Manipulator Dynamics 2

Forward Dynamics

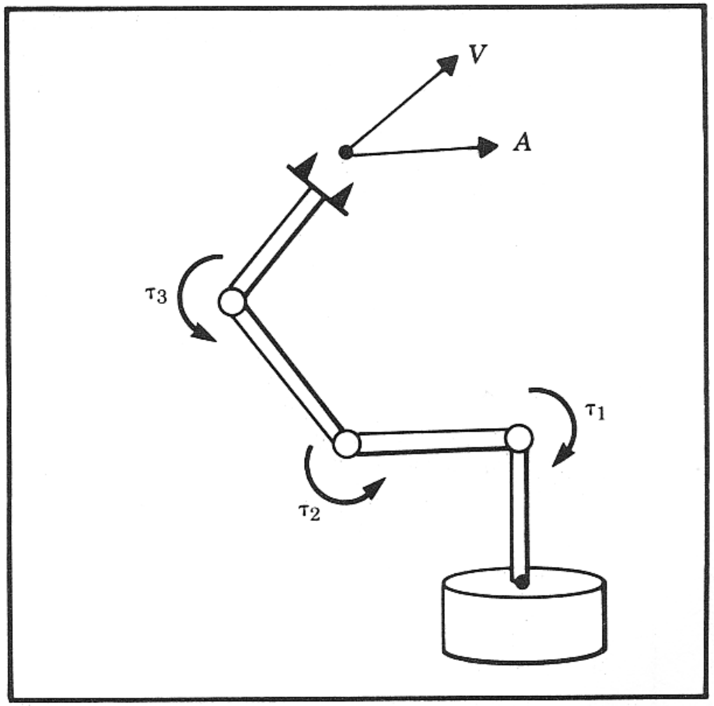
Problem

Given: *Joint torques and links geometry, mass, inertia, friction*

Compute: *Angular acceleration of the links (solve differential equations)*

Solution

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics



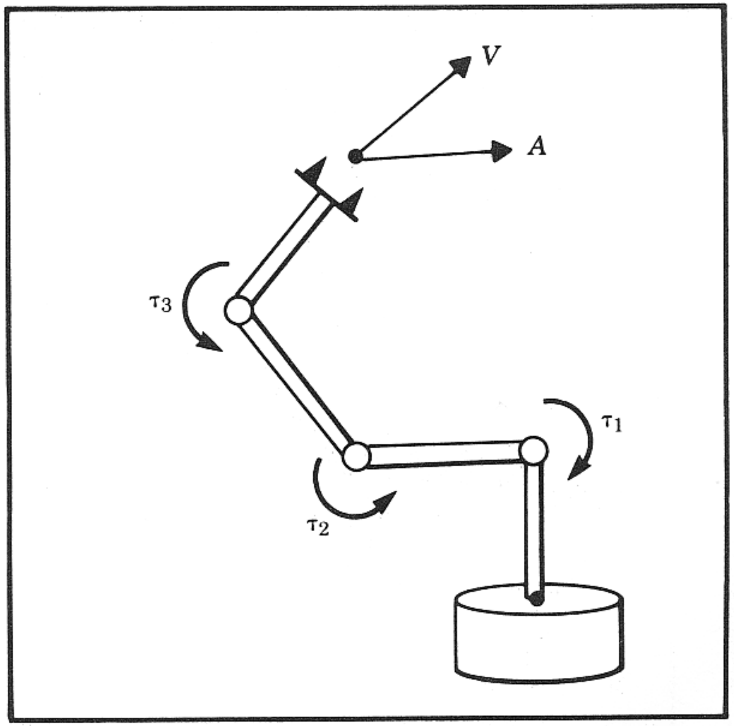
$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

Inverse Dynamics

Problem

Given: *Angular acceleration, velocity and angels of the links in addition to the links geometry, mass, inertia, friction*

Compute: *Joint torques*



Solution

Dynamic Equations - Newton-Euler method or Lagrangian Dynamics

$$\boldsymbol{\tau} = M(\Theta)\ddot{\Theta} + V(\Theta,\dot{\Theta}) + G(\Theta) + F(\Theta,\dot{\Theta})$$

Dynamics - Newton-Euler Equations

- To solve the Newton and Euler equations, we'll need to develop mathematical terms for:

  - $\dot{v}_c$ – The linear acceleration of the center of mass
  - $\dot{\omega}$ – The angular acceleration
  - ${}^{c}I$ – The Inertia tensor (moment of inertia)
  - F - The sum of all the forces applied on the center of mass
  - N - The sum of all the moments applied on the center of mass

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

Iterative Newton-Euler Equations - Solution Procedure

- **Step 1 -** Calculate the link velocities and accelerations iteratively from the robot's base to the end effector

- **Step 2 -** Write the Newton and Euler equations for each link.

Outward iterations: $i : 0 \rightarrow 5$

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1},$$

$$^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1},$$

$$^{i+1}\dot{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}_i\right),$$

$$\begin{aligned} ^{i+1}\dot{v}_{C_{i+1}} = {} & {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{C_{i+1}} \\ & + {}^{i+1}\omega_{i+1} \times \left({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{C_{i+1}}\right) + {}^{i+1}\dot{v}_{i+1}, \end{aligned}$$

$$^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{C_{i+1}},$$

$$^{i+1}N_{i+1} = {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}.$$

Iterative Newton-Euler Equations - Solution Procedure

- **Step 3 -** Use the forces and torques generated by interacting with the environment (that is, tools, work stations, parts etc.) in calculating the joint torques from the end effector to the robot's base.

