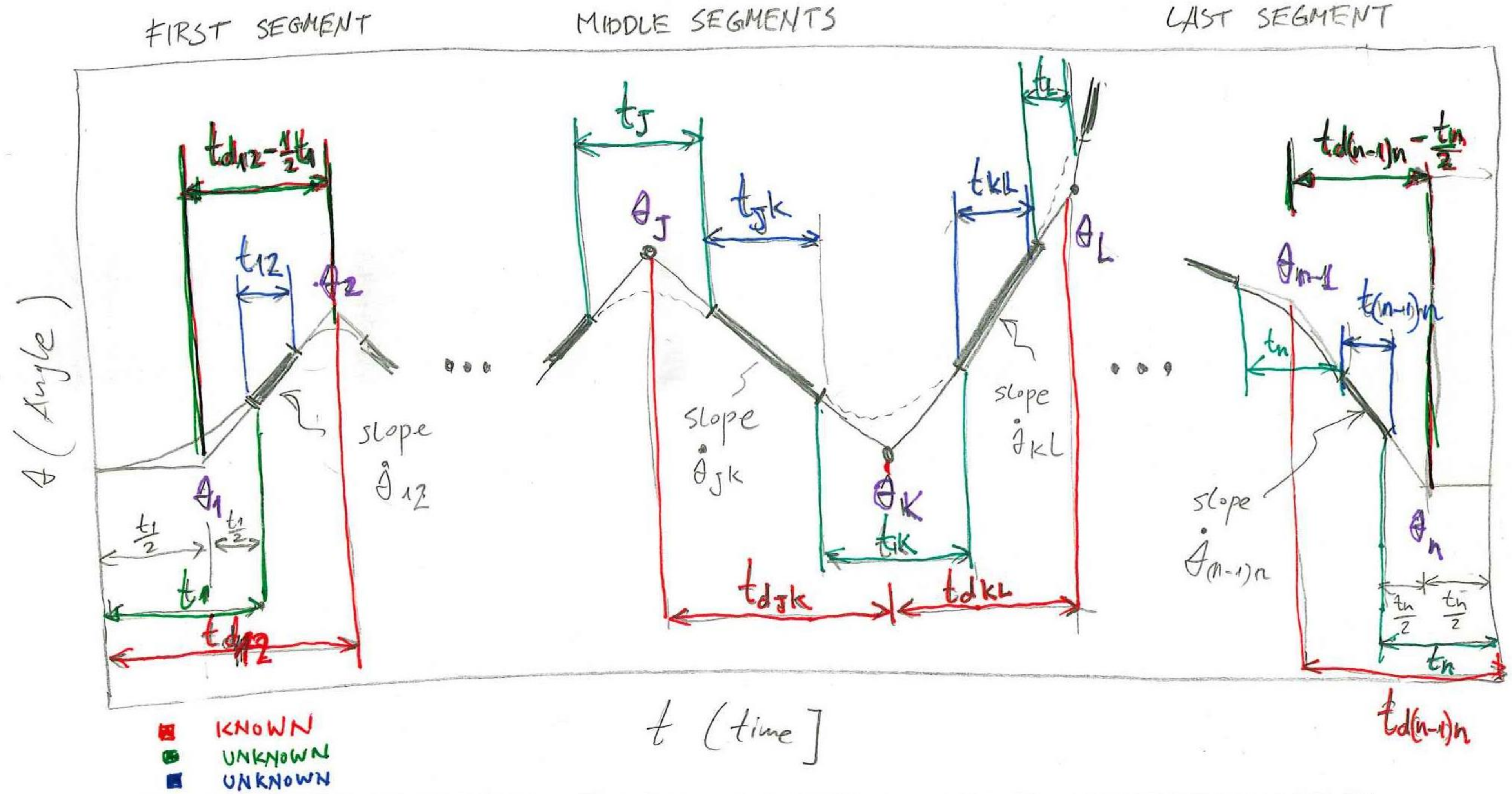


Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend

- The Need
  - Linear path with parabolic blends is used in cases where there are **arbitrary number of via points specified**

- Method Anatomy
  - **Linear Functions** – Connecting the via points
  - **Parabolic Blend** – Connecting the linear functions around the via points

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend



Joint Space Schemes

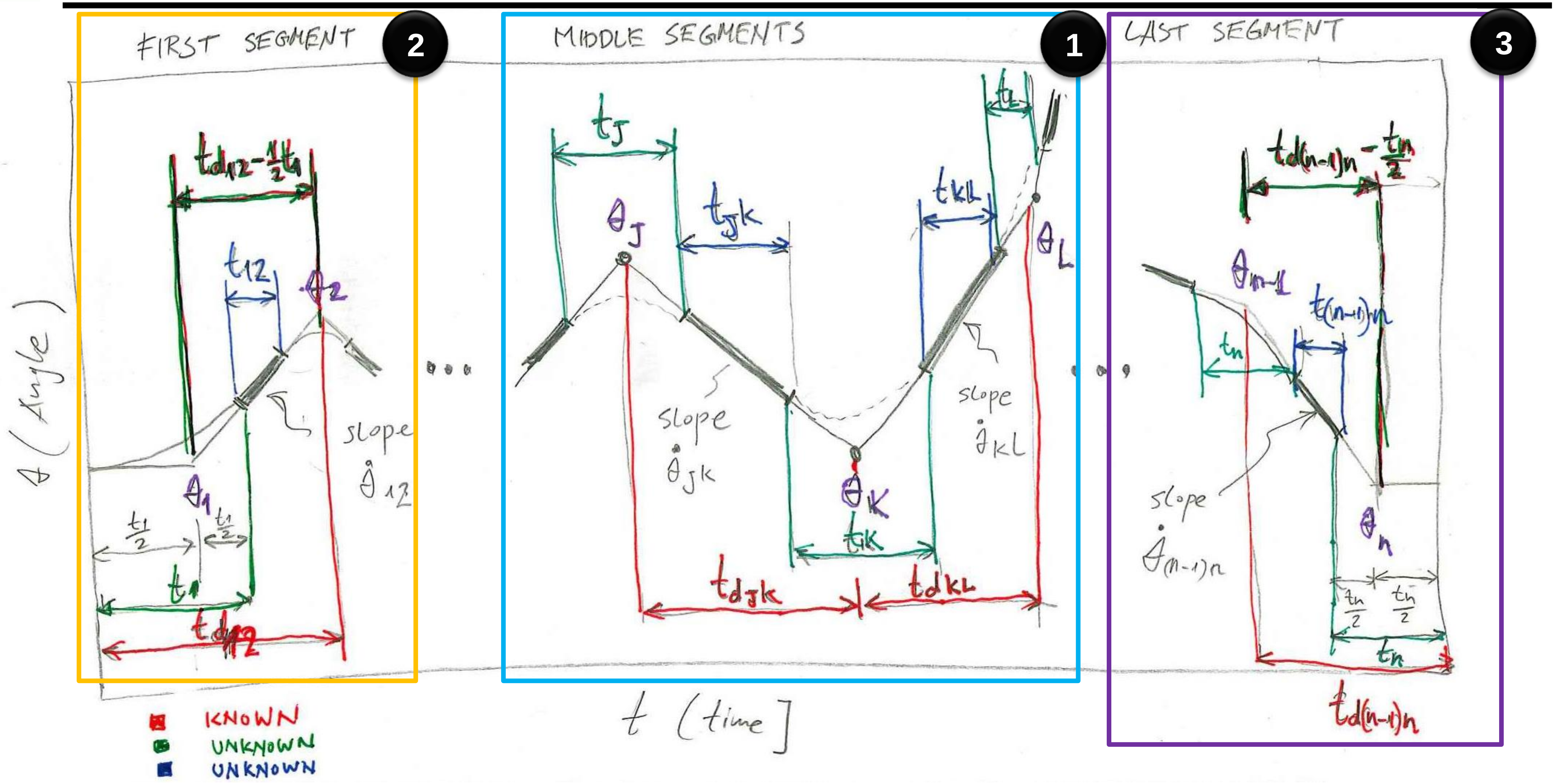
Multiple Time Intervals

Via Point

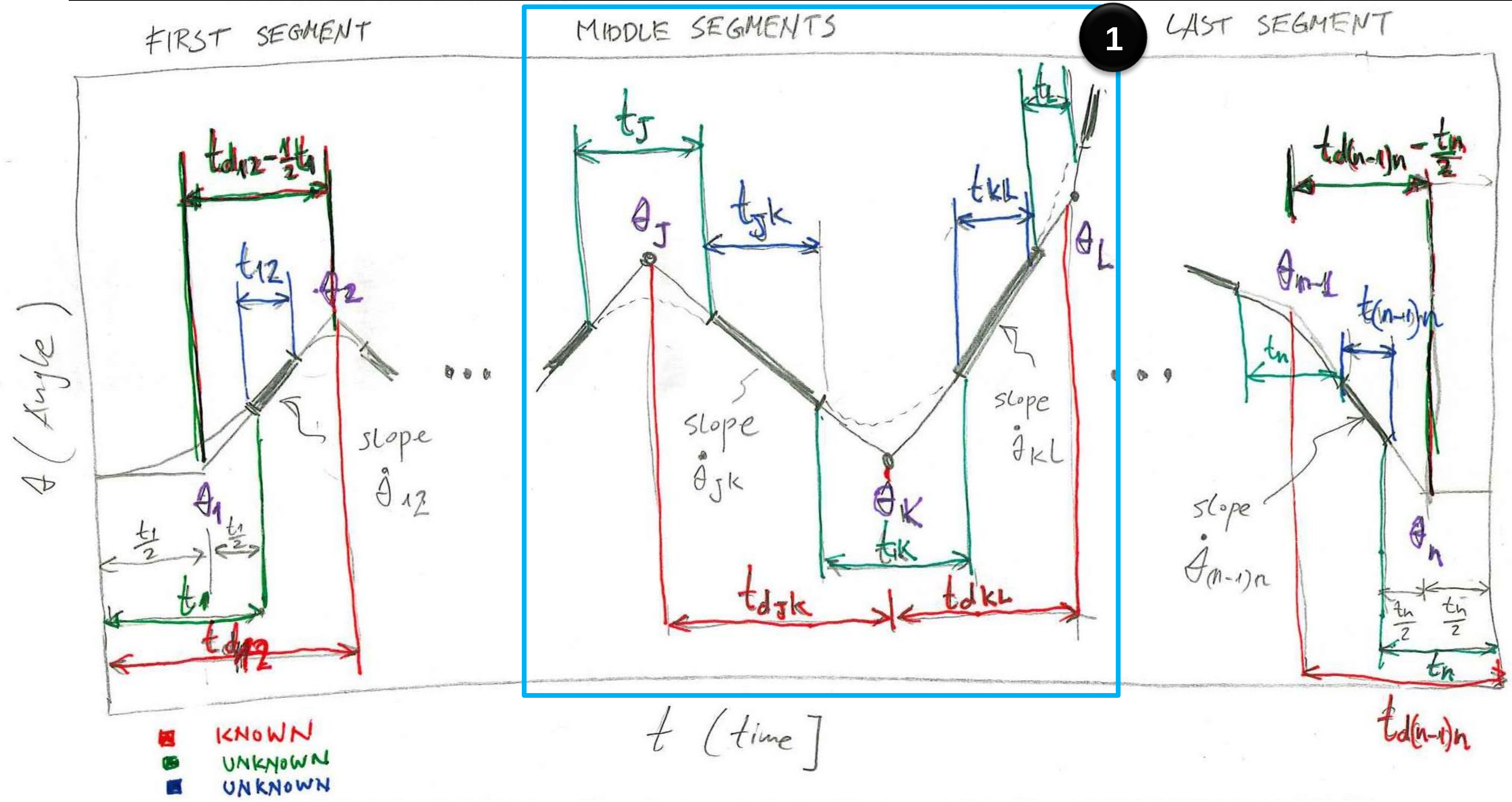
System Defined Function – Linear Function With Parabolic Blend

Time Interval Analysis

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

1

Mid Points (j, k, L)

Given: $\theta_J, \theta_k, \theta_L$

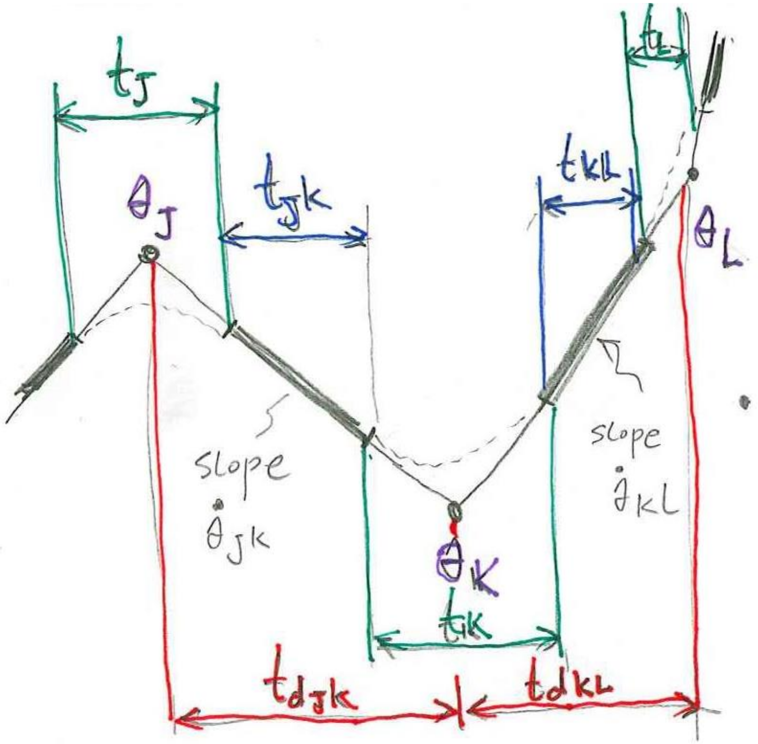
$t_{d_{jk}}, t_{d_{kL}}$

$|\ddot{\theta}_k|$ Desired acceleration at each path point

- Velocity

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_J}{t_{d_{jk}}}$$

$$\dot{\theta}_{kL} = \frac{\theta_L - \theta_k}{t_{d_{kL}}}$$

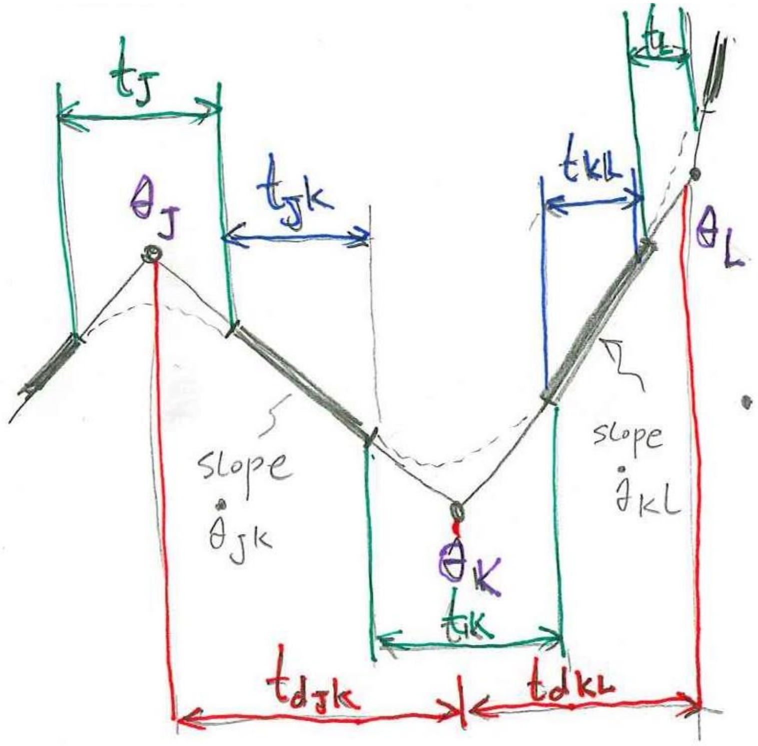


- Acceleration

$$\ddot{\theta}_k = \Big[\overbrace{\underbrace{SGN\left(\dot{\theta}_{kL} - \dot{\theta}_{jk}\right)}_{\ominus\ \text{Deceleration}}}^{\oplus\ \text{Acceleration}}\Big] \left|\ddot{\theta}_k\right|$$

$$t_k = \frac{\dot{\theta}_{kL} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

