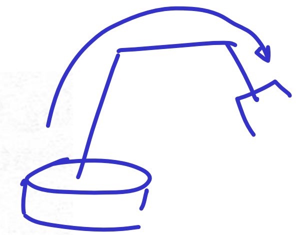
Manipulator Dynamics 3

Iterative Newton-Euler Equations - Solution Procedure

- **Step 1 -** Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

- **Step 2 -** Write the Newton and Euler equations for each link.



Outward iterations: $i : 0 \rightarrow 5$

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\, {}^{i}\omega_i + \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, $$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\, {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\, {}^{i}\omega_i \times \dot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\, {}^{i+1}\hat{Z}_{i+1}, $$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R \left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right) + {}^{i}\dot{v}_i \right), $$

$$ \begin{aligned} {}^{i+1}\dot{v}_{C_{i+1}} = {} & {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ & + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1}, \end{aligned} $$

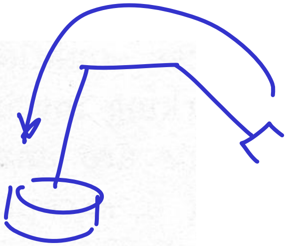
$$ {}^{i+1}F_{i+1} = m_{i+1}\, {}^{i+1}\dot{v}_{C_{i+1}}, $$

$$ {}^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\, {}^{i+1}\omega_{i+1}. $$

Iterative Newton-Euler Equations - Solution Procedure

- **Step 3 -** Use the forces and torques generated by interacting with the environment (that is, tools, work stations, parts etc.) in calculating the joint torques from the end effector to the robot's base.

Inward iterations: $\quad i : 6 \rightarrow 1$



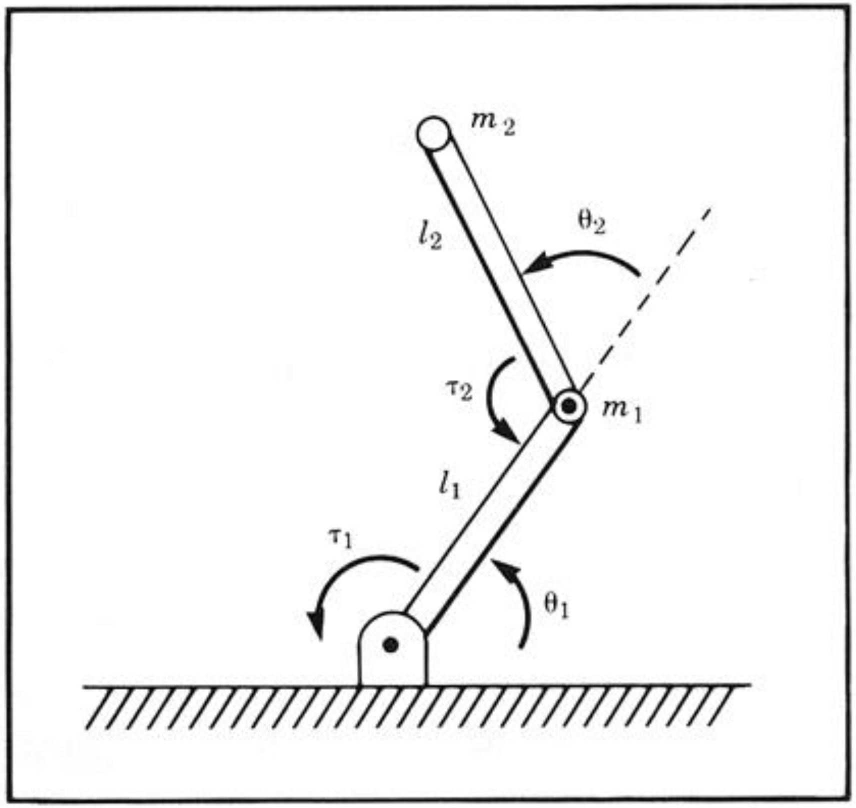
$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} + {}^{i}F_i, \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\
&\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R \; \underline{{}^{i+1}f_{i+1}}, \\
\tau_i &= {}^{i}n_i^T \; {}^{i}\hat{Z}_i.
\end{aligned}
$$

Iterative Newton-Euler Equations - Solution Procedure

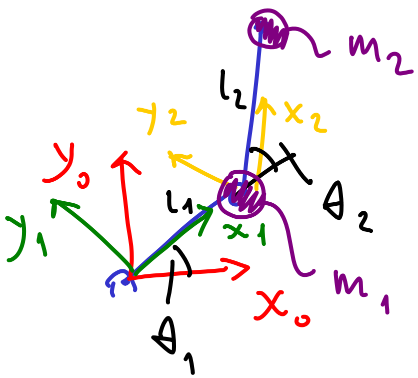
- **Error Checking -** Check the units of each term in the resulting equations

- **Gravity Effect** - The effect of gravity can be included by setting ${}^{0}\dot{v}_0 = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.

