# Manipulator Dynamics 4

# Langrangian Formulation of Manipulator Dynamics 1/

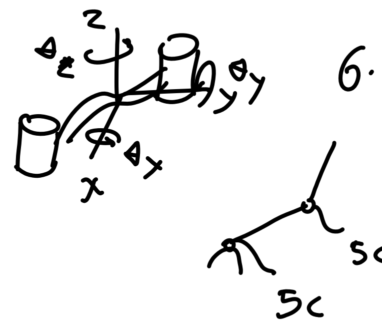
1. Define a set of ***generalized coordinates*** for $i=1,2,3\ldots N$.

   $6 \cdot 2 - 10 = 2$

   These coordinates can be chosen arbitrarily as long as they provide a set of DOF independent variables that map the system in a 1-to-1 manner. The usual variable set for serial manipulators is:

   $6 \cdot m$

$$q_i = \begin{cases} \theta_i & \text{if revolute joint} \\ d_i & \text{if prismatic joint} \end{cases}$$




2. Define a set of ***generalized velocities*** $\dot{q}_i$ for $i=1,2,3\ldots N$

3. Define a set of ***generalized forces (and moments)*** $Q_i$ for $i=1,2,3\ldots N$

   The generalized forces must satisfy

$$Q_i \delta q_i = \delta W$$

where $\delta q_i$ is a small change in the generalized coordinate and $\delta W$ is the work done corresponding to that small change.

# Langrangian Formulation of Manipulator Dynamics 2/

**4.** Write the equations describing the ***kinetic and potential energies*** as functions of the generalized coordinates as well as the resulting Lagrangian.

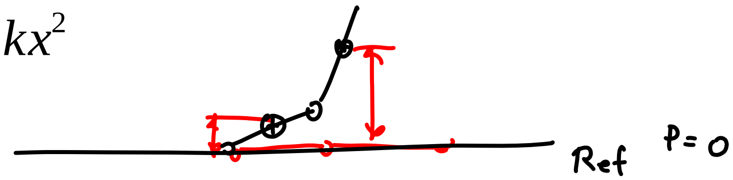
Let $K$ denote the expression describing the kinetic energy. where $K = f\left(q_i, \dot{q}_i, t\right)$

$$k_i = \frac{1}{2} v_{c\,i}^{\;T} m_i v_{c\,i} + \frac{1}{2} {}^{i}\omega_i^{\;c\,i} I_i^{\;i} \omega_i$$

Let $P$ denote the expression describing the potential energy. where $P = f\left(q_i, t\right)$

$$P = \sum m g h_i + \frac{1}{2} k x^2$$

Let $L$ denote the Lagrangian given by:




$$L = K - P$$

# Langrangian Formulation of Manipulator Dynamics 3/

**5.** The equations of motion are given by

$L = K - P$

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \underbrace{\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}} - \cancel{\frac{\partial P}{\partial \dot{q}}}^{\,0}\right)}$$

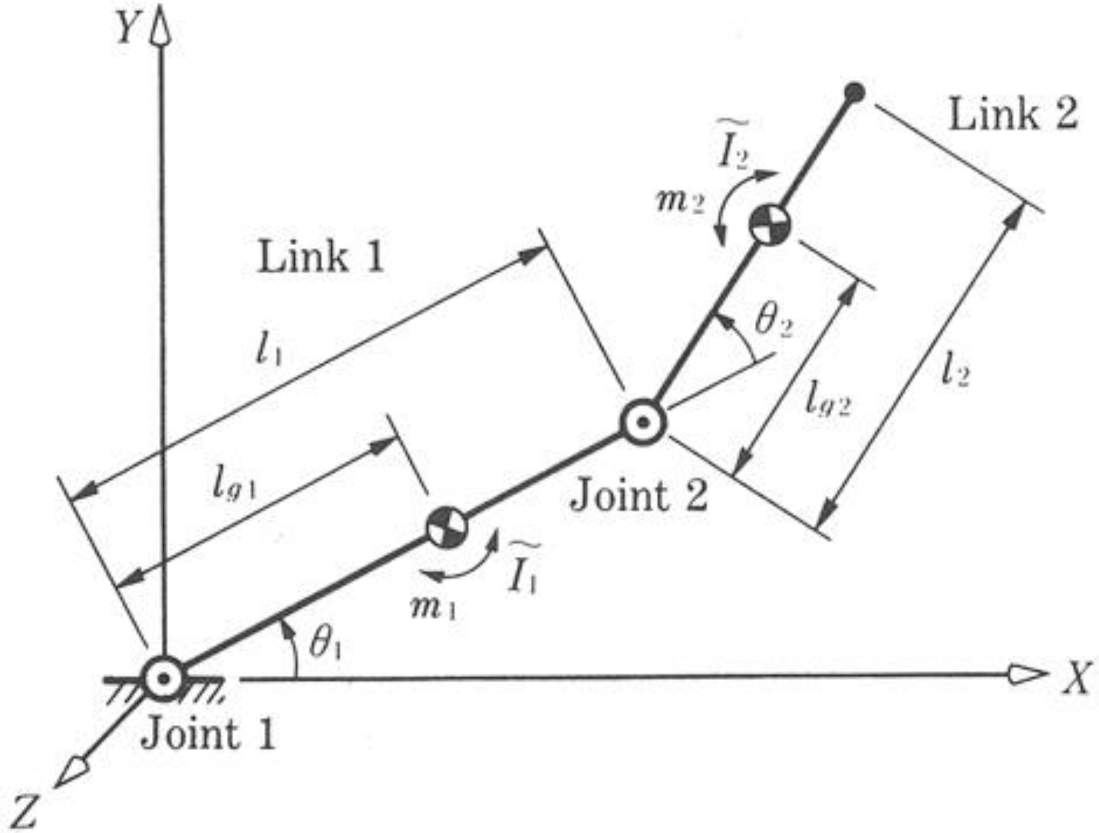
$$-\left[\frac{\partial (K-P)}{\partial q_i}\right]$$

$$\underbrace{-\left[\frac{\partial K}{\partial q_i} - \frac{\partial P}{\partial q_i}\right]}$$

or, more practically, by

$$\boxed{Q_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i}}$$

# Langrangian Formulation - 2R Robot Example



# Langrangian Formulation - 2R Robot Example

**Step 1:** Let $q_1 = \theta_1$ and $q_2 = \theta_2$

**Step 2:** Let $\dot{q}_1 = \dot{\theta}_1$ and $\dot{q}_2 = \dot{\theta}_2$

**Step 3:** Let external forces/torques $Q_i = \tau_i$

**Step 4:**

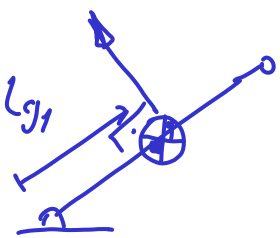
- Kinetic Energy: $k_i = \frac{1}{2} m_i v_{c\,i}{}^T v_{c\,i} + \frac{1}{2} {}^i\omega_i{}^{c\,i} I_i{}^i\omega_i$

$V_{c1} = \omega \times r$

$\dot{\theta} \times l_{g1}$

- For $i=1$

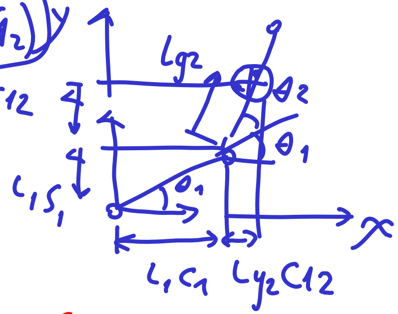
$$\rightarrow \quad k_1 = \frac{1}{2} m_1 \overbrace{L_{g1}{}^2 \dot{\theta}_1{}^2} + \frac{1}{2} I_1 \dot{\theta}_1{}^2$$



# Langrangian Formulation - 2R Robot Example

- To find the velocity of the center of mass of link 2, first consider its position given by

$$ {}^{0}P_{g2} = \begin{bmatrix} L_1 c1 + L_{g2} c12 \\ L_1 s1 + L_{g2} s12 \end{bmatrix} \Rightarrow {}^{0}v_{c2} = \begin{bmatrix} -L_1 s_1 \dot\theta_1 - L_{g2} s_{12}(\dot\theta_1+\dot\theta_2) \\ L_1 c_1 \dot\theta_1 + L_{g2} c_{12}(\dot\theta_1+\dot\theta_2) \end{bmatrix} $$




