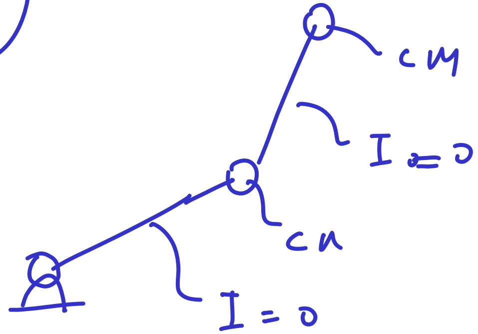
Trajectory Generation 1/2

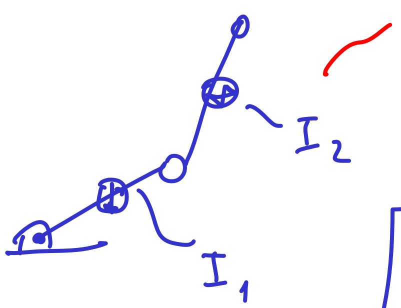
I



In class : Newton Euler

@ Home : Lagrange (SEE NOTES)
(General Form)

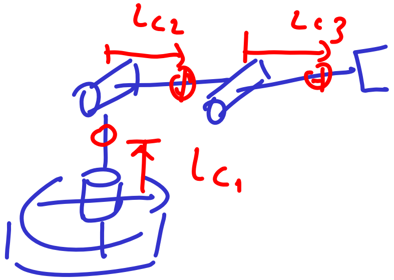
II



$I = \begin{bmatrix} I_x & 0 \\ 0 & I_y \end{bmatrix} I_z$

In class : Lagrange

@ Home : Newton Euler

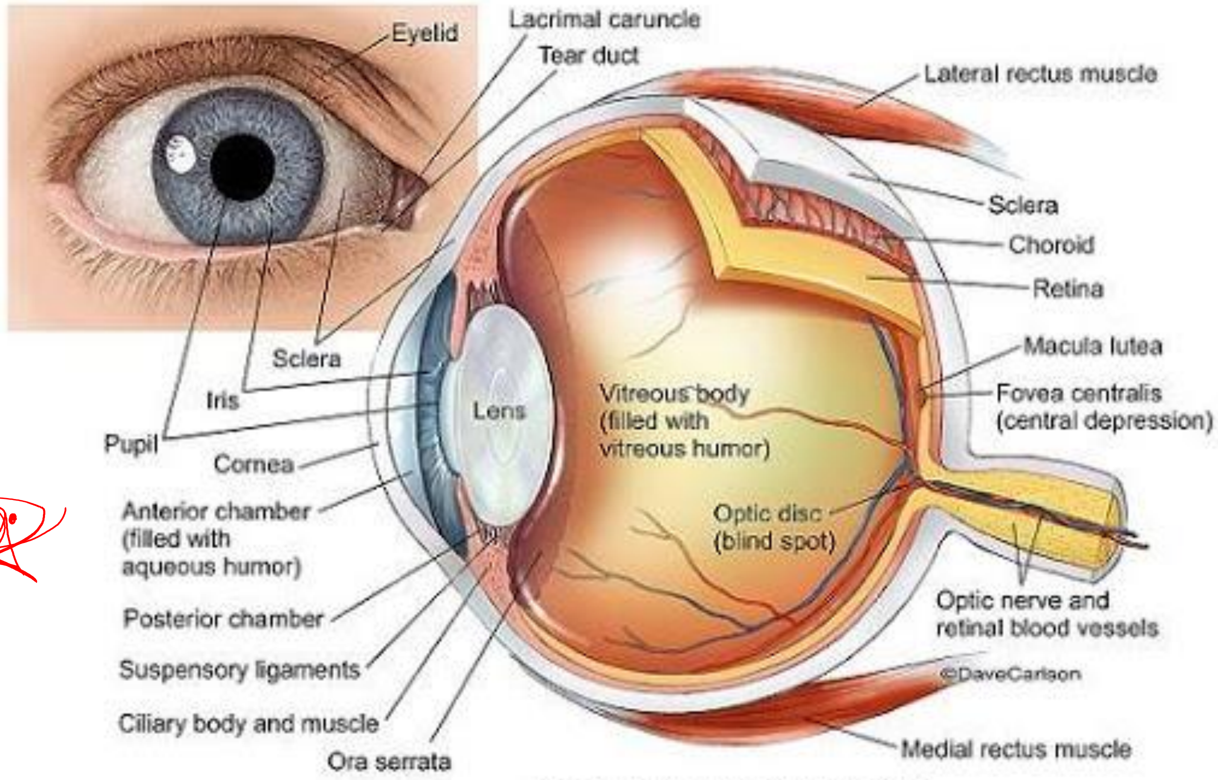


3R Eq. of Motion

- Newton Euler
- Lagrange

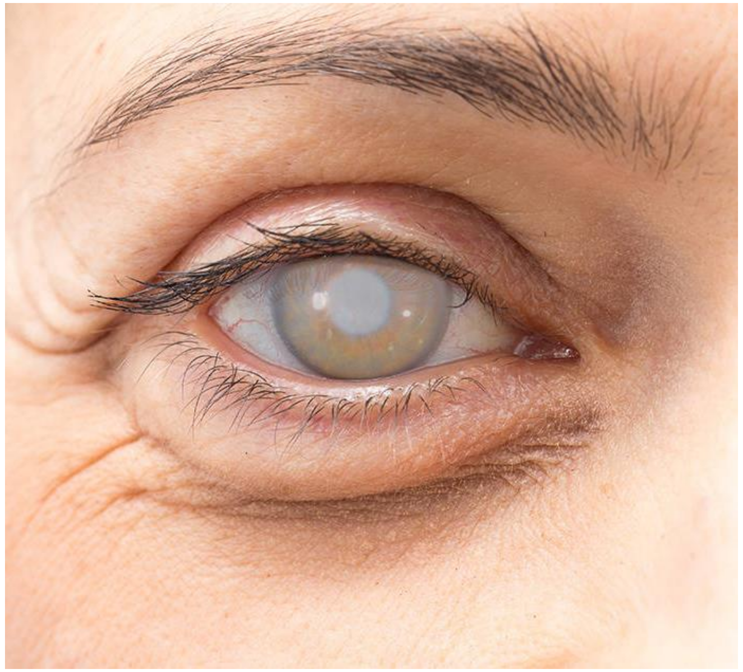
Introduction

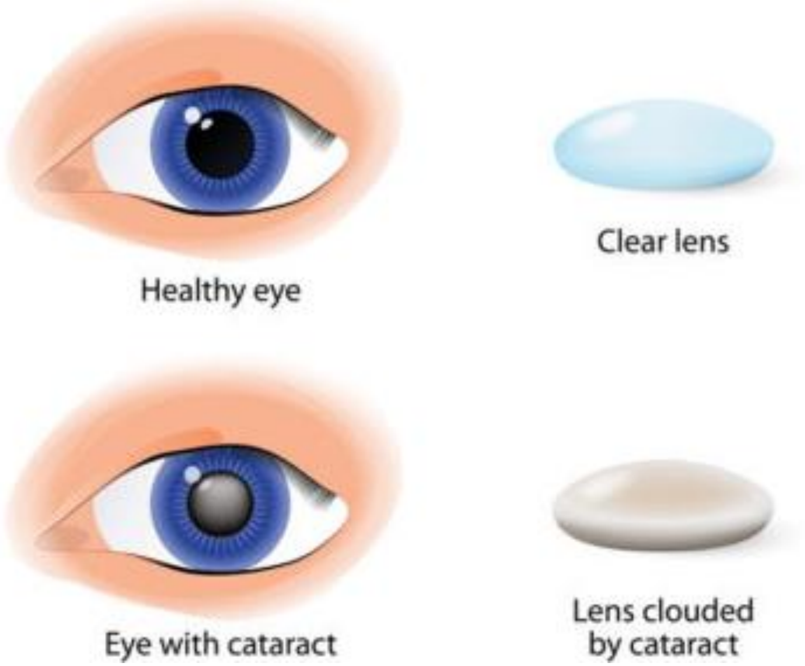
Motion Planning



Right Eye (viewed from above)

Motion Planning





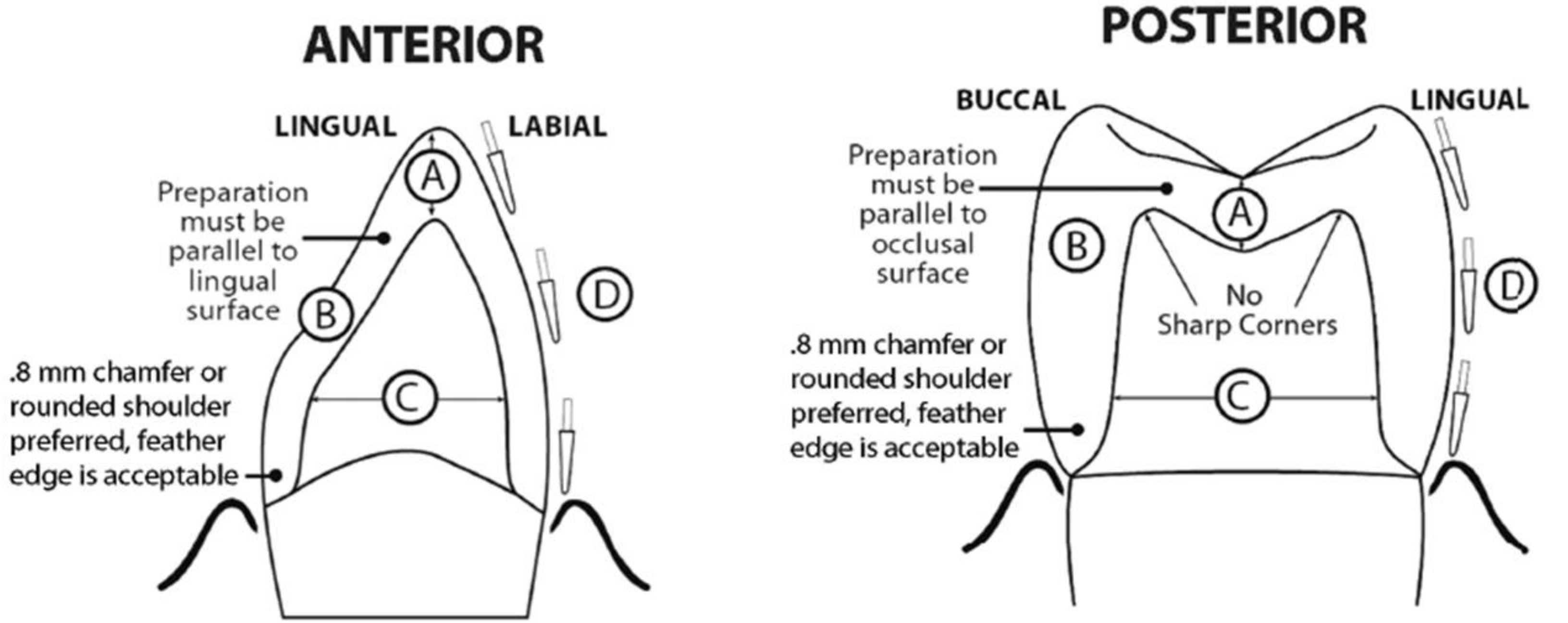
Motion Planning

Routine Cataract Surgery

https://youtu.be/QbeI72QmFAU

Motion Planning

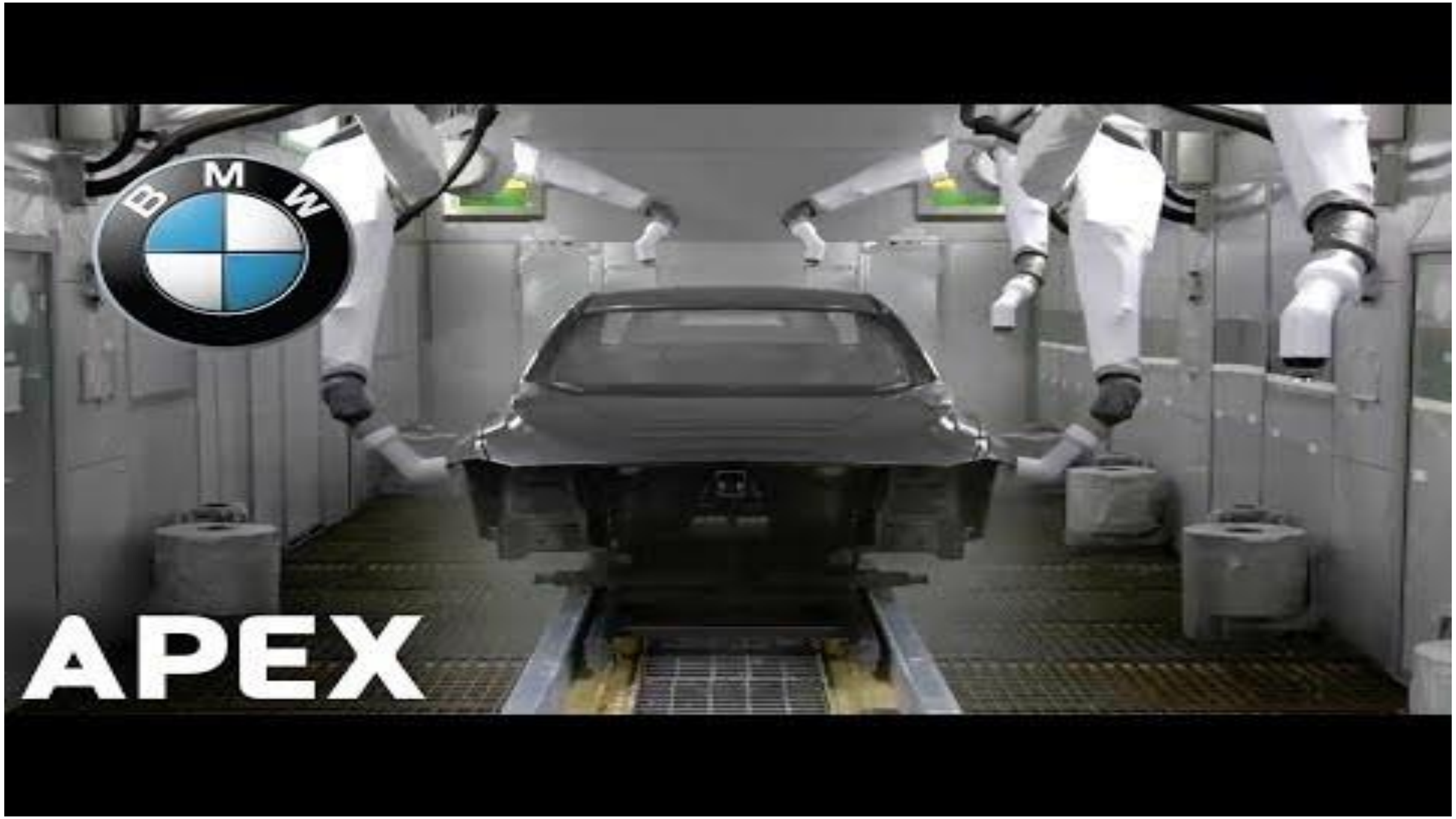
Motion Planning



A. 1.0 mm - 1.5 mm occlusal reduction
B. 1.0 mm middle third reduction
C. Buccal and lingual walls must be convergent
D. Preparation should be cut in three planes

Motion Planning

Motion Planning



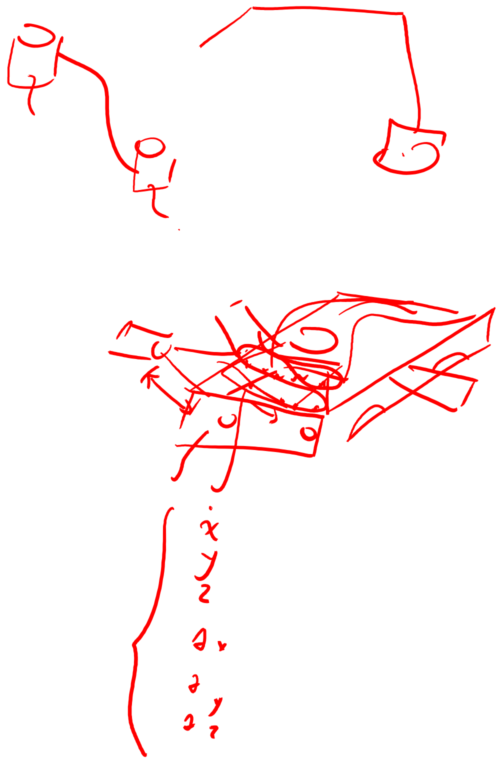
Motion Planning

Motion Planning

Problem Defenition

Motion Planning – Hierarchy

- Trajectory planning is a subset of the overall problem that is ***navigation or motion planning***. The typical hierarchy of motion planning is as follows:
  - **Task planning** – Designing a set of high-level goals, such as “go pick up the object in front of you”.
  - **Path planning** – Generating a feasible path from a start point to a goal point. A path usually consists of a set of connected waypoints.
  - **Trajectory planning** – Generating a time schedule for how to follow a path given constraints such as position, velocity, and acceleration.
  - **Trajectory following** – Once the entire trajectory is planned, there needs to be a control system that can execute the trajectory in a sufficiently accurate manner.
- Q: What’s the difference between path planning and trajectory planning?
- A: A trajectory is a description of how to follow a path over time



Trajectory Generation – Problem Definition

Problem

Given: Manipulator geometry, End Effector Path (via point)