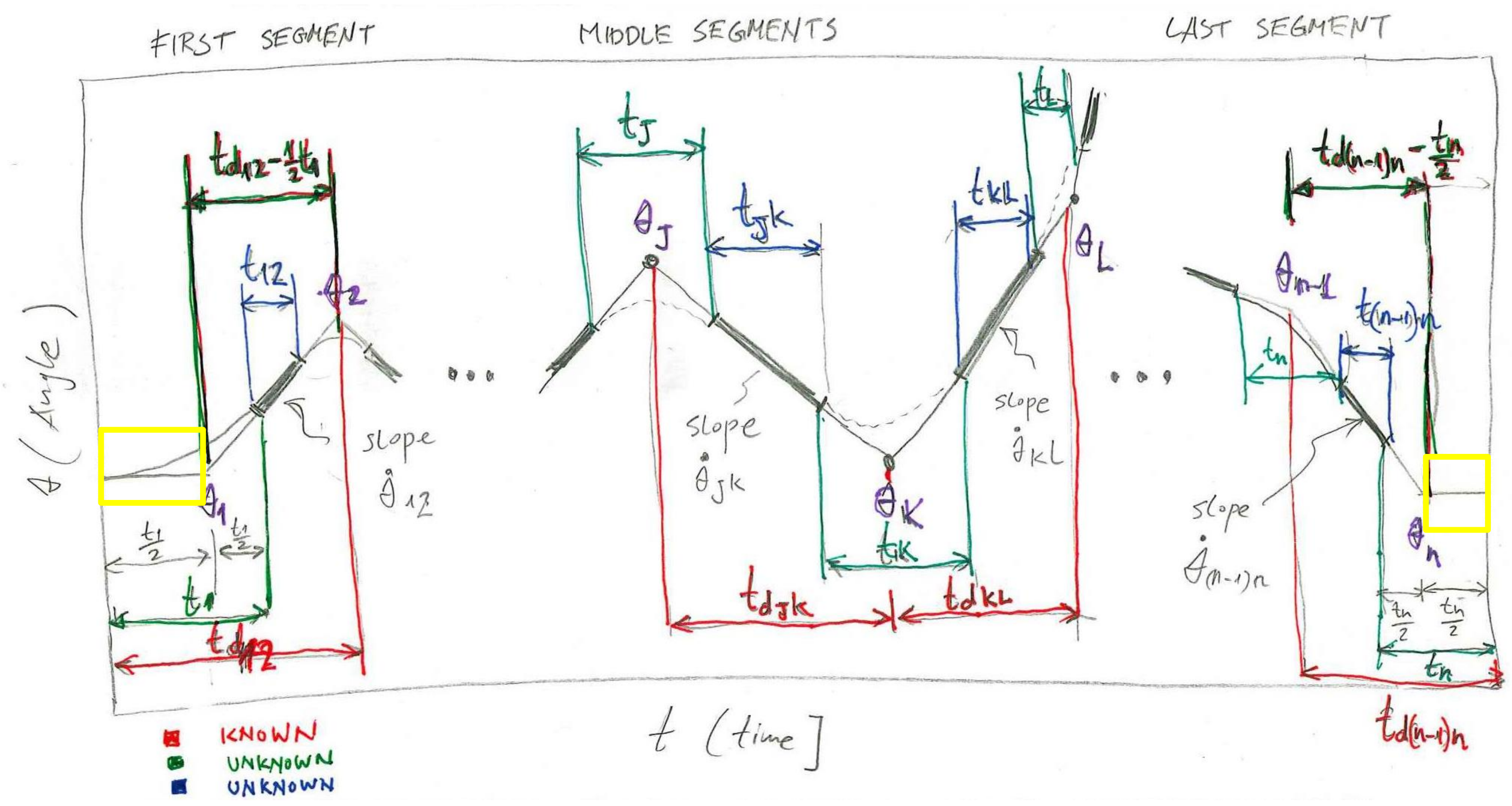
$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k$$



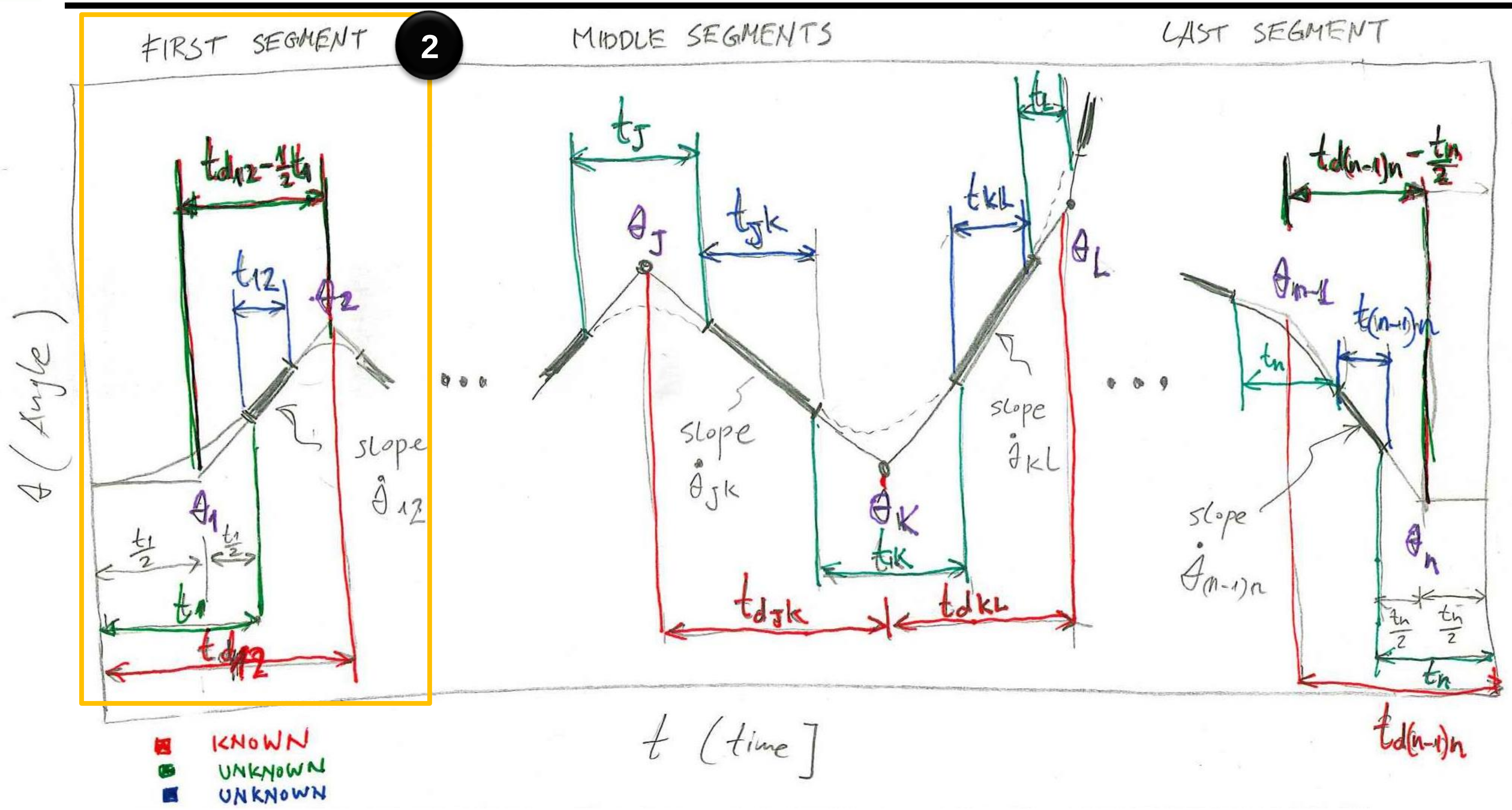
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

- **Note: First and Last Segments** – The first and the last segments must be handled slightly differently because the entire bland regeion at one end of the segment must be counted in the total segment's time duration.

- Note the difference between
  - Mid Points t_{djk} t_{djk}
  - First Point t_{d12}
  - Last Point $t_{d(n-1)n}$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

2

FIRST SEGMENT

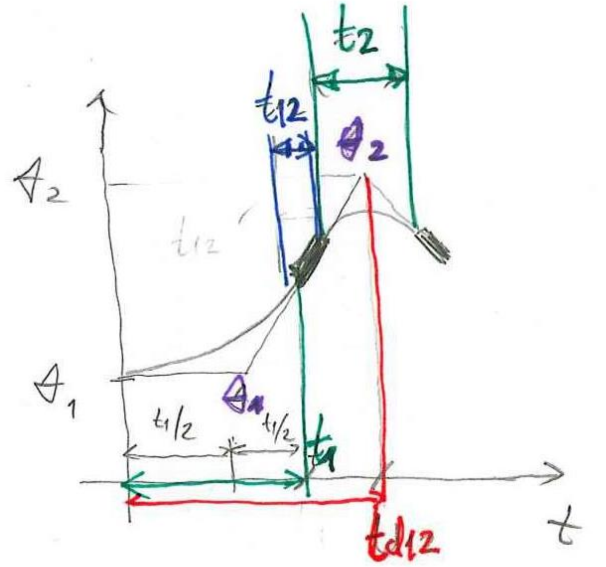
Velocity → $\dfrac{\theta_2-\theta_1}{t_{d12}-\frac{1}{2}t_1} = \underbrace{\ddot{\theta}_1 t_1}_{\to\, V = a \cdot t}$

Acceleration $\ddot{\theta}_1 = \underbrace{SGN}_{\text{SIGN}}(\theta_2-\theta_1)\,|\ddot{\theta}_1|$

$$\theta_2-\theta_1 = \ddot{\theta}\, t_1\left(t_{d12}-\frac{1}{2}t_1\right)$$

$$\left(+\ddot{\theta}_1 \frac{1}{2}\right)t_1^2 - \left(\ddot{\theta}_1 t_{d12}\right)t + (\theta_2-\theta_1) = 0$$

$$t_1 = \frac{\ddot{\theta}_1 t_{d12} \pm \sqrt{\ddot{\theta}_1^2 t_{d12}^2 - \overset{2}{\cancel{4}}\,\ddot{\theta}_1 \cancel{\tfrac{1}{2}}\,(\theta_2-\theta_1)}}{\ddot{\theta}_1}$$



$t_1 \to$ Blending time

$t_{12} \to$ Straight line time

UCLA

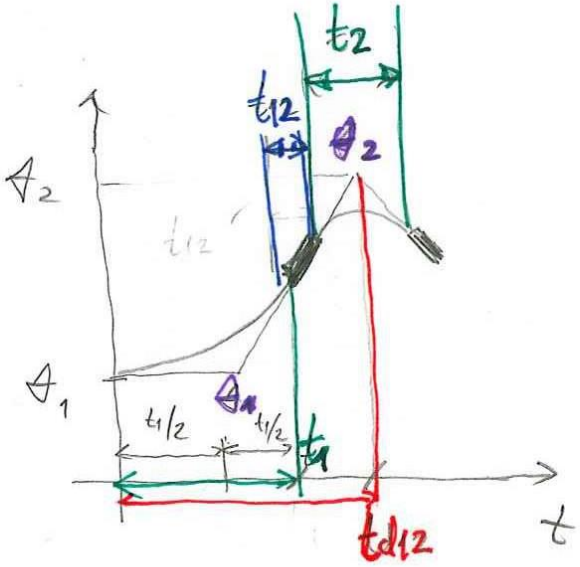
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

$$t_1 = t_{d12} \,\cancel{+}\, - \sqrt{t_{d12}^2 + \frac{2(\theta_1 - \theta_2)}{\ddot{\theta}_1}}$$

Taking only the (−) sign since $t_1 < t_{d12}$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1}$$

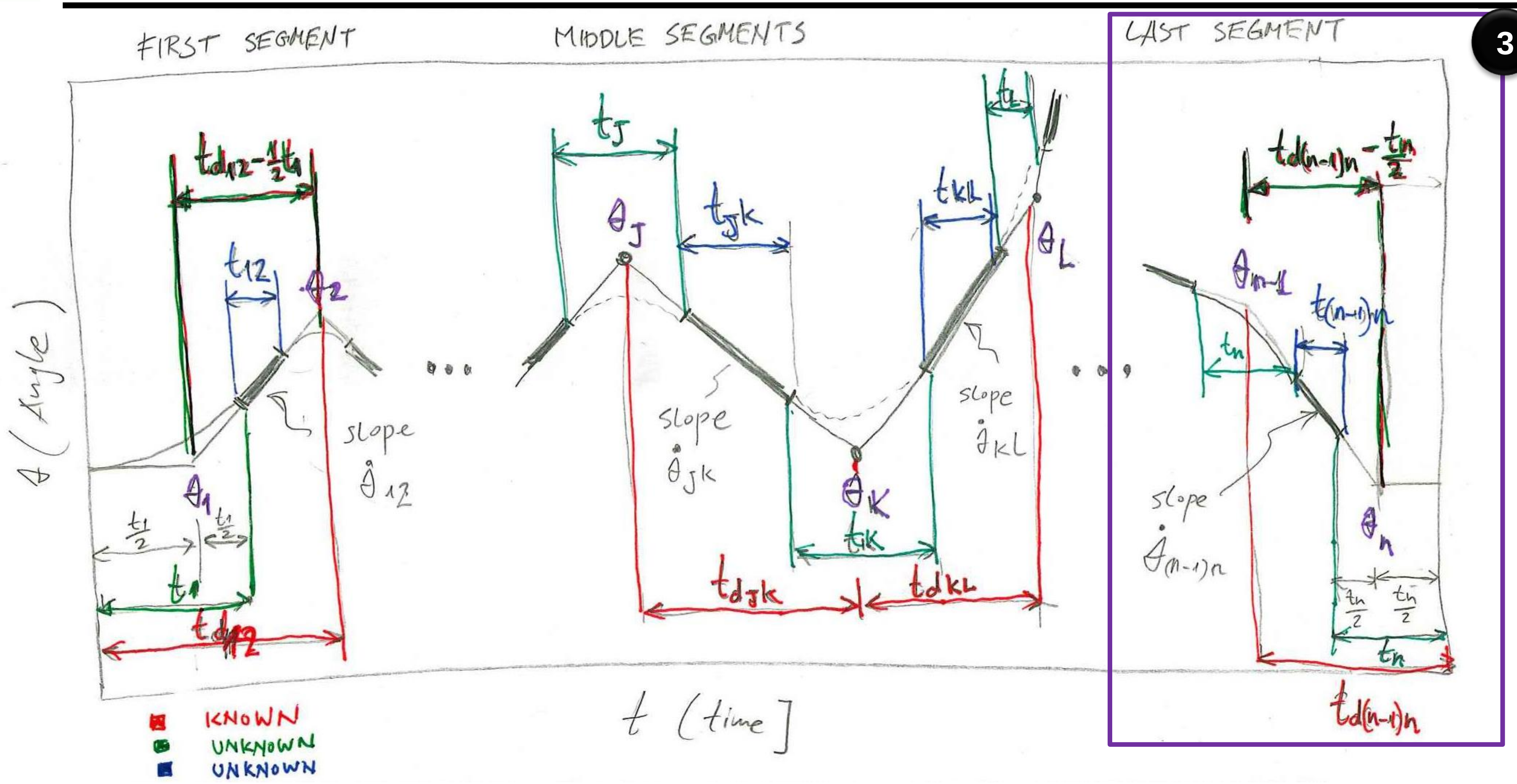
$$t_{12} = t_{d12} - t_1 - \frac{1}{2} t_2$$



$t_1 \rightarrow$ Blending time

$t_{12} \rightarrow$ Straight line time

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

3 LAST SEGMENT

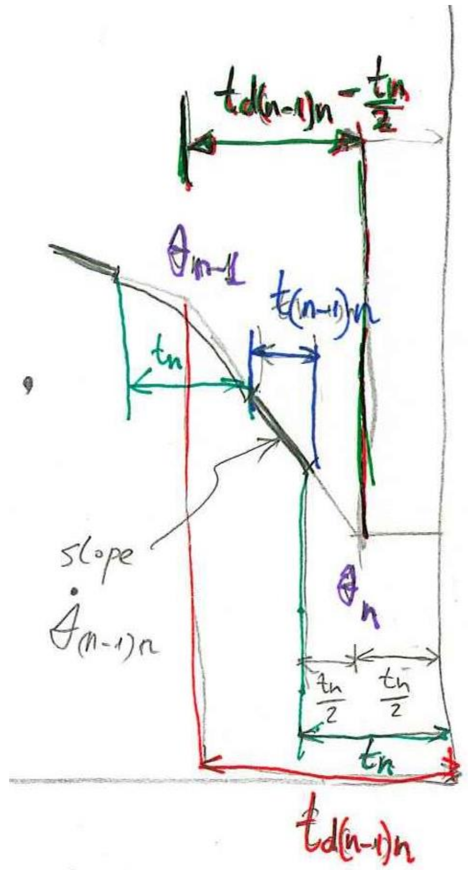
Velocity $\rightarrow$ $$\frac{\theta_{n-1}-\theta_n}{t_{d(n-1)n}-\frac{1}{2}t_n} = \ddot{\theta}_n t_n$$

Acceleration $\rightarrow$ $$\ddot{\theta}_n = SGN\left(\theta_{n-1}-\theta_n\right)\left|\ddot{\theta}_n\right|$$

$$\theta_{n-1}-\theta_n = \ddot{\theta}_n t_n\left(t_{d(n-1)n}-\frac{1}{2}t_n\right)$$

$$+\left(\ddot{\theta}_n\frac{1}{2}\right)t_n^2-\left(\ddot{\theta}_n t_{d(n-1)n}\right)t_n+\left(\theta_{n-1}-\theta_n\right)=0$$

$$t_n = \frac{\ddot{\theta}_n t_{d(n-1)n} \pm \sqrt{\ddot{\theta}_n^2 t_{d(n-1)n}^2 - \overset{2}{\cancel{4}}\ddot{\theta}_n \cancel{\frac{1}{2}}\left(\theta_{n-1}-\theta_n\right)}}{\ddot{\theta}_n}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Time Interval Analysis

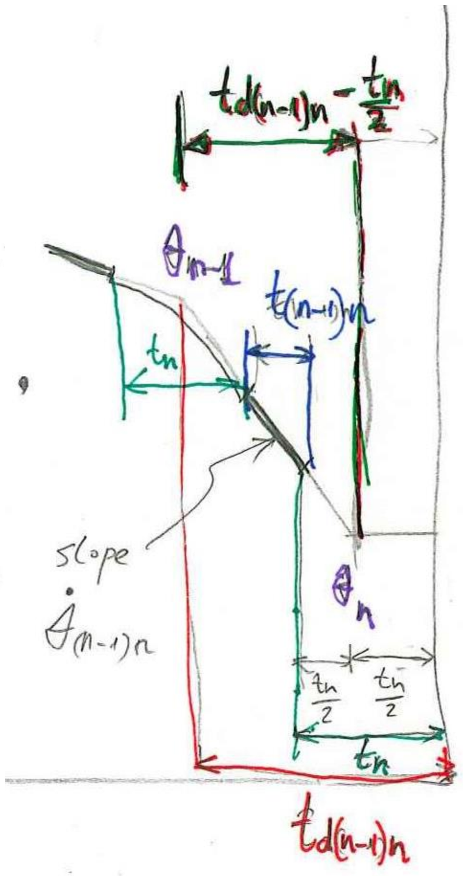
$$t_n = t_{d(n-1)n} \pm \sqrt{t^2_{d(n-1)n} - \frac{2(\theta_{n-1} - \theta_n)}{\ddot{\theta}_n}}$$