Inward iterations: $i : 6 \rightarrow 1$

$$
\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^{i}F_i, \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\
&\quad + {}^{i}P_{i+1} \times {}^{i}_{i+1}R \, {}^{i+1}f_{i+1}, \\
\tau_i &= {}^{i}n_i^T \, {}^{i}\hat{Z}_i.
\end{aligned}
$$

Iterative Newton-Euler Equations - Solution Procedure

- **Error Checking -** Check the units of each term in the resulting equations

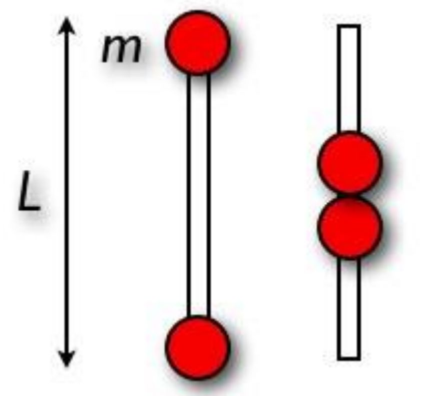
- **Gravity Effect** - The effect of gravity can be included by setting ${}^{0}\dot{v}_{0} = g$. This is the equivalent to saying that the base of the robot is accelerating upward at 1 g. The result of this accelerating is the same as accelerating all the links individually as gravity does.