- The derivative squared gives

+?

$$ L_1^2 \boxed{s_1^2}\,\dot\theta_1^2 + L_{g2}^2 \boxed{s_{12}^2}(\dot\theta_1+\dot\theta_2)^2 - 2L_1L_{g2}\boxed{s_1 s_{12}}\,\dot\theta_1(\dot\theta_1+\dot\theta_2) $$

$c_2$

$$ L_1^2 \boxed{c_1^2}\,\dot\theta_1^2 + L_{g2}^2 \boxed{c_{12}^2}(\dot\theta_1+\dot\theta_2)^2 + 2L_1L_{g2}\boxed{c_1 s_{12}}\,\dot\theta_1(\dot\theta_1+\dot\theta_2) $$

1 1

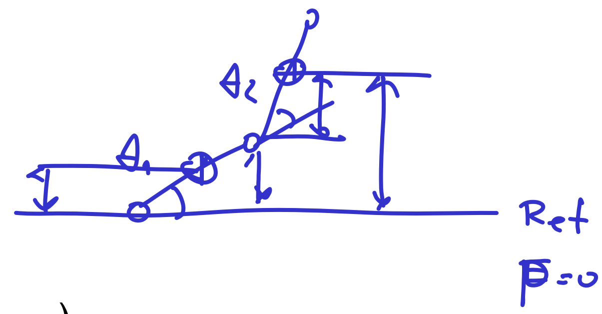
$$ v_{ci}^{T} v_{ci} = L_1^{2}\theta_1^{2} + L_{g2}^{2}\left(\dot\theta_1+\dot\theta_2\right)^2 + 2L_1L_{g2}c2\left(\dot\theta_1^{2}+\dot\theta_1\dot\theta_2\right) $$

- For $i=2$

$$ k_2 = \frac{1}{2} m_2 \left[ L_1^{2}\theta_1^{2} + L_{g2}^{2}\left(\dot\theta_1+\dot\theta_2\right)^2 + 2L_1L_{g2}c2\left(\dot\theta_1^{2}+\dot\theta_1\dot\theta_2\right)\right] + \frac{1}{2} I_2 \left(\dot\theta_1+\dot\theta_2\right)^2 $$

# Langrangian Formulation - 2R Robot Example

- Potential Energy: $p = \sum mgh_i$




- For *i=1* $p_1 = m_1 g L_{g1} s1$
- For *i=2* $p_2 = m_2 g \left( \underbrace{L_1 s1 + L_{g2} s12} \right)$

- Lagrangian: $L = k_1 + k_2 - p_1 - p_2$

# Langrangian Formulation - 2R Robot Example

- **Step 5:** Solving

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

$$Q_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_1}\right) - \frac{\partial K}{\partial \theta_1} + \frac{\partial P}{\partial \theta_1}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\theta}_2}\right) - \frac{\partial K}{\partial \theta_2} + \frac{\partial P}{\partial \theta_2}$$

# Langrangian Formulation - 2R Robot Example

$$\tau_1 = [m_1 L_{g1} + I_1 + m_2 (L_1{}^2 + L_{g2}{}^2 + 2 L_1 L_{g2} c_2) + I_2] \ddot{\theta}_1$$
$$+ [m_2 (L_{g2}{}^2 + L_1 L_{g2} c_2 + I_2] \ddot{\theta}_2$$
$$- m_2 L_1 L_{g2} s_2 (2 \dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2{}^2)$$
$$+ m_1 g L_{g1} c_1 + m_2 g (L_1 c_1 + L_{g2} c_{12})$$

$$\tau_2 = [m_2 (L_{g2}{}^2 + L_1 L_{g2} c_2) + I_2] \ddot{\theta}_1$$
$$+ [m_2 L_{g2} + I_2] \ddot{\theta}_2$$
$$+ m_2 L_1 L_{g2} s_2 \dot{\theta}_1{}^2$$
$$+ m_2 g L_{g2} c_{12}$$

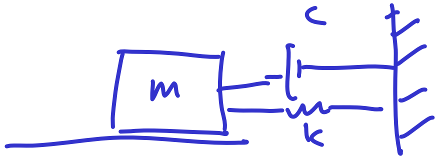
# Gravity Effects - Langrangian Formulation

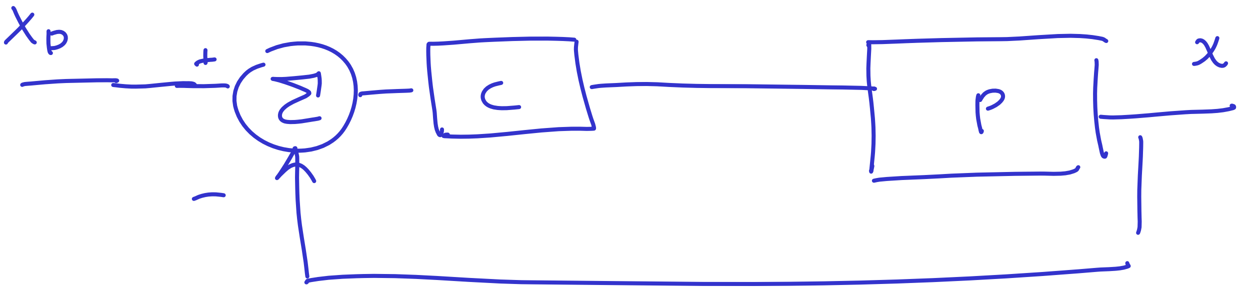
$$\tau_i = \boxed{\frac{d}{dt}\left(\frac{\partial K(\theta,\dot{\theta})}{\partial \dot{\theta}_i}\right) - \frac{\partial K(\theta,\dot{\theta})}{\partial \theta_i}} + \boxed{\frac{\partial P(\theta)}{\partial \theta_i}}$$

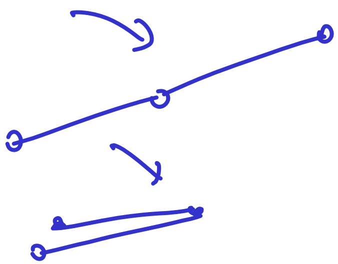
$$\tau = \boxed{M(\theta)\ddot{\theta} + V(\theta,\dot{\theta})} + \boxed{G(\theta)}$$

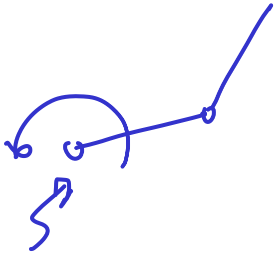
# Manipulators - Control Problem

$$\tau = \underline{M(\theta)}\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$