Taking only the (-) sign since

$$t_n < t_{d(n-1)n}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2} t_n}$$

$$t_{(n-1)(n} = t_{d(n-1)} - t_n - \frac{1}{2} t_{n-1}$$



Joint Space Schemes

Multiple Time Intervals

Via Point

System Defined Function – Linear Function With Parabolic Blend

Linear & Parabolic Spline Analysis

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- Middle Segment

LINEAR SPLINE $t \in t_{Jk}$

$$\rightarrow \quad \theta = \theta_J + \dot{\theta}_{Jk} t$$

$$\dot{\theta} = \dot{\theta}_{Jk}$$

$$\ddot{\theta} = 0$$

PARABOLIC SPLINE $t \in t_k$

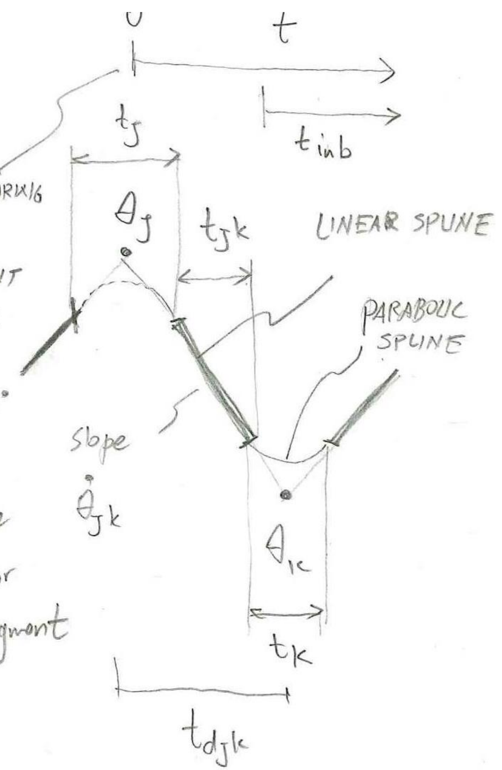
$$t_{inb} = t - \left(\tfrac{1}{2} t_J + t_{Jk}\right)$$

$$\rightarrow \quad \theta = \theta_J + \dot{\theta}_{Jk}(t - t_{inb}) + \frac{1}{2}\ddot{\theta}_k^2 t_{inb}^2$$

$$\dot{\theta} = \dot{\theta}_{Jk} + \ddot{\theta}_k t_{inb}$$

$$\ddot{\theta} = \ddot{\theta}_k$$

E PATH



+ START MEASURING TIME FROM THE VIA POINT θ_J FOR THE SEGMENT Jk

+ Reset the time to 0 at θ_k for the following segment

Note:

$$\dot{\theta}_{Jk} = \frac{\theta_k - \theta_J}{t_{dJk}} \quad ; \quad \ddot{\theta}_k - \text{Max allowed acceleration (Used to define } t_k)$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

- First Segment

LINEAR SPLINE $t \in t_{12}$

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

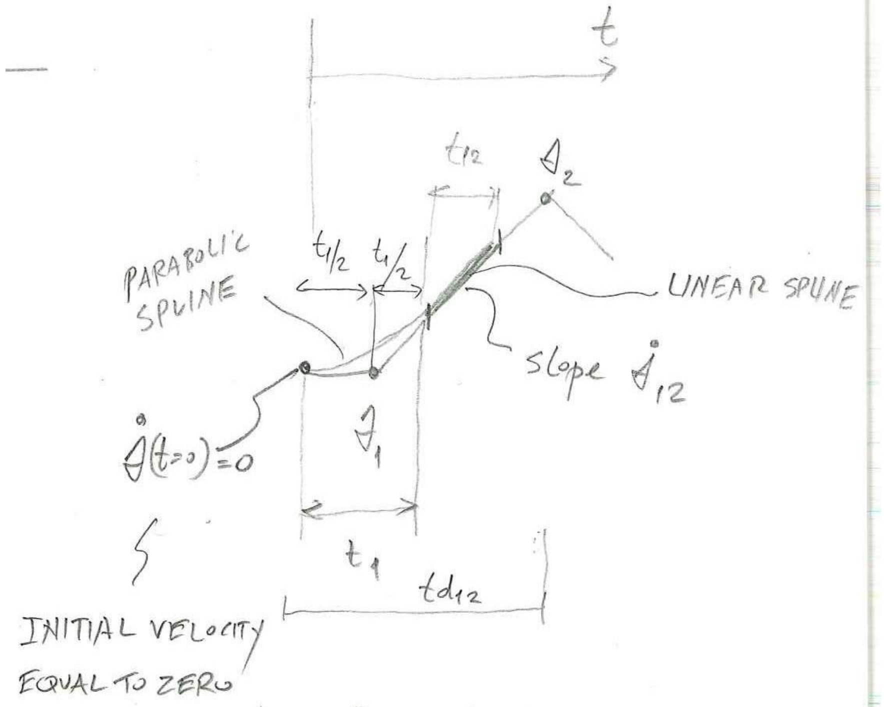
$$\dot{\theta} = \dot{\theta}_{12}$$

$$\ddot{\theta} = 0$$

PARABOLIC SPLINE $t \in t_1$

$$\theta = a_0 + a_1 t + a_2 t^2$$

CONDITIONS
$$\begin{cases} \theta(t=0) = \theta_0 \\ \dot{\theta}(t=0) = 0 \\ \dot{\theta}(t=t_1) = \dot{\theta}_{12} \end{cases}$$



$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

$$\begin{cases} \theta = a_0 + a_1 t + a_2 t^2 \\ \dot{\theta} = a_1 + 2a_2 t \\ \ddot{\theta} = 2a_2 \end{cases}$$

$$\theta(t=0) = \theta_0 = a_0 + a_1 0 + a_2 0 \Rightarrow a_0 = \theta_0$$

$$\dot{\theta}(t=0) = 0 = a_1 + 2a_2 0 \Rightarrow a_1 = 0$$

$$\dot{\theta}(t=t_1) = \dot{\theta}_{12} = +2a_2 t_1 \Rightarrow a_2 = \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1}$$

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

$$\dot{\theta} = \frac{\dot{\theta}_{12}}{t_1} t$$

$$\ddot{\theta} = \frac{\dot{\theta}_{12}}{t_1}$$

- Last Segment

LINEAR SPLINE $t \in t_{(n-1)n}$

$$\theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t$$

$$\dot{\theta} = \dot{\theta}_{(n-1)n}$$

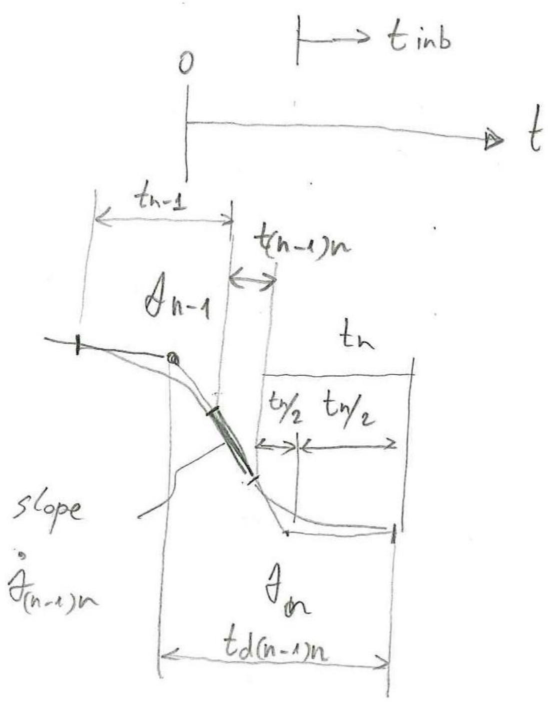
$$\ddot{\theta} = 0$$

PARABOLIC SPLINE

$$t_{inb} = t - \left(\frac{1}{2} t_{(n-1)} + t_{(n-1)n}\right)$$

$$\theta = a_0 + a_1 t_{inb} + a_2 t_{inb}^2$$

CONDITIONS
$$\begin{cases} \theta(t_{inb} = 0) = \theta_{n-1} + \dot{\theta}_{(n-1)n} t_{(n-1)n} = \theta_{inb} \\ \dot{\theta}(t_{inb} = 0) = \dot{\theta}_{(n-1)n} \\ \ddot{\theta}(t = t_{inb}) = 0 \end{cases}$$



$$\dot{\theta}_{(n-1)n} = \frac{\theta_{(n-1)} - \theta_n}{t_{d(n-1)n} - \frac{1}{2} t_n}$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Linear & Parabolic Spline Analysis

$$\begin{cases} \theta = a_0 + a_1 t + a_2 t^2 \\ \dot{\theta} = a_1 + 2a_2 t \\ \ddot{\theta} = 2a_2 \end{cases}$$

$$\theta(t_{inb}=0) = \theta_{inb} = a_0 + a_1 0 + a_2 0 \Rightarrow a_0 = \theta_{inb}$$

$$\dot{\theta}(t_{inb}=0) = \dot{\theta}_{(n-1)n} = a_1 + 2a_2 0 \Rightarrow a_1 = \dot{\theta}_{(n-1)n}$$

$$\dot{\theta}(t=t_{inb}) = 0 = \dot{\theta}_{(n-1)n} + 2a_2 t_{inb} \Rightarrow a_2 = -\frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}}$$

$$\theta = \theta_{inb} + \dot{\theta}_{(n-1)n} t_{inb} - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Calculation Approaches

- **Calculation Approach No. 1**
  - User Defines
    - Via Points
    - Desired time duration of segments
  - System Defines
    - Use default value of acceleration for each joint

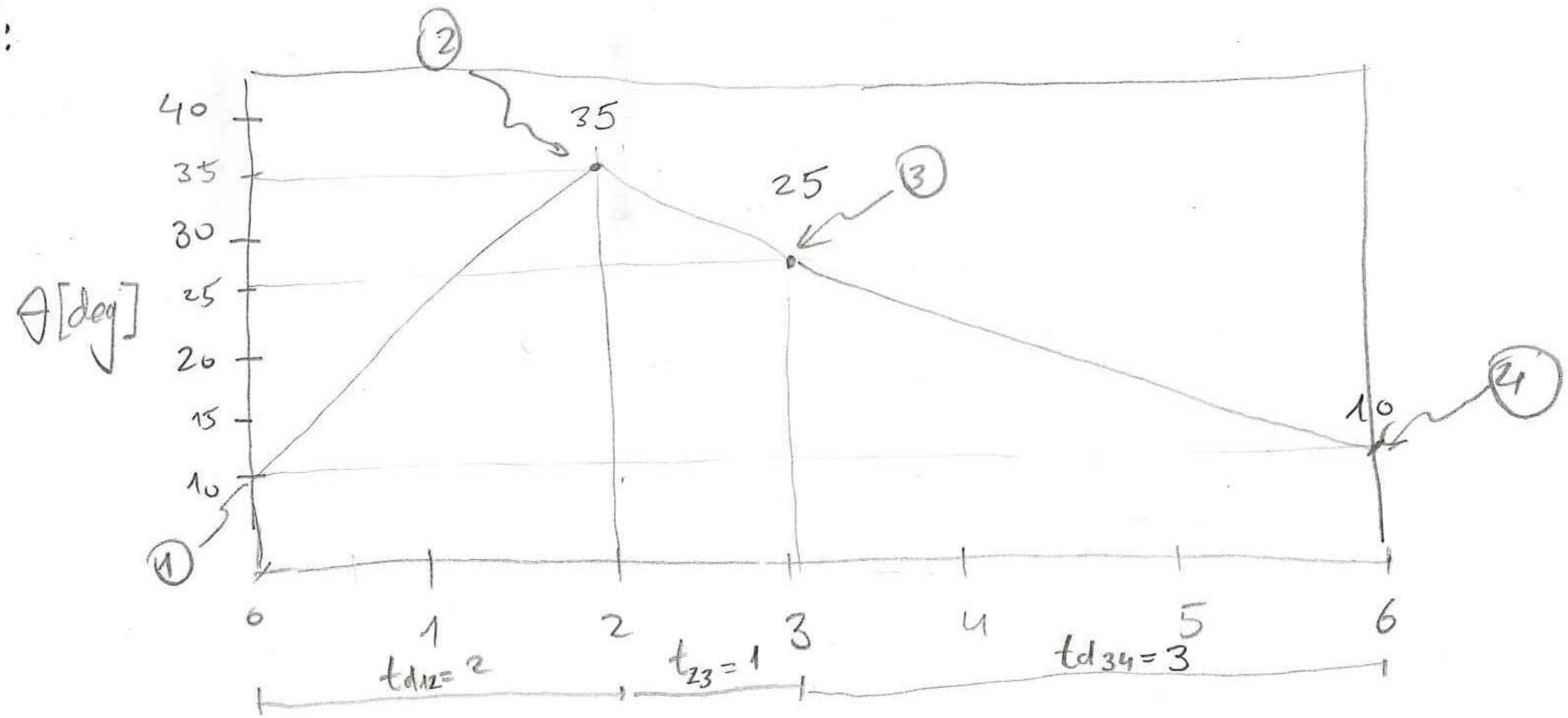
- **Calculation Approach No. 2**
  - System calculate time durations based on default velocities

- **Note for both Approaches –** At all the blends, sufficiently large acceleration must be used so that the system has sufficient time to get into the linear portion of the segment before the next blend region starts

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

The trajectory of a particular joint is specified as follows:

GIVEN:



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- Acceleration – Magnitude of the default acceleration to use at all blend points is

$$|\ddot{\theta}|_{max} = 50 \ Deg/sec^2$$

CALCULATE :
- Segment Velocities $\dot{\theta}$
- Blend times t_i (t_1, t_2, t_3, t_4)
- Linear Segments times

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

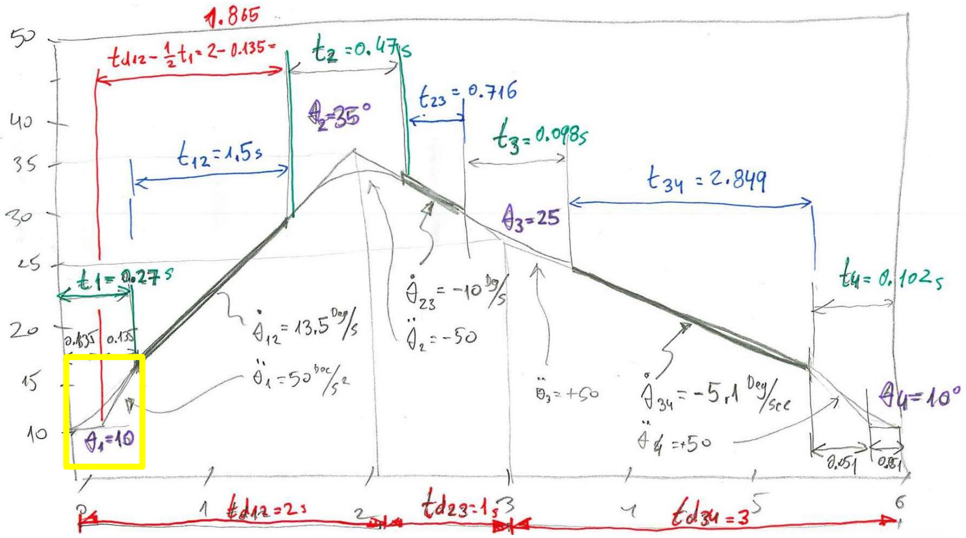
- **Solution Step 1 – Time Interval Analysis**

First Segment (t: 0→2)

$$\ddot{\theta}_1 = 50$$

$$t_1 = t_{d12} - \sqrt{t_{d12}^2 - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}} = 2 - \sqrt{4 - \frac{2(35-10)}{50}} = \boxed{0.27 \text{ sec}}$$

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2} t_1} = \frac{35 - 10}{2 - \frac{1}{2}(0.27)} = 13.50 \ \text{Deg/sec}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Second Segment (t: 2→3)

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{td_{jk}} = \dot{\theta}_{23} = \frac{\theta_3 - \theta_2}{td_{23}} = \frac{25-35}{1} = -10.0 \ \text{Deg/sec}$$

$j=2,\ k=3$

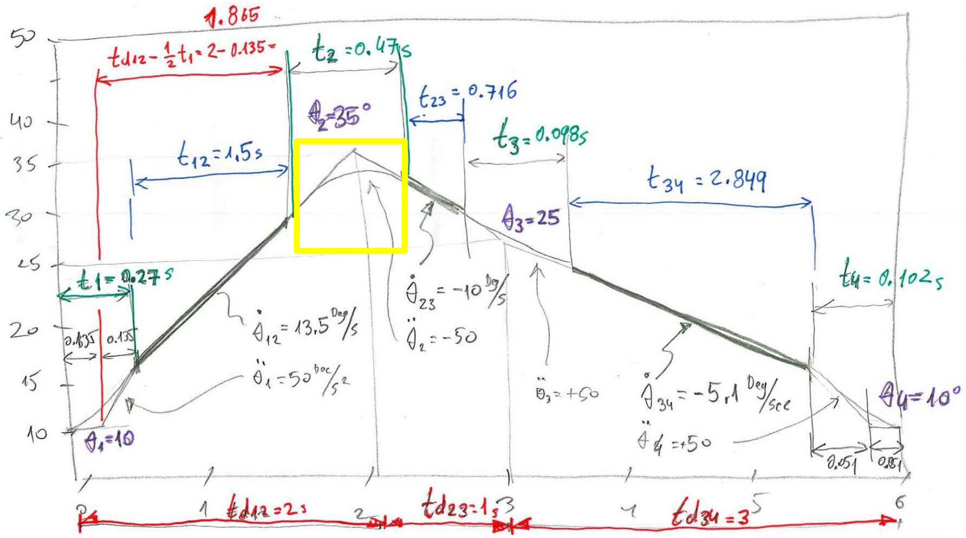
$$\ddot{\theta}_k = SIG(\dot{\theta}_{kl} - \dot{\theta}_{jk})\,|\ddot{\theta}_k| = \ddot{\theta}_2 = SIG(\dot{\theta}_{23} - \dot{\theta}_{12})\,|\ddot{\theta}_2| = SIG(-10-13)\,|\ddot{\theta}_2| = -50 \ \text{Deg/sec}$$