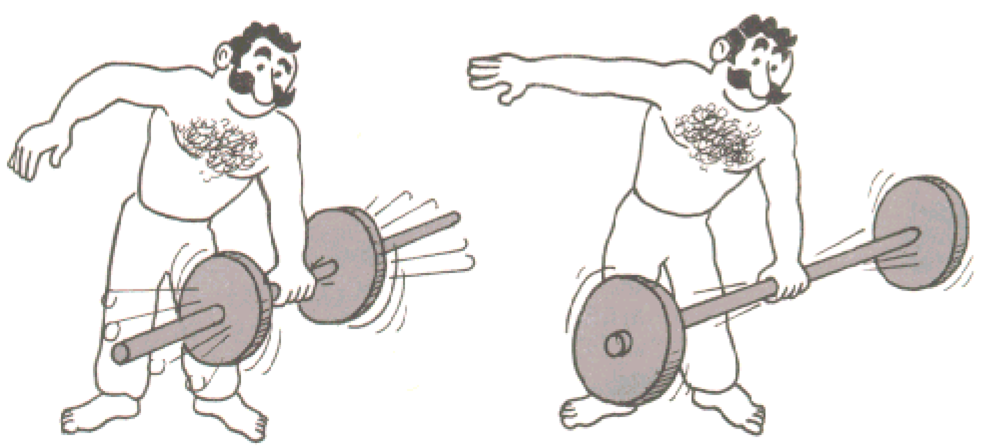
Moment of Inertia / Inertia Tensor

Moment of Inertia – Intuitive Understanding

$$F = m\dot{v}_c$$

$$N = {}^{c}I\dot{\omega} + \omega \times {}^{c}I\omega$$

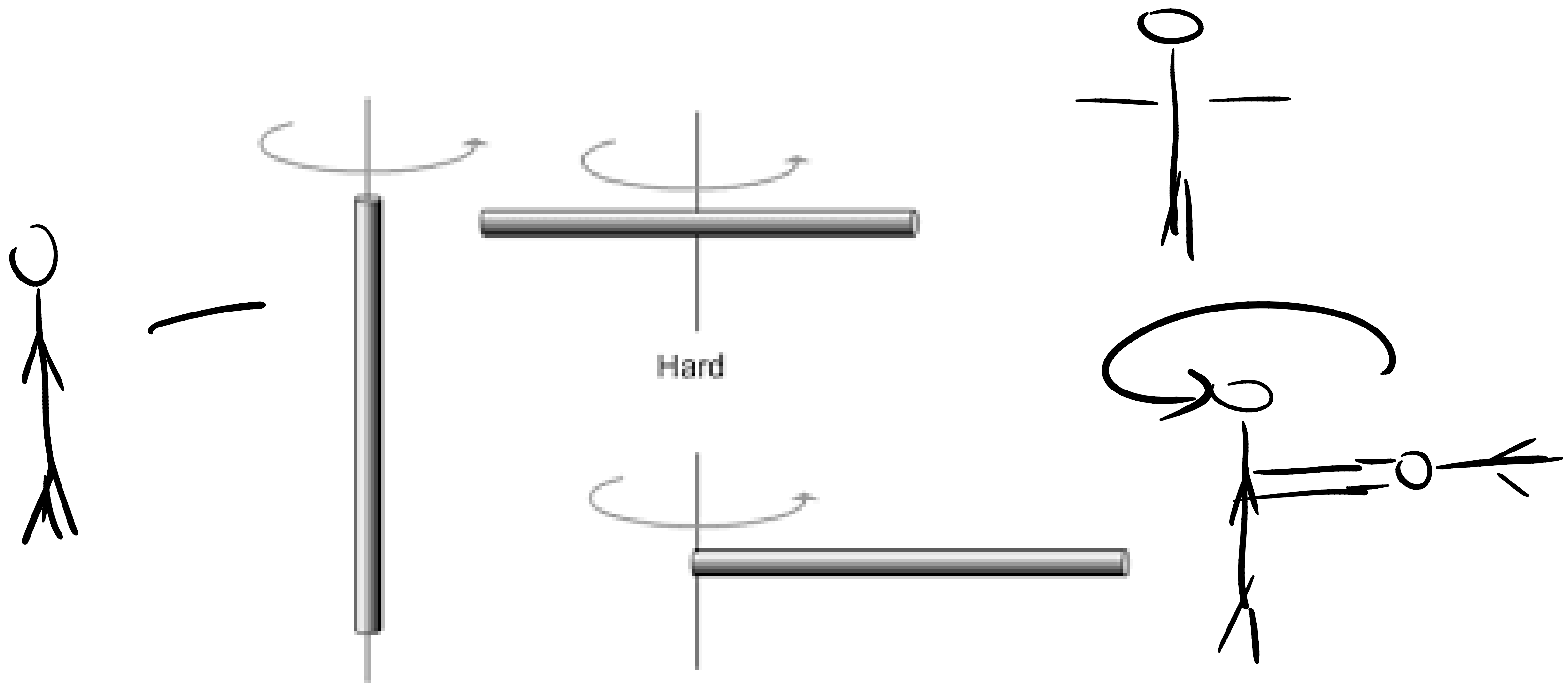




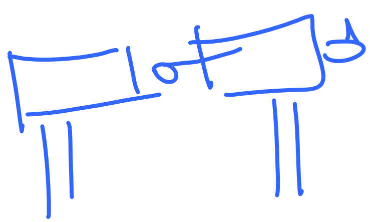
Easier Rotation

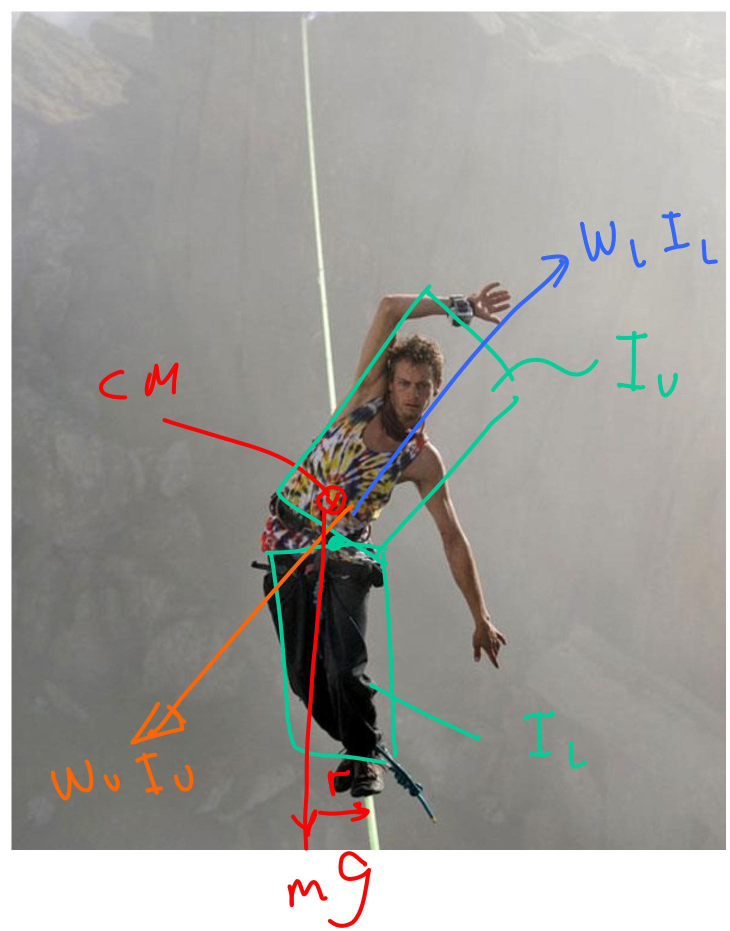
Difficult Rotation

Moment of Inertia – Intuitive Understanding



Moment of Inertia – Intuitive Understanding



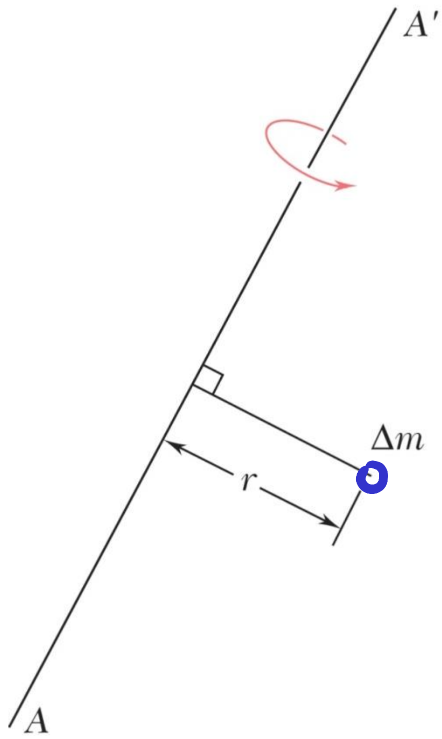


$I_u > I_l$

$w_u < w_l$

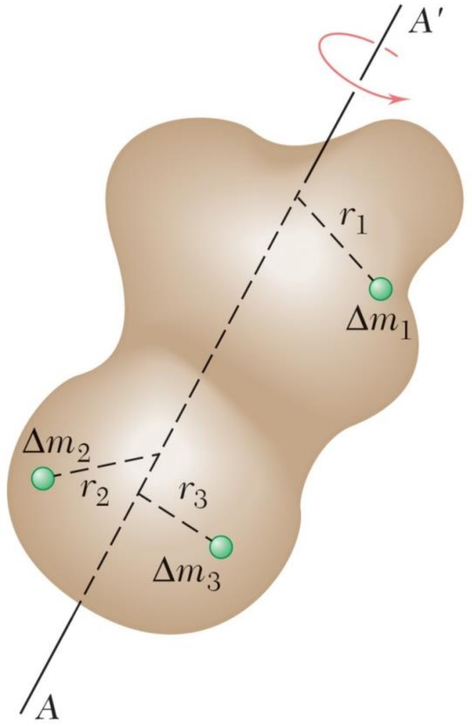
Moment of Inertia – Intuitive Understanding