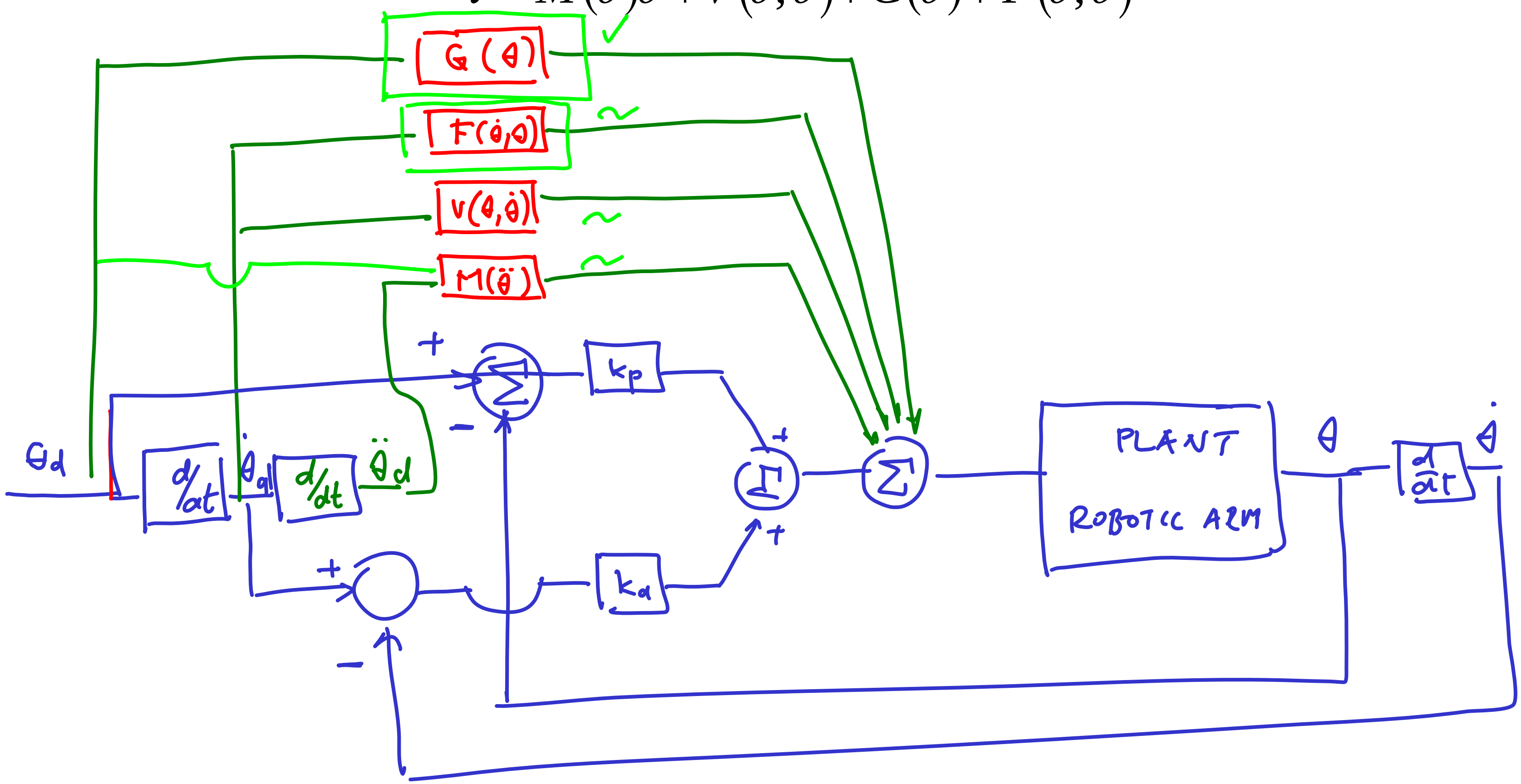





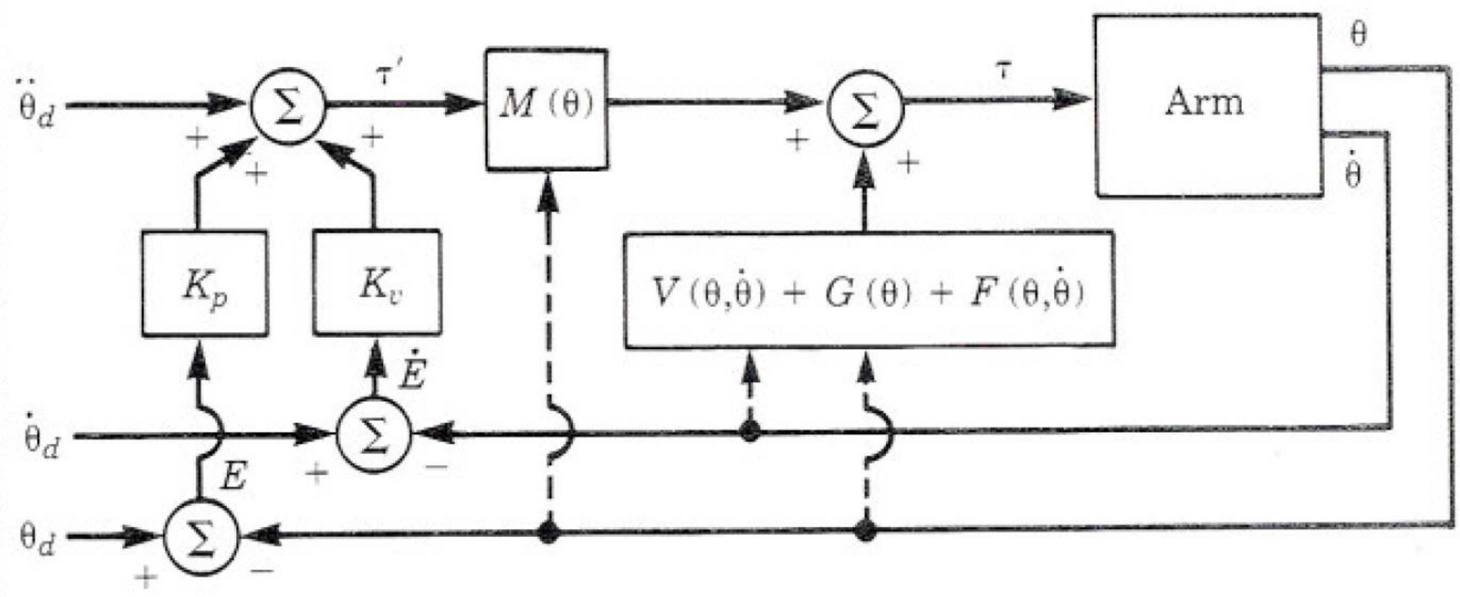
# Manipulators – Non Linear Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$



# Manipulators – Non Linear Control Problem

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$



# Equation of Motion – Non Rigid Body Effects

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

- Viscous Friction $\tau_{friction} = v\dot{\theta}$

- Coulomb Friction $\tau_{friction} = c\,\mathrm{sgn}(\dot{\theta})$

- Model of Friction $\tau_{friction} = v\dot{\theta} + c\,\mathrm{sgn}(\dot{\theta}) = f(\theta,\dot{\theta})$

# LAGRANGIAN FORMULATION OF MANIPULATOR DYNAMICS

Lagrangian function – The differance between kinetic and potential engergy of a mechanical system

$$L = k - U$$

L – Lagrangian
k – kinetic engergy of a mechanical system
U – Potential engergy of a mechanical system

kinetic energy $\quad k = f(P, V)\quad$ – function of the position & velocity of the link

potential energy $\quad U = f(P)\quad$ – function of the position of the link

lagrange's equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

$q$ - vector of generlized coordinates $\quad q = [q_1, q_2, \cdots q_n]^T$

$Q$ - vector of generlized forces $\quad Q = [Q_1, Q_2 \cdots Q_n]^T$

<u>Kinematic Constraint</u> - Imposes some condition on the relative motion between a pair of bodies (e.g. - joint)

KINEMATIC CONSTRAINT

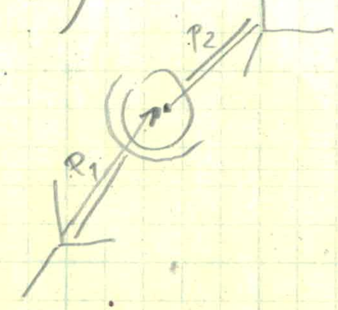
- Holonomic Constraint - if the conditions of the constraint can be expressed as algebraic equations of their coordinates and possibly the time

$$f(x_1, x_2, \cdots, t) = 0$$

$x_i$ - the coordinates of a particle or a rigid body

- Non Holonomic constraint

e.g. joint

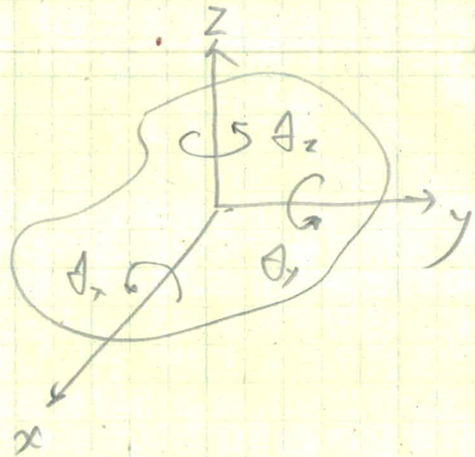



$$\vec{P}_1 = \vec{P}_2$$

—o— Revolute ; —E— presmatic

- The configuration of a mechanical system is known completely if the position and orientation of all the bodies in the system with respect to a referance frame a known