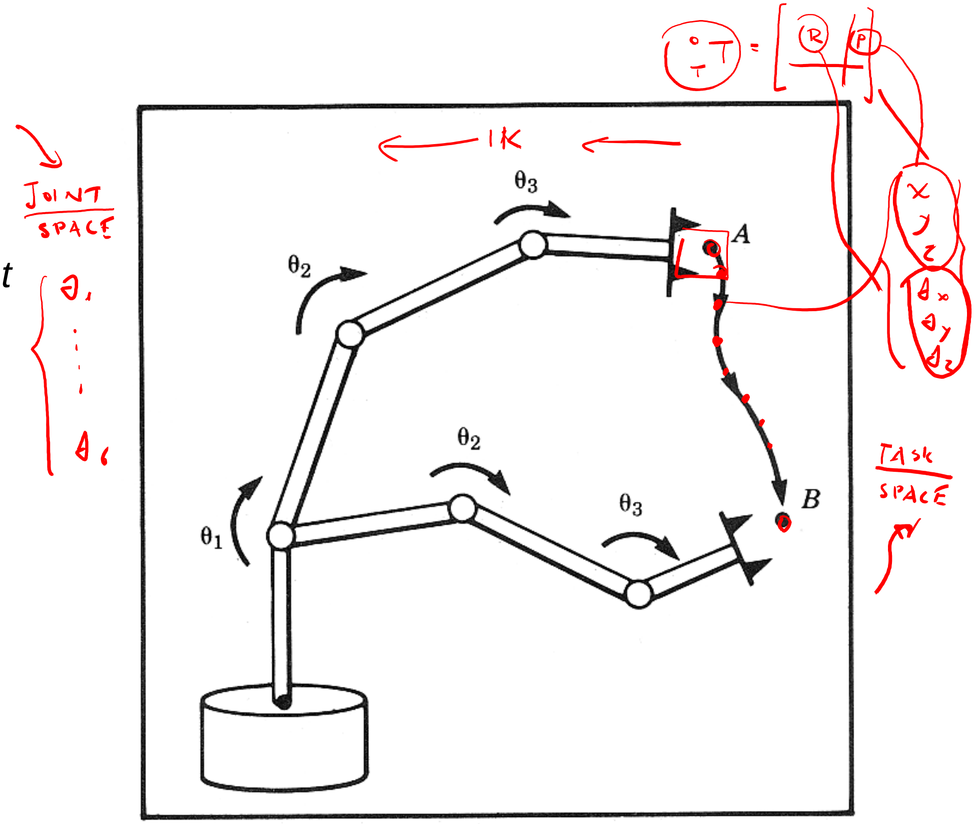
Compute: The trajectory of each joint such that the end effector move in space from point A to Point B

Solution (Domains)

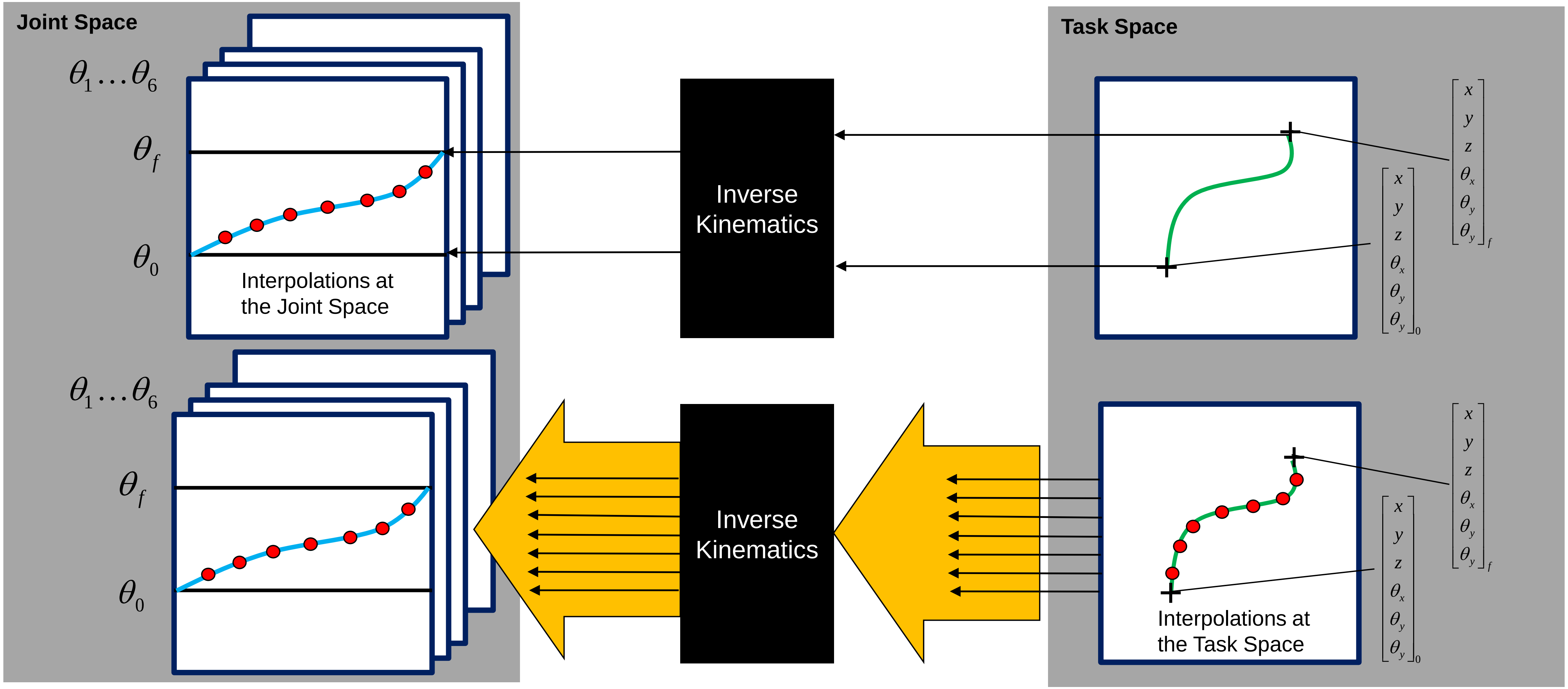
- **Joint space / Task Space**

Definitions

- **Trajectory (Definition)** - Time history of position, velocity, and acceleration for each DOF.
- **Trajectory Generation –** Methods of computing a trajectory that describes the desired motion of a manipulator in a multidimensional space



Task Space Versus Joint Space - Interpolations



Joint Space

$\theta_1 \ldots \theta_6$

θ_f

θ_0

Interpolations at the Joint Space

$\theta_1 \ldots \theta_6$

θ_f

θ_0

Inverse Kinematics

Inverse Kinematics

Task Space

$\begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_y \end{bmatrix}_f$

$\begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_y \end{bmatrix}_0$

$\begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_y \end{bmatrix}_f$

$\begin{bmatrix} x \\ y \\ z \\ \theta_x \\ \theta_y \\ \theta_y \end{bmatrix}_0$

Interpolations at the Task Space

Task Space Versus Joint Space

Interpolation Method	Computational Requirements	Accuracy of the End Effector
Join Space	Low (Advantage)	Low (Disadvantage)
Task Space	High (Disadvantage)	High (Advantage)

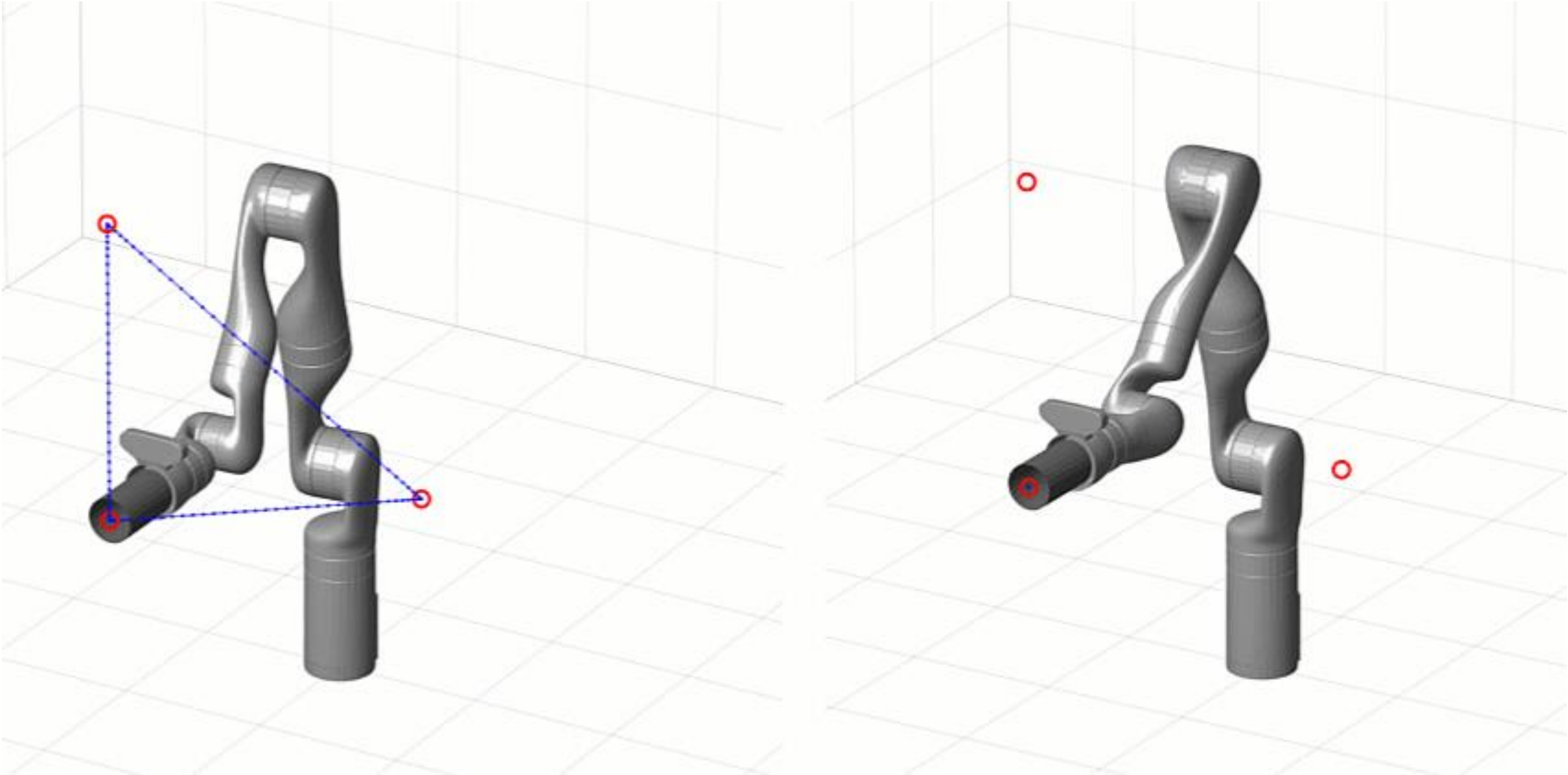
Task Space Versus Joint Space

- **Task space** means the waypoints and interpolation are on the Cartesian pose (position and orientation) of a specific location on the manipulator – usually the end effector.

- **Joint space** means the waypoints and interpolation are directly on the joint positions (angles or displacements, depending on the type of joint)

- **Main Difference** –
  - **Smoothness** - task-space trajectories tend to look more “natural” than joint-space trajectories because the end effector is moving smoothly with respect to the environment even if the joints are not.
  - **IK / Computation** - The big drawback is that following a task-space trajectory involves solving inverse kinematics (IK) more often than a joint-space trajectory, which means *a lot* more computation especially if your IK solver is based on optimization.

Task Space Versus Joint Space



Trajectory Generation – Problem Definition

- **Human Machine interface (requirements)**
  - Human - Specifying trajectories with simple description of the desired motion
    - Example – start / end points position and orientation of the end effectors
  - System - Designing the details of the trajectory
    - Example – Design the exact shape of the path, duration, joint velocity, etc.

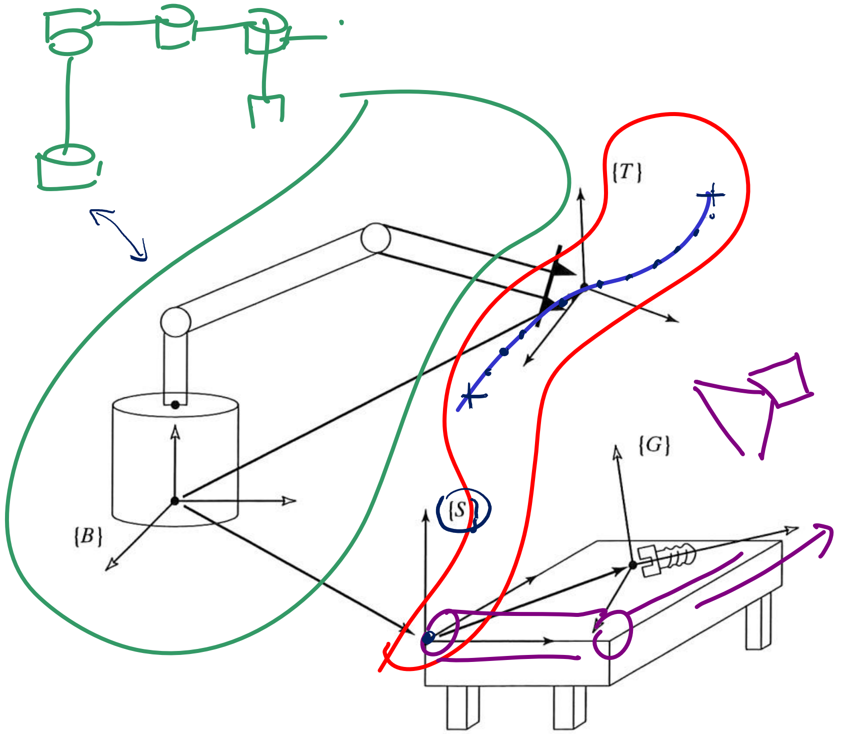
- **Trajectory Representation**
  - Representation of trajectory in the computer after they were planned

- **Trajectory Generation**
  - Generation occurs at runtime (real time) where positions, velocities, and accelerations are computed.
  - Path update rate 60-2000 Hz

General Consideration

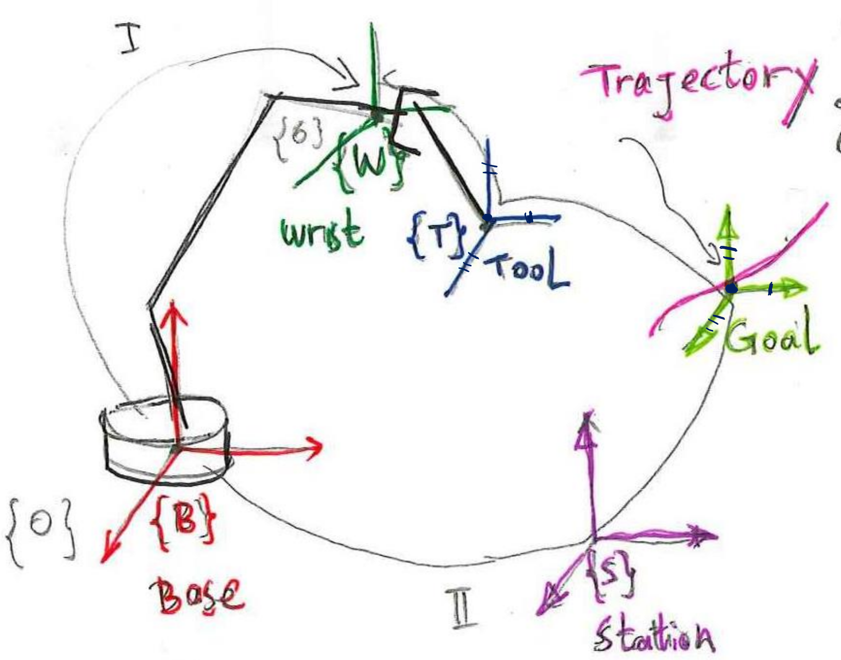
- **General approach for the motion of the manipulator**
  - Specify the path as a motion of the tool frame {*T*} relative to the station frame {*S*}. Frame {*S*} may change it position in time (e.g. conveyer belt)

- **Advantages**
  - Decouple the motion description from any particular robot, end effector, or workspace.
  - Modularity – Use the same path with:
    - Different robot
    - Different tool size



Trajectory Generation & Inverse Kinematics
General Approach

Transforming from the task space to the joint space using invers kinematics



$$ {}^{0}_{6}T = \boxed{\overbrace{{}^{B}_{W}T}^{I} = \overbrace{{}^{B}_{S}T\;{}^{S}_{G}T\;{}^{G}_{T}T\;{}^{T}_{W}T}^{II}} $$

II → Desired position and orientation

Goal (KNOWN)

${}^{0}_{6}T = {}^{B}_{W}T$ – Forward kinematics

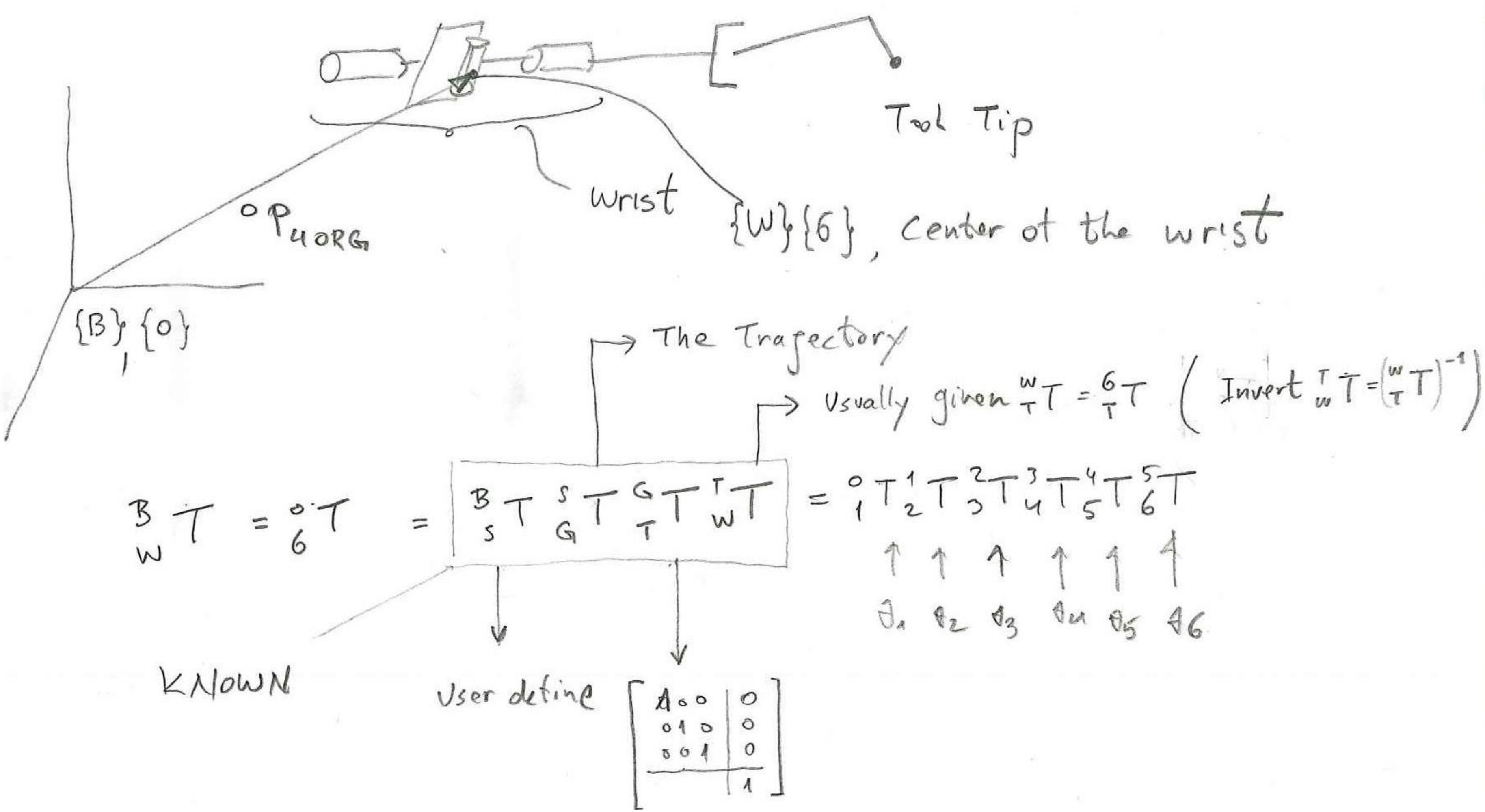
${}^{B}_{S}T$ – Location of the station with respect to the base