Moment of Inertia – Particle – WRT Axis



$$I_{AA'} = r^2 \Delta m$$

Moment of Inertia – Solid – WRT Axis



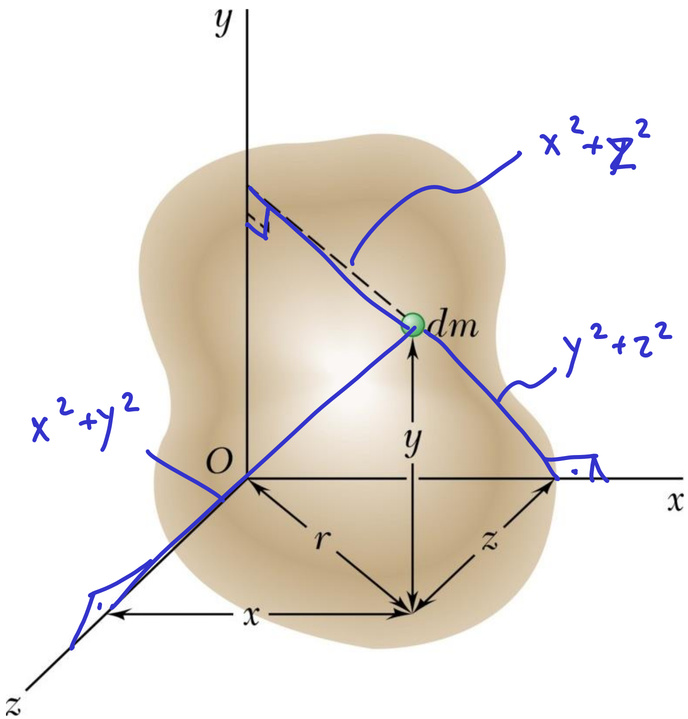
$$I_{AA'} = \sum_i r_i^2 \Delta m_i$$

$$I_{AA'} = \int_V r^2 dm = \iiint_V r^2 \rho dv$$

$dm = \rho dv$

ρ — Desity

Moment of Inertia – Solid – WRT Frame



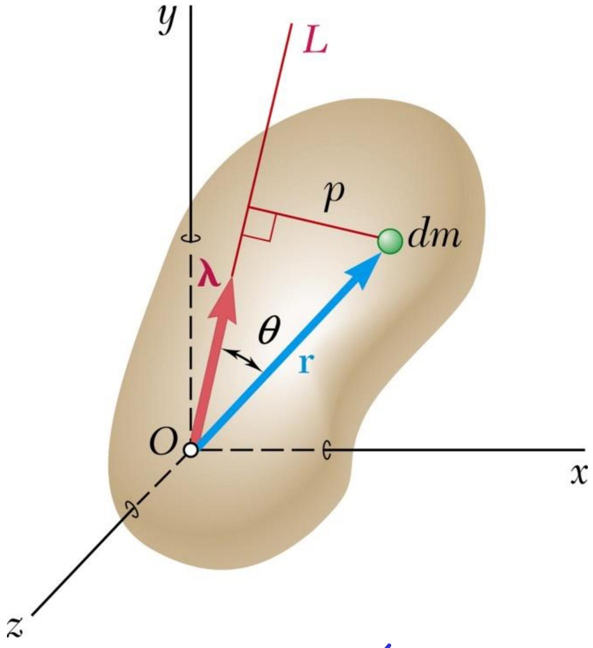
$$I_{yy} = \int r^2\, dm = \int (z^2+x^2)\, dm = \iiint_V (z^2+x^2)\rho\, dv$$

$$I_{xx} = \iiint_V (z^2+y^2)\rho\, dv$$

$$I_{zz} = \iiint_V (x^2+y^2)\rho\, dv$$

Moment of Inertia – Solid – WRT an Arbitrary Axis

$$p = \underbrace{r \sin\theta}_{\lambda \times r}$$



$$I_{OL} = \int p^2 \, dm = \int (\lambda \times r)^2 \, dm$$

$$= \int (\lambda \times r)^T (\lambda \times r) \, dm$$

$$\lambda \times r = \begin{vmatrix} i & j & k \\ \lambda_x & \lambda_y & \lambda_z \\ x & y & z \end{vmatrix} \begin{array}{l} \\ \sim \lambda \\ \sim r \end{array}$$

$$\lambda \times r = i(\lambda_y z - \lambda_z y) + j(\lambda_z x - \lambda_x z) + k(\lambda_x y - \lambda_y x)$$

Moment of Inertia – Solid – WRT an Arbitrary Axis

$$I_{OL} = \int (\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2 \, dm$$

$$I_{OL} = \lambda_x^2 \overbrace{\int (y^2 + z^2)\, dm}^{I_{xx}} + \lambda_y^2 \overbrace{\int (z^2 + x^2)\, dm}^{I_{yy}} + \lambda_z^2 \overbrace{\int (x^2 + y^2)\, dm}^{I_{zz}}$$

$$- 2\lambda_x \lambda_y \underbrace{\int xy\, dm}_{I_{xy}} - 2\lambda_y \lambda_z \underbrace{\int yz\, dm}_{I_{yz}} - 2\lambda_z \lambda_x \underbrace{\int zx\, dm}_{I_{zx}}$$

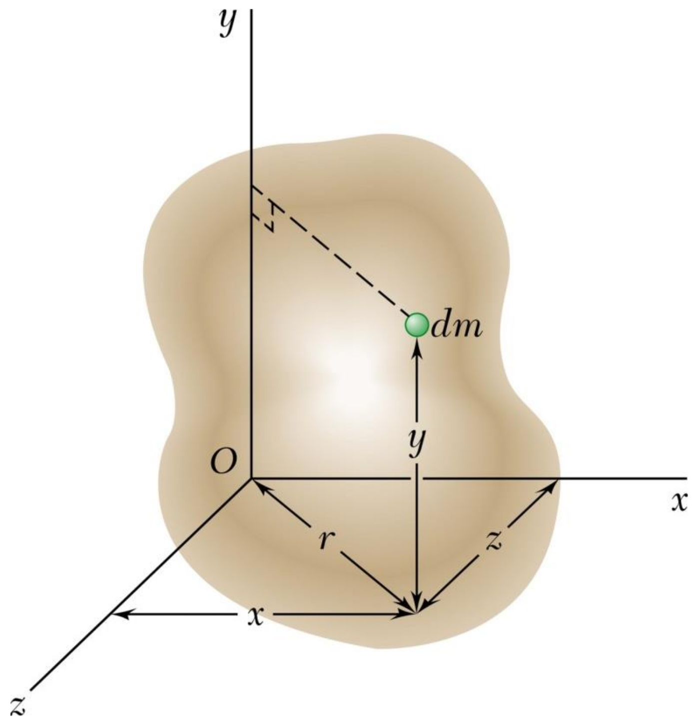
$$I_{OL} = I_{xx}\lambda_x^2 + I_{yy}\lambda_z^2 + I_{zz}\lambda_z^2 - 2 I_{xy}\lambda_x\lambda_y - 2 I_{yz}\lambda_y\lambda_z - 2 I_{zx}\lambda_z\lambda_x$$