- Rigid body 6 DOF $x, y, z, \theta_x, \theta_y, \theta_z$




- Mechanical system with (m) moving bodies requires (6m) coordinates to specify its configuration completely in 3D space

- Robotic Arm - link are subject to mechanical constrains imposed by the joint the (6m) coordinates are no longer independent

- Most of the constrains encountered in a robotic system are holonomic

$C$ - No of holonomic constrains

$n$ - DOF

$$n = 6m - C$$

The ($n$) independent variables are a set of independent generalized coordinates

$$\left[\begin{array}{c}\text{\# independent generalized} \\ \text{coordinates}\end{array}\right] = \left[\ \text{\# DOF}\ \right]$$

* Lagrangian Equations of the Second Type

- All the forces of constraint in the joint do not appear in the equations,
- # of eq = # DOF
- Applicable to mechanical system with holonomic constraints
- Applicable to serial manipulators

* Lagrangian Equations of the First Type

- \# nonindependent coordinates > # DOF
- coordinates are no longer independent
- { Eq of motion
  { set of constraint Eq. (e.g. Lagrangian Multipliers)

- nonindependent coordinates - Redundant Lagrangian Coordinates
- sutable for paralles manipulators
- Applicable to mechanical systems with both holonamic and nonholonamic constraints

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- FOR SERIAL MANIPULATOR

$q = [q_1, q_2 \cdots q_n]^T$   $q$ < joint angle (revolute joint) / translational distance (prismatic joint)

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DIMENSIONS

$$q_i \begin{cases} [RAD] \\ [m] \end{cases}$$

$$Q_i \begin{cases} [Nm] \\ [N] \end{cases}$$

$$Q_i^T q_i \ [J]$$

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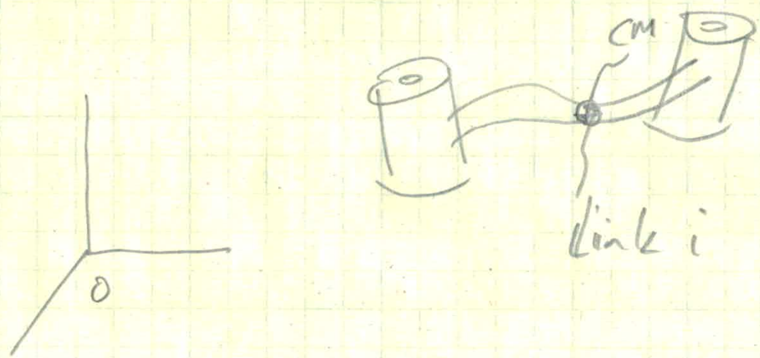
KINETIC ENERGY




$$K_i = \frac{1}{2} V_{ci}^T m_i V_{ci} + \frac{1}{2} W_i \, {}^0I_{ic} \, W_i$$

${}^0I_{ic}$ = The inertia matrix of link $i$ about its CM and expressed in the base frame

${}^iI_{ic}$ – ________________, ________________ — " — link frame

$$^{0}I_{iCM} = {}^{0}R_i \, {}^{i}I_{ic} \left[{}^{0}R_i\right]^T$$

NOTE : – ${}^{i}I_{ic}$ – Time Invariant

${}^{0}I_{ic}$ – Depends on the robot arm posture because it is expressed in the base frame and the orientation of link $i$ with respect to the base is a function of joint variables

Methods for expressing the velocity at the center of mass ${}^{0}V_{ci}$

- Recursive method
- Jacobian (instantaneous screw motion)

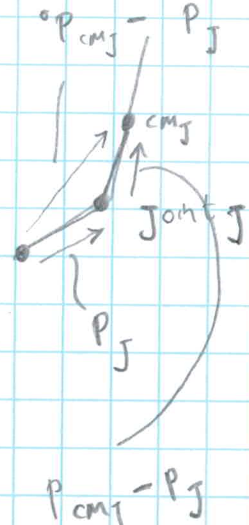
$${}^{0}V_{ci} = {}^{0}J_i \, \dot{\theta}_i$$

$${}^{0}V_{ci} = \begin{bmatrix} {}^{i}V_{ci} \\ \omega_i \end{bmatrix} \qquad J = \begin{bmatrix} {}^{0}J_{vi} \\ {}^{0}J_{wi} \end{bmatrix}$$

$^0J_i$ – A $6 \times n$ matrix that maps the instantaneous joint rates into the instantaneous velocity of the center of mass and the angular velocity of the link $i$ – link Jacobian matrix

$^0J_{vi}$, $^0J_{wi}$ – Two $3 \times n$ submatrices of $^0J_i$ – link Jacobian sub matrices of $^0J_i$

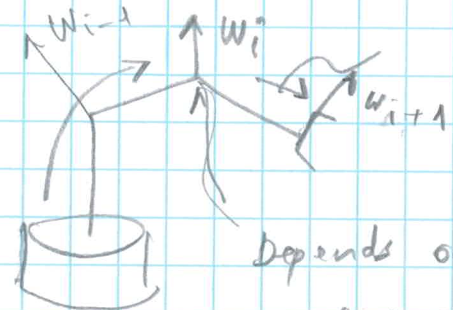
$$
\begin{array}{c}
\downarrow \text{ colum vector } J \\[4pt]
{}^0J_i = \left[ \begin{array}{c}
J^{J}_{vi} = \begin{cases} {}^0Z_J \times ({}^0P_{cmJ} - P_J) & \text{Revolute joint} \\ {}^0Z_J & \text{Prismatic joint} \end{cases} \\ \hline
J^{J}_{wi} = \begin{cases} {}^0Z_J & \text{Revolute joint} \\ 0 & \text{Prismatic Joint} \end{cases}
\end{array} \right]
\end{array}
$$



${}^{J}P_{c_i}$ - Position vector defined from the origine of the $J$ link frame to the center of mass of link $i$ and expressed in the base frame

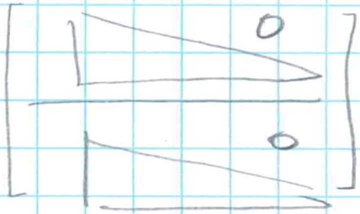
$J_{v_i}^{J}, J_{\omega_i}^{J}$ - The partial rate of change of the velocity of the center of mass and the angular velocity of link $i$ with respect to the joint $J^{th}$ joint motion

Since the motion of link $i$ depends only on the joint 1 through $i$ the two vectors in the matrix are set to zero for $J > i$




Depends on the previous joints

$$ {}^0J_i \begin{bmatrix} \boxed{J^1_{v_i}} & \boxed{J^2_{v_i}} & \boxed{J^3_{w_i}} & \dots & J^i_{v_i}, 0, 0, 0 \dots & 0 \\ \hline \boxed{J^1_{w_i}} & \boxed{J^2_{w_i}} & \boxed{J^3_{w_i}} & \dots & w^J_{v_i}, 0, 0, 0 \dots & 0 \end{bmatrix} $$



$$ k = \frac{1}{2} \sum_{i=1}^{n} \left( {}^0v_{c_i}^T m_i \, {}^0v_{c_i} + w_i^T \, {}^0I_i \, {}^0w_i \right) $$

$$ = \frac{1}{2} \sum_{i=1}^{n} \left[ \left( {}^0J_{v_i} \dot{q} \right)^T m_i \, {}^0J_{v_i} + \left( {}^0J_{w_i} \dot{q} \right)^T {}^0I_i \left( {}^0J_{w_i} \dot{q} \right) \right] $$

$$ = \frac{1}{2} \dot{q}^T \left[ \underbrace{\sum_{i=1}^{n} \left( {}^0J_{v_i}^T m_i J_{v_i} + {}^0J_{w_i}^T \, {}^0I_i \, {}^0J_{w_i} \right)}_{M} \right] \dot{q} $$

Define an $n \times n$ manipulation inertia matrix as

$$ M = \sum_{i=1}^{n} \left( {}^0J_{v_i}^T m_i \, {}^0J_{v_i} + J_{w_i}^T \, {}^0I_i \, J_{w_i} \right) $$

The total kinetic energy of a robot arm can be expressed in terms of the manipulator inertia matrix and the vector of joint rates

$$K = \frac{1}{2} \dot{q}^T M \dot{q}$$

M is configuration dependent because $J_v$ & $J_w$ are configuration dependent as well

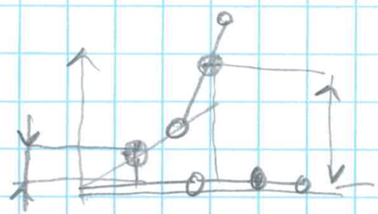
M - symatric
- positive definite

The quadratic form of the equation indicates that the kinetic energy is always positive unless the system is at rest.