${}^{S}_{G}T$ – Defenition of the Trajectory

${}^{G}_{T}T = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$ – Full allignment (position, orientation)

${}^{T}_{W}T$ = Location of the wrist with respect to the tool → usualy given ${}^{W}_{T}T$

Trajectory Generation & Inverse Kinematics
General Approach



Usually given ${}^W_T T = {}^6_T T$ (Invert ${}^T_W T = ({}^W_T T)^{-1}$)

$$
{}^B_W T = {}^0_6 T = \boxed{{}^B_S T\; {}^S_G T\; {}^G_T T\; {}^T_W T} = {}^0_1 T\; {}^1_2 T\; {}^2_3 T\; {}^3_4 T\; {}^4_5 T\; {}^5_6 T
$$

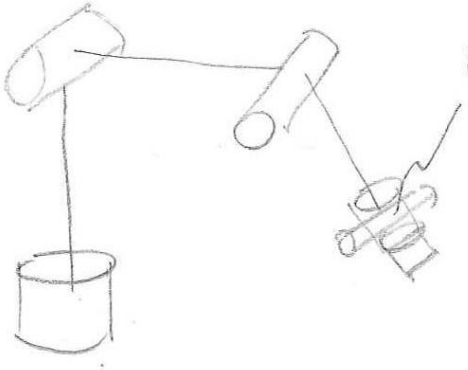
$$
\begin{array}{cccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6
\end{array}
$$

User define

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\hline
 & & & 1
\end{array}\right]
$$

Trajectory Generation & Inverse Kinematics
General Approach

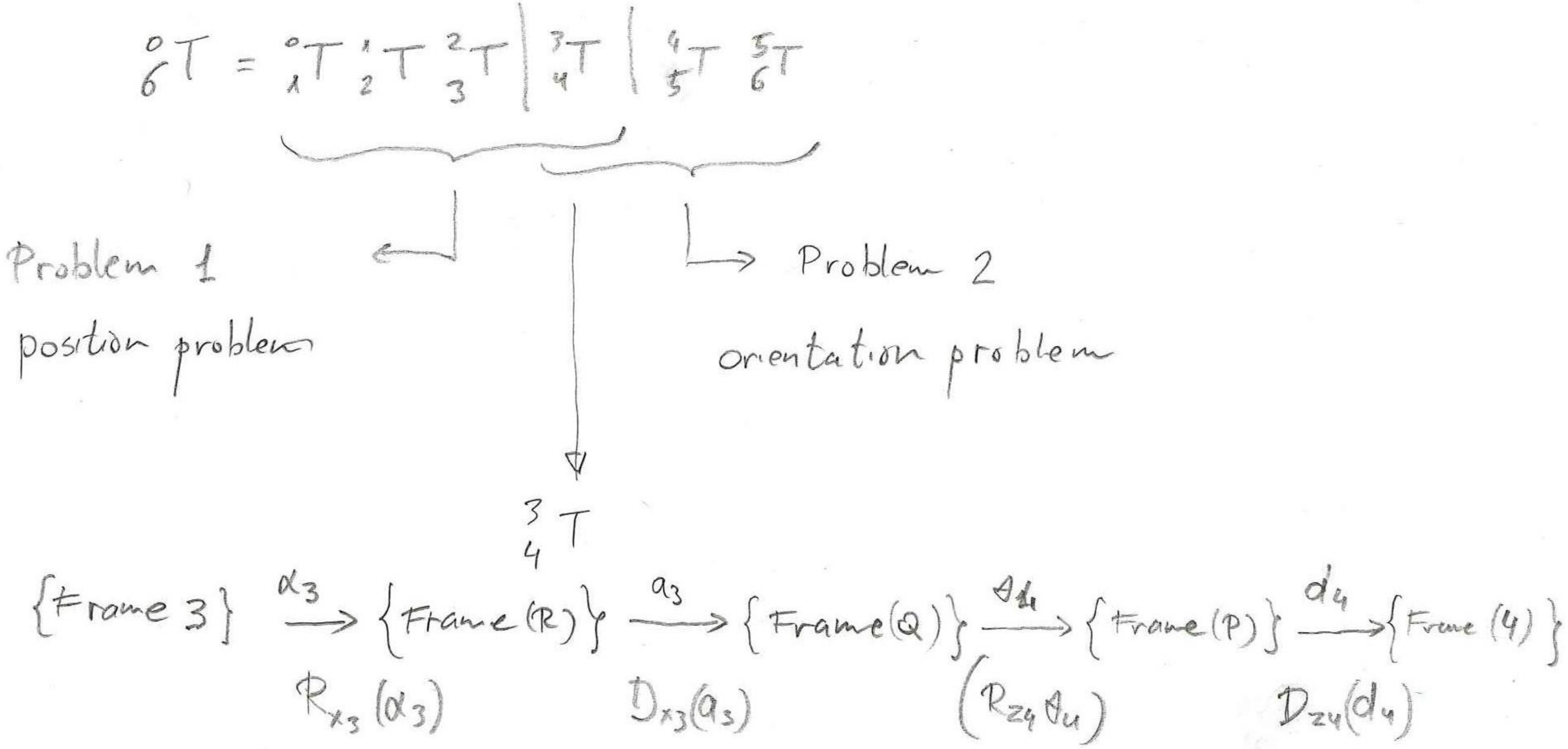
$${}^0_6T = \underset{\theta_1}{\underset{\downarrow}{{}^0_1T}}\ \underset{\theta_2}{\underset{\downarrow}{{}^1_2T}}\ \underset{\theta_3}{\underset{\downarrow}{{}^2_3T}}\ \underset{\theta_4}{\underset{\downarrow}{{}^3_4T}}\ \underset{\theta_5}{\underset{\downarrow}{{}^4_5T}}\ \underset{\theta_6}{\underset{\downarrow}{{}^5_6T}} = \underbrace{\left[\begin{array}{c|c} {}^0_6R & {}^0P_6 \\ \hline & \end{array}\right]}_{\text{GIVEN}}$$



Given 3 intersecting axis 4,5,6 (origines of 4,56 are at the same point)

$${}^0P_6 = {}^6P_4$$

Trajectory Generation & Inverse Kinematics
General Approach



$${}^{0}_{6}T = {}^{0}_{1}T \; {}^{1}_{2}T \; {}^{2}_{3}T \; \Big| \; {}^{3}_{4}T \; \Big| \; {}^{4}_{5}T \; {}^{5}_{6}T$$

Problem 1

position problem

Problem 2

orientation problem

$${}^{3}_{4}T$$

$$\{\text{Frame } 3\} \xrightarrow{\alpha_3} \{\text{Frame}(R)\} \xrightarrow{a_3} \{\text{Frame}(Q)\} \xrightarrow{\theta_4} \{\text{Frame}(P)\} \xrightarrow{d_4} \{\text{Frame}(4)\}$$

$$R_{x_3}(\alpha_3) \qquad D_{x_3}(a_3) \qquad (R_{z_4}\theta_4) \qquad D_{z_4}(d_4)$$

Trajectory Generation & Inverse Kinematics
General Approach

$$ {}^6_6T = \underbrace{{}^0_1T\,{}^1_2T\,{}^2_3T \;\vdots\; R_{x3}(\alpha_\alpha)\, D_{x3}(a_3)}_{\substack{\text{Problem 1}\\ \text{position problem}}} \;\Big|\; \underbrace{R_{z4}(\theta_4)\, D_{z4}(d_4)\, {}^4_5T\, {}^5_6T}_{\substack{\text{Problem 2}\\ \text{Oriantation problem}}} $$

$$ {}^0_6T = {}^0_1T\; {}^1_2T\; {}^2_3T\; \underset{\substack{\uparrow\\ \text{set } \theta_4=0}}{{}^3_4T}\Big|_{\theta_4=0} \;\Bigg|\; \underset{\substack{\uparrow\\ \text{set } \alpha_3=0}}{{}^3_4T}\Big|_{\alpha_3=0}\; {}^4_5T\; {}^5_6T $$

Trajectory Generation & Inverse Kinematics
General Approach

$${}^0_6R = {}^0_3R \; {}^3_4R \; {}^4_6R$$

$$\downarrow$$

$${}^0_6R = {}^0_3R \left(R(\alpha_3) \; \underset{\underset{D_{x_3}(a_3)}{\uparrow}}{I} \;\Big|\; R(\theta_4) \; \underset{\underset{D_{z_4}(d_4)}{\uparrow}}{I} \right) {}^4_6R$$

$${}^0_6R = \left[{}^0_3R \; R_{x_3}(\alpha_3) \right] \left[R_{z_4}(\theta_4) \; {}^4_6R \right]$$

Trajectory Generation & Inverse Kinematics
General Approach

Solving for $\theta_4, \theta_5, \theta_6$

$$\underbrace{R_{Z4}(\theta_4)\,{}^4_6R} = \underbrace{\left[{}^0_3R\,R_{X3}(\alpha_3)\right]^{-1}}_{\text{Solved in Problem 1; Known } \theta_1, \theta_2, \theta_3}\;\underbrace{{}^0_6R}_{\text{Desired oriantation given for every point on the trajectory}}$$

$$R_{Z4}(\theta_4)\,{}^4_5R(\theta_5)\,{}^5_6R(\theta_6) = \bar{R}_D \longrightarrow$$

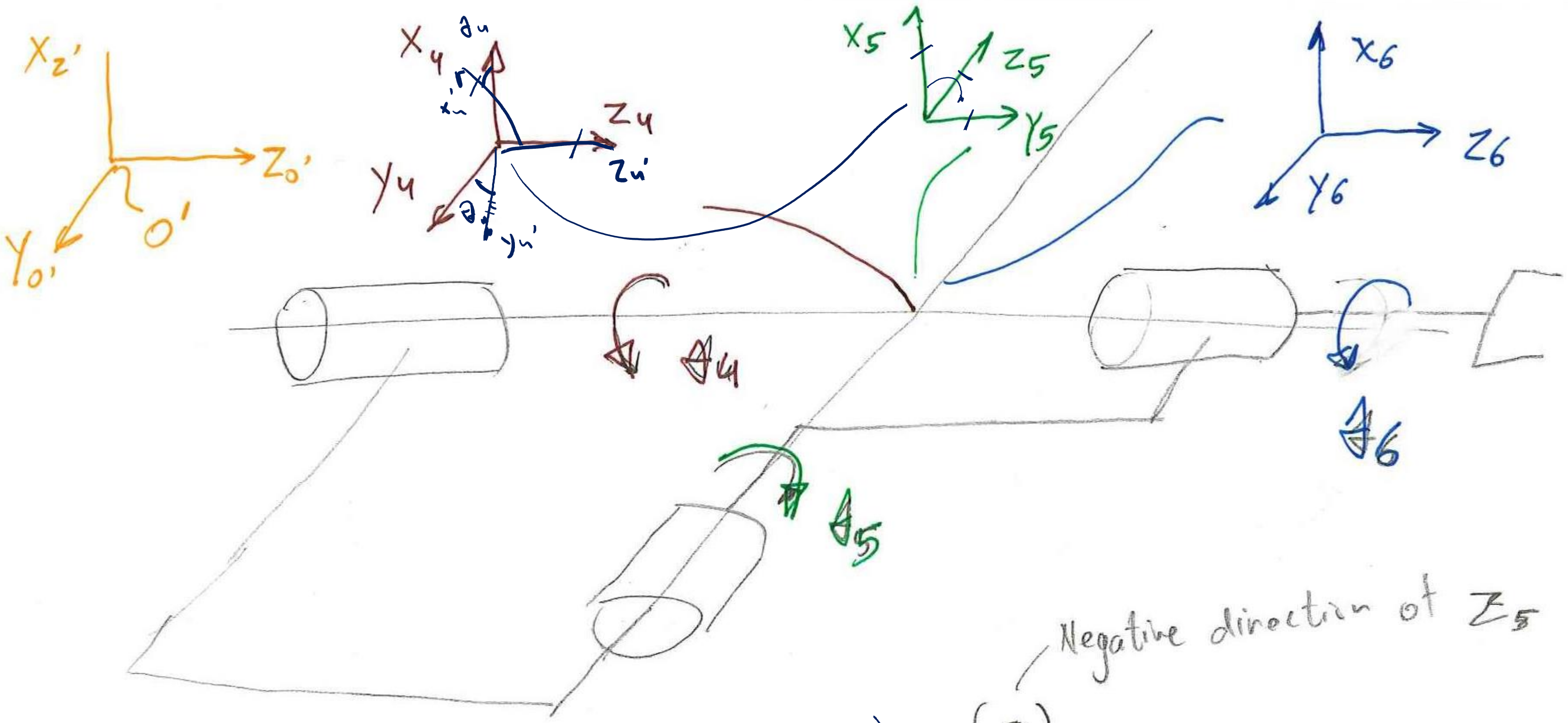
[Desired orintation of the wrist taking into account the contribution of the first 3 angles to the orintation]

Trajectory Generation & Inverse Kinematics
General Approach

$$R_{z4}(\theta_4)\,{}^4_5R(\theta_5)\,{}^5_6R(\theta_6) = \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}^{\bar{R}_D}$$

Solve for $\theta_4, \theta_5, \theta_6$ using the Z-Y-Z problem

Trajectory Generation & Inverse Kinematics
General Approach



Negative direction of Z_5

(-)

$$R_{Euler} = R_z(\theta_4)\ R_{y4}(\theta_5)\ R_{z4}(\theta_6)$$

Trajectory Generation & Inverse Kinematics
General Approach

$$R_{ZYZ}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} C\alpha & -S\alpha & 0 \\ S\alpha & C\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} C\gamma & -S\gamma & 0 \\ S\gamma & C\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\theta_4 \quad -\theta_5 \quad \theta_6$$

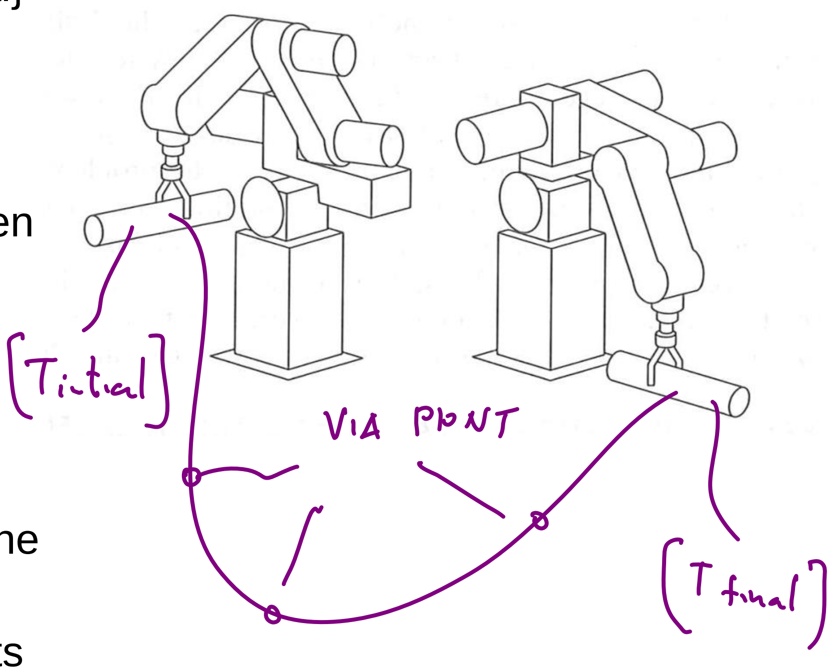
Note · $\underset{0}{\nabla}$

$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

General Consideration – Via Points

- **Basic Problem –** Move the tool frame {T} from its initial position / orientation {T_initial} to the final position / orientation {T_final}.

- **Specific Description**
  - **Via Point** – Intermediate points between the initial and the final end- effector locations that the end-effector mast go through and match it position and orientation along the trajectory.
  - Each via point is defined by a **frame defining the position/orintataion** of the tool with respect to the station frame
  - **Path Points** – includes all the via points along with the initial and final points
  - **Point (Frame)** – Every point on the trajectory is define by a frame (spatial description)

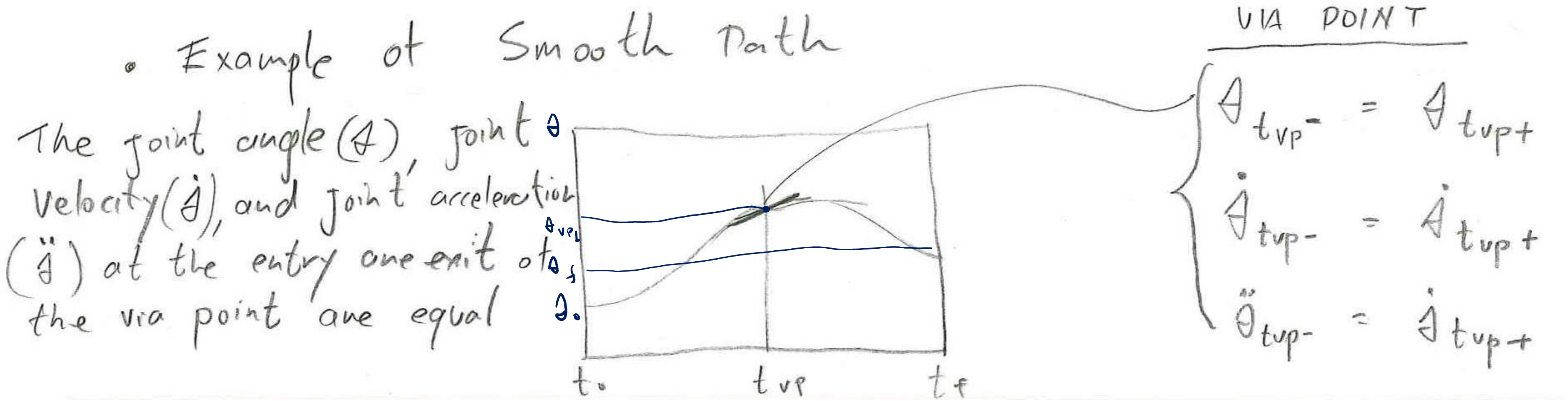


$$T = \left[\begin{array}{c|c} {}^{S}_{T}R & {}^{S}P_{T} \\ \hline & \end{array}\right]$$

General Consideration – Smooth Path

- **“Smooth” Path or Function**
  - Continuous path / function with first and second derivatives.
  - Add constrains on the spatial and temporal qualities of the path between the via-points

- **Implications of non-smooth path**
  - Increase wear in the mechanism (rough jerky movement)
  - Vibration – exciting resonances.