Moment of Inertia – Solid – WRT an Arbitrary Axis

- For a rigid body that is free to move in a 3D space there are infinite possible rotation axes
- The intertie tensor characterizes the mass distribution of the rigid body with respect to a specific coordinate system

Inertia Tensor

- For a rigid body that is free to move in a 3D space there are infinite possible rotation axes
- The intertie tensor characterizes the mass distribution of the rigid body with respect to a specific coordinate system
- The intertie Tensor relative to frame {A} is express as a matrix



$$
{}^{A}I = \begin{bmatrix} Ixx & -Ixy & -Ixz \\ -Ixy & Iyy & -Iyz \\ -Ixz & -Iyz & Izz \end{bmatrix}
$$

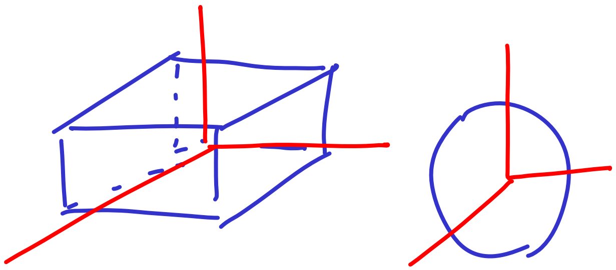
Inertia Tensor

$$
{}^{A}I = \begin{bmatrix} Ixx & -Ixy & -Ixz \\ -Ixy & Iyy & -Iyz \\ -Ixz & -Iyz & Izz \end{bmatrix}
$$

$$
\left.\begin{aligned}
I_{xx} &= \iiint_V (y^2 + z^2)\rho dv \\
I_{yy} &= \iiint_V (x^2 + z^2)\rho d \\
I_{zz} &= \iiint_V (x^2 + y^2)\rho d
\end{aligned}\right\} \text{Mass moments of inertia}
$$

$$
\left.\begin{aligned}
I_{xy} &= \iiint_V xy\rho d \\
I_{xz} &= \iiint_V xz\rho d \\
I_{yz} &= \iiint_V yz\rho d
\end{aligned}\right\} \text{Mass products of inertia}
$$

Tensor of Inertia – Example



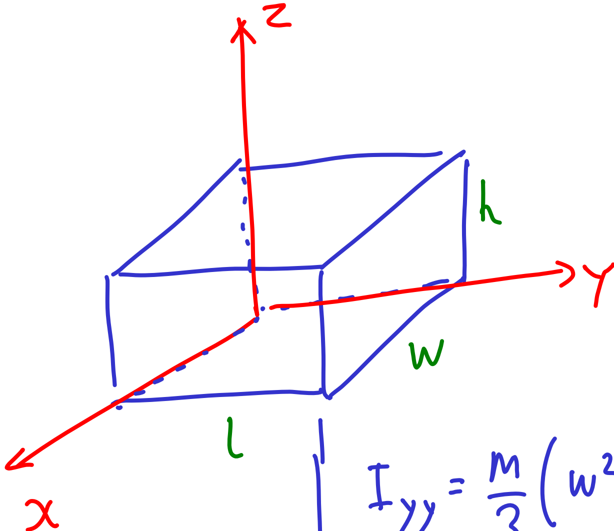
$$
{}^{A}I = \begin{bmatrix} Ixx & -Ixy & -Ixz \\ -Ixy & Iyy & -Iyz \\ -Ixz & -Iyz & Izz \end{bmatrix}
$$

- This set of six independent quantities for a given body, depend on the **position and orientation** of the frame in which they are defined
- We are free to choose the orientation of the reference frame. It is possible to cause the product of inertia to be zero

$$
\left.\begin{matrix} I_{xy} = 0 \\ I_{xz} = 0 \\ I_{yz} = 0 \end{matrix}\right\} \text{Mass products of inertia} \qquad {}^{A}I = \begin{bmatrix} Ixx & 0 & 0 \\ 0 & Iyy & 0 \\ 0 & 0 & Izz \end{bmatrix}
$$

- The axes of the reference frame when so aligned are called the **principle axes** and the corresponding mass moments are called the principle **moments of intertie**

Tensor of Inertia – Example



$$I_{xx} = \int_0^h \int_0^L \int_0^w (y^2+z^2)\,\rho\, dx\, dy\, dz$$

$$= \int_0^h \int_0^L (y^2+z^2)\, w\rho\, dy\, dz$$

$$= \int_0^h \left(\frac{L^3}{3} + z^2 L\right) w\rho\, dz$$

$$= \left(\frac{hL^3w}{3} + \frac{h^3Lw}{3}\right)\rho = \underbrace{\rho h L w}_{m}\,\frac{L^2}{3} + \underbrace{\rho h l w}_{m}\,\frac{h^2}{3}$$

$$= \frac{m}{3}\left(L^2 + h^2\right)$$

$$I_{yy} = \frac{m}{3}\left(w^2 + h^2\right)$$

$$I_z = \frac{m}{3}\left(l^2 + w^2\right)$$

$x: \; 0 \to w$

$y: \; 0 \to l$

$z: \; 0 \to h$

Tensor of Inertia – Example

$$I_{xy} = \underbrace{\int_0^h}_{z} \underbrace{\int_0^l}_{y} \underbrace{\int_0^w}_{x} xy\rho \, dx\,dy\,dz = \int_0^h \int_0^l \frac{w^2}{2} y\rho \, dy\,dz$$