# POTENTIAL ENERGY

- POTENTIAL ENERGY stored in a link is defined as the amount of work required to raise the center of mass of link $i$ from the horizontal reference plane to it present position under the influence of gravity
- with reference to the inertial frame (frame 0) the work required to displace link (i) to position $p_{ci}$ is given by $m_i g^T p_{ci}$. The total potential energy stored in a robot arm is

$$U = -\sum_{i=1}^{n} m_i g^T p_{ci}$$



# GENERALIZED FORCES

FORCES — Gravitation forces
— Inertial forces
— All the rest — Gravitation forces

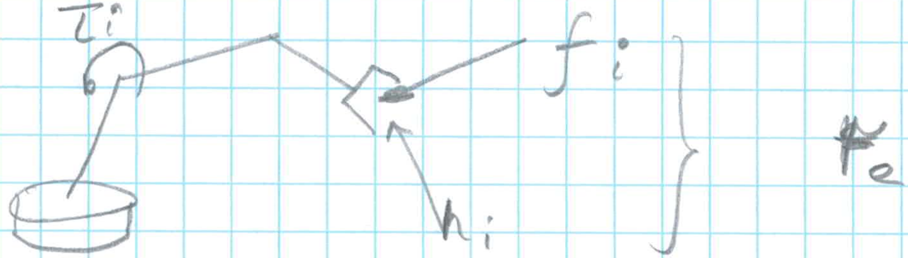
- All the forces acting on a robot arm that consistent with the mechanical constraints

- The vector of generalized forces $Q = [Q_1, Q_2, \ldots Q_n]^T$ is defined by the principle of virtual work as

$$\delta W = Q^T \delta q$$

Actuators $\longrightarrow$ Force / Torque at the joint

External Forces / Moment $\longrightarrow$ END EFFECTOR




$$\delta W = \tau^T \delta q + F_e^T \delta x$$

$\tau = [\tau_1, \dots \tau_n]^T$ - n dimentional vector of joint torques generated by the actuators

$F_e = [f_e^T \; n_e^T]$ - six dimentional vector of resultant force and moment exerted at the end effector

$\delta W$ - virtual work

$\delta x$ - six dimentional virtual displacement vector of the end effector substituting $\qquad \delta x = J\,\delta q$

$$\underbrace{Q^T \delta q}_{\delta W} = \tau^T \delta q + F^T \underbrace{J \delta q}_{\delta x}$$

$$Q^T = \tau^T + F_e^T J$$

$$(Q^T)^T = (\tau^T + F_e^T J)^T$$

$$Q = \tau + J^T F_e$$

- <u>Joint Friction</u>
  - highly nonlinear
  - Grease / oil lubricated bearing
    * static friction
    * boundary lubrication
    * partial fluid lubrication
    * full fluid lubrication

- Full fluid lubrication

Frication force $\propto$ relative velocity

$$f_{r_i} = -b_i \dot{q}_i$$

- The virtual work contributed by this type of frication

$$\delta W = -f_r^T \delta q$$

- $f_r = [b_1 \dot{q}_1, b_2 \dot{q}_2 \cdots b_n \dot{q}_n]$ - The frictional torques or forces in the joints. The minus sign indicates that the direction of the fricational torque or force is always opposite to the joint velocity

$$Q = \tau + J^T F_e - f_r$$

# GENERAL FORM OF DYNAMICAL EQUATIONS

$$L = K - U$$

$$L = \frac{1}{2} \dot{q}^T M \dot{q} + \sum_{i=1}^{n} m_i g^T p_{ci}$$

Expand the term for the kinetic energy into a sum of scalars

$M_{ij}$ - The $(i,j)$ element of the manipulator inertia matrix $M$

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n} m_i g^T p_{ci}$$

The partial derivative of $L$ with respect to $\dot{q}_i$:
Note that the potential energy does not depend on $\dot{q}_i$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} M_{ij} \dot{q}_j$$

Taking the derivative of $\partial L / \partial \dot{q}_i$ with respect to time

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} M_{ij} \ddot{q}_j + \sum_{j=1}^{n} \left(\frac{dM_{ij}}{dt}\right) \dot{q}_j = \sum_{j=1}^{n} M_{ij} \ddot{q}_j + \sum_{j=1}^{n} \left[\sum_{k=1}^{n} \frac{\partial M_{ij}}{\partial q_k} \dot{q}_k\right] \dot{q}_j$$

$(gh)' = gh' + g'h$

$\frac{dM}{dq} \frac{dq}{dt}$

Taking the partial derivative $\partial L / \partial q_i$

$$\frac{\partial L}{\partial q_i} = \frac{1}{2} \frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n} \sum_{k=1}^{n} M_{jk} \dot{q}_j \dot{q}_k\right) + \sum_{j=1}^{n} m_j g^T \left(\frac{\partial p_{cj}}{\partial q_i}\right)$$

The $i^{th}$ column vector of the link Jacobian sub matrix

${}^{0}J_{vi}$

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k + \sum m_i g^T \, {}^{0}J_{vj}$$

substitute all the equation into the Lagrange Eq.

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{\displaystyle\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j\right)} - \frac{\partial L}{\partial q_i} = Q_i$$

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j\right) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k - \sum_{j=1}^{n} m_j g^T J^i_{vj}$$

$$\underbrace{\sum_{j=1}^{n} M_{ij}\ddot{q}_j}_{} + \underbrace{V_i}_{} + \underbrace{G_i}_{} = Q_i$$

Inertia

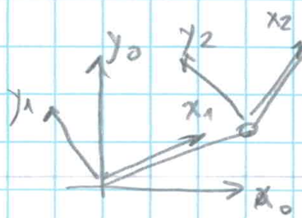
Coriolis & centrifugal

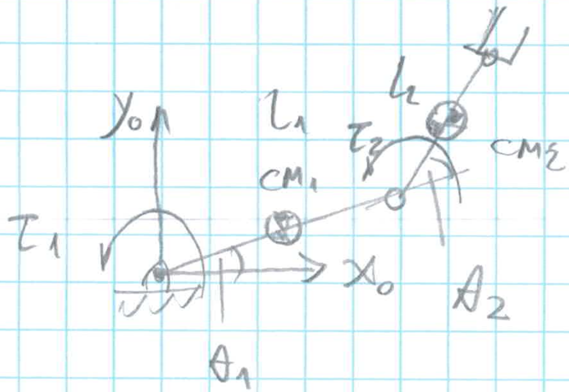
$$\left[ V_i = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k \right.$$

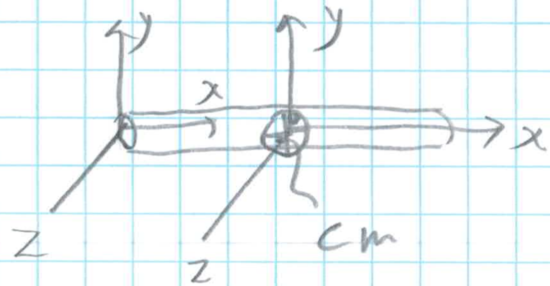
gravitational effects

$$\left[ G_i = -\sum_{j=1}^{n} m_j g^T J^i_{vj} \right.$$

# EXAMPLE 2 DOF



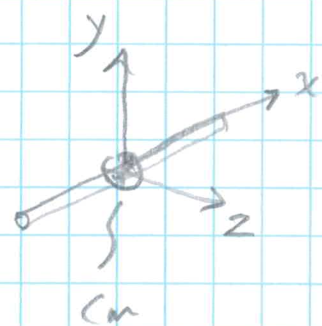




- Note - the link coordinate axes are aligned with the principle axes of each link