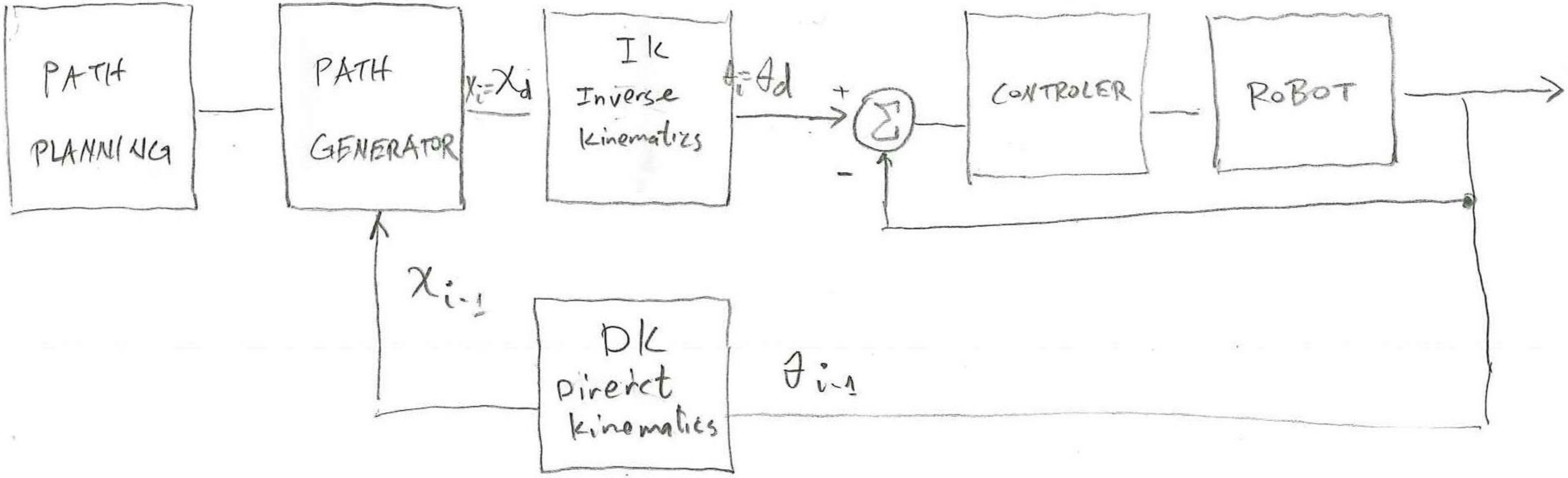
- Example of Smooth Path

The joint angle (θ), joint velocity ($\dot{\theta}$), and joint acceleration ($\ddot{\theta}$) at the entry and exit of the via point are equal

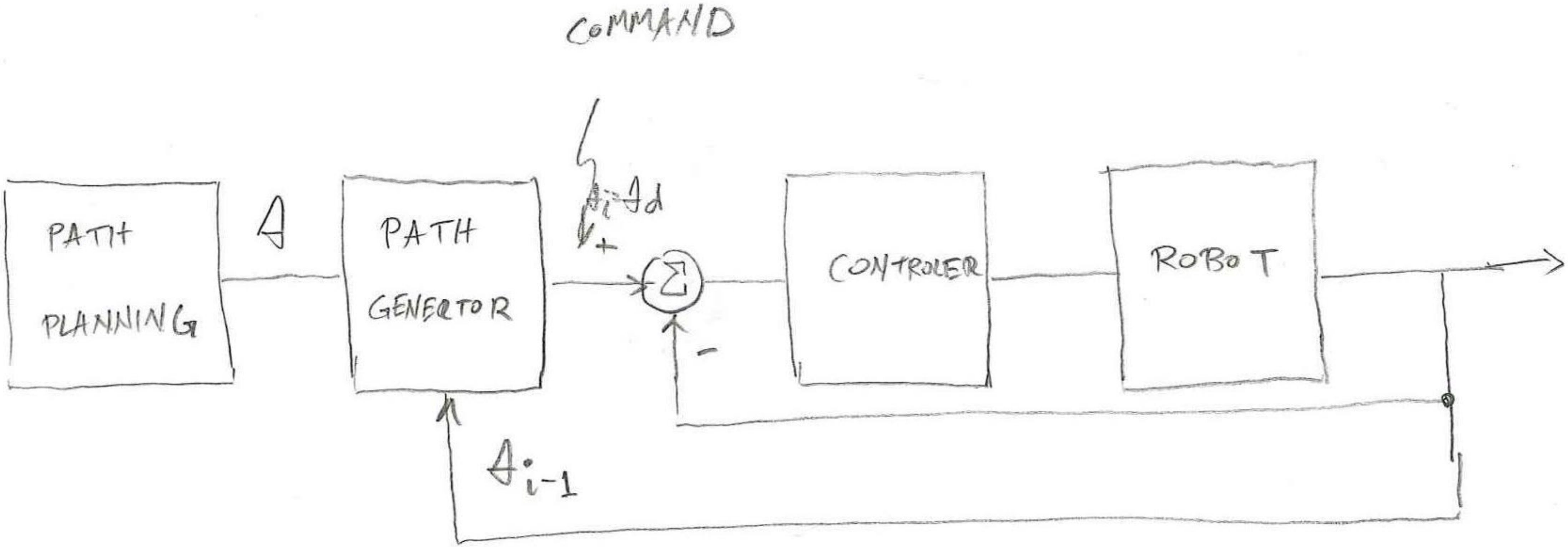
VIA POINT

$$\theta_{tvp-} = \theta_{tvp+}$$
$$\dot{\theta}_{tvp-} = \dot{\theta}_{tvp+}$$
$$\ddot{\theta}_{tvp-} = \dot{\theta}_{tvp+}$$

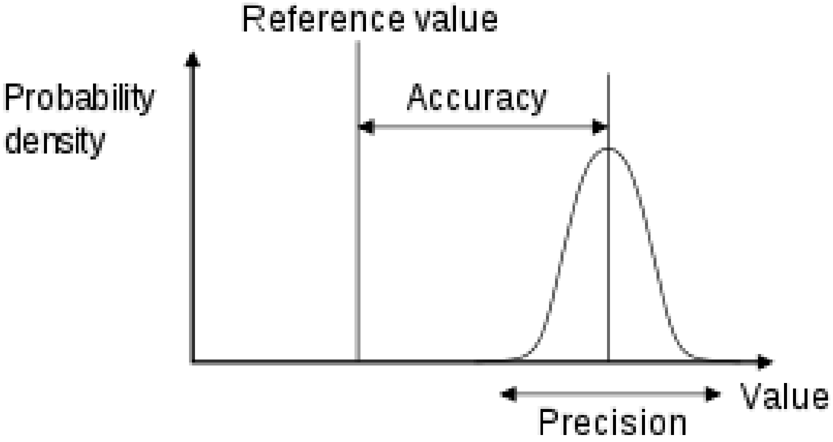
Trajectory Generation – Task Space Control

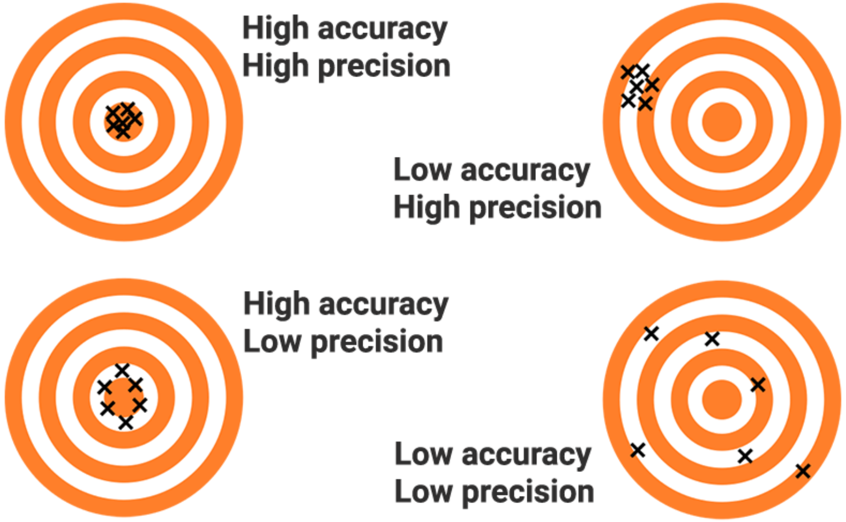


Trajectory Generation – Joint Space Space Control

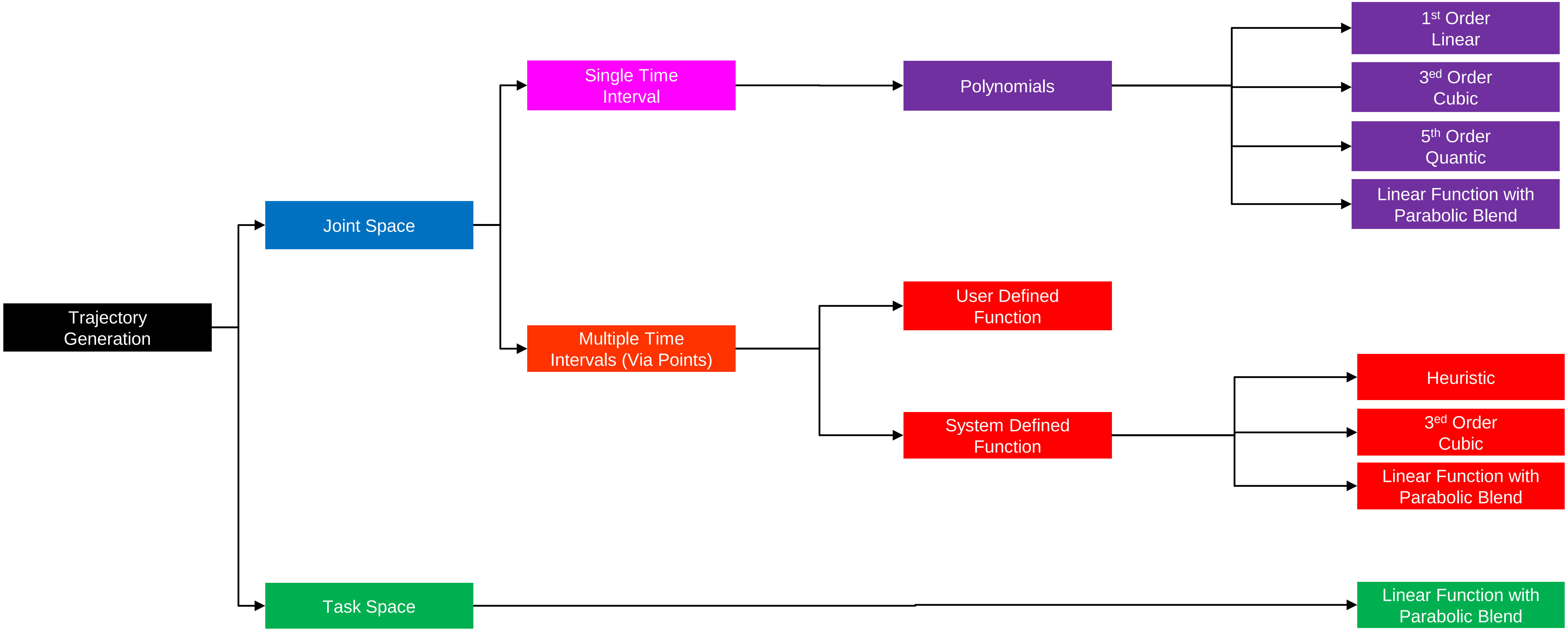


Precision versus Accuracy





Trajectory Generation – Roadmap Diagram



Trajectory Generation

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3ed Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

System Defined Function

Heuristic

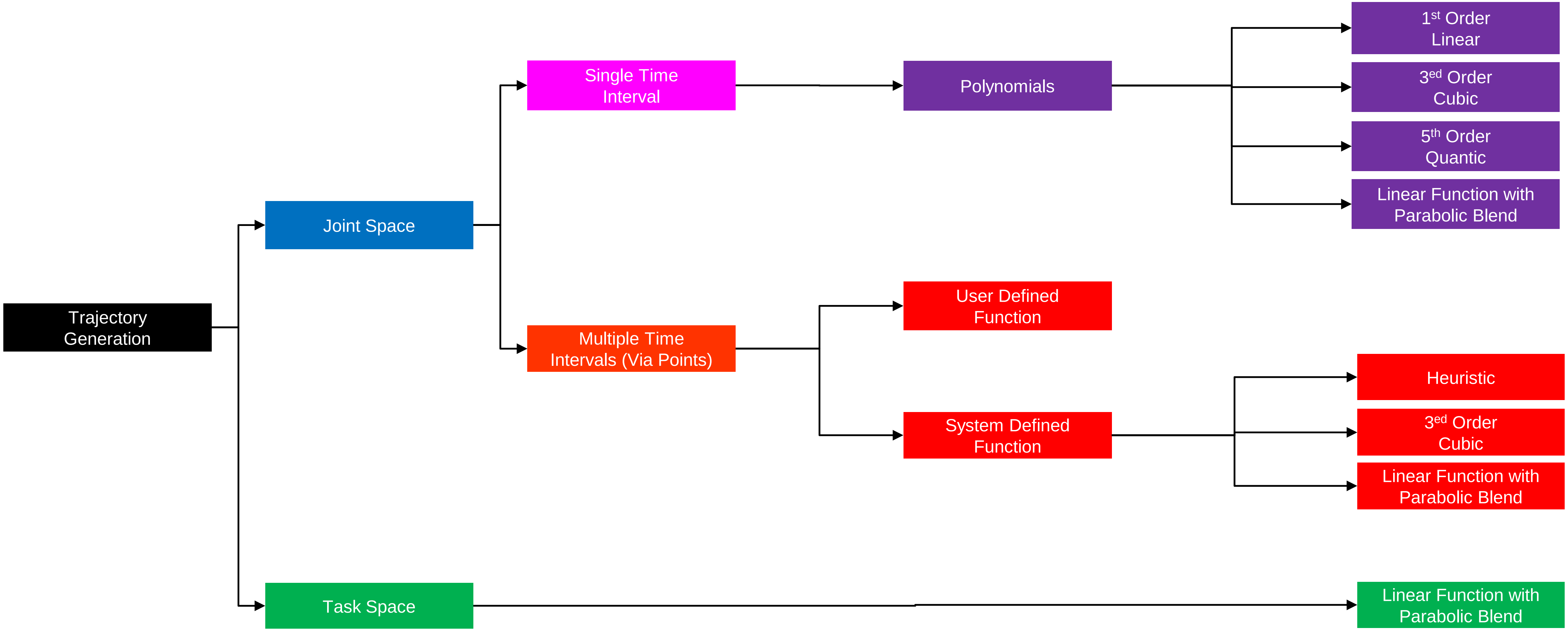
3ed Order Cubic

Linear Function with Parabolic Blend

Task Space

Linear Function with Parabolic Blend

Trajectory Generation – Roadmap Diagram



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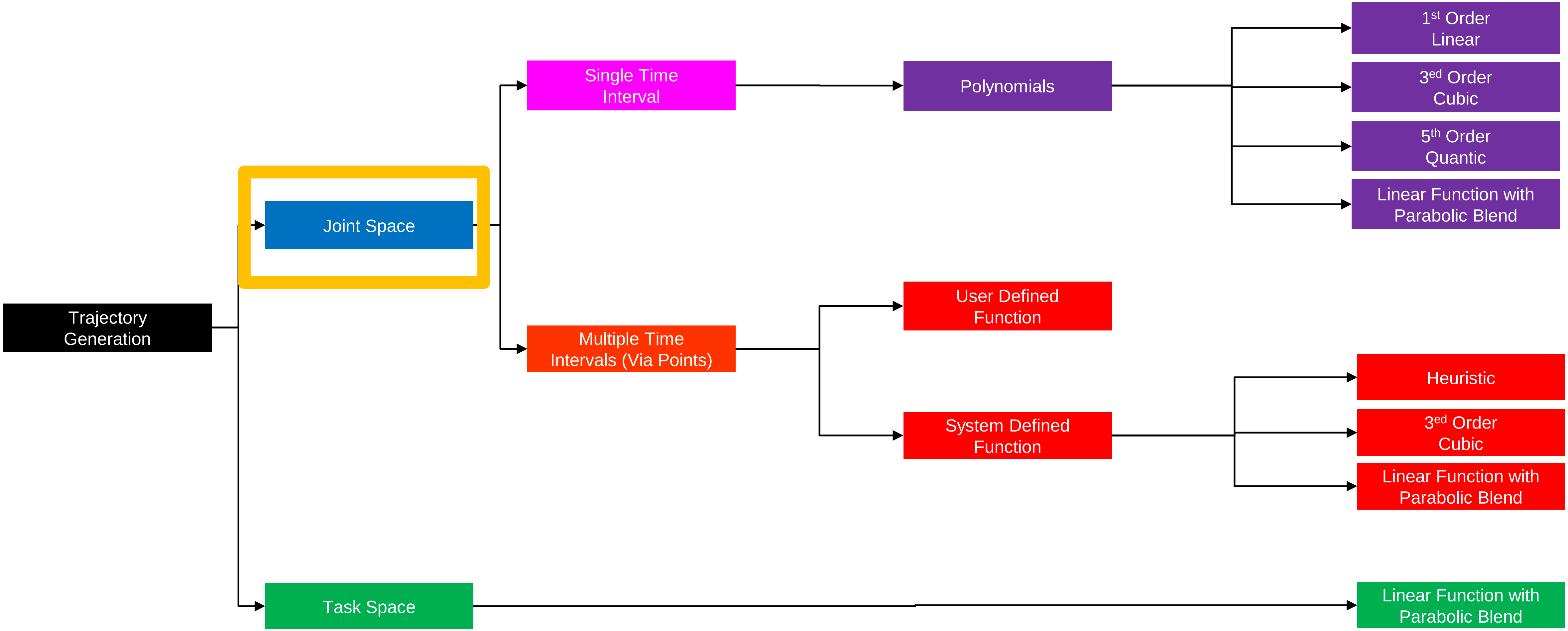
Joint Space Schemes

Single Time Interval

Trajectory Generation – Roadmap Diagram



Trajectory Generation

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Task Space

Linear Function with Parabolic Blend

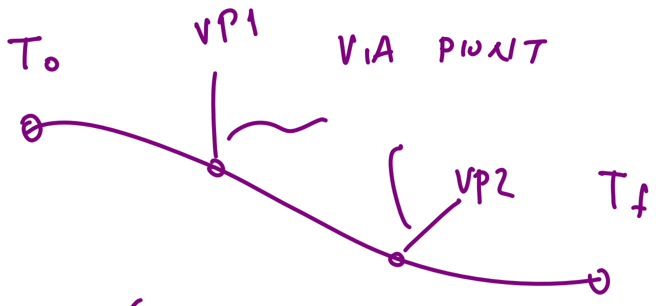
Joint Space Schemes

- **Joint space Schemes** – Path shapes (in space and in time) are described in terms of functions in the joint space.