Iterative Newton-Euler Equations - 2R Robot Example



Iterative Newton-Euler Equations - 2R Robot Example



$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Vectors locates the center of mass for each link

$$
{}^1P_{c_1} = l_1 \hat{X}_1 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \qquad {}^2P_{c_2} = l_2 \hat{X}_2 = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{cases} {}^{c_1}I_1 = 0 \\ {}^{c_2}I_2 = 0 \end{cases}
$$

Iterative Newton-Euler Equations - 2R Robot Example

- No external force / torques on the end effector

$$\begin{cases} f_3 = 0 \\ n_3 = 0 \end{cases}$$

- The base of the robot is not rotating

$$\begin{cases} \omega_0 = 0 \\ \dot{\omega}_0 = 0 \end{cases}$$

- To include gravity

$${}^0\dot{v}_0 = g\,\hat{Y}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $\underline{i = 0}$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{1}\omega_1 = {}^{1}_{0}R\,{}^{0}\overset{0}{\cancel{\omega}}_0 + \dot{\theta}_1\,\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $\underline{i=0}$

$$ {}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{\ i}R\ {}^{i}\dot{\omega}_{i} + {}^{i+1}_{\ i}R\ {}^{i}\omega_{i} \times \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} $$

$$ {}^{1}\dot{\omega}_{1} = \underbrace{{}^{1}_{0}R\ {}^{0}\cancel{\dot{\omega}}_{0}}_{=0} + \underbrace{{}^{1}_{0}R\ {}^{0}\cancel{\omega}_{0} \times \dot{\theta}_{1}\ {}^{1}\hat{Z}_{1}}_{=0} + \ddot{\theta}_{1}\ {}^{1}\hat{Z}_{1} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_{1} \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i=0$

$$ {}^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right) $$

$$ {}^{1}\dot{v}_{1} = {}^{1}_{0}R\left(\underbrace{{}^{0}\cancel{\dot{\omega}_0} \times {}^{0}P_{1}}_{=0} + \underbrace{{}^{0}\cancel{\omega_0} \times \left({}^{0}\cancel{\omega_0} \times {}^{0}P_{1}\right)}_{=0} + {}^{0}\dot{v}_0\right) = $$

$$ = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$\overset{\text{cm}}{}$$

$$^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1}$$

$${}^1\dot{v}_{c_1} = {}^1\dot{w}_1 \times {}^1P_{c_1} + {}^1w_1 \times \left({}^1w_1 \times {}^1P_{c_1} \right) + {}^1\dot{v}_1$$

$$= \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + {}^1w_1 \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ L_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ L_1 \ddot{\theta}_1 \\ 0 \end{bmatrix} + \underbrace{{}^1w_1 \times \begin{bmatrix} 0 \\ L_1 \dot{\theta}_1 \\ 0 \end{bmatrix}}_{} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

$$= \begin{bmatrix} 0 \\ L_1 \ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & L_1 \dot{\theta}_1 & 0 \end{vmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ L_1 \ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -L_1 \dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}$$

$$^1\dot{v}_{c_1} = \begin{bmatrix} -L_1 \dot{\theta}_1^2 + g s_1 \\ L_1 \ddot{\theta} + g c_1 \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $\underline{i = 0}$

$$ {}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}} $$

$$ {}^1F_{1x} $$

$$ {}^1F_1 = m_1\,{}^1\dot{v}_{c_1} = m_1 \begin{bmatrix} -l_1\dot{\theta}_1^2 + g s_1 \\ l_1\ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix} $$

$$ {}^1F_{1y} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 0$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

$$^{1}N_{1} = \underbrace{{}^{C_1}\cancelto{0}{I_1}\,{}^{1}\dot{\omega}_{1}}_{=0} + \underbrace{{}^{1}\omega_{1} \times {}^{C_1}\cancelto{0}{I_1}\,{}^{1}\omega_{1}}_{=0} = 0$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i=1$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{2}\omega_{2} = {}^{2}_{1}R\,{}^{1}\omega_1 + \dot{\theta}_{2}\,{}^{2}\hat{Z}_{2}$$

$$= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i=1$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\omega_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$${}^{2}\dot{\omega}_{2} = {}^{2}_{1}R\,{}^{1}\dot{\omega}_{1} + {}^{2}_{1}R\,\underbrace{{}^{1}\omega_{1} \times \dot{\theta}_{2}\,{}^{2}\hat{Z}_{2}} + \ddot{\theta}_{2}\,{}^{2}\hat{Z}_{2}$$