## (a) Link inertia matrices

small $\quad l >> r$

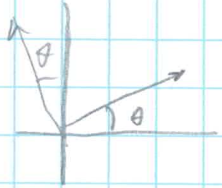
$$ {}^{i}I_{ci} = \frac{1}{12} m l_i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ {}^{0}I_i = {}^{0}_{i}R \; {}^{i}I_{c_1} \left({}^{0}R_i\right)^T $$

$\underline{i = 1}$

$$ {}^{0}I_1 = {}^{0}_{1}R \; {}^{1}I_{c_1} \left({}^{0}_{1}R\right)^T $$




$$ {}^{0}I_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m l_1^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ = \frac{1}{12} m l_1^2 \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = $$

$$ = \frac{1}{12} m l_1^2 \begin{bmatrix} +s_1^2 & -c_1 s_1 & 0 \\ -c_1 s_1 & c_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$i = 2$

$${}^{0}_{2}R = {}^{0}_{1}R\,{}^{1}_{2}R = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$= \begin{bmatrix} C_1C_2 - S_1S_2 & -C_1S_2 - S_1C_2 & 0 \\ S_1C_2 + C_1S_2 & -S_1S_2 + C_1C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C12 & -S12 & 0 \\ S12 & C12 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}I_2 = \begin{bmatrix} C12 & -S12 & 0 \\ S12 & C12 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{12} m l_2^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C12 & S12 & 0 \\ -S12 & C12 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{12} m l_2^2 \begin{bmatrix} C12 & -S12 & 0 \\ S12 & C12 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -S12 & C12 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{12} m l_2^2 \begin{bmatrix} +S^2 12 & -S12\,C12 & 0 \\ -S12\,C12 & C^2 12 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Link Jacobian Matrices

position of the CM

$$ {}^0P_{c1} = \begin{bmatrix} 1/2\, l_1 c_1 \\ 1/2\, l_1 s_1 \\ 0 \end{bmatrix} \qquad {}^0P_2 = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix} $$

$$ {}^2P_{c2} = \begin{bmatrix} 1/2\, l_2 c_2 \\ 1/2\, l_2 s_2 \\ 0 \end{bmatrix} \quad ; \quad {}^0P_{c2} = \begin{bmatrix} l_1 c_1 + 1/2\, l_2 c_{12} \\ l_1 s_1 + 1/2\, l_2 s_{12} \\ 0 \end{bmatrix} $$

$$ {}^0P_{c2} = {}^0_2T\, {}^2P_{c2} = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|c} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} 1/2\, l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ \left[\begin{array}{ccc|c} c_{12} & -s_{12} & 0 & l_1 c_1 \\ s_{12} & c_{12} & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1/2\, l_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2\, l_2 c_{12} + l_1 c_1 \\ 1/2\, l_2 s_{12} + l_1 s_1 \\ 0 \\ 1 \end{bmatrix} $$

$$\begin{matrix} v_{cm1} \\ v_{cm2} \end{matrix} = \left[ {}^0J_v \right] \begin{matrix} \theta_1 \\ \theta_2 \end{matrix}$$

$${}^0J^j_{v_i} = \begin{cases} {}^0_jZ_{j-1} \times {}^0P_{cm_i} - {}^0P_j & rev \\ {}^0_jZ_{j-1} & pri \end{cases}$$

$${}^0J^j_{w_i} = \begin{cases} Z_j & Rev \\ 0 & pri \end{cases}$$

$i = 1$

$${}^0J_{v1} = \left[ \begin{matrix} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{z_0} \times \underbrace{\begin{pmatrix} 1/2\, l_1 c_1 \\ 1/2\, l_1 s_1 \\ 0 \end{pmatrix}}_{{}^0P_{c1} - {}^0P_1} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{matrix} \right] \qquad \left[ \underbrace{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}}_{{}^0Z_1} \quad \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$$

position of the CM in frame 0

$$= \left[ \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 1/2\, l_1 c_1 & 1/2\, l_1 s_1 & 0 \end{vmatrix} \quad \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$$

$$\begin{bmatrix} -1/2\, l_1 s_1 & 0 \\ +1/2\, l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}$$


$i = 2$

$$J=1 \qquad {}^0P_{CMi} - {}^0P_j \qquad J=2 \qquad {}^0P_{CMi} - {}^0P_j$$

$$
{}^0J_{v2} = \left[\begin{array}{c|c}
{}^0Z_0 \times \left(\begin{bmatrix} L_1c_1 + L_2c_{12} \\ L_1s_1 + L_2s_{12} \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) & {}^0Z_1 \times \left(\begin{bmatrix} L_1c_1 + L_2c_{12} \\ L_1s_1 + L_2s_{12} \\ 0 \end{bmatrix} - \begin{bmatrix} L_1c_1 \\ L_1s_1 \\ 0 \end{bmatrix}\right)
\end{array}\right]
$$

$$
\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ L_1c_1 + L_2c_{12} & L_1s_1 + L_2s_{12} & 0 \end{vmatrix}
\qquad
\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{2}L_1c_{12} & \frac{1}{2}L_1s_{12} & 0 \end{vmatrix}
$$

$$
\left[\begin{array}{c|c}
-l_1s_1 - l_2s_{12} & -\frac{1}{2}l_1s_{12} \\
-l_1c_1 - l_2s_{12} & +\frac{1}{2}l_2c_{12} \\
0 &
\end{array}\right]
$$

$$
{}^0J_{w2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}
$$

(c) Manipulation Inertia Matrix

$$M = J_{v1}^T m_1 J_{v1} + J_{w1}^T I_1 J_{w1} + J_{v2}^T m_2 J_{v2} + J_{w2}^T I_2 J_{w2}$$

$$= \begin{bmatrix} -1/2\, l_1 s_1 & 1/2\, l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} m_1 \begin{bmatrix} -1/2\, l_1 s_1 & 0 \\ 1/2\, l_2 c_1 & 0 \\ 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} 1/12\, m_1 l_1^2 \begin{bmatrix} s_1^2 & -s_1 c_1 & 0 \\ -s_1 c_1 & c_1^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} -l_1 s_1 - 1/2\, l_2 s_{12} & l_1 c_1 + 1/2\, l_2 c_{12} & 0 \\ -1/2\, l_2 s_{12} & 1/2\, l_2 c_{12} & 0 \end{bmatrix} m_2 \begin{bmatrix} -l_1 s_1 - 1/2\, l_2 s_{12} & -1/2\, l_2 s_{12} \\ l_1 c_1 + 1/2\, l_2 c_{12} & 1/2\, l_2 c_{12} \\ 0 & 0 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} 1/2\, m_2 l_2^2 \begin{bmatrix} s_{12}^2 & -s_{12} c_{12} & 0 \\ -s_{12} c_{12} & c_{12}^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{1}{3} m_1 l_1^2 + m_2 \left( l_1^2 + l_1 l_2 c_2 + \frac{1}{3} l_2^2 \right) & m_2 ( \frac{1}{2} l_1 l_2 c_2 + \frac{1}{3} l_2^2 \\ m_2 \left( \frac{1}{2} l_1 l_2 c_2 + \frac{1}{3} l_2^2 \right) & \frac{1}{3} m_2 l_2^2 \end{bmatrix}$$