$-$

$j=1,\ k=2,\ l=3$

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k} = t_2 = \frac{\dot{\theta}_{23} - \dot{\theta}_{12}}{\ddot{\theta}_2} = \frac{(-10)-(13.5)}{-50} = \boxed{0.47 \ \text{sec}}$$

$j=1,\ k=2,\ l=3$



$$t_{12} = td_{12} - t_1 - \frac{1}{2}t_2 = 2 - 0.27 - \frac{1}{2}(0.47) = 1.5$$

Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

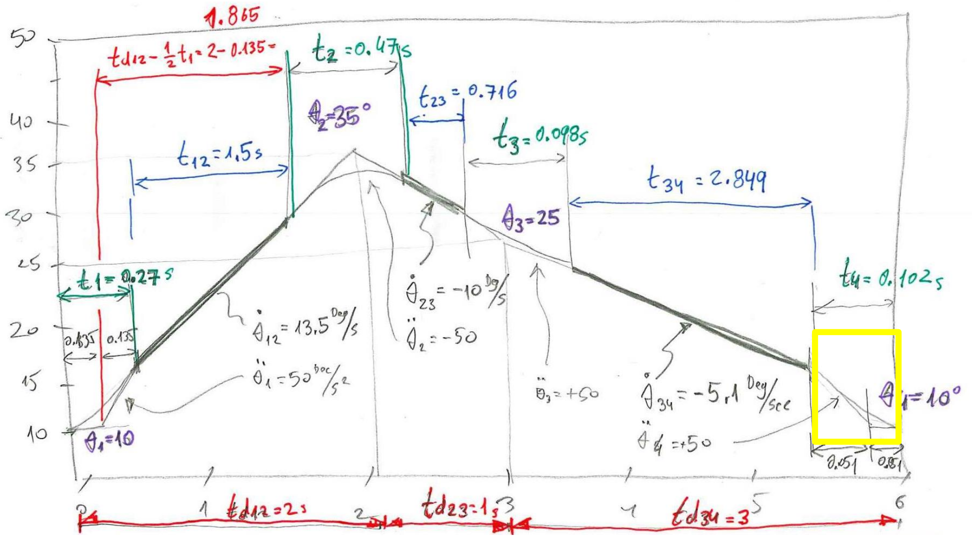
- **Third Segment (t: 3→6)**

$$\ddot{\theta}_n = SGN(\theta_{n-1}-\theta_n)|\ddot{\theta}_n| \underset{\uparrow\, n=4}{=} SIG(\theta_3-\theta_4)|\ddot{\theta}_4| = (25-10)|50| = +50\ Deg/s^2$$

$$\ddot{\theta}_4 = 50\ Deg/s^2$$

$$t_n = t_{d(n-1)n} - \sqrt{t^2_{d(n-1)n} - \frac{2(\theta_{n-1}-\theta_n)}{\ddot{\theta}_n}} \underset{\uparrow\, n=4}{=} t_4 = t_{d34} - \sqrt{t^2_{d34} - \frac{2(\theta_3-\theta_4)}{\ddot{\theta}_4}} = 3 - \sqrt{9 - \frac{2(25-10)}{50}} = \boxed{0.102\ sec}$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n} \underset{\uparrow\, n=4}{=} \dot{\theta}_{34} = \frac{\theta_4 - \theta_3}{t_{d34} - \frac{1}{2}t_4} = \frac{10-25}{3-\frac{1}{2}0.102} = -5.10\ Deg/sec$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

- **Third Segment (t: 3→6) Cont.**

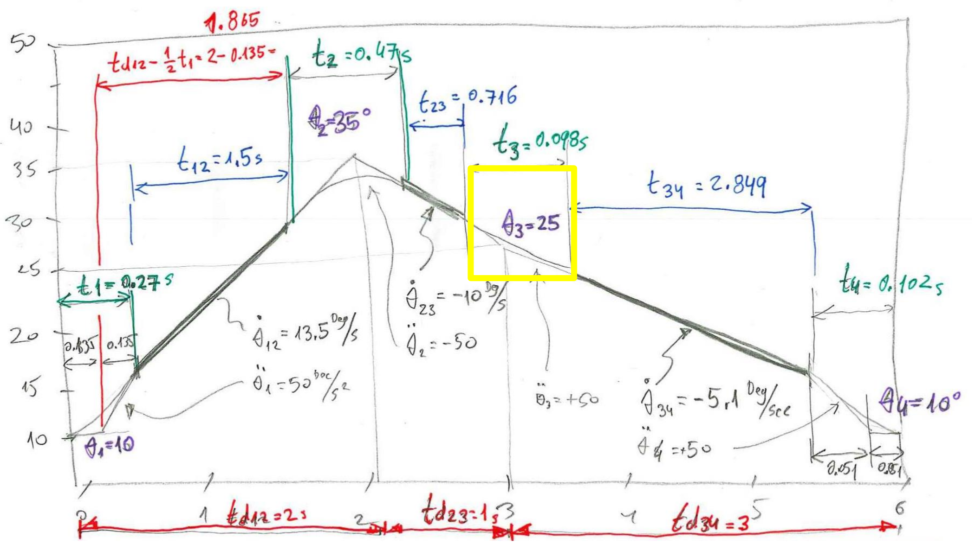
$$\ddot{\theta}_k = SGN(\dot{\theta}_{kL} - \dot{\theta}_{jk})(\ddot{\theta}_k) \Rightarrow \ddot{\theta}_3 = SGN(\dot{\theta}_{34} - \dot{\theta}_{23})|\ddot{\theta}_3| = SGN((-5.10)-(-10))|50|$$

J = 2
K = 3
L = 4

$$\ddot{\theta}_3 = +50 \ \mathrm{Deg/sec^2}$$

$$t_k = \frac{\dot{\theta}_{kL} - \dot{\theta}_{jk}}{\ddot{\theta}_k} \Rightarrow t_3 = \frac{\dot{\theta}_{34} - \dot{\theta}_{23}}{\ddot{\theta}_3} = \frac{(-5.10)-(-10)}{50} = \boxed{0.098 \ \mathrm{sec}}$$

J = 2
K = 3
L = 4



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

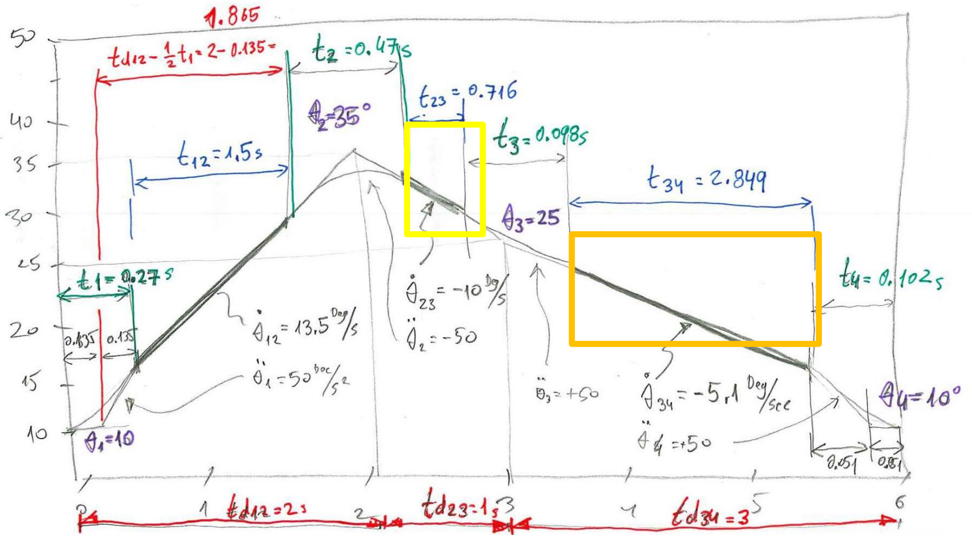
- **Third Segment (t: 3→6) Cont.**

$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k \underset{\substack{j=2\\k=3}}{=} t_{23} = t_{d23} - \frac{1}{2}t_2 - \frac{1}{2}t_3 = 1 - \frac{1}{2}(0.47) - \frac{1}{2}(0.098)$$

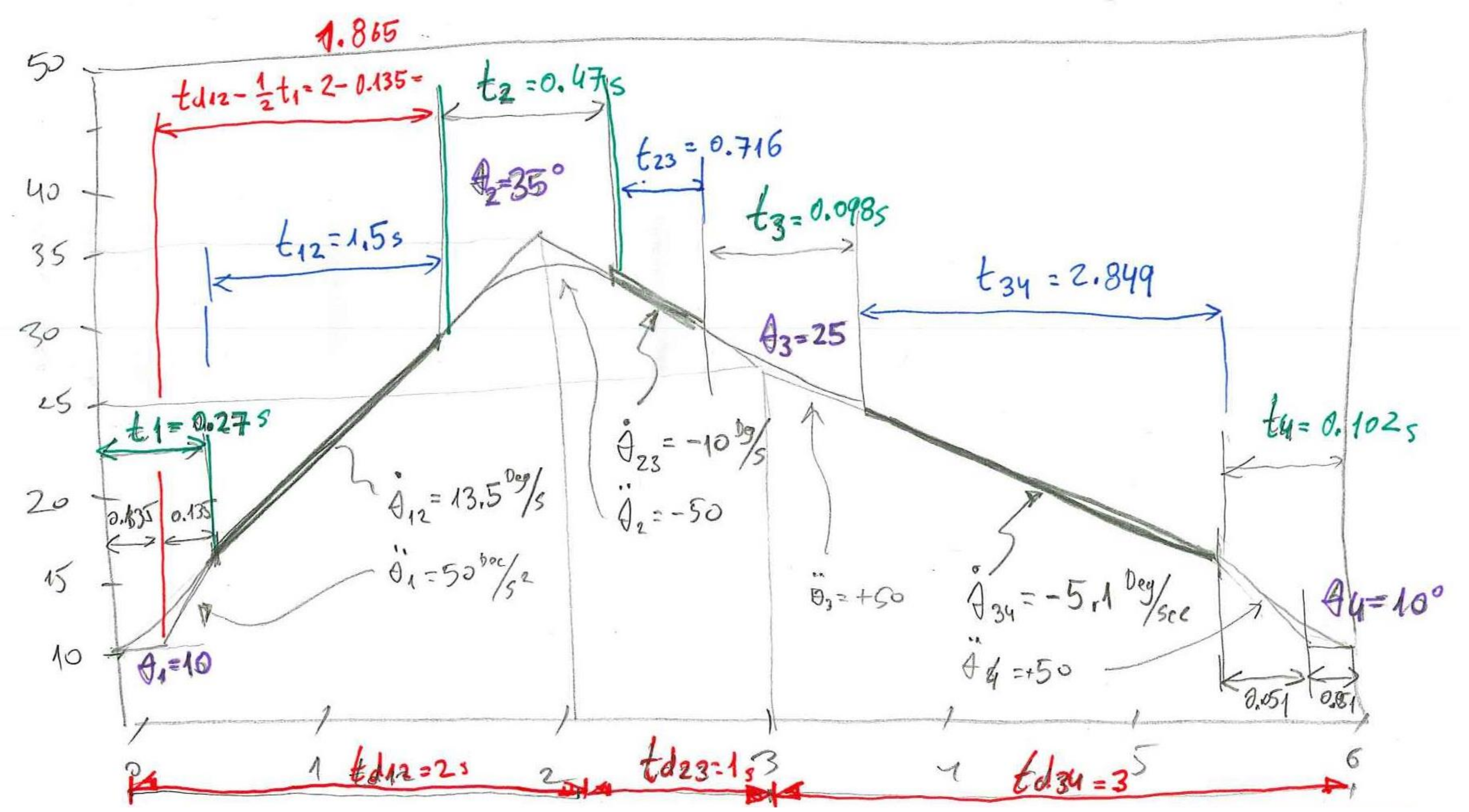
$$= \boxed{0.716\ sec}$$

$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2}t_{n-1} \underset{n=4}{=} t_{34} = t_{d34} - t_4 - \frac{1}{2}t_3 =$$

$$= 1 - 0.102 - \frac{1}{2}\,0.098 = \boxed{2.849\ sec}$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example



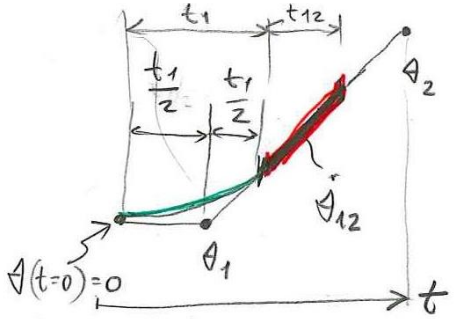
Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear / parabolic Splines – Summary Reminder

FIRST SEGMENT

$$\theta = \theta_1 + \dot{\theta}_{12} t$$

$$\theta = \theta_0 + \frac{1}{2}\frac{\dot{\theta}_{12}}{t_1} t^2$$

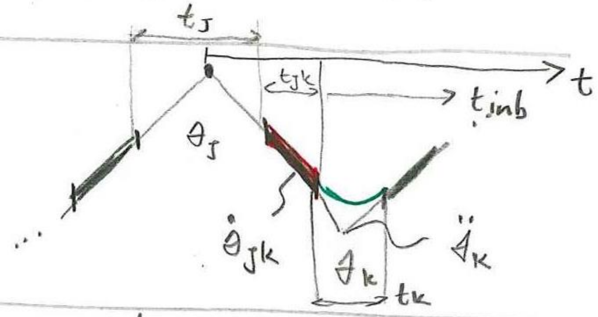


MID SEGMENT

$$\theta = \theta_J + \dot{\theta}_{JK} t$$

$$\theta = \theta_J + \dot{\theta}_{JK}(t - t_{inb}) + \frac{1}{2}\ddot{\theta}_K^2 t_{inb}$$

$$t_{inb} = t\left(\frac{1}{2}t_J + t_{JK}\right)$$

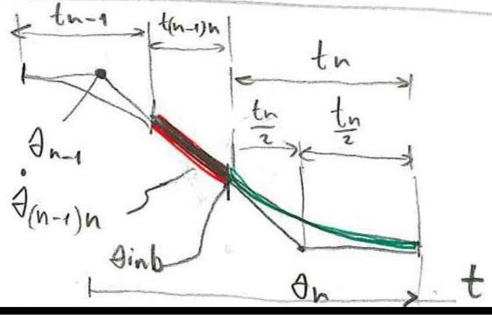


LAST SEGMENT

$$\theta = \theta_{n-1} + \dot{\theta}_{(n-1)n} t$$

$$\theta = \theta_{inb} + \dot{\theta}_{(n-1)n} t - \frac{1}{2}\frac{\dot{\theta}_{(n-1)n}}{t_{inb}} t^2$$

$$t_{inb} = t - \left(\frac{1}{2}t_{(n-1)} + t_{(n-1)n}\right)$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Linear Polynomial

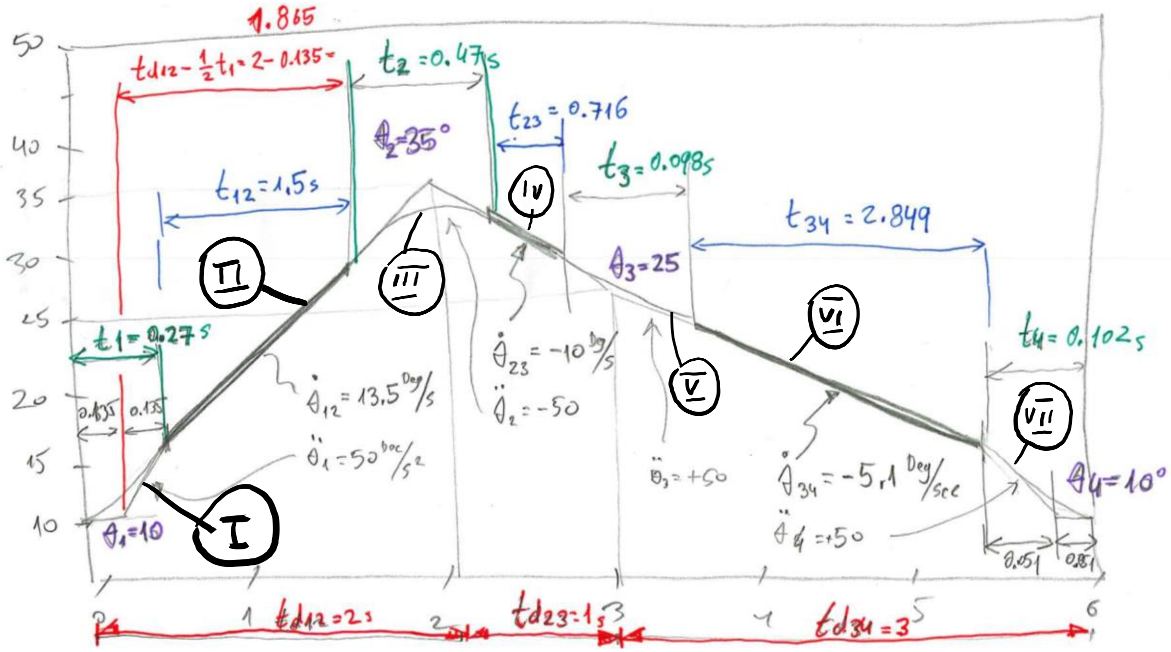
LINEAR SEGMENTS

II $\theta = 10 + 13.5t$

IV $\theta = 35 - 10t$

VI $\theta = 25 - 5.1t$

$$\theta_{inb} = 25 - 5.1\left(\frac{0.098}{2} + 2.849\right)$$



Joint Space Schemes – Multiple Time Intervals – Via Points – Linear Function With Parabolic Blend – Example