$$= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} + {}^{2}_{1}R \cancelto{0}{\begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & 0 & \dot{\theta}_2 \end{vmatrix}} + \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $\underline{i=1}$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right)$$

$$^{2}\dot{v}_{2} = {}^{2}_{1}R\left({}^{1}\dot{\omega}_1 \times {}^{1}P_{2} + {}^{1}\omega_1 \times \left({}^{1}\omega_1 \times {}^{1}P_{2}\right) + {}^{1}\dot{v}_1\right)$$

$$\begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(\begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_2 \\ l_1 & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \underbrace{\begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix}} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}\right)$$

$$\left[\quad\downarrow\quad\right]\left(\begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ 0 & l_1\dot{\theta}_1 & 0 \end{vmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} c_2 & s_1 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix}\right)$$

Iterative Newton-Euler Equations - 2R Robot Example

$$= \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L_1 \dot{\theta}_1^2 + g s_1 \\ L_1 \ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} =$$

$$^2\dot{v}_2 = \begin{bmatrix} L_1 \ddot{\theta}_1 s_2 - L_1 \dot{\theta}_1^2 c_2 + g s_{12} \\ L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + g c_{12} \\ \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i=1$

$$ {}^{i+1}\dot{v}_{C_{i+1}} = {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}} \right) + {}^{i+1}\dot{v}_{i+1} $$

$$ {}^{2}\dot{v}_{c_2} = {}^{2}\dot{\omega}_{2} \times {}^{2}P_{c_2} + {}^{2}\omega_{2} \times \left({}^{2}\omega_{2} \times {}^{2}P_{c_2} \right) + {}^{2}\dot{v}_{2} = $$

$$ = \begin{vmatrix} i & j & k \\ 0 & 0 & \ddot{\theta}_1 + \ddot{\theta}_2 \\ L_2 & 0 & 0 \end{vmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ L_2 & 0 & 0 \end{vmatrix} + \begin{bmatrix} L_1 \ddot{\theta}_1 s_2 - L_1 \dot{\theta}_1^2 c_2 + g s_{12} \\ L_1 \ddot{\theta}_1 c_2 + L_1 \dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} 0 \\ L_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \underbrace{\begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 + \dot{\theta}_2 \\ 0 & L_2(\dot{\theta}_1 + \dot{\theta}_2) & 0 \end{vmatrix}} + \begin{bmatrix} \ \\ \ \\ \ \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

$$
{}^{2}\dot{v}_{c_2} = \begin{bmatrix} 0 \\ L_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -L_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} L_1\ddot{\theta}_1 s_2 - L_1\dot{\theta}_1^2 c_2 + g\, s_{12} \\ L_1\ddot{\theta}_1 c_2 + L_1\dot{\theta}_2 s_2 + g\, c_{12} \\ 0 \end{bmatrix}
$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}}$$

$$^{2}F_{2} = m_2\,{}^{2}\dot{v}_{C_2} =$$

$$= \begin{bmatrix} m_2 l_1 \ddot{\theta}_1 s_2 - m_2 l_1 \dot{\theta}_1^2 c_2 - m_2 l_1 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 g s_{12} \\ m_2 l_1 \ddot{\theta}_1 c_2 + m_2 l_1 \dot{\theta}_1^2 s_2 + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 g c_{12} \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Outward Iteration $i = 1$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}$$

$$^{2}N_{2} = \underbrace{\cancelto{0}{{}^{C_2}I_2}\;{}^{2}w_{2}}_{=0} + \underbrace{{}^{2}w_{2} \times \cancelto{0}{{}^{C_2}I_2}\;{}^{2}w_{2}}_{=0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $\underline{i = 2}$

$i : 2 \rightarrow 1$

$${}^{i}f_{i} = {}^{i}_{i+1}R\ {}^{i+1}f_{i+1} + {}^{i}F_{i}$$

$${}^{2}f_{2} = {}^{2}_{3}R\ \cancelto{0}{{}^{3}f_{3}} + {}^{2}F_{2} = {}^{2}F_{2}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $\quad i = 2$

$$ {}^i n_i = {}^i N_i + {}^{i}_{i+1}R\;{}^{i+1}n_{i+1} + {}^iP_{C_i} \times {}^iF_i + {}^iP_{i+1} \times {}^{i}_{i+1}R\;{}^{i+1}f_{i+1} $$

$$ {}^2n_2 = \underbrace{\cancelto{0}{{}^2N_2}}_{=0} + \underbrace{{}^2_3R\;\cancelto{0}{{}^3n_3}}_{=0} + \underbrace{{}^2P_{c_2} \times {}^2F_2} $$

$$ = \begin{vmatrix} i & j & k \\ l_2 & 0 & 0 \\ {}^2F_{2x} & {}^2F_{2y} & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2\,{}^2F_{2y} \end{bmatrix} = $$