## (d) Velocity Coupling Vector

$$V_1 = \sum_{j=1}^{2} \sum_{k=1}^{2} \left( \frac{\partial M_{1j}}{\partial \theta_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \theta_1} \right) \dot{\theta}_j \dot{\theta}_k$$

$$\overset{j=1 \;\; k=1}{=} \left( \frac{\partial M_{11}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_1} \right) \dot{\theta}_1 \dot{\theta}_1 + \overset{j=1 \;\; k=2}{\left( \frac{\partial M_{11}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{12}}{\partial \theta_1} \right)} \dot{\theta}_1 \dot{\theta}_2 +$$

$$\overset{j=2 \;\; k=1}{\left( \frac{\partial M_{12}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{21}}{\partial \theta_1} \right)} \dot{\theta}_2 \dot{\theta}_1 + \overset{j=2 \;\; k=2}{\left( \frac{\partial M_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_1} \right)} \dot{\theta}_2 \dot{\theta}_2 =$$

$$\left( 0 - \cancelto{0}{1/2 \, 0} \right) \dot{\theta}_1 \dot{\theta}_1 + \left( -m_2 l_1 l_2 s_2 - \cancelto{0}{1/2 \, 0} \right) \dot{\theta}_1 \dot{\theta}_2 +$$

$$\left( 0 - \cancel{1/2 \, 0} \right) \dot{\theta}_2 \dot{\theta}_1 + \left( -m_2 \, 1/2 \, l_1 l_2 s_2 - \cancelto{0}{1/2 \, 0} \right) \dot{\theta}_2^2$$

$$V_1 = -m_2 l_1 l_2 s_2 \left( \dot{\theta}_1 \dot{\theta}_2 - 1/2 \, \dot{\theta}_2^2 \right)$$

$$V_2 = \sum_{j=1}^{2} \sum_{k=1}^{2} \left( \frac{\partial M_{2j}}{\partial \theta_k} - \frac{1}{2} \frac{\partial M_{jk}}{\partial \theta_2} \right) \dot\theta_j \dot\theta_k$$

$$= \overbrace{\left( \frac{\partial M_{21}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{11}}{\partial \theta_2} \right)}^{j=1 \quad k=1} \dot\theta_1 \dot\theta_2 + \overbrace{\left( \frac{\partial M_{21}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{12}}{\partial \theta_2} \right)}^{j=1 \quad k=2} \dot\theta_1 \dot\theta_2$$

$$\overbrace{\left( \frac{\partial M_{22}}{\partial \theta_1} - \frac{1}{2} \frac{\partial M_{21}}{\partial \theta_2} \right)}^{j=2 \quad k=1} \dot\theta_2 \dot\theta_1 + \overbrace{\left( \frac{\partial M_{22}}{\partial \theta_2} - \frac{1}{2} \frac{\partial M_{22}}{\partial \theta_2} \right)}^{j=2 \quad k=2} \dot\theta_2 \dot\theta_2$$

$$\left[ 0 - \frac{1}{2} \left( -m_2 l_1 l_2 s_2 \right) \right] \dot\theta_1 \dot\theta_1 + \left[ -m_2 \tfrac{1}{2} l_1 l_2 s_2 + \frac{1}{2} m_2 \tfrac{1}{2} l_1 l_2 s_2 \right] \dot\theta_1 \dot\theta_2$$

cancel

$$\left[ 0 + \tfrac{1}{2} \left( m_2 l_1 l_2 s_2 \right) \right] \dot\theta_2 \dot\theta_1 + \left[ 0 - \frac{1}{2} \left( 0 \right) \right] \dot\theta_2 \dot\theta_2$$

$$V_2 = \frac{1}{2} m_2 l_1 l_2 s_2 \dot\theta_1^2$$

## (e) Gravitational vector

$$G_i = -\sum_{J=1}^{n} m_J \, g^T \, J_{vJ}^{i}$$

$i=1$ $\quad G_1 = -\overset{J=1}{m_1 g^T J_{v1}^1} - \overset{J=2}{m_2 g^T J_{v2}^1}$

FIRST column of $J_{v1}$

First column of $J_{v2}$

$$= -m_1 [0 \; -g \; 0] \begin{bmatrix} -1/2\, l_1 s_1 \\ 1/2\, l_1 c_1 \\ 0 \end{bmatrix} - m_2 [0 \; -g \; 0] \begin{bmatrix} -l_1 s_1 - 1/2\, l_2 s_{12} \\ l_1 c_1 + \frac{1}{2} a_2 c_{12} \\ 0 \end{bmatrix}$$

$$G_1 = +\frac{1}{2} m_1 g l_1 c_1 + m_2 g l_1 c_1 + 1/2\, m_2 g l_2 s_{12}$$

---

$i=2$ $\quad G_2 = -\overset{J=1}{m_1 g^T J_{v1}^2} - \overset{J=2}{m_2 g^T J_{v2}^2}$

$$-m_1 [0 \; -g \; 0] \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - m_2 [0 \; -g \; 0] \begin{bmatrix} -1/2\, l_2 s_{12} \\ 1/2\, l_2 c_{12} \end{bmatrix}$$

$$G_2 = 1/2\, m_2 g l_2 c_{12}$$

$$M\ddot{q} + V + G = Q$$

$$(1/3 m_1 + m_2) l_1^2 + m_2 l_1 l_2 c_2 + 1/3 m_2 l_2^2$$

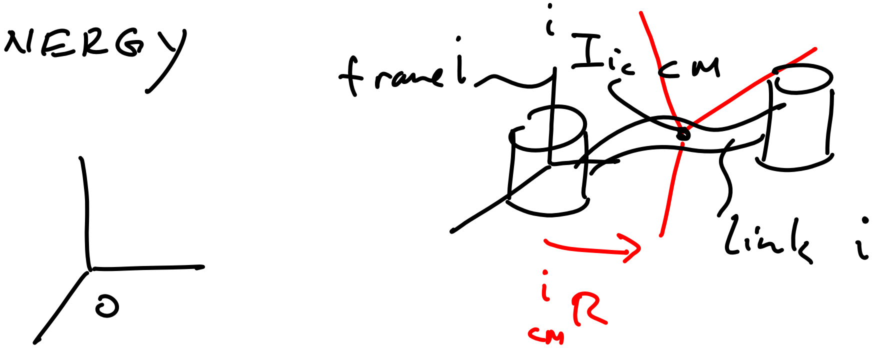
$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \left[\begin{array}{c|c} 1/3 m_1 l_1^2 + m_2 \left( l_1^2 + l_1 l_2 c_2 + \frac{1}{3} l_2^2 \right) & m_2 (1/2 l_1 l_2 c_2) + 1/3 l_2^2 m_2 \\ \hline m_2 \left( 1/2 l_1 l_2 c_2 + 1/3 l_2^2 \right) & 1/3 m_2 l_2^2 \end{array}\right] \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$\left[\begin{array}{c|c} 0 & 1/2 m_2 l_1 l_2 s_2 \\ \hline 1/2 m_1 l_1 l_2 s_2 & 0 \end{array}\right] \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -m_2 l_1 l_2 s_2 \\ 0 \end{bmatrix} \dot{\theta}_1 \dot{\theta}_2 +$$

$$g \begin{bmatrix} 1/2 m_1 l_1 c_1 + m_2 (l_1 c_1 + 1/2 l_2 s_{12}) \\ 1/2 m_2 l_2 c_{12} \end{bmatrix}$$

# Langrangian Formulation of Manipulator Dynamics

KINETIC ENERGY




$$k_i = \frac{1}{2} V_{c_i}^T m_i V_{c_i} + \frac{1}{2} \omega_i \, {}^{0}I_{i_c} \, \omega_i$$

${}^{0}I_{i_c}$ – The inertia matrix of link i about it cm and expressed in the base frame

${}^{i}I_{i_c}$ = ______________ " ______________
______ " ______ link frame

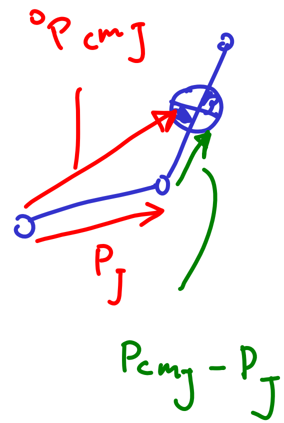
# Langrangian Formulation of Manipulator Dynamics

$$ {}^0I_{i\,cm} = {}^0_iR \; {}^iI_{ic} \left[ {}^0_iR \right]^T $$

$$ {}^0v_{c_i} = {}^0J_i \; \dot{\theta}_i $$

$$ \begin{bmatrix} {}^iv_{c_i} \\ w_i \end{bmatrix} \qquad \begin{bmatrix} {}^0J_{v_i} \\ {}^0J_{w_i} \end{bmatrix} $$