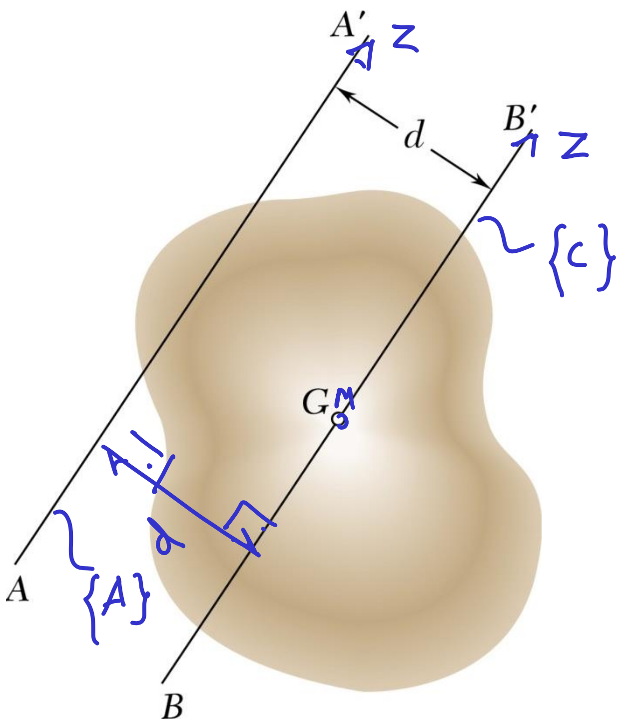
$$= \int_0^h \frac{w^2 l^2}{4} \rho \, dz = \frac{w^2 l^2 h}{4} \rho = \big(\overbrace{wh l\rho}^{m}\big) \frac{wl}{4} = \frac{m}{4} wl$$

$$I_{xz} = \frac{m}{4} hw \quad ; \quad I_{yz} = \frac{m}{4} hl$$

$$^{A}I = \begin{bmatrix} \frac{m}{3}(l^2+h^2) & -\frac{m}{4} wl & -\frac{m}{4} hw \\ -\frac{m}{4} wl & \frac{m}{3}(w^2+h^2) & -\frac{m}{4} hl \\ -\frac{m}{4} hw & -\frac{m}{4} hl & \frac{m}{3}(l^2+w^2) \end{bmatrix}$$

Parallel Axis Theorem – 1D

- The inertia tensor is a function of the position and orientation of the reference frame
- **Parallel Axis Theorem** – How the inertia tensor changes under translation of the reference coordinate system

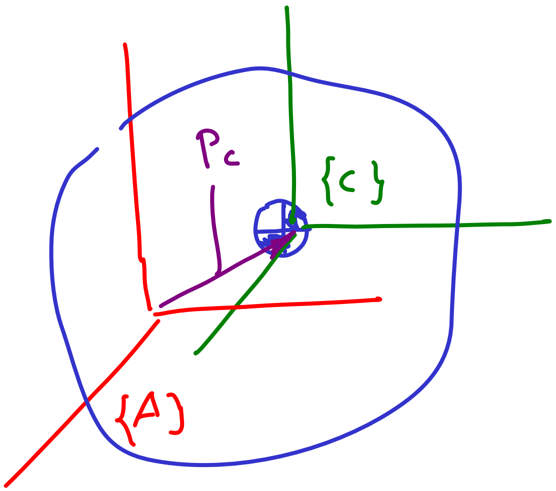


Frame {C} - Is located at the CM

Frame {A} - An arbitrarily translated frame

$${}^{A}I_{zz} = {}^{C}I_{zz} + md^2$$

Parallel Axis Theorem – 3D



$$ {}^{A}I_{zz} = {}^{C}I_{zz} + m\left(x_c^2 + y_c^2\right) $$

xx	xx	z_c	y_c
yy	yy	x_c	z_c

$$ {}^{A}I_{xy} = {}^{C}I_{xy} - m\,x_c\,y_c $$

xz	xz	x_c	z_c
yz	yz	y_c	z_c

$$ P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} $$ – location of the CM (origine of c) relative to frame [A]

Parallel Axis Theorem – 3D

$$ {}^{A}I = {}^{C}I + m\left[(P \cdot P) I_3 - P \otimes P \right] $$

OUTER PRODUT

$$ (x_c^2 + y_c^2 + z_c^2) \qquad \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \qquad \begin{bmatrix} P_x P_x & P_x P_y & P_x P_z \\ P_y P_x & P_y P_y & P_y P_z \\ P_z P_x & P_z P_y & P_z P_z \end{bmatrix} $$

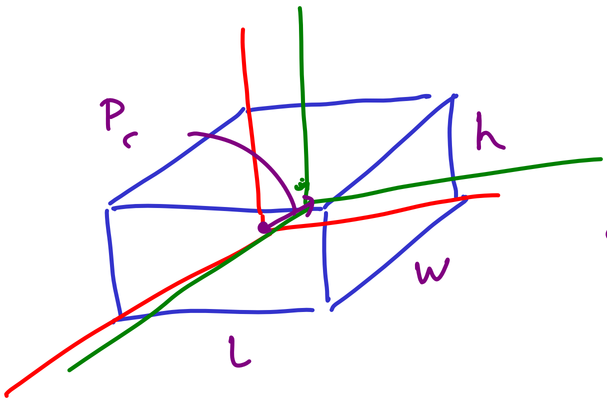
$$ \begin{bmatrix} x_c^2 + y_c^2 + z_c^2 & 0 & 0 \\ 0 & - \,''\, - & 0 \\ 0 & 0 & - \,''\, - \end{bmatrix} $$