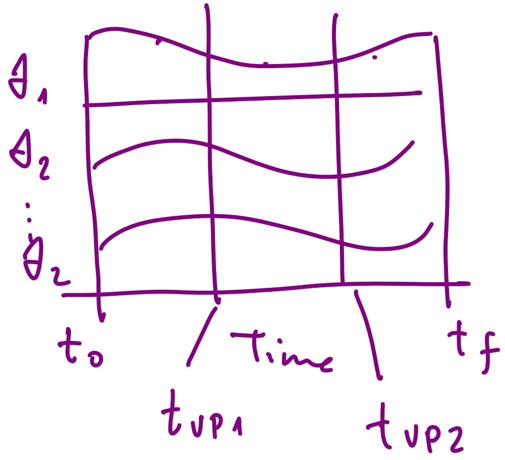
- **General process (Steps) given initial and target P/O**
  1. Select a path point or via point (desired position and orientation of the tool frame {T} with respect to the base frame {s})
  2. Convert each of the “via point” into a set of joint angles using the invers kinematics
  3. Find a smooth function for each of the *n* joints that pass trough the via points, and end the goal point.

Note 1: The **time required to complete each segment is the same for each joint** such that the all the joints will reach the via point at the same time. Thus resulting in the position and orientation of the frame {T} at the via point.

Note 2: The **joints move independently** with only one time restriction (Note 1)



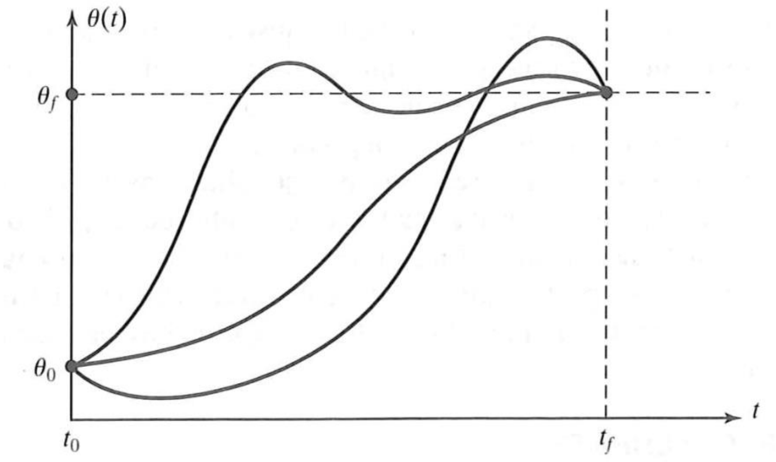
$IK(T_i) \rightarrow \theta_1 \dots \theta_n$

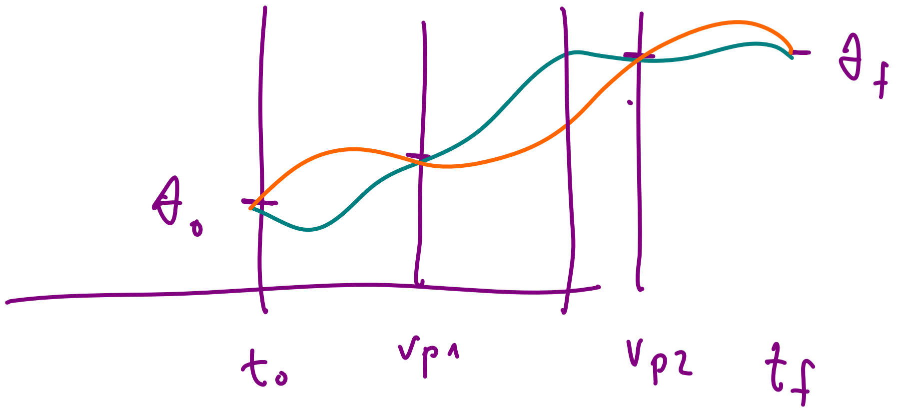


Joint Space Schemes

- Define a function for each joint such that value at t_0 is the **initial position** of the joint and whose value at t_f is the **desire goal position** of the joint

- There are many smooth functions $\theta(t)$ that may be used to interpolate the joint value.





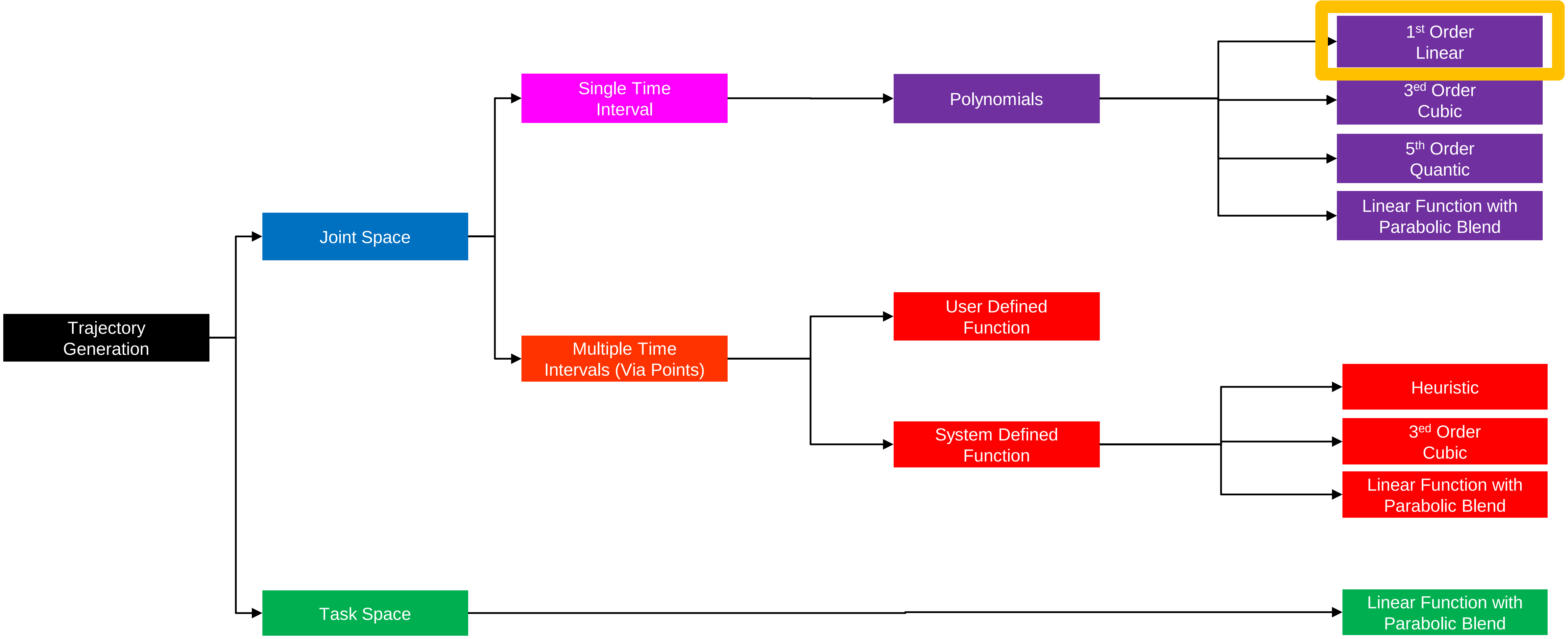
Joint Space Schemes

Single Time Interval

Polynomials

First Order Polynomial

Trajectory Generation – Roadmap Diagram



Trajectory Generation

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3ed Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

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System Defined Function

Heuristic

3ed Order Cubic

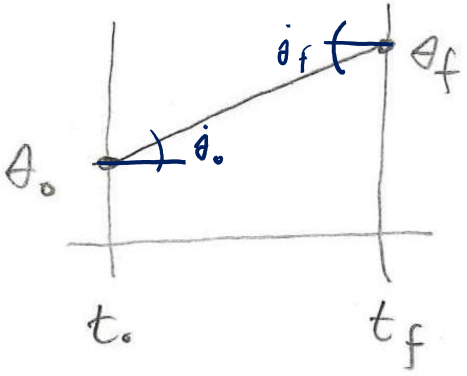
Linear Function with Parabolic Blend

Task Space

Linear Function with Parabolic Blend

Joint Space Schemes – Linear Polynomials

- Problem – Define a function for each joint such that its Value at
  - – t_0 is the initial position of the joint
  - t_f is the desired goal position of the joint



- Given – Constrains on $\theta(t)$

$$\begin{cases} \theta(0) = \theta_0 \\ \theta(t_f) = \theta_f \end{cases}$$

Joint Space Schemes – Linear Polynomials

- Solution – The two constraints can be satisfied by a first order polynomial

$$\theta(t) = a_0 + a_1 t$$

- Combined with the two desired constrains yields two equations in two unknown

$$\begin{cases} \theta_0(0) = a_0 \\ \theta_f(t) = a_0 + a_1 t_f \end{cases} \Rightarrow \begin{cases} \theta_0 = a_0 \\ a_1 = \dfrac{\theta_f - \theta_0}{t_f} \end{cases}$$

$$\theta = \theta_0 + \left(\frac{\theta_f - \theta_0}{t_f} \right) t$$

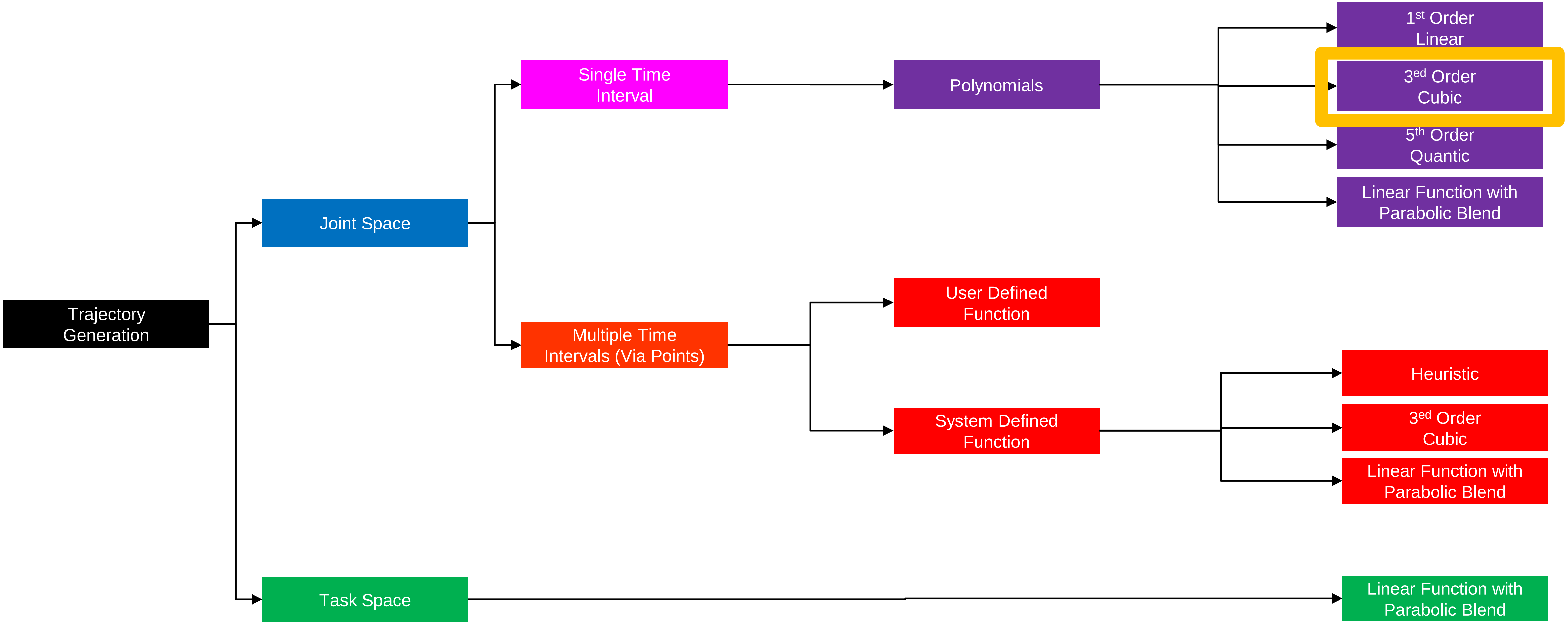
Joint Space Schemes

Single Time Interval

Polynomials

Cubic Order Polynomial

Trajectory Generation – Roadmap Diagram



Trajectory Generation

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3ed Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

System Defined Function

Heuristic

3ed Order Cubic

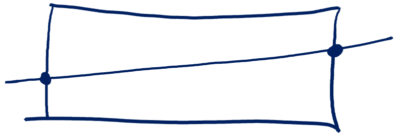
Linear Function with Parabolic Blend

Task Space

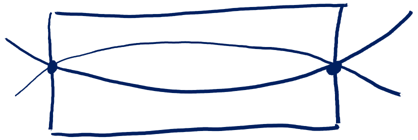
Linear Function with Parabolic Blend

Joint Space Schemes – Order of the Polynomials

$\theta(t) = a_0 + a_1 t$



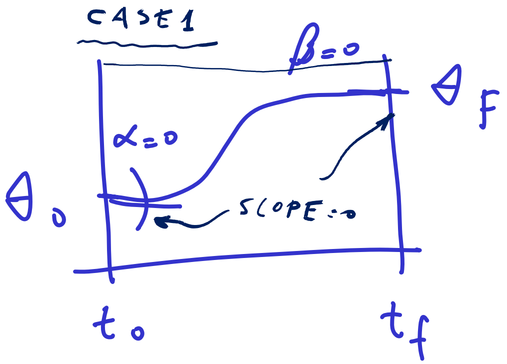
$\theta(t) = a_0 + a_1 t + a_2 t^2$



$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

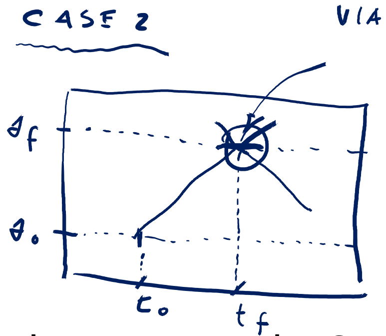
- Problem - Define a function for each joint such that it value at
  - t_0 is the **initial position** of the joint and at
  - t_f is the **desired goal position** of the joint



- Given - Constrains on $\theta(t)$

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \theta(0)=\theta_0 \\ \theta(t_f)=\theta_f \end{array}\right. \\ \left\{ \begin{array}{l} \dot{\theta}(0)=0 \\ \dot{\theta}(t_f)=0 \end{array}\right. \end{array}\right.$$

$$\theta(t) = \underbrace{\underset{\uparrow}{a_0} + \underset{\uparrow}{a_1} t + \underset{\uparrow}{a_2} t^2 + \underset{\uparrow}{a_3} t^3}$$



VIA POINT : - change in direction
- setting the speed to be 0

- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial of at least third degree

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

- The joint velocity and acceleration

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$$\begin{array}{ll} \theta(0) = \theta_0 & \theta_0 = a_0 \\ \theta(t_f) = \theta_f & \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\ \dot{\theta}(0) = 0 & 0 = a_1 \\ \dot{\theta}(t_f) = 0 & 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 \end{array}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$\begin{bmatrix}\theta_0\\ \theta_f\\ 0\\ 0\end{bmatrix} = \begin{array}{c}\begin{array}{cccc} a_0 & a_1 & a_2 & a_3 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}\\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix}\end{array} \begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$$\begin{cases} \theta_f = \theta_0 + a_2 t_f^2 + a_3 t_f^3 \\ 0 = 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

$$\begin{cases} a_2\left[t_f^2\right] + a_3\left[t_f^3\right] = \theta_f - \theta_0 \\ a_2\left[2t_f\right] + a_3\left[3t_f^2\right] = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} \theta_f - \theta_0 & t_f^3 \\ 0 & 3t_f^2 \end{vmatrix}}{\Delta} = \frac{3t_f^2(\theta_f - \theta_0)}{t_f^4}$$

$$= \frac{3(\theta_f - \theta_0)}{t_f^2}$$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & \theta_f - \theta_0 \\ 2t_f & 0 \end{vmatrix}}{\Delta} = \frac{-2t_f(\theta_f - \theta_0)}{t_f^4}$$

$$= \frac{-2(\theta_f - \theta_0)}{t_f^3}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0)$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$\dot{\theta}_{max}$ – Max angular velocity at $t_f/2$

$$\dot{\theta}_{max}\left(t = t_f/2\right) = \frac{6}{t_f^{\cancel{2}}}(\theta_f - \theta_0)\left[\frac{\cancel{t_f}}{2}\right] - \frac{6}{t_f^3}(\theta_f - \theta_0)\left[\frac{t_f}{2}\right]^2$$

$$= \frac{3(\theta_f - \theta_0)}{t_f} - \frac{\overset{3}{\cancel{6}}}{\underset{2}{\cancel{4}}}\frac{(\theta_f - \theta_0)}{t_f}$$