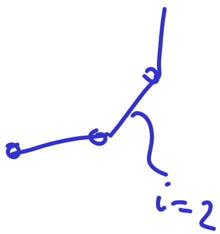
c.v. $J$




$$ {}^0J_i = \begin{array}{c} J^J_{v_i} \\ \\ J^J_{w_i} \\ \\ \end{array} \left[ \begin{array}{c} {}^0Z_J \times \underline{\left( {}^0P_{cm_J} - P_J \right)} \\ {}^0Z_J \\ \hline {}^0Z_J \\ 0 \end{array} \right] \begin{array}{l} - R \\ - P \\ - R \\ - P \end{array} $$

# Langrangian Formulation of Manipulator Dynamics

$$
{}^0J_i = \left[\begin{array}{cccccccccccc}
J_{v_i}^1 & J_{v_i}^2 & J_{v_i}^3 & \cdots & J_{v_i}^i & 0 & 0 & 0 & 0 & 0 & \\
\hline
J_{w_i}^1 & J_{w_i}^2 & J_{w_i} & \cdots & J_{w_i}^i & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$




$$
K = \frac{1}{2}\sum_{i=1}^{n}\left({}^0V_{c_i}^T\, m_i\, {}^0V_{c_i} + W_i^T\, {}^0I_i\, {}^0W_i\right)
$$

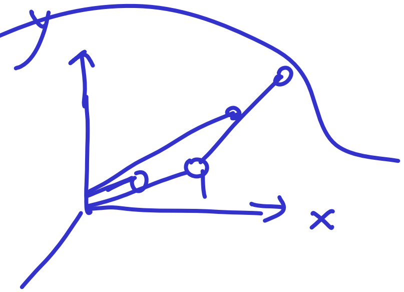
$$
\frac{1}{2}\sum_{i=1}^{n}\left[\left(J_{v_i}\dot q\right)^T m_i \left({}^0J_{v_i}\dot q\right) + \left(J_{w_i}\dot q\right)^T {}^0I_{i\,cm}\left(J_{w_i}\dot q\right)\right]
$$

$$
\boxed{K = \frac{1}{2} q^T \underbrace{\left[\sum_{i=1}^{n}\left(J_{v_i}^T m_i J_{v_i} + J_{w_i}^T\, {}^0I_{cm_i}\, J_{w_i}\right]\right.}_{M} \dot q = \frac{1}{2}\dot q^T M \dot q}
$$

# Langrangian Formulation of Manipulator Dynamics

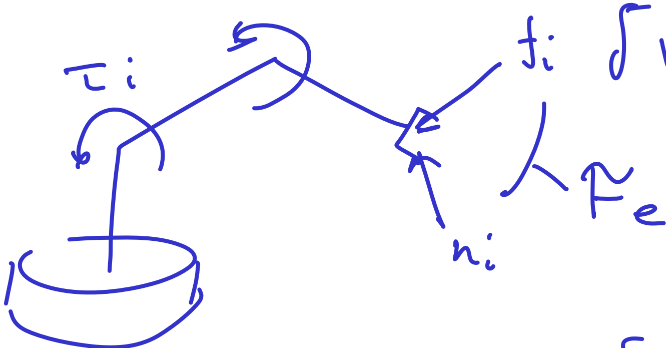
Potential Energy

$$\boxed{U = -\sum_{i=1}^{n} m_i g^T p_{ci}}$$




$$g = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}$$

GENERAL FORCES




$$\delta W = \underbrace{Q^T}\, \delta q$$

$$\underbrace{\delta x = J\, \delta q}$$

$$\underbrace{\delta W = \tau^T \delta q + F^T \delta x}$$

# Langrangian Formulation of Manipulator Dynamics

$$\underbrace{Q^T \cancel{\delta q}}_{\delta W} = \tau^T \cancel{\delta q} + F^T \underbrace{J \cancel{\delta q}}_{\delta x}$$

$$Q^T = \tau^T + F_e^T J$$

$$(Q^T)^T = (\tau^T + F_e^T J)^T$$

$$\rightarrow \boxed{Q = \underbrace{\tau}_{\text{Joint Torque}} + \underbrace{J^T F_e}_{\text{external forces}} - f_r}$$

# Langrangian Formulation of Manipulator Dynamics

$$L = K - U$$

$$L = \frac{1}{2} \underbrace{\dot{q}^T M \dot{q}} + \sum_{i=1}^{n} m_i g^T p_{ci}$$

$$L = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \overbrace{\dot{q}_i \dot{q}_j}^{+} + \sum_{i=1}^{n} m_i g^T p_{ci}$$

$$\rightarrow \quad \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}_{*} - \underbrace{\frac{\partial L}{\partial q_i}}_{**} = Q_i$$

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*

$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} = \sum_{j=1}^{n} M_{ij}\,\ddot{q}_j + \sum_{j=1}^{n}\left(\frac{dM_{ij}}{dt}\right)\dot{q}_j =$$

$(gh)' = gh' + g'h$

$\frac{\partial M}{\partial q}\frac{\partial q}{dt}$

$$= \sum_{j=1}^{n} M_{ij}\,\ddot{q}_j + \sum_{j=1}^{n}\left[\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\right]\dot{q}_j$$

**

$$\frac{dL}{\partial q_i} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} M_{jk}\,\dot{q}_j\,\dot{q}_k\right) + \sum_{j=1}^{n} m_j\, g^T\left(\frac{\partial P_{c_j}}{\partial q_i}\right)$$

${}^0J_{v_i}$

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$$\underbrace{\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j\right)}_{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)} \quad V_i$$

$$\underbrace{-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k - \sum_{j=1}^{n} m_j g^T J_{v_j}^{i}}_{\frac{\partial L}{\partial q_i}} = Q_i$$

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \boxed{V_i} + \boxed{G_i} = Q_i$$

# Dynamic Equations - State Space Equation

- It is often convenient to express the dynamic equations of a manipulator in a single equation

$$\boxed{\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)}$$

where

$M(\theta)$ - Mass matrix (includes inertia terms) - ***nxn Matrix***

$V(\theta,\dot{\theta})$ - Centrifugal (square of joint velocity) and Coriolis (product of two different joint velocities) terms - ***nx1 Vector***

$G(\theta)$ - gravitational terms - ***nx1 Vector***.

# Dynamic Equations - Configuration Space Equation

- By rewriting the velocity dependent term $V(\theta,\dot{\theta})$ in a different form, we can write the dynamic equations as

$$\boxed{\tau = M(\theta)\ddot{\theta} + B(\theta)\left[\dot{\theta}\,\dot{\theta}\right] + C(\theta)\left[\dot{\theta}^2\right] + G(\theta)}$$

where

$B(\theta)$ - Centrifugal coefficients(square of joint velocity)

$C(\theta)$ - Coriolis coefficients (product of two different joint velocities)

- This form can be useful for applications using force control. Each of the matrices is a function of manipulator configuration only (that is, joint position) and can be updated at a rate depending on the magnitude of joint changes.

# Dynamic Equations - Cartesian State Space Equation

- It can sometimes be desirable to have a relationship between the end effector's Cartesian accelerations and the joint torques. Beginning from the Configuration Space equation

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$$

- we can substitute the joint moments using our definition of the Jacobian matrix:

$$\tau = J^T(\theta)F \qquad F = J^{-T}(\theta)\tau$$

$$\dot{x} = J(\theta)\dot{\theta}$$

- By differentiation, we find

$$\ddot{x} = \dot{J}(\theta)\dot{\theta} + J(\theta)\ddot{\theta}$$

# Dynamic Equations - Cartesian State Space Equation

- Solving for joint acceleration gives

$$\ddot{\theta} = J^{-1}\ddot{x} - J^{-1}\dot{J}\dot{\theta}$$

- Substitution yields

$$F = J^{-T}\tau = J^{-T}M(\theta)J^{-1}\ddot{x} - J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta}) + J^{-T}G(\theta)$$

$$\boxed{F = M_x(\theta)\ddot{x} + V_x(\theta,\dot{\theta}) + G_x(\theta)}$$

Where

$$M_x(\theta) = J^{-T}M(\theta)J^{-1}$$

$$V_x(\theta,\dot{\theta}) = J^{-T}M(\theta)J^{-1}\dot{J}\dot{\theta} + J^{-T}V(\theta,\dot{\theta})$$

$$G_x(\theta) = J^{-T}G(\theta)$$

- This equation relates the forces and moments at the end effector to the Cartesian accelerations of the end effector and the manipulator joint positions and velocities.