Parabolic Blend

I $\theta = 10 + \frac{1}{2}\frac{13.5}{0.27}t^2$ (FIRST)

III $\theta = 10 + 13.5(t - t_{inb}) + \frac{1}{2}(50)^2 t_{inb}$

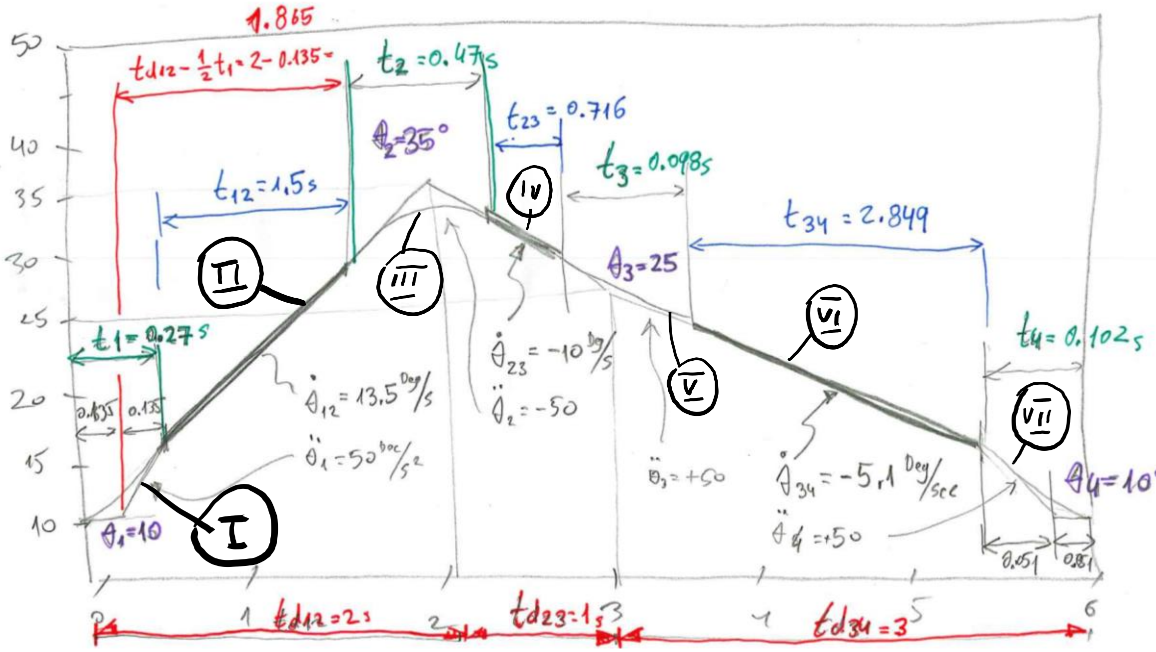
$t_{inb} = t\,(\frac{1}{2}0.27 + 1.5)$

V $\theta = 35 - 10(t - t_{inb}) + \frac{1}{2}(50)^2 t_{inb}$

$t_{inb} = t\,(\frac{1}{2}0.47 + 0.716)$

VII $\theta = \theta_{inb} + 35t - \frac{1}{2}\frac{(-5.1)}{t_{inb}}t^2$ (LAST)

$t_{inb} = t - (\frac{1}{2}0.098 + 2.849)$



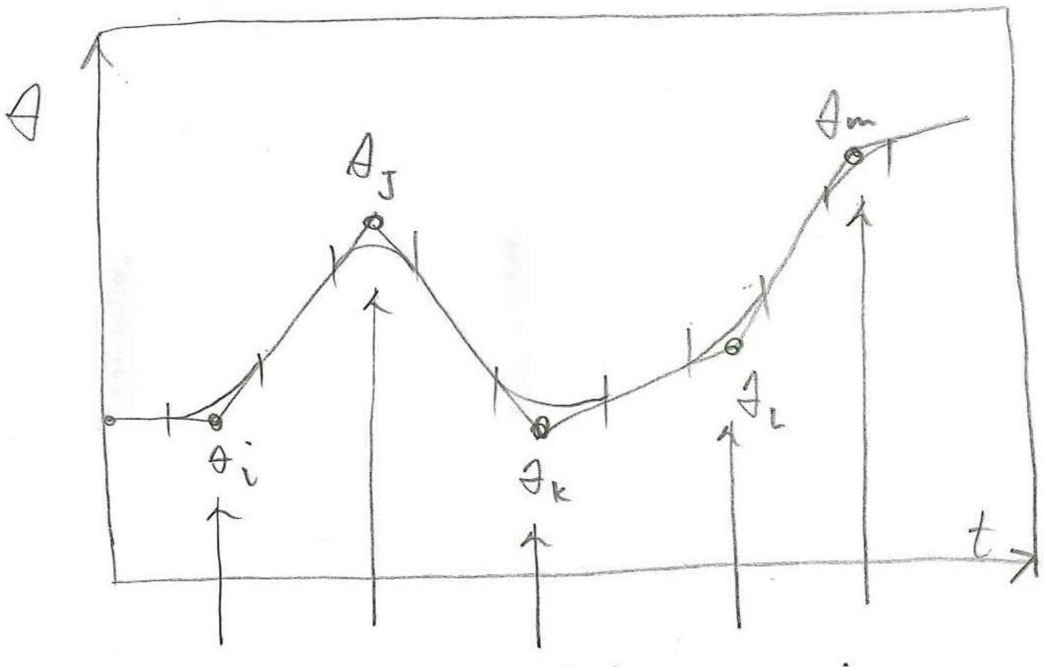
Joint Space Schemes

Multiple Time Intervals

Pseudo Via Point

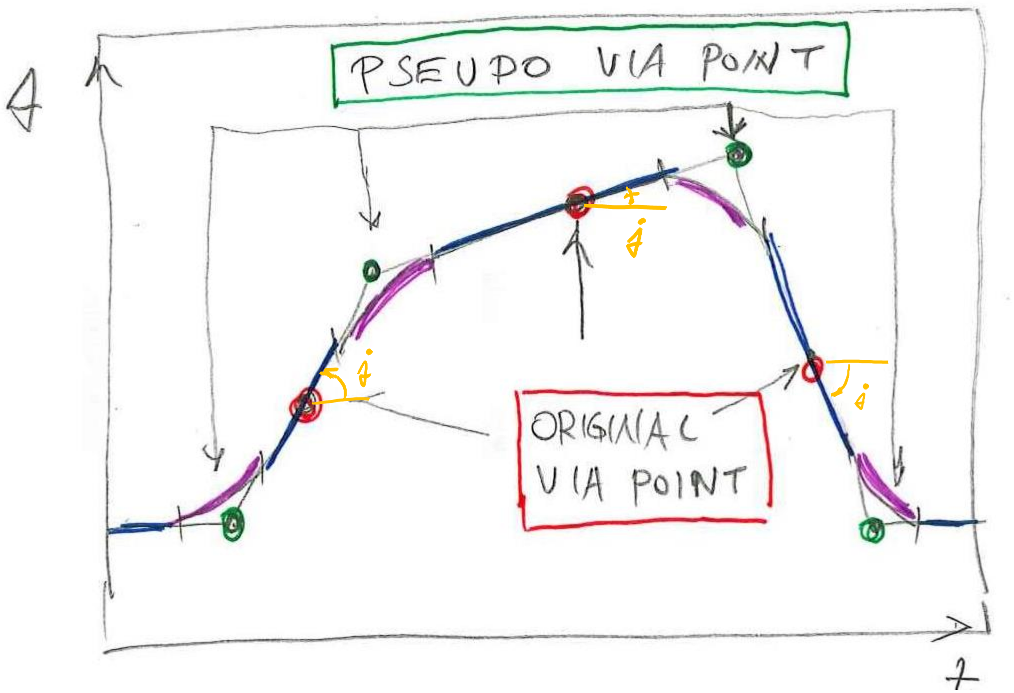
Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

- **Problem**:
  - In the linear parabolic blend spline, note that the via points are not actually reached unless the manipulator come to a stop
  - Often when the acceleration is sufficiently high the path will come quite close to the desire via point



Joint Space Schemes – Multiple Time Intervals – Pseudo Via Points

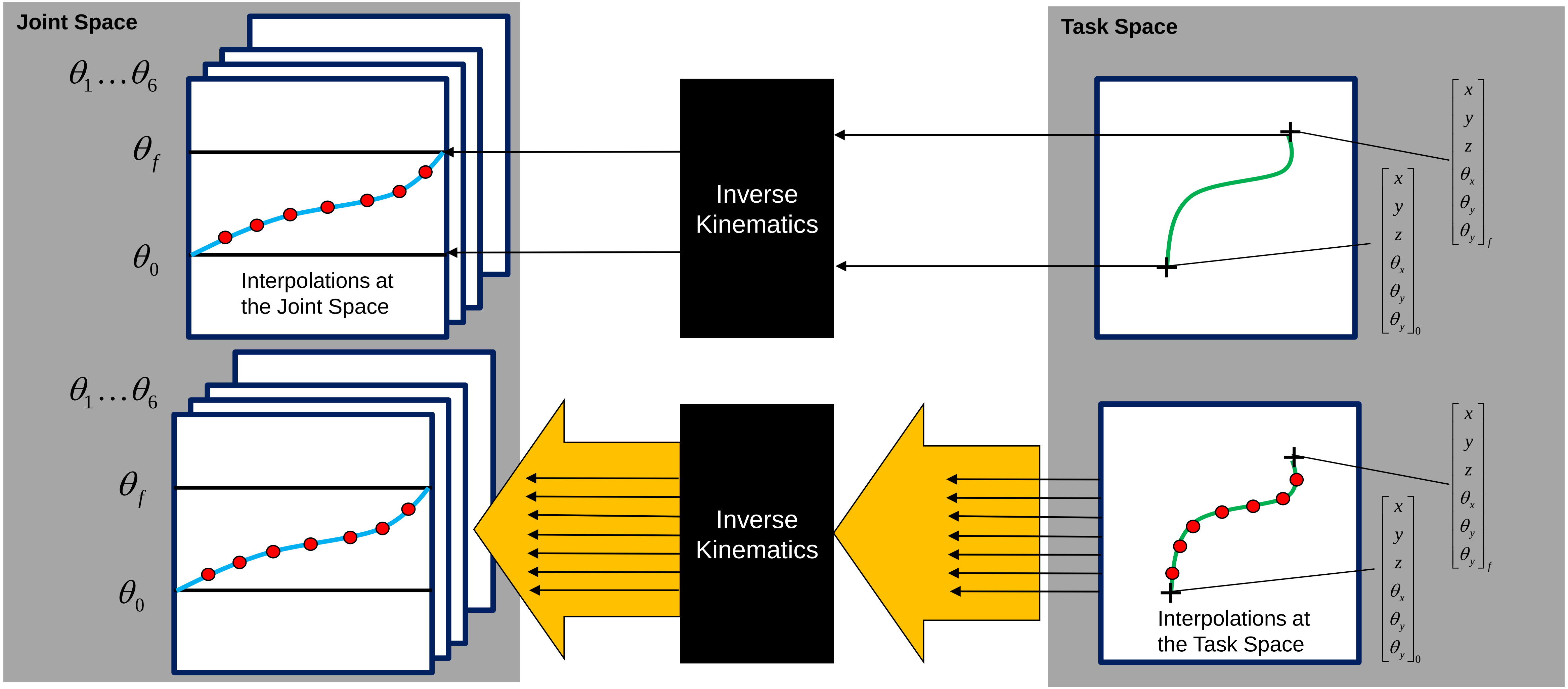
- **Solution**
  - **Define Pseudo via Points** - The system automatically replace the via point through which we wish the manipulator to pass through with two pseudo via points one on each side of the original.
  - The original via point will now lie in the linear region of the path connecting the two pseudo via points
  - **Define Velocity at the original Via Points** - In addition to requesting that the manipulator pass exactly through the via point, the user can also request that it pass through with a certain velocity. If the user does not specify this velocity the system chooses it by means of suitable heuristic
  - **Define Through Point** - Through Point rather than via point is used to specify a path through which we force the manipulator to pass exactly through



Task Space Schemes

General Discussion

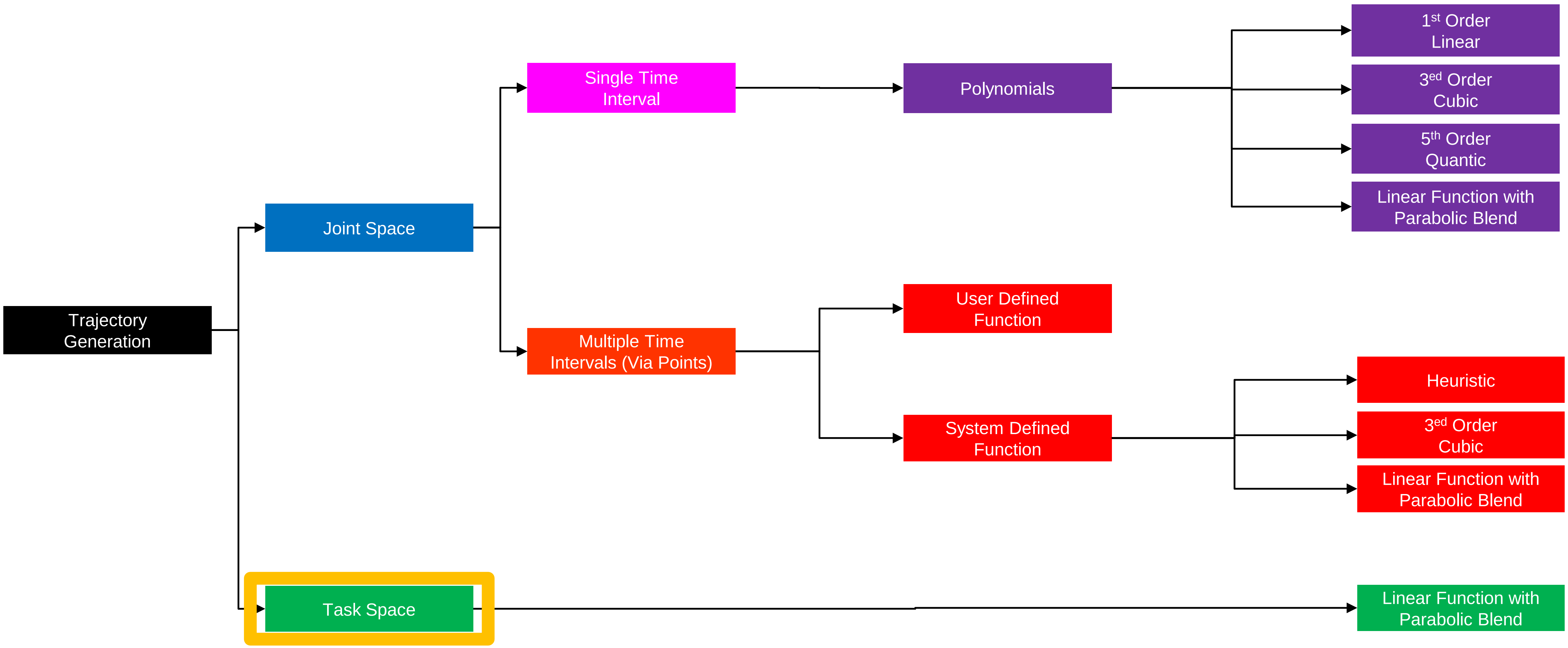
Task Space Versus Joint Space - Interpolations



Join Space Versus Task Space – Comparison

Parameter	Joint Space	Task Space
Interpolation Space intermediate points along the trajectory	Joint Space	Task Space
Tool Trajectory Type / Length	Curved Line / Long	Straight Lines / Short
Invers Kinematics (IK) Usage	Low	High
Computation Expense (IK)	Low (IK for Start/Finish & Via Points)	High (IK for every single point / time steo on the trajectory)
Passing through Via Points	No (Correction by establishing Pseudo Points)	Yes
Via Points Defined in the Task Space	No	Yes
Path Dependency on a Specific Manipulator	Yes	No

Trajectory Generation – Roadmap Diagram



Task Space Scheme – Problem Definition
Position / Orientation Problem

- **General Approach (continue)**
  - Every point along the path is defined by position and orientation of the end effector

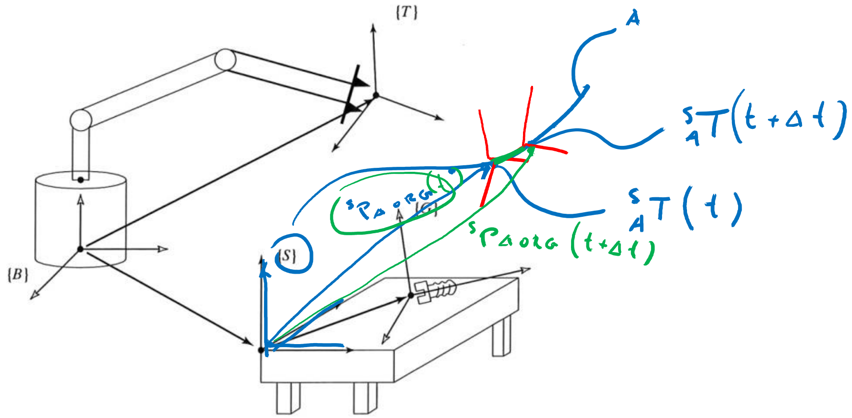
(t+Δt)

(t)

$$ {}^S_A T = \left[\begin{array}{ccc|c} & {}^S_A R & & {}^S P_{AORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \qquad \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} $$