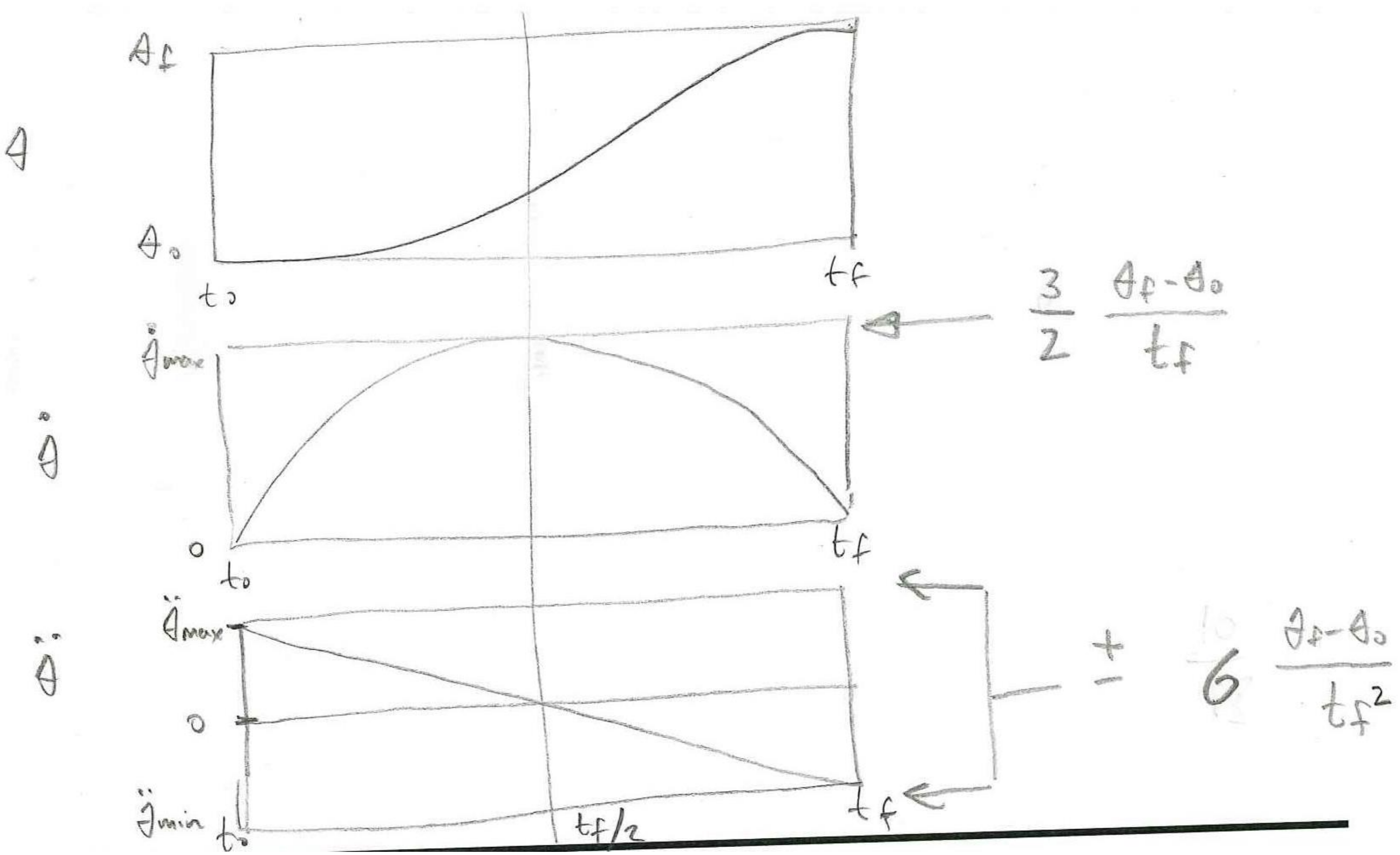
$$= \frac{3}{2}\frac{\theta_f - \theta_0}{t_f}$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

$\ddot{\theta}_{max}$ - Max angular acceleration at $t=0$ and $t=t_f$

$$\ddot{\theta}_{max} = \frac{6}{t_f^2}(\theta_f - \theta_0)$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity



Joint Space Schemes – Cubic Polynomials - Zero Velocity

- Example – A single-link robot with a rotary joint is motionless at $\theta_0 = 15$ degrees. It is desired to move the joint in a smooth manner to $\theta_f = 75$ degrees in 3 seconds. Find the coefficient of the cubic polynomial that accomplish this motion and brings the manipulator to rest at the goal

$$\theta(0) = 15$$

$$\theta(t_f) = 75$$

$$\dot{\theta}(0) = 0$$

$$\dot{\theta}(t_f) = 0$$

$$a_0 = \theta_0 = 15$$

$$a_1 = 0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) = \frac{3}{9}(75 - 15) = 20$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) = -\frac{2}{27}(75 - 15) = -4.44$$

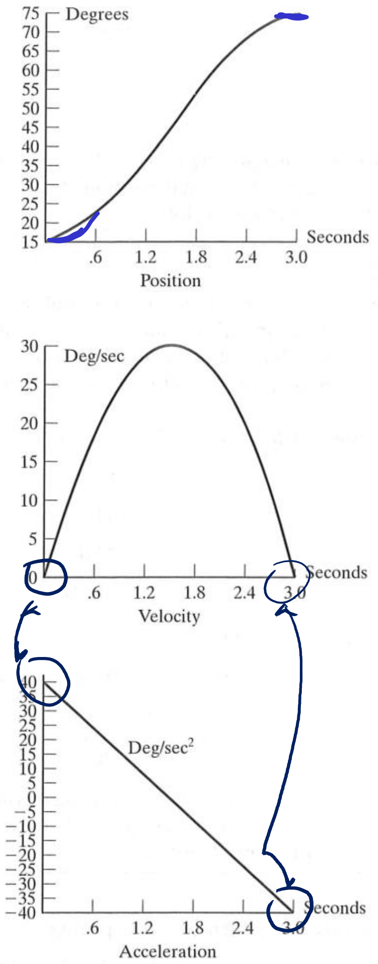
$$\theta(t) = 15 + 20t^2 - 4.44t^3$$

$$\dot{\theta}(t) = 40t - 13.33t^2$$

$$\ddot{\theta}(t) = 40 + 26.66t$$

Joint Space Schemes – Cubic Polynomials - Zero Velocity

- The velocity profile of any cubic function is a parabola
- The acceleration profile of any cubic function is linear

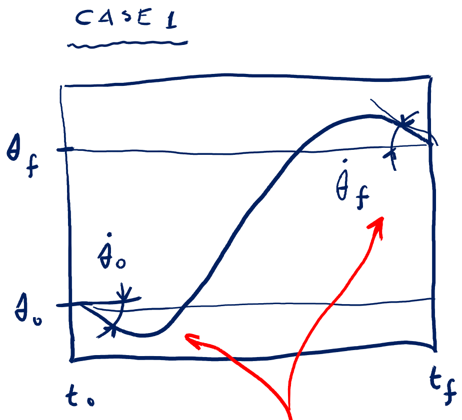


ZERO VELOCITY

HIGHEST ACCELERATION

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

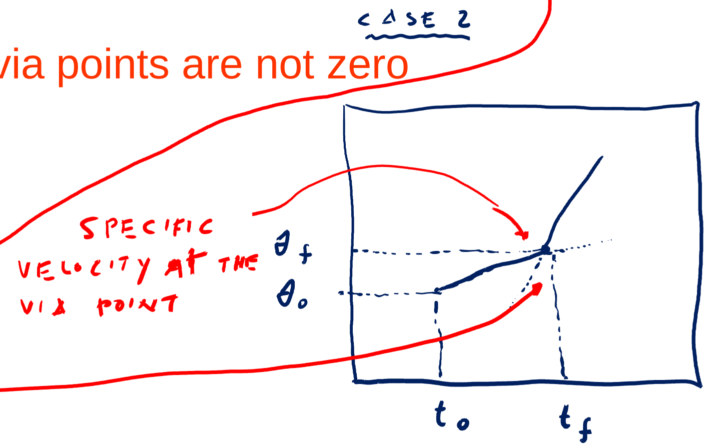
- Previous Method - The manipulator comes to rest at each via point
- General Requirement - Pass through a point without stopping
- Problem - Define a function for each joint such that it value at
  - t_0 is the **initial position** of the joint and at
  - t_f is the **desire goal position** of the joint
- Given - Constrains on $\theta(t)$ such that the velocities at the via points are not zero but rather some known velocities



$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

$$\boxed{\begin{aligned}\dot{\theta}(0) &= \dot{\theta}_0 \\ \dot{\theta}(t_f) &= \dot{\theta}_f\end{aligned}}$$



Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solution - The four constraints can be satisfied by a polynomial

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t$$

- Combined with the four desired constraints yields four equations in four unknowns

$$\theta(0) = \theta_0 \qquad \theta_0 = a_0$$

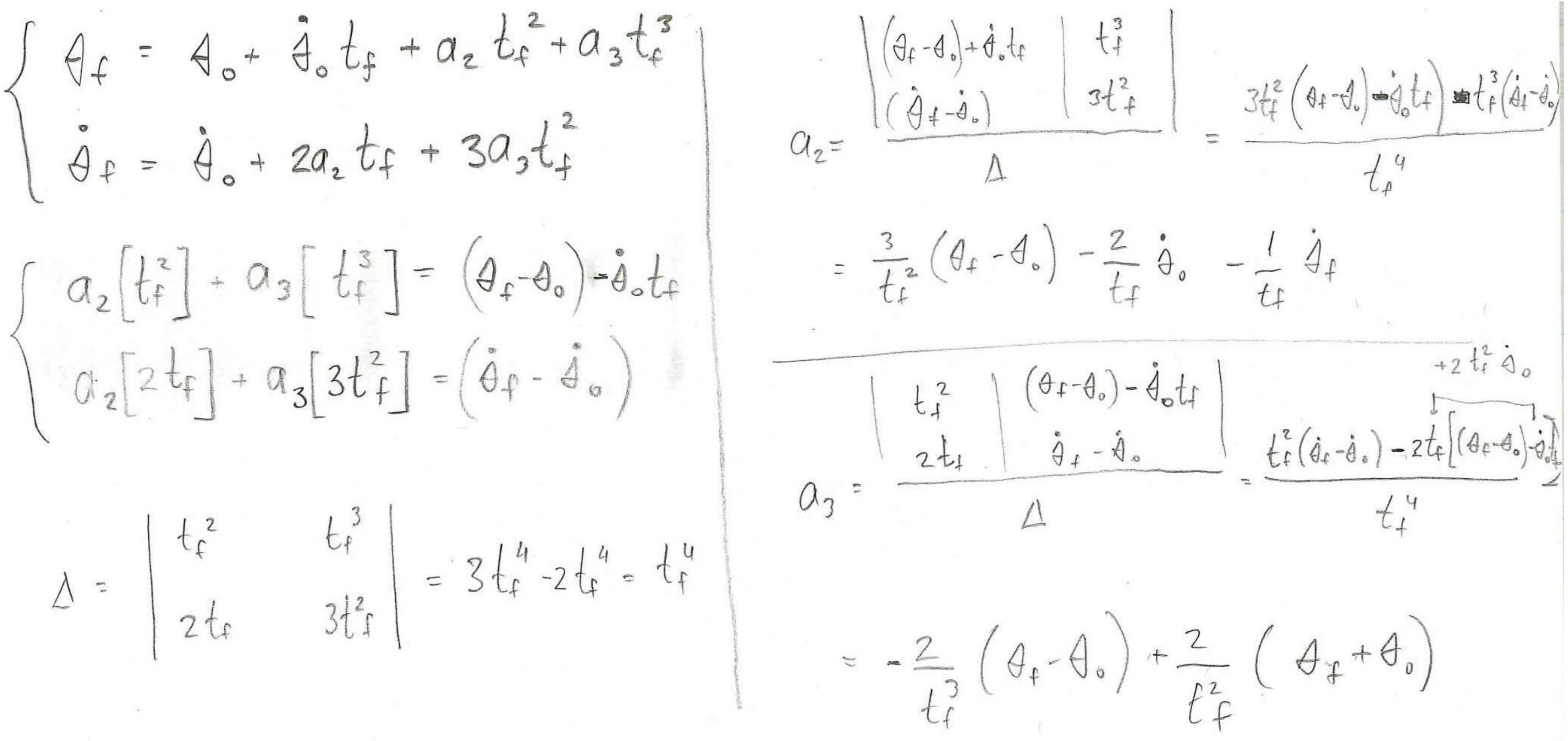
$$\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$

$$\boxed{\begin{array}{ll} \dot{\theta}(0) = \dot{\theta}_0 & \dot{\theta}_0 = a_1 \\ \dot{\theta}(t_f) = \dot{\theta}_f & \dot{\theta}_f = a_1 t_f + 2a_2 t_f + 3a_3 t_f^2 \end{array}}$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

$$\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 0 & 0 \\ 0 & t_f & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity



$$\begin{cases} \theta_f = \theta_0 + \dot{\theta}_0 t_f + a_2 t_f^2 + a_3 t_f^3 \\ \dot{\theta}_f = \dot{\theta}_0 + 2a_2 t_f + 3a_3 t_f^2 \end{cases}$$

$$\begin{cases} a_2 [t_f^2] + a_3 [t_f^3] = (\theta_f - \theta_0) - \dot{\theta}_0 t_f \\ a_2 [2t_f] + a_3 [3t_f^2] = (\dot{\theta}_f - \dot{\theta}_0) \end{cases}$$

$$\Delta = \begin{vmatrix} t_f^2 & t_f^3 \\ 2t_f & 3t_f^2 \end{vmatrix} = 3t_f^4 - 2t_f^4 = t_f^4$$

$$a_2 = \frac{\begin{vmatrix} (\theta_f - \theta_0) + \dot{\theta}_0 t_f & t_f^3 \\ (\dot{\theta}_f - \dot{\theta}_0) & 3t_f^2 \end{vmatrix}}{\Delta} = \frac{3t_f^2\left((\theta_f - \theta_0) - \dot{\theta}_0 t_f\right) - t_f^3(\dot{\theta}_f - \dot{\theta}_0)}{t_f^4}$$

$$= \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = \frac{\begin{vmatrix} t_f^2 & (\theta_f - \theta_0) - \dot{\theta}_0 t_f \\ 2t_f & \dot{\theta}_f - \dot{\theta}_0 \end{vmatrix}}{\Delta} = \frac{t_f^2(\dot{\theta}_f - \dot{\theta}_0) - 2t_f \overset{+2t_f^2\dot{\theta}_0}{\left[(\theta_f - \theta_0) - \dot{\theta}_0 t_f\right]}}{t_f^4}$$

$$= -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\theta_f + \theta_0)$$

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{2}{t_f}\dot{\theta}_0 - \frac{1}{t_f}\dot{\theta}_f$$

$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{2}{t_f^2}(\dot{\theta}_f + \dot{\theta}_0)$$

- Given - velocities at each via point are
- Solution - Apply these equations for each segment of the trajectory.
- Note: The Cubic polynomials ensures the continuity of velocity but not the acceleration. Practically, the industrial manipulators are sufficiently rigid so this this continuity in acceleration

Joint Space Schemes – Cubic Polynomials – Non Zero Velocity

- Note:
  - The Cubic polynomials ensures the continuity of velocity but not the acceleration.
  - Practically, the industrial manipulators are sufficiently rigid so this discontinuity in acceleration is filtered by the mechanical structure
  - Therefore this trajectory is generally satisfactory for most applications

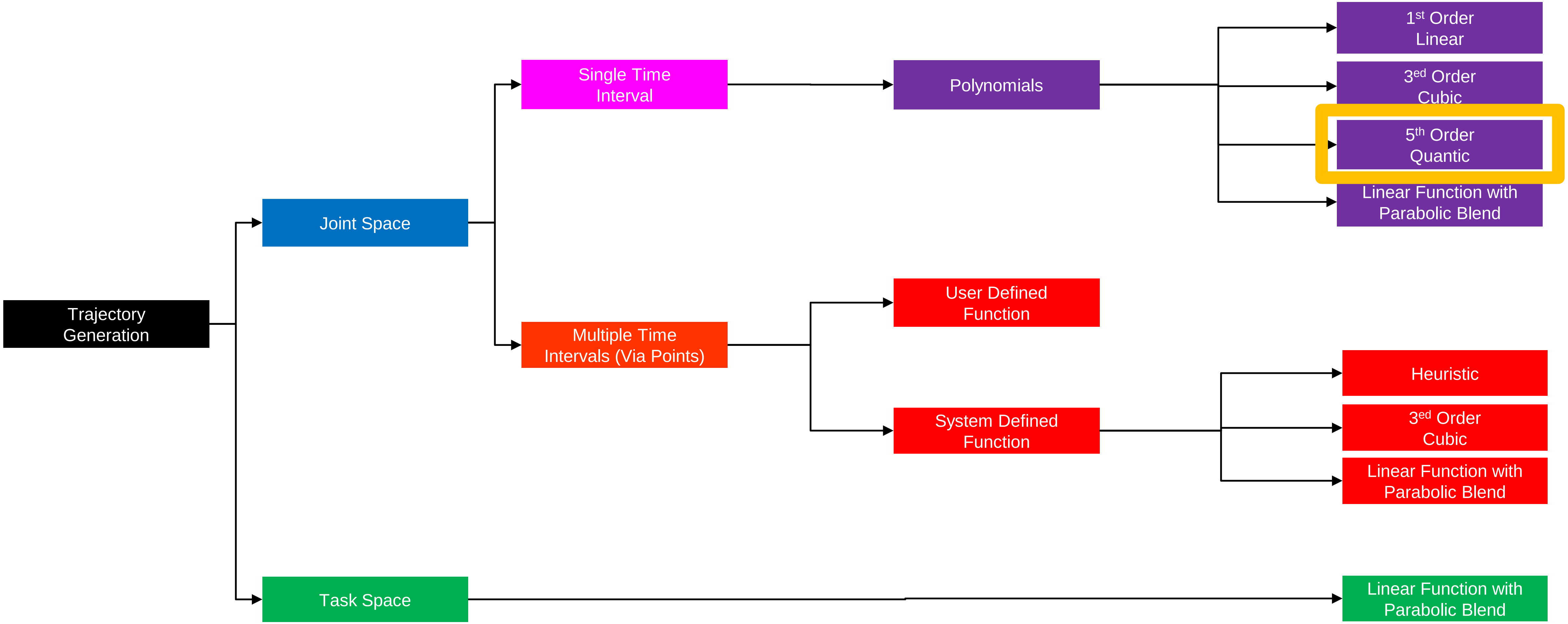
Joint Space Schemes

Single Time Interval

Polynomials

Quantic Order Polynomial

Trajectory Generation – Roadmap Diagram



Trajectory Generation

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3ed Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

System Defined Function

Heuristic

3ed Order Cubic

Linear Function with Parabolic Blend

Task Space

Linear Function with Parabolic Blend

Joint Space Schemes – Quantic Polynomials

- Rational for Quantic Polynomials (high order)
  - High Speed Robot
  - Robot Carrying heavy/delicate load
  - Non Rigid links
  - For high speed robots or when the robot is handling heavy or delicate loads. It is worth insuring the continuity of accelerations as well as avoid excitation of the resonance modes of the mechanism

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**

- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment

$$\theta(0) = \theta_0$$

$$\theta(t_f) = \theta_f$$

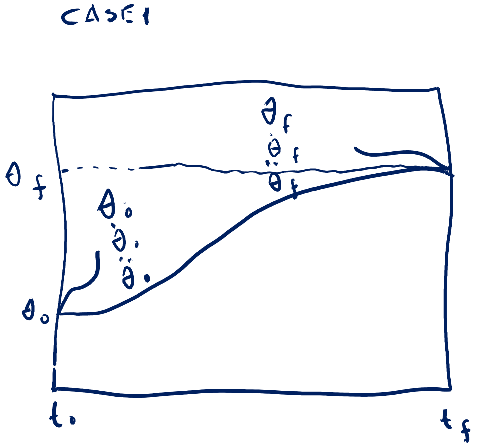
$$\dot{\theta}(0) = \dot{\theta}_0$$

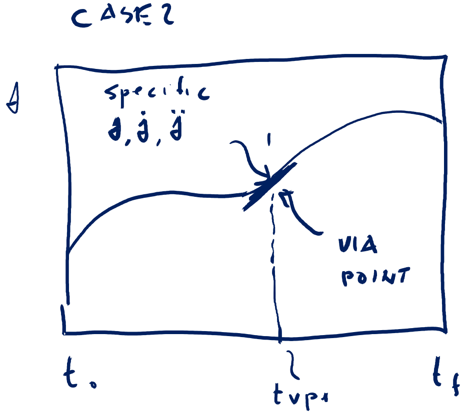
$$\dot{\theta}(t_f) = \dot{\theta}_f$$

$$\ddot{\theta}(0) = \ddot{\theta}_0$$

$$\ddot{\theta}(t_f) = \ddot{\theta}_f$$

- What should be the order of the polynomial function to meet these constrains?





Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

(1) $\theta(0) = \theta_0 \qquad \theta_0 = a_0$

(2) $\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$

(3) $\dot{\theta}(0) = \dot{\theta}_0 \qquad \dot{\theta}_0 = a_1$

(4) $\dot{\theta}(t_f) = \dot{\theta}_f \qquad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$

(5) $\ddot{\theta}(0) = \ddot{\theta}_0 \qquad \ddot{\theta}_0 = 2a_2$

(6) $\ddot{\theta}(t_f) = \ddot{\theta}_f \qquad \ddot{\theta}_f = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$\begin{bmatrix} \theta_0 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \\ \ddot{\theta}_0 \\ \ddot{\theta}_f \end{bmatrix} = \begin{array}{c} \begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{array}\right] \end{array} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

$$(2) \qquad \theta_f = \theta_0 + \dot{\theta}_0 t_f + \frac{\ddot{\theta}_0}{2} t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$$

$$(4) \qquad \dot{\theta}_f = \dot{\theta}_0 + \cancel{2} \frac{\ddot{\theta}_0}{\cancel{2}} t_f + 3 a_3 t_f^3 + 4 a_4 t_f^4 + 5 a_5 t_f^4$$

$$(6) \qquad \ddot{\theta}_f = \cancel{2} \frac{\ddot{\theta}_0}{\cancel{2}} + 6 a_3 t_f + 12 a_4 t_f^2 + 20 a_5 t_f^3$$

$$\begin{bmatrix} \theta_f - \theta_0 - \dot{\theta} t_f - \frac{\ddot{\theta}_0}{2} t_f^2 \\ \dot{\theta}_f - \dot{\theta}_0 - \ddot{\theta}_0 t_f \\ \ddot{\theta}_f - \ddot{\theta}_0 \end{bmatrix} \begin{bmatrix} t_f^3 & t_f^4 & t_f^5 \\ 3 t_f^3 & 4 t_f^4 & 5 t_f^4 \\ 6 t_f & +12 t_f^2 & 20 t_f^3 \end{bmatrix} \begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

Joint Space Schemes – Cubic Polynomials - Non Zero Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2}$$

$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\theta_f + 12\theta_0)t_f - (3\theta_0 - \theta_f)t_f^2}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\theta_f + 16\theta_0)t_f + (3\theta_0 - 2\theta_f)t_f^2}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\theta_f + 6\theta_0)t_f - (\theta_0 - \theta_f)t_f^2}{2t_f^5}$$

Joint Space Schemes – Quantic Polynomials - Non Zero Acceleration

For a generalized case where $t_0 \neq 0$

$T = t_f - t_0$; $h = \theta_f - \theta_0$

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{1}{2} a_0$$

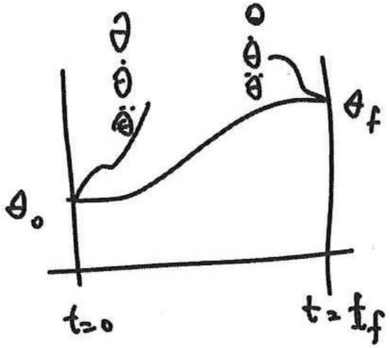
$$a_3 = \frac{1}{2T^3}\left[20h - (8\dot{\theta}_f + 12\dot{\theta}_0)T - (3a_0 - a_1)T^2\right]$$

$$a_4 = \frac{1}{2T^4}\left[-30h + (14\dot{\theta}_f + 16\dot{\theta}_0)T + (3a_0 - 2a_1)T^2\right]$$

$$a_5 = \frac{1}{2T^5}\left[12h - 6(\dot{\theta}_1 - \dot{\theta}_0)T + (a_1 - a_0)T^2\right]$$

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment



$$\theta(0)=\theta_0 \qquad \dot{\theta}(0)=\dot{\theta}_0 \qquad \boxed{\ddot{\theta}(0)=0}$$

$$\theta(t_f)=\theta_f \qquad \dot{\theta}(t_f)=\dot{\theta}_f \qquad \boxed{\ddot{\theta}(t_f)=0}$$

- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\theta(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

(1) $\theta(0) = \theta_0 \qquad \theta_0 = a_0$

(2) $\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$

(3) $\dot{\theta}(0) = \dot{\theta}_0 \qquad \dot{\theta}_0 = a_1$

(4) $\dot{\theta}(t_f) = \dot{\theta}_f \qquad \dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$

(5) $\ddot{\theta}(0) = 0 \qquad 0 = 2a_2$

(6) $\ddot{\theta}(t_f) = 0 \qquad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$

Joint Space Schemes – Quantic Polynomials - Zero Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = \dot{\theta}_0$$

$$a_2 = \frac{\ddot{\theta}_0}{2} = 0$$

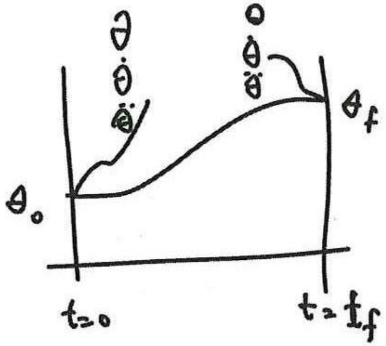
$$a_3 = \frac{20\theta_f - 20\theta_0 - (8\dot{\theta}_f + 12\dot{\theta}_0)t_f}{2t_f^3}$$

$$a_4 = \frac{30\theta_0 - 30\theta_f + (14\dot{\theta}_f + 16\dot{\theta}_0)t_f}{2t_f^4}$$

$$a_5 = \frac{12\theta_f - 12\theta_0 - (6\dot{\theta}_f + 6\dot{\theta}_0)t_f}{2t_f^5}$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Problem - Define a function for each joint such that it value at
  - t_0 is the time at the **initial position**
  - t_f is the time at the **desired goal position**
- Given - Constrains on the position velocity and acceleration at the beginning and the end of the path segment



$$\theta(0)=\theta_0 \qquad \theta(t_f)=\theta_f$$

$$\boxed{\begin{matrix}\dot{\theta}(0)=0\\ \dot{\theta}(t_f)=0\end{matrix}} \quad \boxed{\begin{matrix}\ddot{\theta}(0)=0\\ \ddot{\theta}(t_f)=0\end{matrix}}$$

- What should be the order of the polynomial function to meet these constrains?

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solution - The six constraints can be satisfied by a polynomial of at least fifth order

$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\theta(t) = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

- Combined with the six desired constraints yields six equations with six unknowns

(1) $\theta(0) = \theta_0 \qquad \theta_0 = a_0$

(2) $\theta(t_f) = \theta_f \qquad \theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 + a_4 t_f^4 + a_5 t_f^5$

(3) $\dot{\theta}(0) = 0 \qquad 0 = a_1$

(4) $\dot{\theta}(t_f) = 0 \qquad 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 + 4a_4 t_f^3 + 5a_5 t_f^4$

(5) $\ddot{\theta}(0) = 0 \qquad 0 = 2a_2$

(6) $\ddot{\theta}(t_f) = 0 \qquad 0 = 2a_2 + 6a_3 t_f + 12a_4 t_f^2 + 20a_5 t_f^3$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

- Solving these equations for the a_i we obtain

$$a_0 = \theta_0$$

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = \frac{20\theta_f - 20\theta_0}{2t_f^3} = \frac{10\theta_f - 10\theta_0}{t_f^3} = 10\left[\frac{\theta_f - \theta_0}{t_f^3}\right]$$

$$a_4 = \frac{30\theta_0 - 30\theta_f}{2t_f^4} = -\frac{15\theta_f - 15\theta_0}{t_f^4} = -15\left[\frac{\theta_f - \theta_0}{t_f^4}\right]$$

$$a_5 = \frac{12\theta_f - 12\theta_0}{2t_f^5} = \frac{6\theta_f - 6\theta_0}{t_f^5} = 6\left[\frac{\theta_f - \theta_0}{t_f^5}\right]$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\theta = \overset{\overset{a_0}{\downarrow}}{\theta_0} + \quad + \overset{\overset{a_3}{\downarrow}}{\left(10\,\frac{\theta_f-\theta_0}{t_f^3}\right)} t^3 - \overset{\overset{a_4}{\downarrow}}{\left(15\,\frac{\theta_f-\theta_0}{t_f^4}\right)} t^4 + \overset{\overset{a_5}{\downarrow}}{\left(6\,\frac{\theta_f-\theta_0}{t_f^5}\right)} t^5$$

$$\theta = a_0 + \quad + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\dot{\theta} = \quad + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$\ddot{\theta} = 6a_3 t + 12a_4 t^2 + 20a_5 t^3$$

$$\theta = \theta_0 + 10\left[\frac{\theta_f-\theta_0}{t_f^3}\right] t^3 - 15\left[\frac{\theta_f-\theta_0}{t_f^4}\right] t^4 + 6\left[\frac{\theta_f-\theta_0}{t_f^5}\right] t^5$$

$$\dot{\theta} = 30\left[\frac{\theta_f-\theta_0}{t_f^3}\right] t^2 - 60\left[\frac{\theta_f-\theta_0}{t_f^4}\right] t^3 + 30\left[\frac{\theta_f-\theta_0}{t_f^5}\right] t^4$$

$$\ddot{\theta} = 60\left[\frac{\theta_f-\theta_0}{t_f^3}\right] t - 180\left[\frac{\theta_f-\theta_0}{t^4}\right] t^2 + 120\left[\frac{\theta_f-\theta_0}{t_f^5}\right] t^3$$

$$\dddot{\theta} = 60\left[\frac{\theta_f-\theta_0}{t_f^3}\right] - 360\left[\frac{\theta_f-\theta_0}{t_f^4}\right] t + 360\left[\frac{\theta_f-\theta_0}{t_f^5}\right] t^2$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

$$\dot{\theta}_{max} \Longrightarrow \text{at} \quad t = t_f/2$$

$$\dot{\theta}_{max} = +3\overbrace{\left[10\frac{\theta_f-\theta_0}{\underset{t_f}{\cancel{t_f^3}}}\right]}^{a_3}\frac{\cancel{t_f^2}}{4} - 4\overbrace{\left[15\frac{\theta_f-\theta_0}{\underset{t_f}{\cancel{t_f^4}}}\right]}^{a_4}\frac{\cancel{t_f^3}}{8} + 5\overbrace{\left[6\frac{\theta_f-\theta_0}{\underset{t_f}{\cancel{t_f^5}}}\right]}^{a_5}\frac{\cancel{t_f^4}}{16}$$

$$\frac{\overset{15}{\cancel{30}}}{\underset{2}{\cancel{4}}}\left[\frac{\theta_f-\theta_0}{t_f}\right] - \frac{\overset{15}{\cancel{60}}}{\underset{2}{\cancel{8}}}\left[\frac{\theta_f-\theta_0}{t_f}\right] + \frac{\overset{15}{\cancel{30}}}{\underset{8}{\cancel{16}}}\left[\frac{\theta_f-\theta_0}{t_f}\right]$$

$$= \frac{15}{8}\,\frac{\theta_f-\theta_0}{t_f} \qquad \checkmark$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration

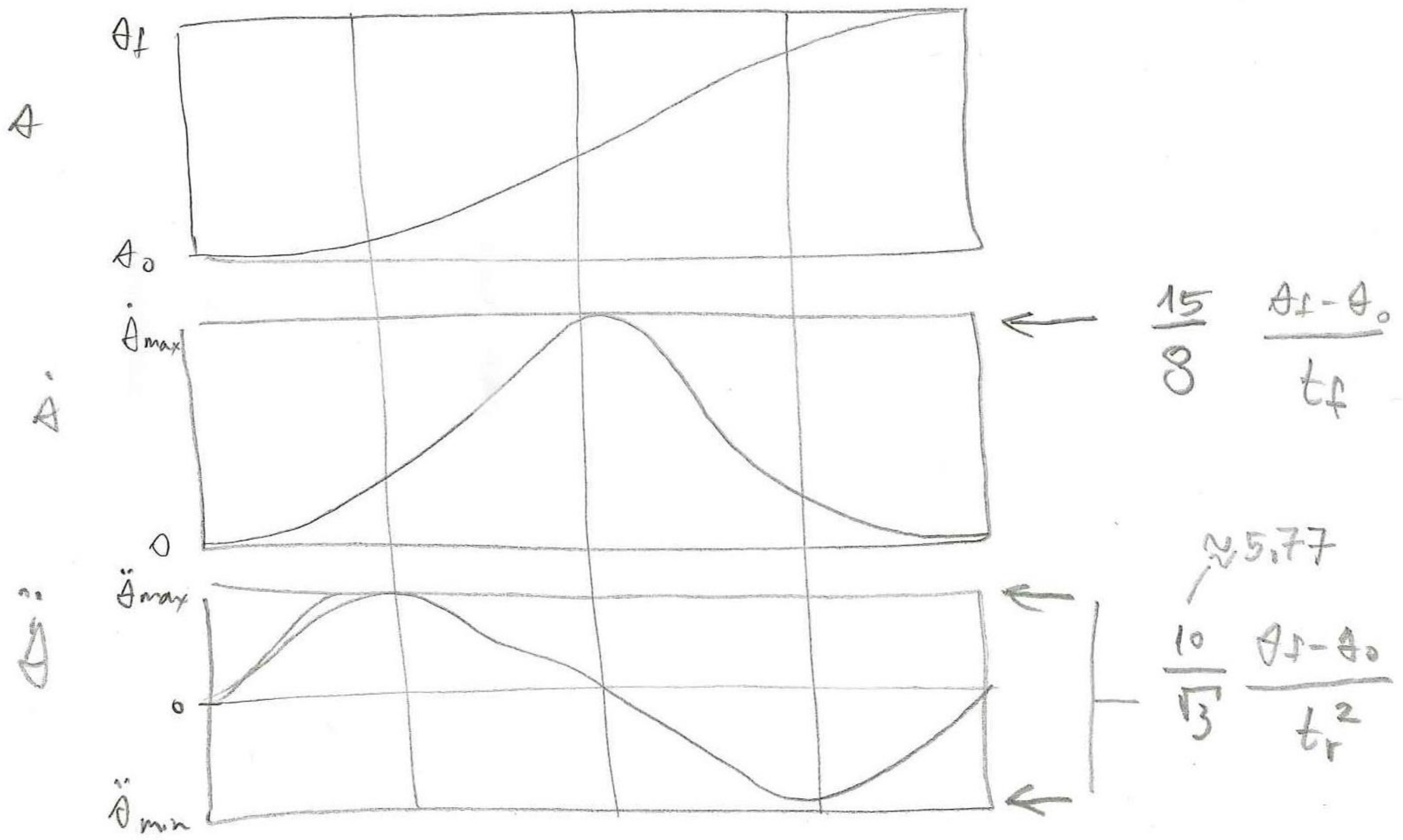
$$\ddot{\theta}_{max} \rightarrow \text{at} \quad t = t_f/4$$

$$\ddot{\theta}_{max} = 6\left[10\,\frac{\theta_f-\theta_0}{\underset{t_f^2}{\cancel{t_f^3}}}\right]\frac{\cancel{t_f}}{4} - \overset{3}{\cancel{12}}\left[15\,\frac{\theta_f-\theta_0}{t_f^4}\right]\frac{t_f^2}{\underset{4}{\cancel{16}}} + \overset{5}{\cancel{20}}\left[6\,\frac{\theta_f-\theta_0}{t_f^5}\right]\frac{t_f^3}{\cancel{4}\cdot 16}$$

$$= 15\left[\frac{\theta_f-\theta_0}{t_f^2}\right] - \overset{\frac{90}{8}}{\frac{45}{4}}\left[\frac{\theta_f-\theta_0}{t_f^2}\right] + \frac{15}{8}\left[\frac{\theta_f-\theta_0}{t_f^2}\right] =$$

$$= \underbrace{\left[15 - \frac{75}{8}\right]}_{5.625}\frac{\theta_f-\theta_0}{t_f^2}$$