Inertia Tensor

$$
{}^{A}I = \begin{bmatrix} {}^{C}I_{xx} + m(y_c^2 + z_c^2) & {}^{C}I_{xy} - m x_c y_c & {}^{C}I_{xz} - m x_c z_c \\ {}^{C}I_{xy} - m x_c y_c & {}^{C}I_{yy} + m(x_c^2 + z_c^2) & {}^{C}I_{yz} - m y_c z_c \\ {}^{C}I_{xz} - m x_c y_c & {}^{C}I_{yz} - m y_c z_c & {}^{C}I_{zz} + m(x_c^2 + y_c^2) \end{bmatrix}
$$

Tensor of Inertia – Example

$$P_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \frac{1}{2}\begin{bmatrix} w \\ l \\ h \end{bmatrix}$$



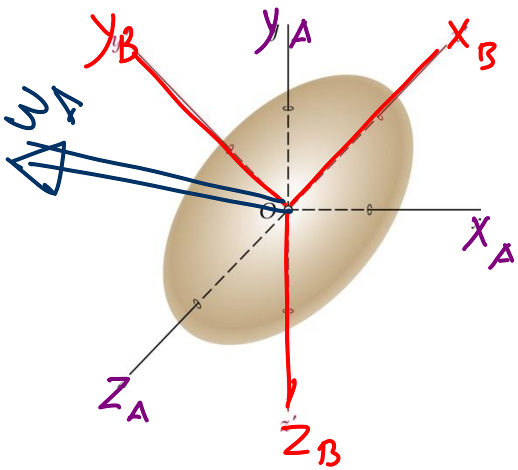
$$^{C}I_{zz} = {^{A}I_{zz}} - m\left(x_c^2 + y_c^2\right) = \frac{m}{3}\left(l^2 + w^2\right) - \frac{m}{4}\left(w^2 + l^2\right)$$

$$= \frac{m}{12}\left(w^2 + l^2\right)$$

$$^{C}I = \frac{m}{12}\begin{bmatrix} h^2+l^2 & 0 & 0 \\ 0 & w^2+h^2 & 0 \\ 0 & 0 & l^2+w^2 \end{bmatrix}$$

$$^{C}I_{xy} = {^{A}I_{xy}} + m x_c y_c = -\frac{mwl}{4} + m\frac{1}{2}w\frac{1}{2}l = 0$$

Rotation of the Inertia Tensor



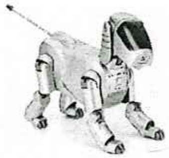
- Given:
  - The inertia tensor of the a body expressed in frame A
  - Frame B is rotate with respect to frame A
- Find
  - The inertia tensor of frame B
- The angular Momentum of a rigid body rotating about an axis passing through is

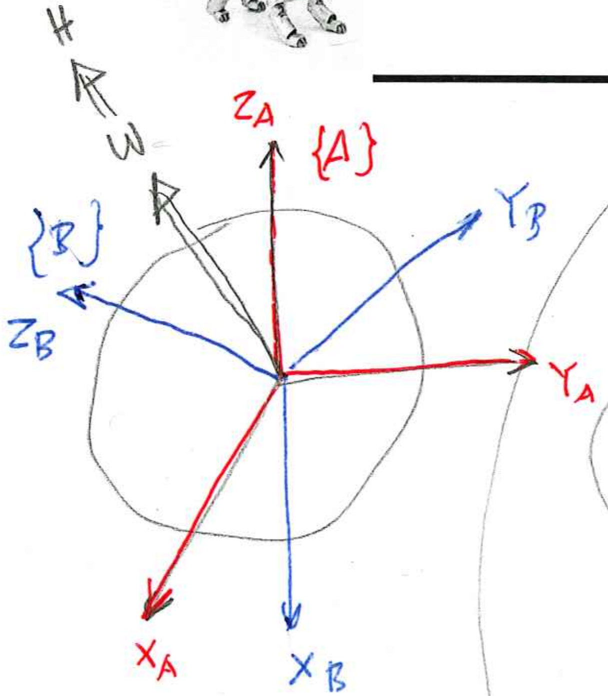
$$H_A = I_A \omega_A$$

- Let's transform the angular momentum vector to frame B

$$H_B = {}^A_B R H_A$$

ROTATION OF THE INERTIA TENSOR





$${}^{A}H = {}^{A}I\,{}^{A}W$$

$${}^{B}H = {}^{B}I\,{}^{B}W$$

${}^{A}W, {}^{A}H$ – Angular velocity and Momentum expressed in frame A

${}^{B}W, {}^{B}H$ – Angular Velocity and Momentum expressed in frame B

$${}^{A}H = {}^{A}_{B}R\,{}^{B}H \quad ; \quad {}^{A}W = {}^{A}_{B}R\,{}^{B}W$$

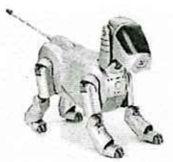
$${}^{A}H = {}^{A}_{B}R\,{}^{B}I\,{}^{B}W$$

$${}^{A}H = {}^{A}_{B}R\,{}^{B}I\,\underbrace{{}^{A}_{B}R^{-1}\,{}^{A}_{B}R}_{I}\,{}^{B}W \qquad {}^{A}W$$