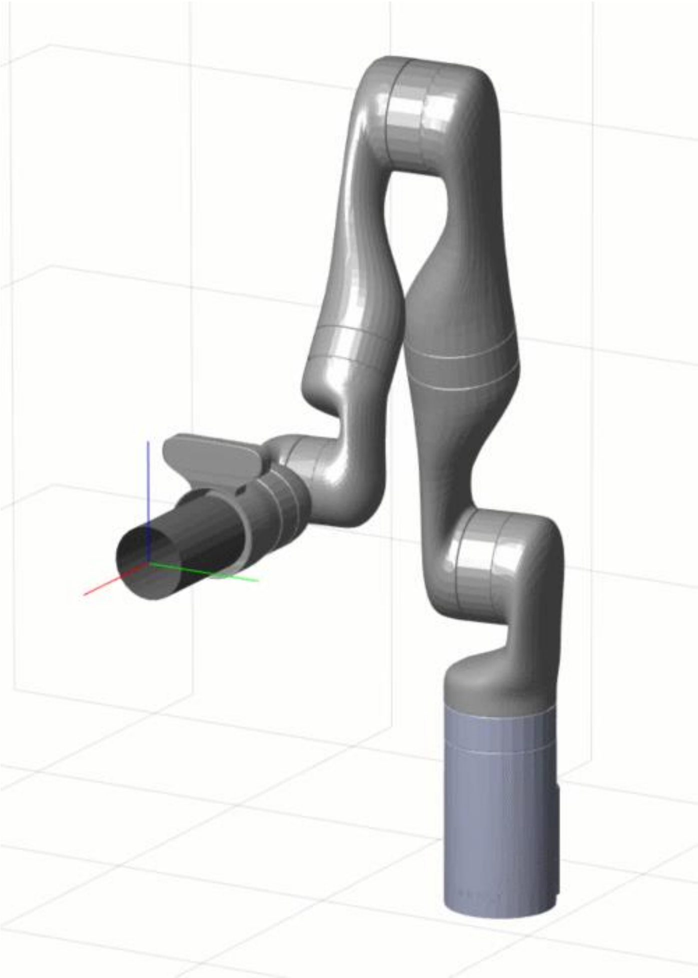
x

α β γ

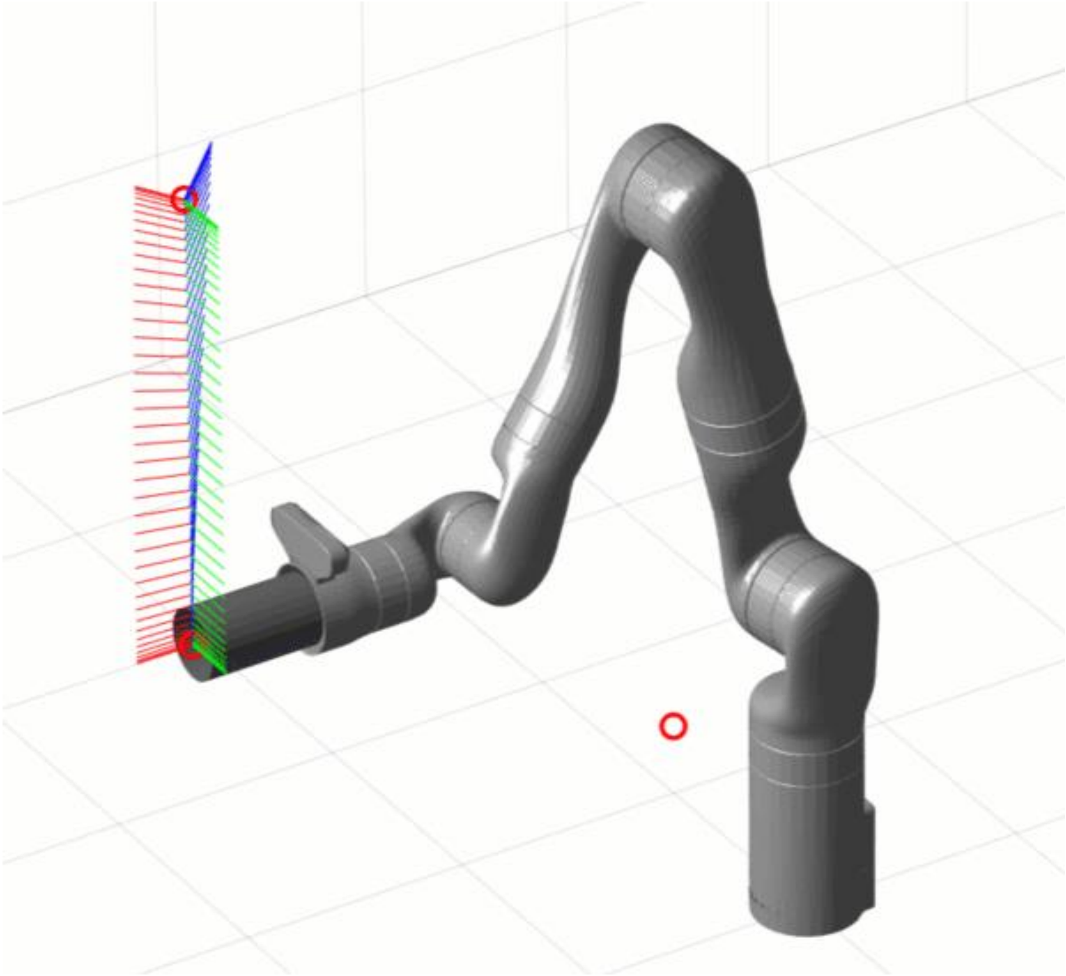


  - **End Effector Position** – Vector – Easy interpolation
  - **End Effector Ordination** – Matrix – Impossible to interpolate (interpolating the individual elements of the matrix violate the requirements that all column of the matrix must be orthogonal)

Task Space Scheme – Problem Definition
Orientation Problem

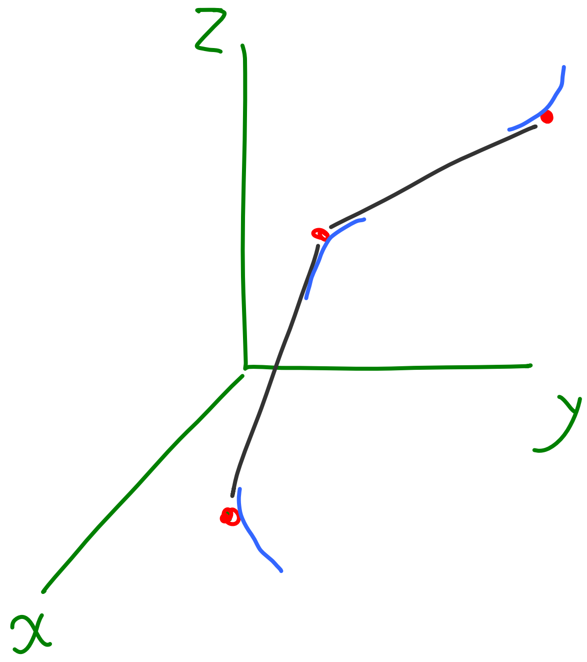


Task Space Scheme – Problem Definition
Position / Orientation Problem – Trapezoid Velocity



Task Space Scheme – Problem Definition
Position Problem

- **General Approach** - Define the path (in the Cartesian space) as
  - straight lines (linear functions)
  - Velocity Trapezoid method
  - Parabolic lines (blends)



Task Space Scheme – Problem Definition
Orientation Problem - Equivalent Angle – Axis Representation

- Start with the frame coincident with a know frame {A}; then rotate frame {B} about a vector ${}^{A}\hat{K}$ by an angle θ according to the right hand rule.

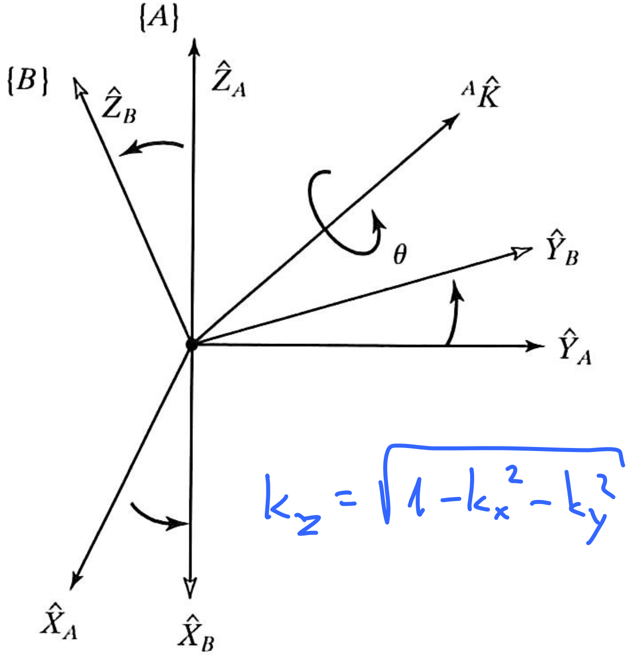
- Equivalent Angle – Axis Representation

$$ {}^{A}_{B}R(\hat{K},\theta) \quad or \quad R_{K}(\theta) $$

- Vector ${}^{A}\hat{K}$ is called the equivalent axis of a finite rotation.

- The specification of ${}^{A}\hat{K}$ requires two parameters since it length is always 1.

- The angle specify the third parameter



$k_z = \sqrt{1 - k_x^2 - k_y^2}$

$\hat{k}_x, \hat{k}_y, \hat{k}_z \qquad \sqrt{k_x^2 + k_y^2 + k_z^2} = 1$

Task Space Scheme – Problem Definition
Orientation Problem - Equivalent Angle – Axis Representation

- **Conversion 1** - Conversion for single angle axis representation to rotation matrix representation

$$R_K(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta - k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

$$c\theta = \cos\theta$$
$$s\theta = \sin\theta$$
$$v\theta = 1 - \cos\theta$$

- **Conversion 2** – Compute ${}^{A}\hat{K}$ and θ given a rotation a matrix

$$R_K(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{K} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

Task Space Scheme – Problem Definition
Position / Orientation Problem - Equivalent Angle – Axis Representation

- Combining the angle-axis representation of orientation with the 3x1 Cartesian position representation we have a 6x1 representation of Cartesian position and orientation.

- Consider a via point specified relative to a station point frame as ${}^S_A T$

$Rot({}^S K_A, \theta_{SA})$

$$ {}^S_A T = \begin{bmatrix} {}^S_A R & {}^S P_{AORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} $$

${}^S P_{AORG\,x}$

${}^S P_{AORG\,y}$

${}^S P_{AORG\,z}$

  - Frame {A} specifies a via point
    - Position of the end effector given by ${}^S P_{AORG}$
    - Orientation of the end effector given by ${}^S_A R$

Task Space Scheme – Problem Definition
Position / Orientation Problem - Equivalent Angle – Axis Representation

- Convert the rotation matrix into an angle axis representation

$$ {}^S_A R = ROT({}^S\hat{K}_A, \theta_{SA}) = {}^S K_A \qquad \Big(\underbrace{k_x \hat{i} + k_y \hat{j} + \sqrt{1-k_x^2-k_y^2}\,\hat{k}}_{\text{UNIT VECTOR}}\Big)\theta_{SA} $$

- Use the symbol χ to represent 6x1 position and orientation

$$ {}^S\chi_A = \begin{bmatrix} {}^S P_{AORG} \\ {}^S K_A \end{bmatrix} \longrightarrow \begin{bmatrix} {}^S P_{Ax} \\ {}^S P_{Ay} \\ {}^S P_{Az} \\ {}^S K_{Ax} \\ {}^S K_{Ay} \\ {}^S K_{Az} \end{bmatrix} $$

- Where ${}^S K_A$ is formed by scaling the unite vector ${}^S\hat{K}_A$ by the amount of rotation θ_{SA}

Task Space Scheme – Cartesian Straight Line

- Process - For a given trajectory we describe a spline function that smoothly vary these six quantities from path point to path point as a function of time.

$$ {}^{S}\chi_{A} = \begin{bmatrix} {}^{S}P_{AORG} \\ {}^{S}K_{A} \end{bmatrix} $$

- Linear Spline with parabolic bland
  - Path shape between via points will be linear
  - When via points are passed, the linear and angular velocity of the end effector are changed smoothly

Task Space Scheme – Cartesian Straight Line

- **Complication –** The angle-axis representation is not unique

$$({}^S\hat{K}_B, \theta_{SB}) = ({}^S\hat{K}_B, \theta_{SB} \pm n360)$$

. . . . -2, -1, 0, 1, 2 . . .

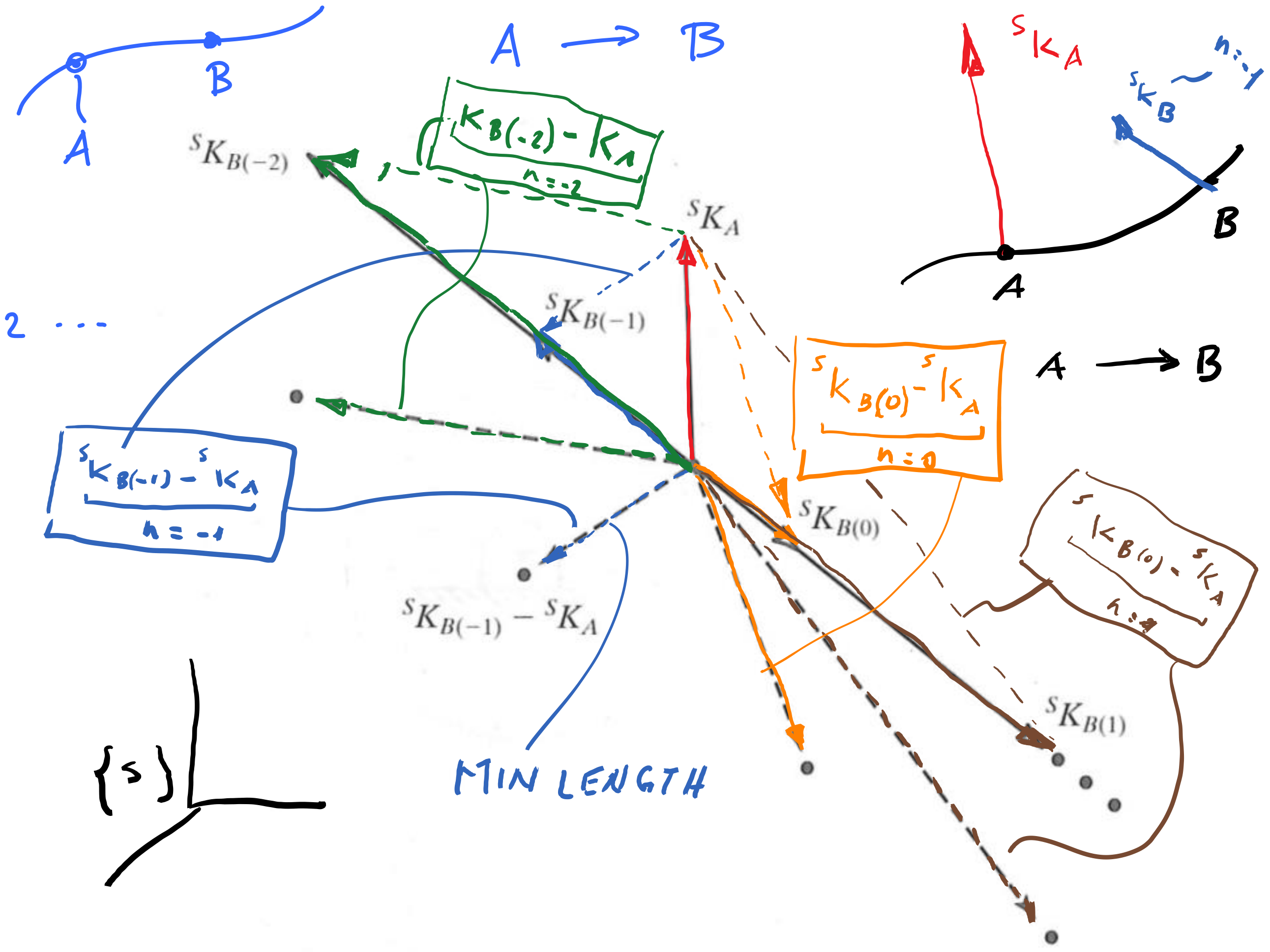
- In going from via point {A} to a via point {B}, the total amount of rotation should be minimized

- Choose ${}^S\hat{K}_B$ such that

$$\underbrace{\min\left|{}^S\hat{K}_B - {}^S\hat{K}_A\right|} \rightarrow {}^S K_B$$

$\uparrow$

$n=-1$

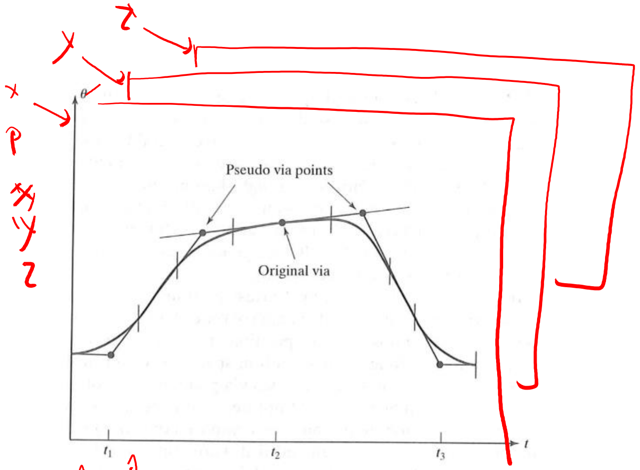


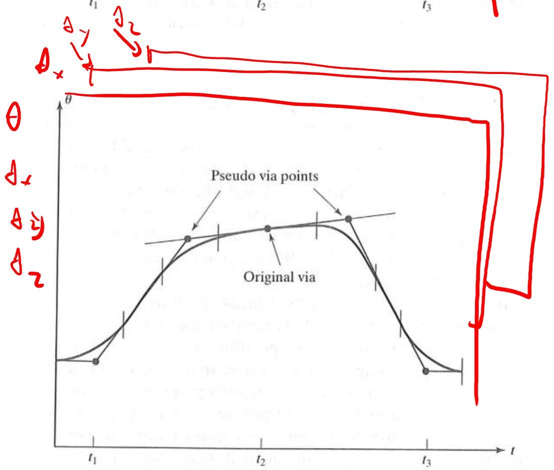
Task Space Scheme – Cartesian Straight Line