Joint Space Schemes – Quantic Polynomials - Zero Velocity & Acceleration



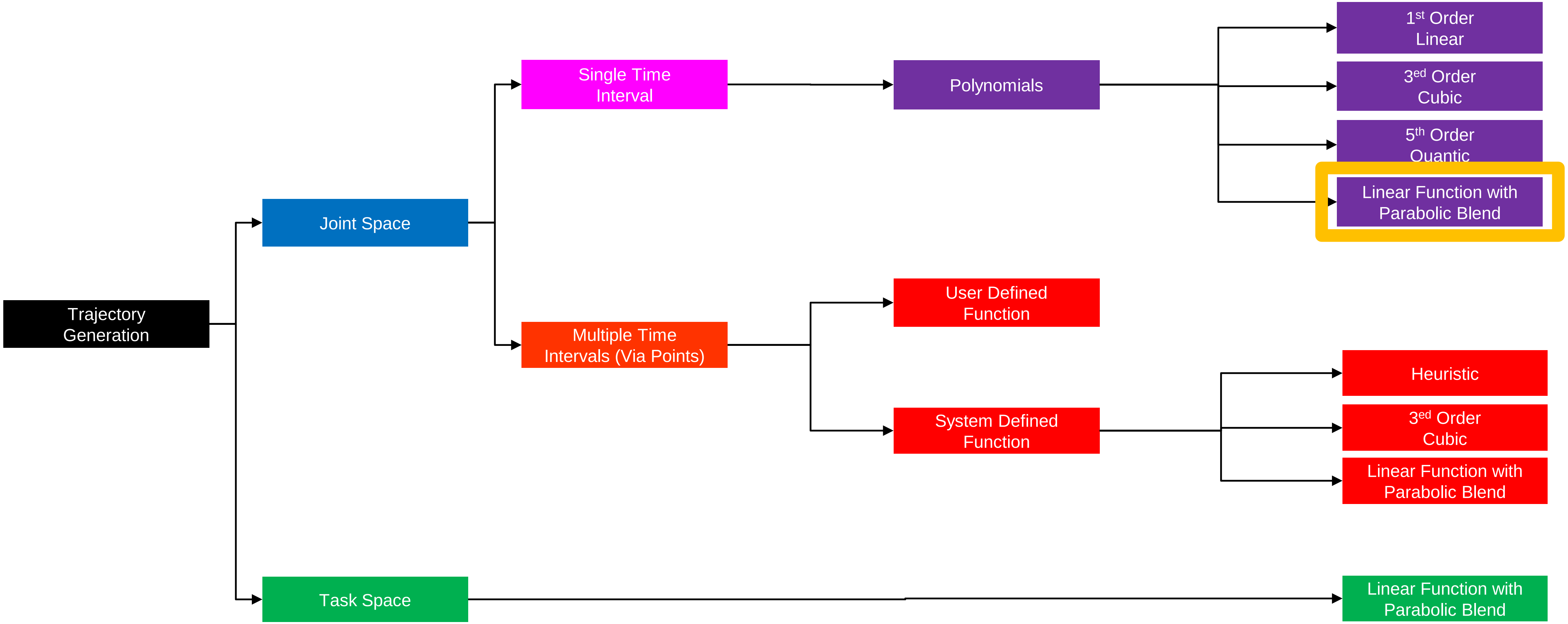
Joint Space Schemes

Single Time Interval

Polynomials

Linear Function with Parabolic Blend (Trapezoid Velocity Method)

Trajectory Generation – Roadmap Diagram



Trajectory Generation

Joint Space

Single Time Interval

Polynomials

1st Order Linear

3ed Order Cubic

5th Order Quantic

Linear Function with Parabolic Blend

Multiple Time Intervals (Via Points)

User Defined Function

System Defined Function

Heuristic

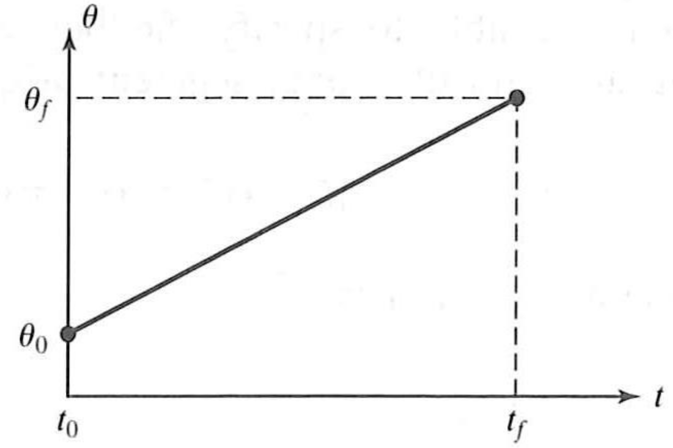
3ed Order Cubic

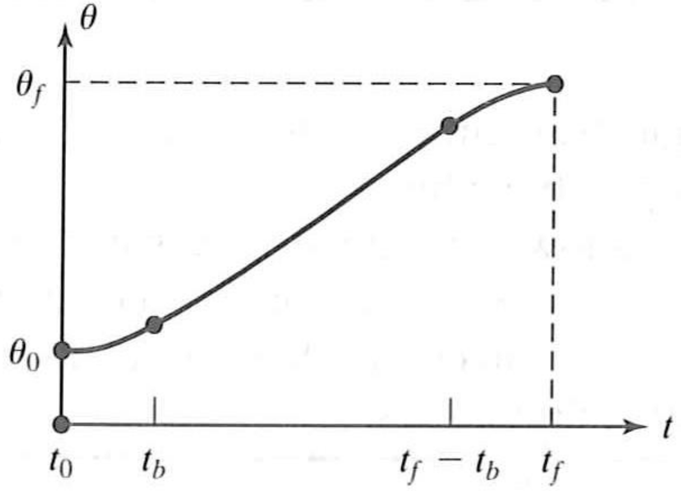
Linear Function with Parabolic Blend

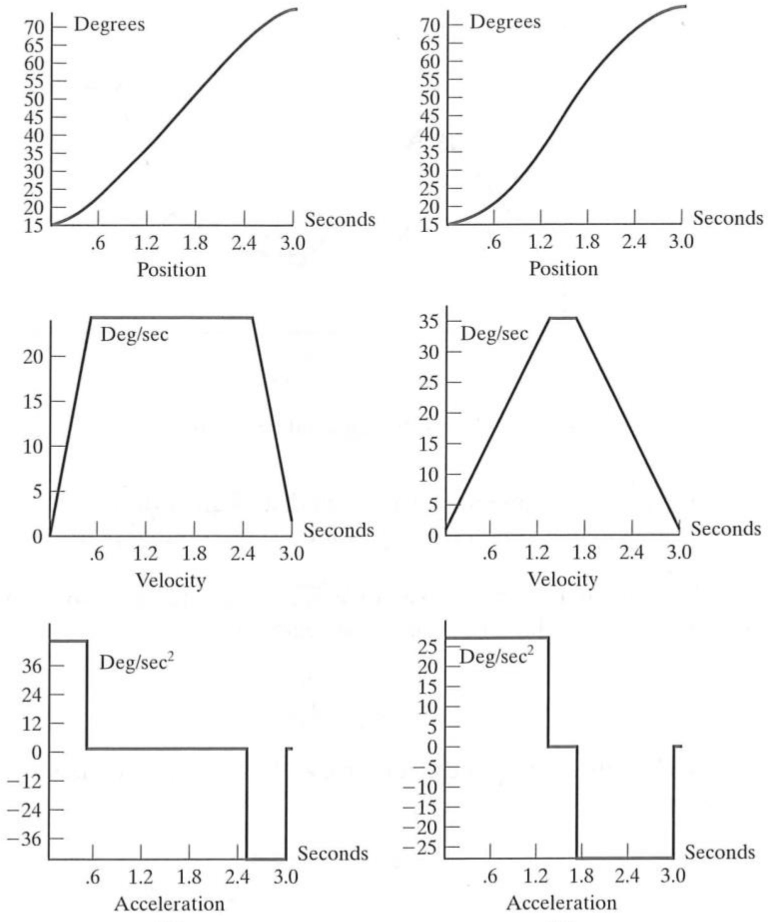
Task Space

Linear Function with Parabolic Blend

Joint Space Schemes – Linear Function With Parabolic Blend



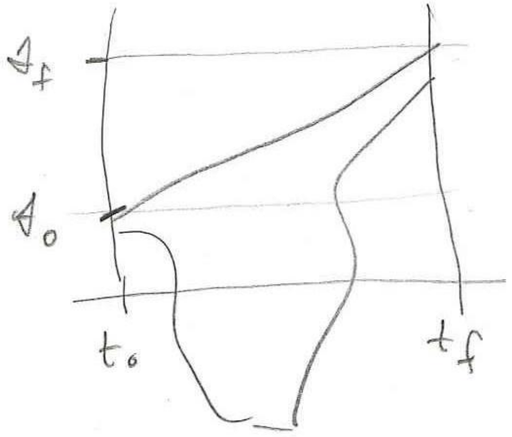




(a) (b)

Joint Space Schemes –
Linear Function With Parabolic Blend

- Linear interpolation to move from the present joint position $\theta_0 (t = t_0)$ to the final position $\theta_f (t = t_f)$



Discontinous Velocity

- Note: Although to motion of each joint is linear the EE in general does not move in a straight line is space

- Problem: Linear interpolation would couse the velocity to be discontinous at the begining /end

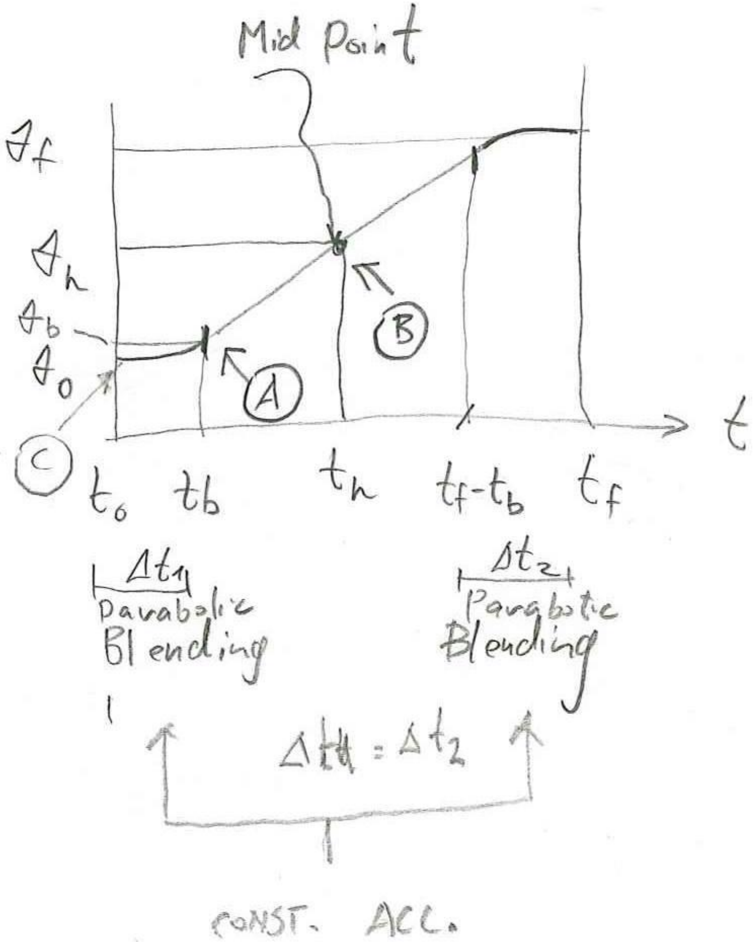
- Solution: Parabolic blend region.

Joint Space Schemes – Linear Function With Parabolic Blend

- During the blend – Constant Acceleration to change the velocity smoothly
- Assumptions (1) the parabolic blend segments (Δt_1, Δt_2) have the same duration

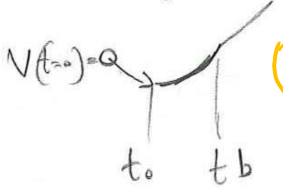
  $\Delta t_1 = \Delta t_2$

  (2) The same constant acceleration is used during both blends



Joint Space Schemes – Linear Function With Parabolic Blend

- Conditions : Point (A)



(I) - Constant acceleration during the blend (Point C to A)

- Intial velocity is zero (Point (C))

$$(1)\quad \underbrace{\boxed{\theta_b = \theta_0 + \overset{0}{\underline{V_0}}\, t + \frac{1}{2}\ddot{\theta}\, t_b^2}}_{I} \longrightarrow \text{Constant Acceleration}$$

Intial velocity (=0) II

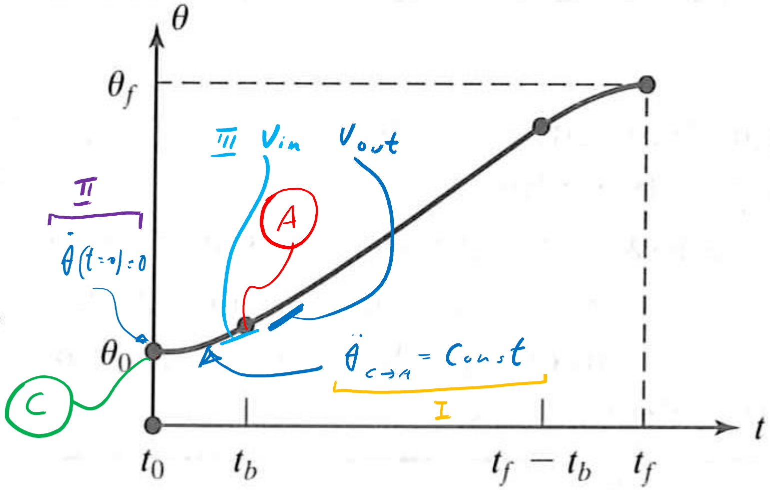
II - The slope at point (A) must be equal on both sides

$V_{in} = V_{out}$

Acc ←   → time

$$(2)\quad \boxed{\underbrace{\ddot{\theta}\ t_b}_{III} = \frac{\theta_h - \theta_b}{\underbrace{t_h - t_b}_{III}}}$$

Velocity from the left (V_{in})   Velocity from the right (V_{out})



Joint Space Schemes – Linear Function With Parabolic Blend

Point Ⓑ

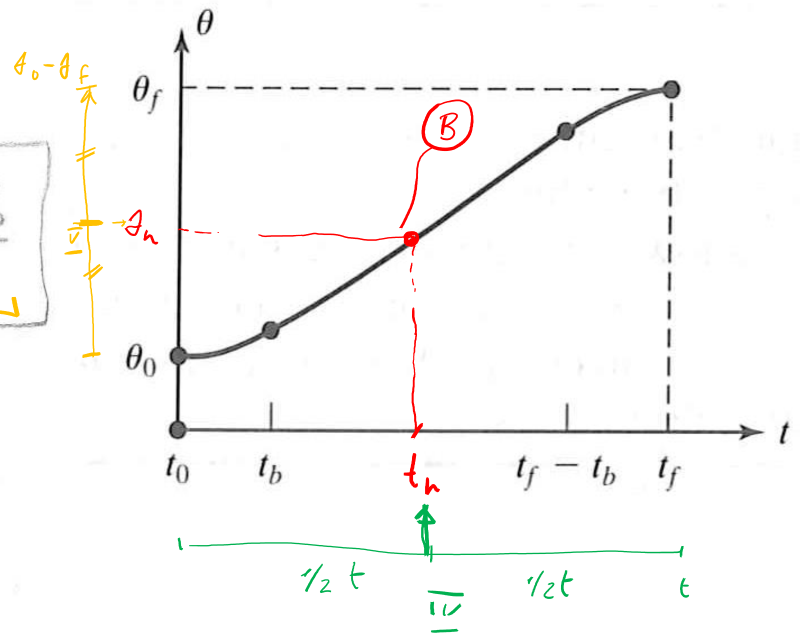
- Point Ⓑ is at the midle of the segment

$$(3)\quad \boxed{\underbrace{t_h = \frac{t}{2}}_{\text{IV}}}$$

$$(4)\quad \boxed{\theta_h = \frac{\theta_f - \theta_0}{2} + \theta_0 = \frac{\theta_f - \theta_0 + 2\theta_0}{2} = \underbrace{\frac{\theta_f + \theta_0}{2}}_{\text{V}}}$$

plug Eq (4) into Eq(2) and Eq(3) into Eq(2)

$$\ddot{\theta} t_b = \frac{\boxed{\theta_h} - \theta_b}{\boxed{t_h} - t_b} = \frac{\boxed{\left(\frac{\theta_f + \theta_0}{2}\right)} - \theta_b}{\boxed{\frac{t}{2}} - t_b}$$



Joint Space Schemes – Linear Function With Parabolic Blend

$$\ddot{\theta}\, t_b\left(\frac{t}{2}-t_b\right) = \frac{\theta_f+\theta_0}{2} - \theta_b$$

$$\ddot{\theta}\, t_b\left(\frac{t-2t_b}{\cancel{2}}\right) = \frac{\theta_f+\theta_0-2\theta_b}{\cancel{2}}$$

$$\ddot{\theta}\, t_b t - 2\ddot{\theta}\, t_b^2 = \theta_f+\theta_0-2\theta_b$$

$$\ddot{\theta}(t_b t) - 2\ddot{\theta}\, t_b^2 - \theta_f - \theta_0 + 2\theta_b = 0$$

↑ plug Eq (1)

$$\ddot{\theta}(t_b t) - 2\ddot{\theta}\, t_b^2 - \theta_f - \theta_0 + 2\theta_0 + \ddot{\theta}\, t_b^2 = 0$$

$$\ddot{\theta}(t_b t) - \ddot{\theta}\, t_b^2 + \theta_f + \theta_0 = 0$$

Joint Space Schemes –
Linear Function With Parabolic Blend

(5) $$\underbrace{\left(\ddot{\theta}\right)}_{a} t_b^2 + \underbrace{\left(\ddot{\theta}\, t\right)}_{b} t_b + \underbrace{\left(\theta_f - \theta_0\right)}_{c} = 0$$

option 1
- Given : θ_f, θ_0, t, t_b (desired duration of motion)
- Calculate: $\ddot{\theta}$ (Eq.5)

option 2
- Given : $\ddot{\theta}$ (chosen), t, θ_f, θ_0
- Calculate: t_b

$$t_b = \frac{+\ddot{\theta} t \pm \sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}} = \frac{t}{2} \pm \sqrt{\frac{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)}{2\ddot{\theta}}}$$

Joint Space Schemes – Linear Function With Parabolic Blend

- Constraint on the acceleration used in the blend

$$\sqrt{\ddot{\theta}^2 t^2 - 4\ddot{\theta}(\theta_f - \theta_0)} > 0$$

$$\ddot{\theta}^{\cancel{2}} t^2 > 4\cancel{\ddot{\theta}}(\theta_f - \theta_0)$$

$$\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t^2}$$

If equal

$$t_b = \frac{t}{2} \pm \frac{\sqrt{0}}{2\ddot{\theta}} \Rightarrow t_b = \frac{t}{2}$$

Joint Space Schemes –
Linear Function With Parabolic Blend

- The length of the linear portion and the parabolic portion may vary

- High Acceleration($\ddot{\theta}$) $\rightarrow$ Short Blend

  Low Acceleration($\ddot{\theta}$) $\rightarrow$ Long Blend

Joint Space Schemes –
Linear Function With Parabolic Blend