Iterative Newton-Euler Equations - 2R Robot Example

$$\begin{bmatrix} 0 \\ 0 \\ \underbrace{m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2^2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2)}_{{}^2n_2 \hat{Z}} \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $\quad i=1$

$$ {}^{i}f_i = {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} + {}^{i}F_i $$

$$ {}^{1}f_1 = {}^{1}_{2}R \; {}^{2}f_2 + {}^{1}F_1 = $$

$$ \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -m_1 l_1 \dot{\theta}^2 + m_1 g s_1 \\ m_1 l_2 \ddot{\theta}_1 + m_2 g c_1 \\ 0 \end{bmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $\quad i=1$

$$ {}^{i}n_{i} = {}^{i}N_{i} + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_{i} + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ {}^{1}n_{1} = {}^{1}\cancel{N}_{1}^{\,0} + \underbrace{{}^{1}_{2}R\,{}^{2}n_{2}}_{} + \underbrace{{}^{1}P_{C_1} \times {}^{1}F_{1}}_{} + {}^{1}P_{2} \times \overbrace{{}^{1}_{2}R\,{}^{2}f_{2}}^{} $$

$$ = \underbrace{\begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ {}^{2}n_{2}\hat{z} \end{bmatrix}}_{} + \underbrace{\begin{vmatrix} i & j & k \\ l_1 & 0 & 0 \\ {}^{1}F_{1x} & {}^{1}F_{1y} & 0 \end{vmatrix}}_{} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{2}f_{2x} \\ {}^{2}f_{2y} \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} {}^{2}n_{2}\hat{z} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1\,{}^{1}F_{1y} \end{bmatrix} + \begin{vmatrix} i & j & k \\ l_1 & 0 & 0 \\ c_2\,{}^{2}f_{2x} - s_2\,{}^{2}f_{2y} & s_2\,{}^{2}f_{2x} + c_2\,{}^{2}f_{2y} & 0 \end{vmatrix} $$

Iterative Newton-Euler Equations - 2R Robot Example

$$\left[{}^2n_2\right] + \begin{bmatrix} 0 \\ 0 \\ l_1\,{}^1F_{1y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1\left(s_2\,{}^2f_{2x} + c_2\,{}^2f_{2y}\right) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_2 g c_1 \end{bmatrix}$$

$$\begin{bmatrix} l_1 s_2 \left[m_2 l_1 \ddot{\theta}_1 s_2 - m_2 l_1 \dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \right] + \\ l_1 c_2 \left[m_2 l_1 \ddot{\theta}_1 c_2 + m_2 l_1 \dot{\theta}_1^2 s_2 + m_2 g c_{12} - m_2 l_1 (\ddot{\theta}_1 + \ddot{\theta}_2) \right] \end{bmatrix}$$

Iterative Newton-Euler Equations - 2R Robot Example

- Inward iteration $i = 1 \quad i = 2$

$$\tau_i = {}^i n_i^T \; {}^i\hat{Z}_i$$

$$\left\{ \begin{aligned} \tau_2 &= m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} \\ &\quad + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ \tau_1 &= m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 \\ &\quad - m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + (m_1 + m_2) l_1 g c_1 \\ &\quad + m_2 l_2 g c_{12} \end{aligned} \right.$$

Iterative Newton-Euler Equations - 2R Robot Example

$$[\tau] = \underbrace{[M(\theta)]}_{\text{Mass/Inertia Matrix}}[\ddot{\theta}] + \underbrace{V(\theta, \dot{\theta})}_{C(\theta)[\dot{\theta}^2] + B(\theta)[\dot{\theta}\dot{\theta}]} + \underbrace{G(\theta)}_{\text{gravity}}$$

Coriolis Centrafugal

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_2 l_2^2 + 2 m_2 l_1 l_2 c_2 + (m_1 + m_2) l_1^2 & m_2 l_2^2 + m_2 l_1 l_2 c_2 \\ m_2 l_2^2 + m_2 l_1 l_2 c_2 + m l_2^2 & 2 m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & -m_2 l_1 l_2 s_2 \\ m_2 l_1 l_2 s_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} -2 m_2 l_1 l_2 s_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{\theta}_2]$$

$$+ \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

Equation of Motion – Non Rigid Body Effects

$$\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta) + F(\theta,\dot{\theta})$$

- Viscous Friction $\tau_{friction} = v\dot{\theta}$
- Coulomb Friction $\tau_{friction} = c\,\mathrm{sgn}(\dot{\theta})$
- Model of Friction $\tau_{friction} = v\dot{\theta} + c\,\mathrm{sgn}(\dot{\theta}) = f(\theta,\dot{\theta})$