- The splines are composed of linear and parabolic blend section

- **Constrain**
  - The transition between the linear segment and the parabolic segment for all the DOF must take place at the same time.

$$ {}^{S}\chi_{A} = \begin{bmatrix} {}^{S}P_{AORG} \\ {}^{S}K_{A} \end{bmatrix} \qquad {}^{S}\chi_{B} = \begin{bmatrix} {}^{S}P_{BORG} \\ {}^{S}K_{B} \end{bmatrix} $$





Task Space Schemes

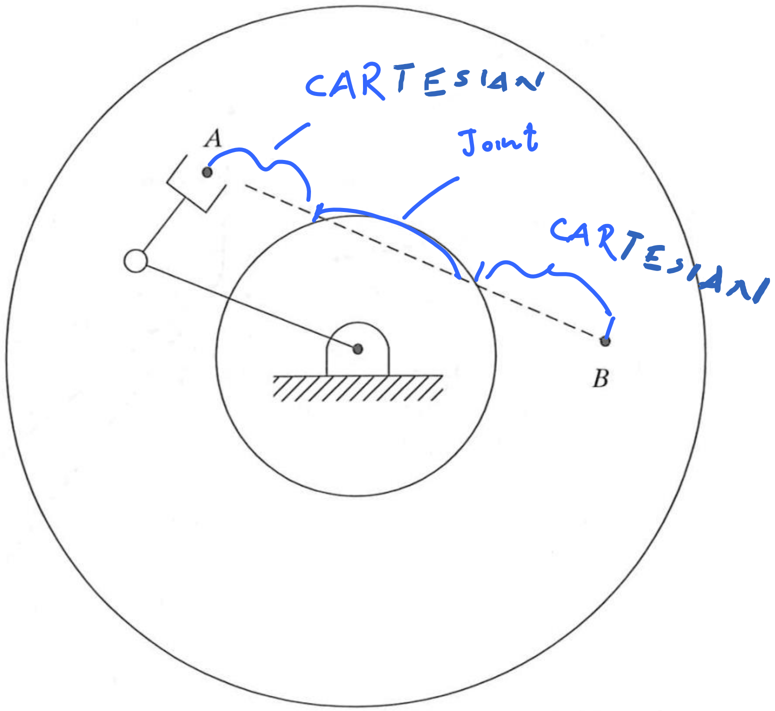
Geometric Problems with Paths in Task Space

Geometric Problems – Cartesian Paths

- **Problem Type 1 – Unreachable Intermediate Points**

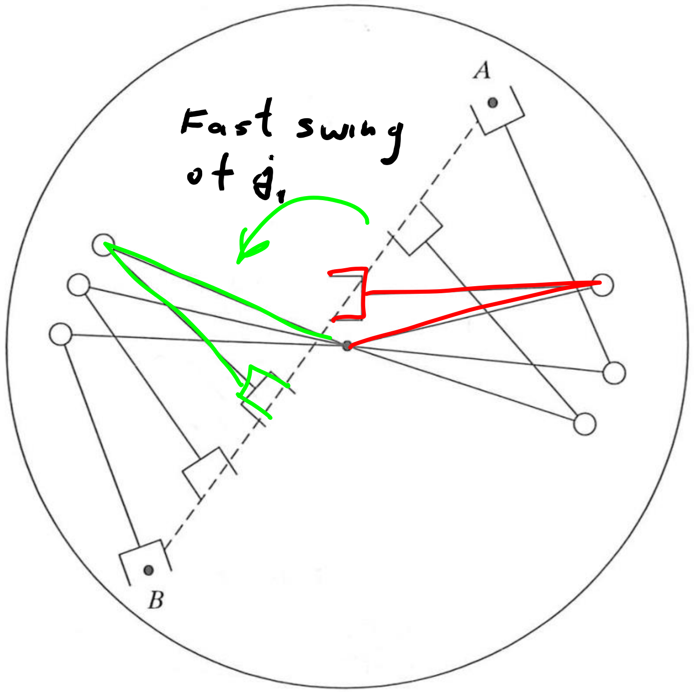
- The initial and the final point are in the reachable workspace however some point along the path may be out of the workspace.

- **Solution**
  - Joint space path – unreachable
  - Cartesian straight Path – reachable

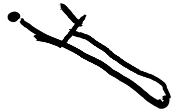


Geometric Problems – Cartesian Paths

- **Problem Type 2 – High Joint Rate Near Singularity.**
- In singularity the velocity of one or more joint approach infinity.
- The velocity of the mechanism are upper bounded, approaching singularity results in the manipulator's deviation form the desired path.
- **Solution**
  - Slow down the velocity such that all the joint velocities will remain in their bounded velocities



singularity



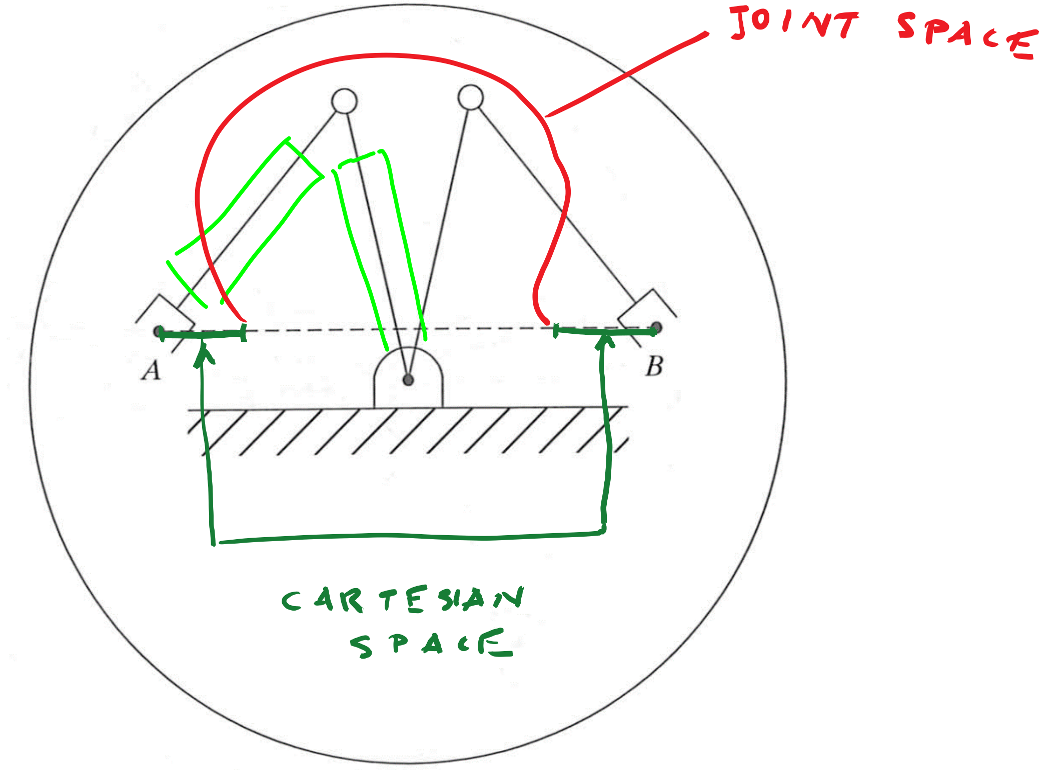
$\dot{\theta}_1 \to \infty$

Geometric Problems – Cartesian Paths

- **Problem Type 3 – Start and Goal reachable in different solutions**

- Joint limits may restrict the number of solutions that the manipulator may use given a goal point.

- Solution
  - Switch between joint space (default) and Cartesian space trajectories (used only if needed)

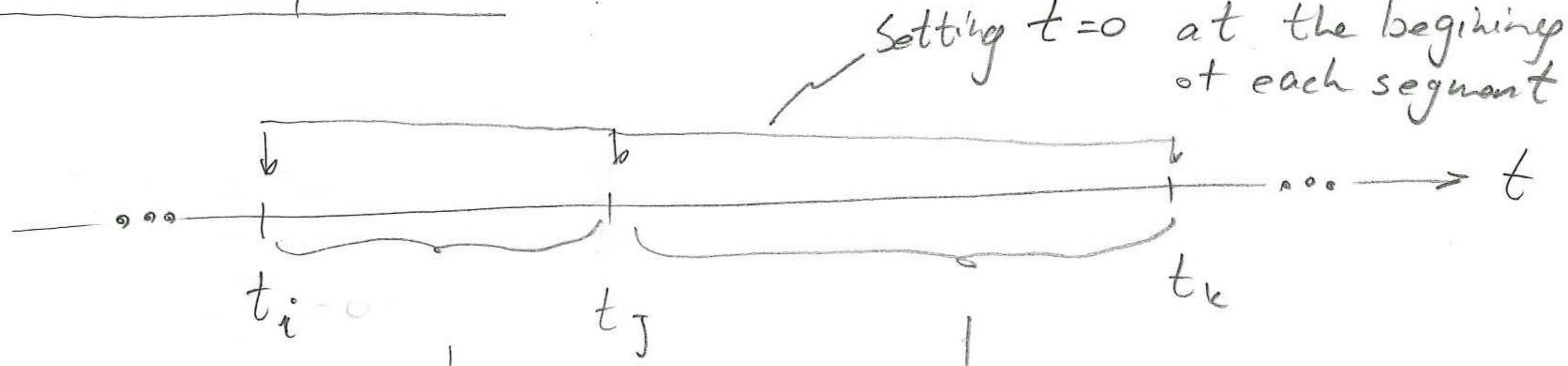


Path Generation & Run Time – Summary

Joint Space

Task Generation at Run Time – Join Space

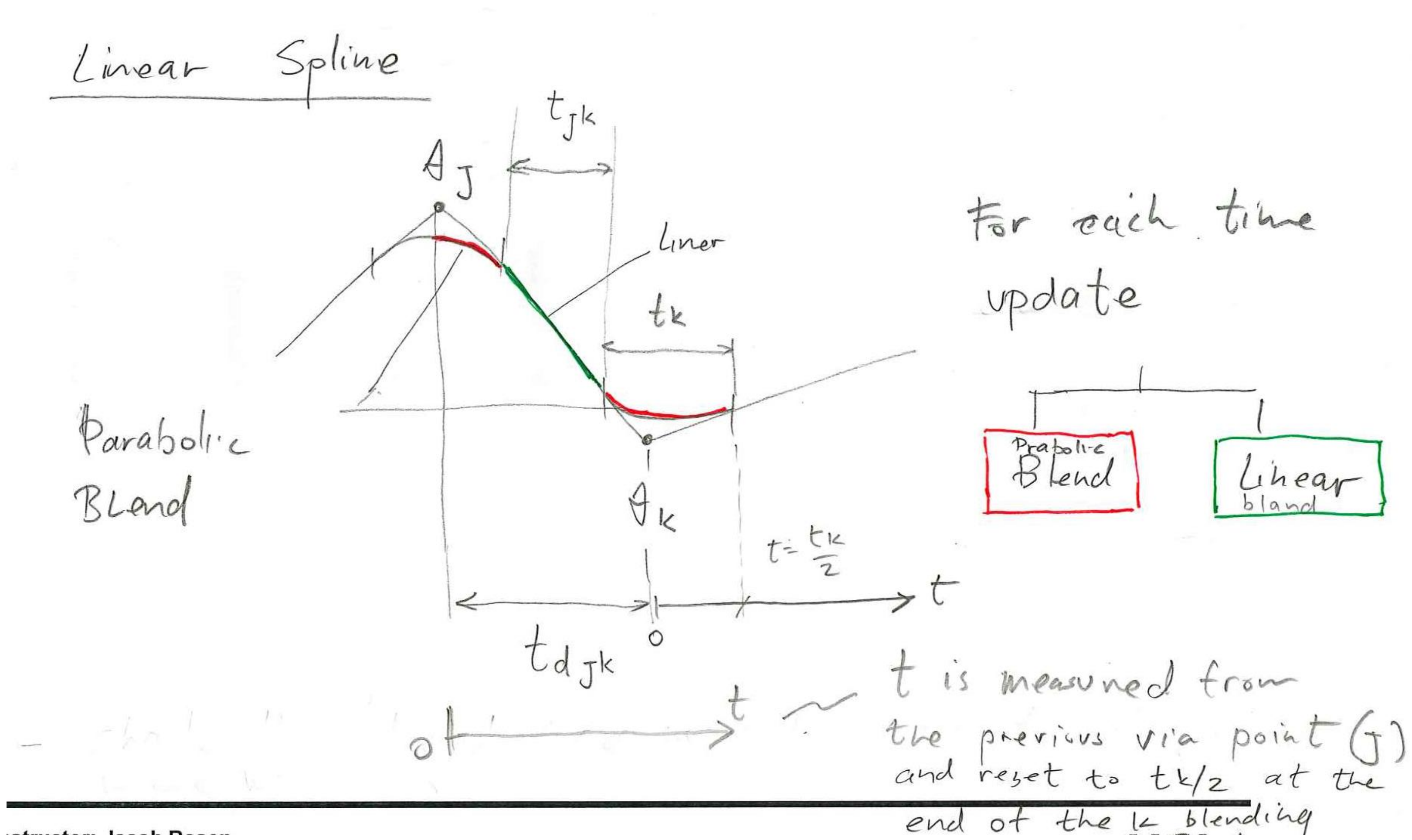
Cubic Splines



$$\theta(t: t_i \rightarrow t_J) = a_{0ij} + a_{1ij}t + a_{2ij}t^2 + a_{3ij}t^3$$

$$\theta(t: t_J \rightarrow t_k) = a_{0jk} + a_{1jk}t + a_{2jk}t^2 + a_{3jk}t^3$$

Task Generation at Run Time – Join Space

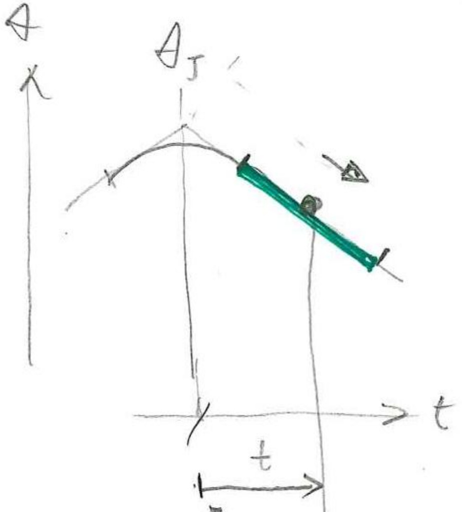


Task Generation at Run Time – Join Space

Linear

In the linear portion, the trajectory of each joint is calculated as

$$\begin{cases} \theta = \theta_J + \dot{\theta}_{Jk}\, t \\ \dot{\theta} = \dot{\theta}_{Jk} \\ \ddot{\theta} = 0 \end{cases}$$



calculated during the path planing $\dot{\theta}_{Jk} = \dfrac{\theta_k - \theta_J}{t_{dJk}}$

Time since the J via point

Task Generation at Run Time – Join Space

Parabolic Blend

In the blend region the trajectory of each joint is calculated as

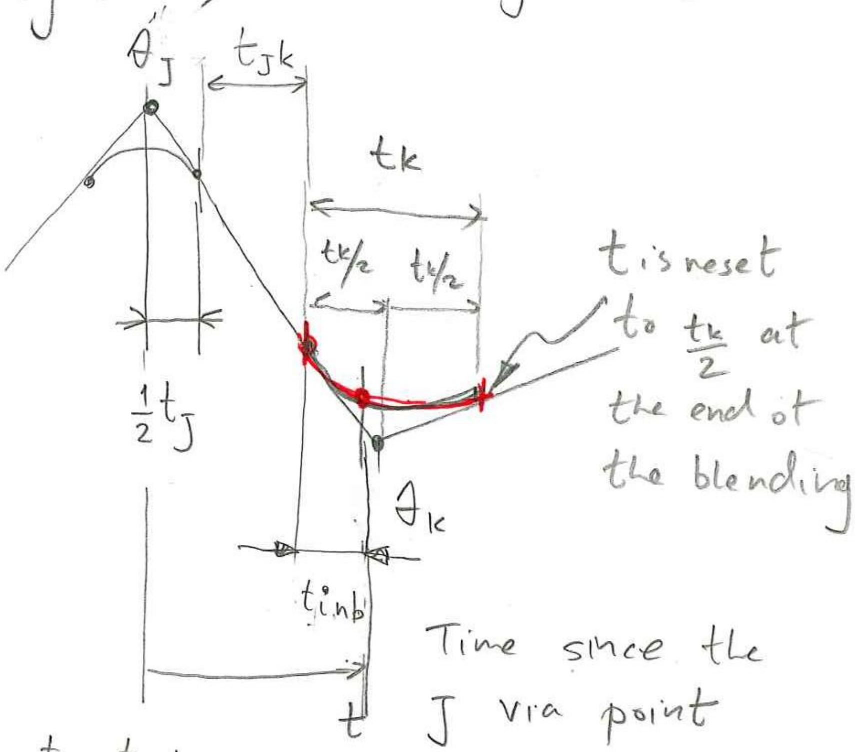
$$t_{inb} = t - \left(\frac{1}{2} t_J + t_{Jk}\right)$$

$$\theta = \theta_J + \dot{\theta}_{Jk}(t - t_{inb}) + \frac{1}{2}\ddot{\theta}_k t_{inb}^2$$

$$\dot{\theta} = \dot{\theta}_{Jk} + \ddot{\theta}_k t_{inb}$$

$$\ddot{\theta} = \ddot{\theta}_k$$

$$t - t_{inb} = t - \left[t - \left(\frac{1}{2} t_J + t_{Jk}\right)\right] = \frac{1}{2} t_J + t_{Jk}$$



– See previously eq. for $\dot{\theta}_{Jk}, \ddot{\theta}_k, t_J, t_{Jk}$

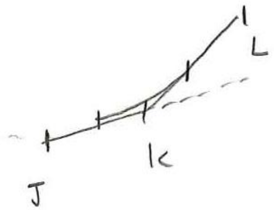
Task Generation at Run Time – Join Space

$$\dot{\theta}_{JK} = \frac{\theta_K - \theta_J}{t_{dJK}}$$

$$\ddot{\theta}_K = \underbrace{SGN(\dot{\theta}_{KL} - \dot{\theta}_{JK})}\,\overbrace{|\ddot{\theta}_K|}^{\text{user selection}}$$

SGN - Determine the sign

(+) Positive Acceleration



$$t_K = \frac{\dot{\theta}_{KL} - \dot{\theta}_{JK}}{\ddot{\theta}_K}$$

$$t_{JK} = t_{dJK} - \frac{1}{2} t_J - \frac{1}{2} t_K$$

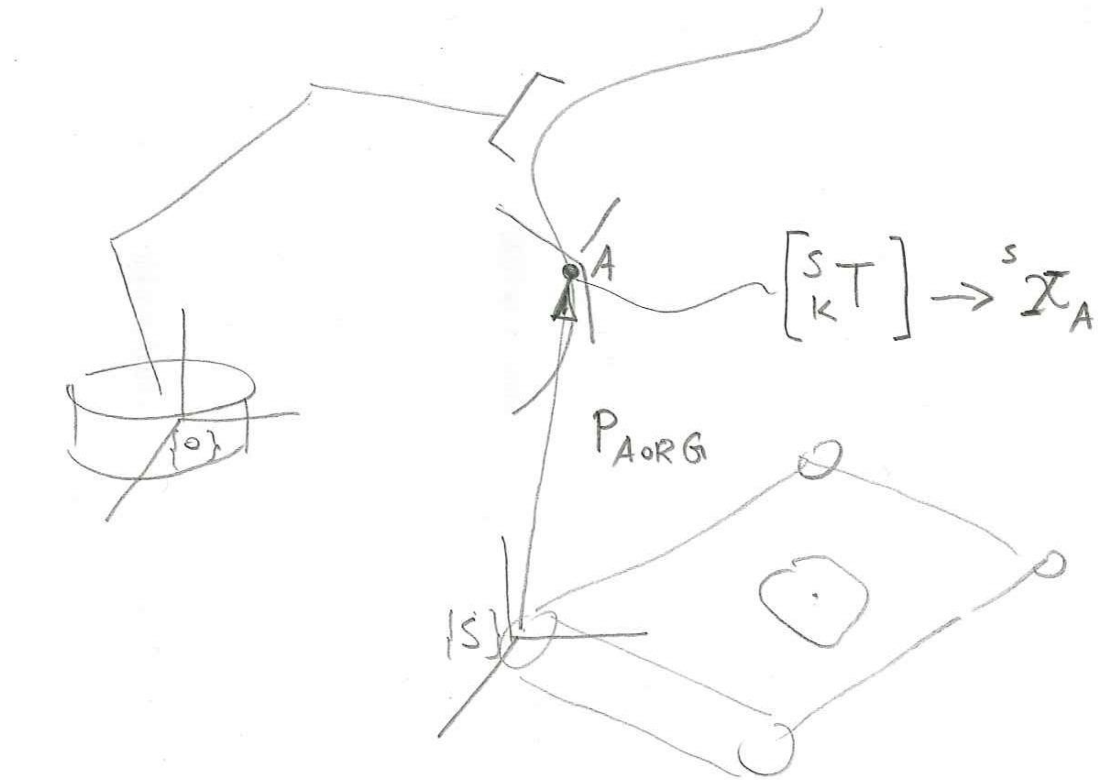
(-) Negative Acceleration

Note: See special case for first/Last segment

Path Generation & Run Time – Summary

Task Space

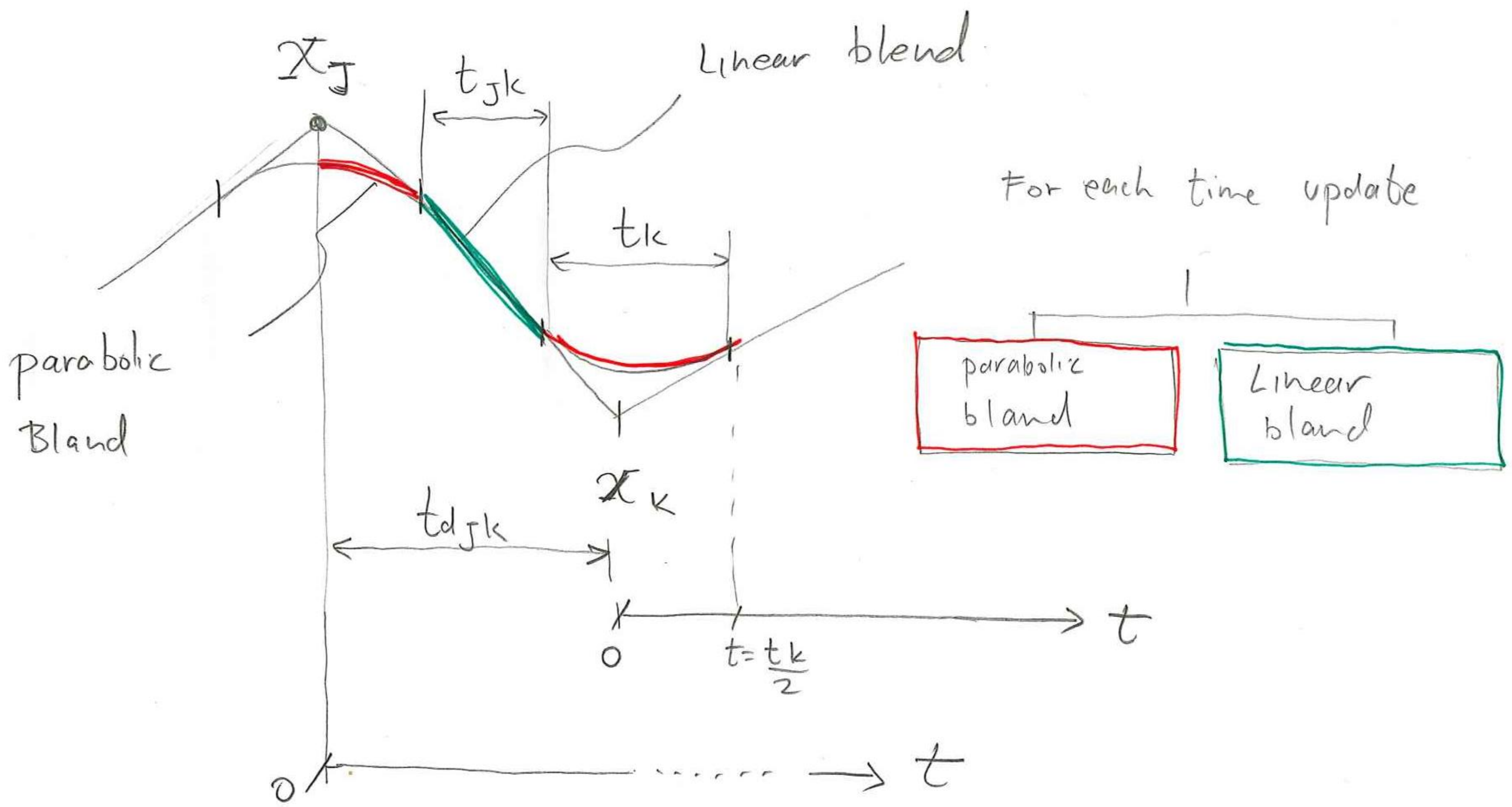
Task Generation at Run Time – Task Space



$$\left[{}^S_K T\right] \rightarrow {}^S\mathcal{X}_A = \begin{bmatrix} P_{AORG} \\ {}^S K_A \end{bmatrix} = \overset{6\times 1}{\begin{bmatrix} P_{AORG\,x} \\ P_{AORG\,y} \\ P_{AORG\,z} \\ {}^S K_{Ax} \\ {}^S K_{Ay} \\ {}^S K_{Ay} \end{bmatrix}}$$

Scaled unit vector by θ of the equ. vector

Task Generation at Run Time – Task Space

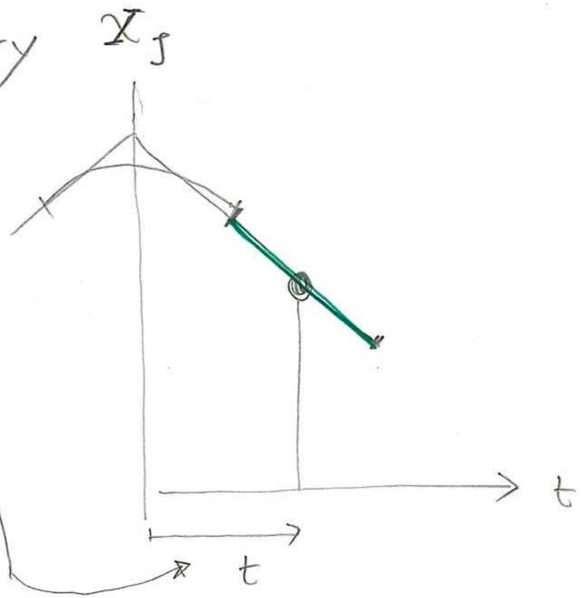


Task Generation at Run Time – Task Space

Linear Blend

In the linear portion, the trajectory of each coordinate of x

$$\begin{cases} x = x_J + \dot{x}_{Jk}\, t \\ \dot{x} = x_{Jk} \\ \ddot{x} = 0 \end{cases}$$



Time since the J via point

calculated during the path planning

$$\dot{P}_{Jk\,ORG}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \frac{P_{K.ORG}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} - P_{J\,ORG}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}}{td_{Jk}}$$

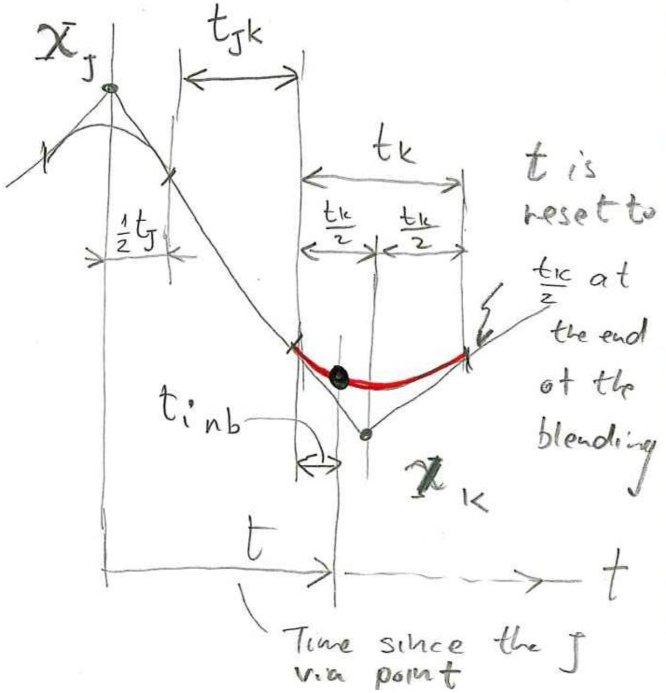
$$\dot{K}_{Jk}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \frac{K_{k}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} - K_{J}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}}{td_{Jk}}$$

Task Generation at Run Time – Task Space

Parabolic Blend

In the parabolic blend region the end effector trajectory of each coordinate of the X

$$\begin{cases} t_{inb} = t - \left(\frac{1}{2}t_J + t_{Jk}\right) \\ X = X_J + \dot{X}_{Jk}(t - t_{inb}) + \frac{1}{2}\ddot{X}_k t^2 \\ \dot{X} = \dot{X}_{Jk} + \ddot{X}_k t_{inb} \\ \ddot{X} = \ddot{X}_k \end{cases}$$

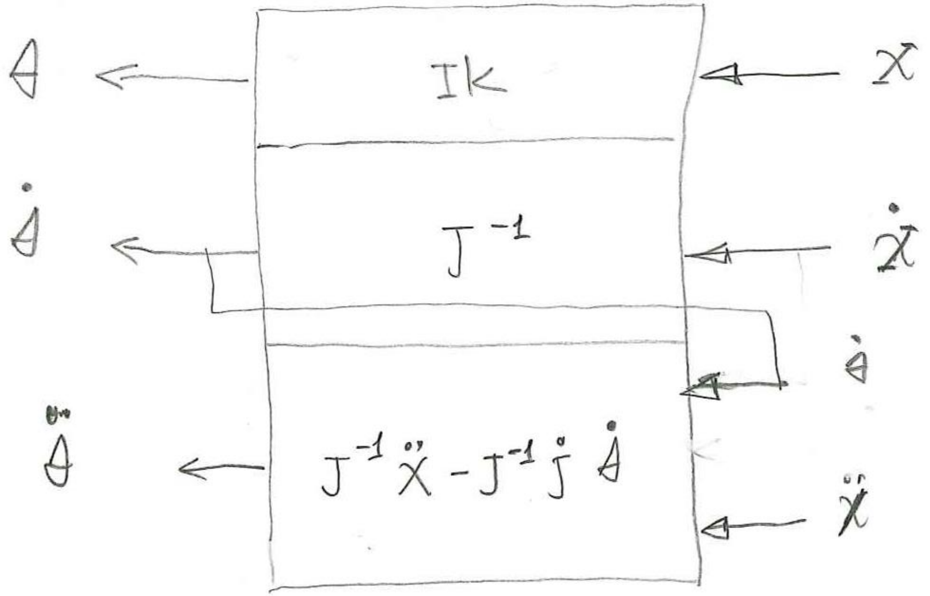


where $\dot{X}_{Jk}$, $\ddot{X}_k$, t_J and t_{Jk} are computed at plan time, just as in the joint space

Task Generation at Run Time – Task/Joint Space Mapping

Mapping $x, \dot{x}, \ddot{x}$ from the task space to the joint space

<u>Option 1</u> – Time derivative are done at the task space



$$\dot{\theta} = J^{-1}\dot{x}$$

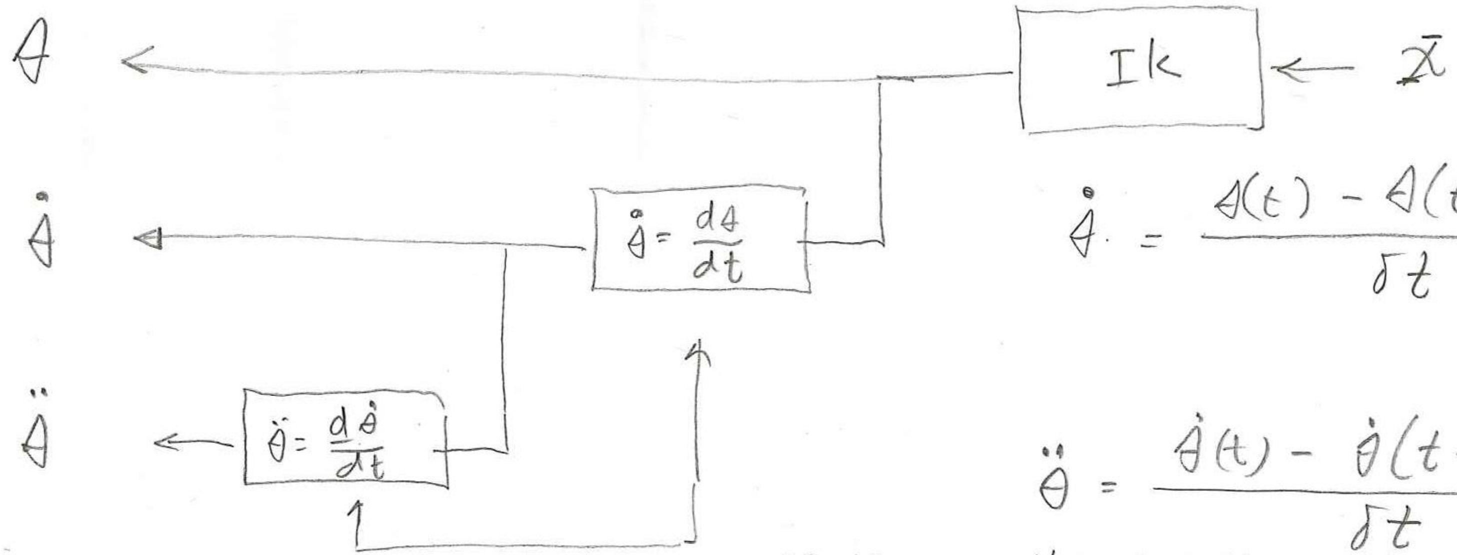
$$\dot{x} = J\dot{\theta}$$

$$\ddot{x} = \dot{J}\dot{\theta} + J\ddot{\theta}$$

$$\ddot{\theta} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{\theta}$$

Task Generation at Run Time – Task/Joint Space Mapping

Option 2 – Time derivative of $\dot{\theta}, \ddot{\theta}$ are done at the joint space



$$\dot{\theta} = \frac{\theta(t) - \theta(t - \delta t)}{\delta t}$$

$$\ddot{\theta} = \frac{\dot{\theta}(t) - \dot{\theta}(t - \delta t)}{\delta t}$$

- Note + This differentiation can be done off-line resulting in better quality of $\dot{\theta}, \ddot{\theta}$
- + Many control systems do not require a $\ddot{\theta}$ input