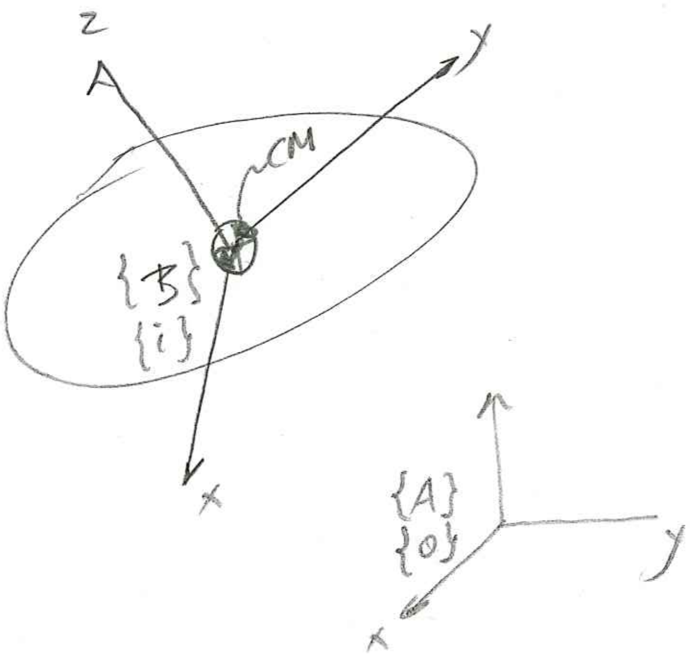
$${}^{A}H = \boxed{{}^{A}_{B}R\,{}^{B}I\,{}^{A}_{B}R^{-1}}\,W_A$$

$${}^{A}I$$

$$\boxed{I_A = {}^{A}_{B}R\,{}^{B}I\,{}^{A}_{B}R^{-1}}$$



ROTATION OF THE INERTIA TENSOR



$$^{A}I = {}^{A}_{B}R \; {}^{B}I \; {}^{A}_{B}R^{T}$$

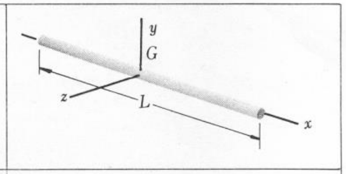
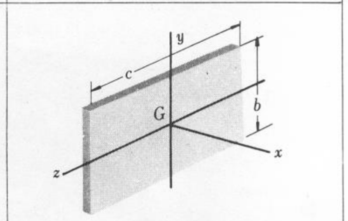
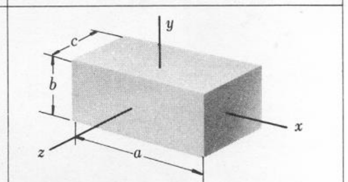
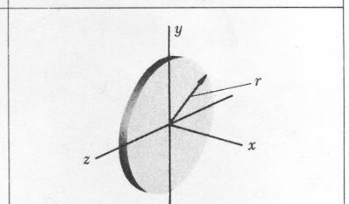
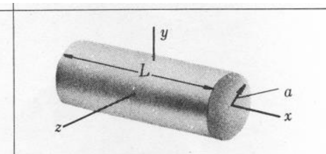
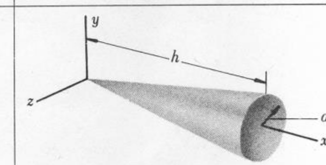
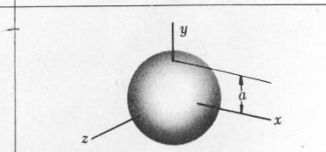
$$^{0}I_{CM} = {}^{0}_{i}R \; {}^{i}I_{cm} \; {}^{0}_{i}R^{T}$$

$A \rightarrow 0$

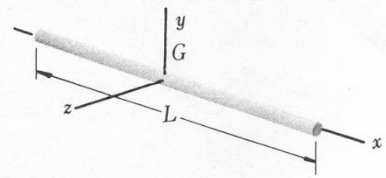
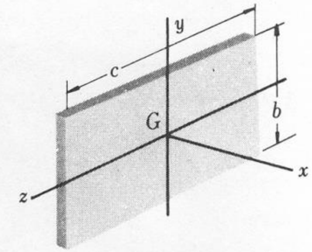
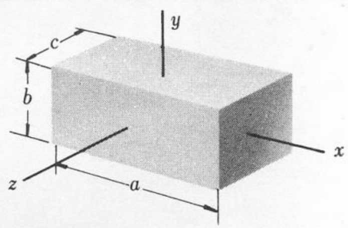
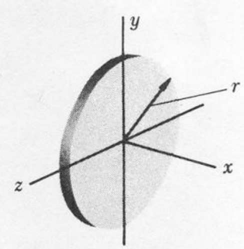
$B \rightarrow i$

Inertia Tensor 2/

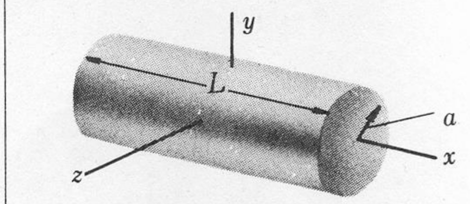
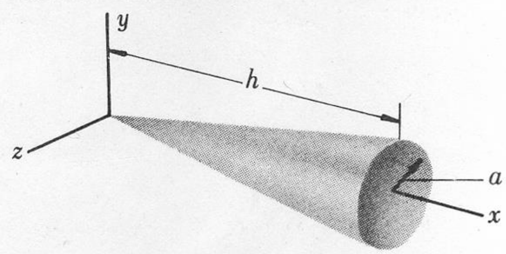
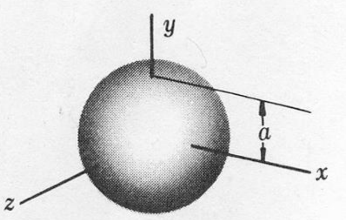
- The elements for relatively simple shapes can be solved from the equations describing the shape of the links and their density. However, most robot arms are far from simple shapes and as a result, these terms are simply measured in practice.

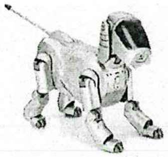
Slender rod $I_y = I_z = \frac{1}{12} m L^2$		
Thin rectangular plate $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m c^2$ $I_z = \frac{1}{12} m b^2$		
Rectangular prism $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m (c^2 + a^2)$ $I_z = \frac{1}{12} m (a^2 + b^2)$		
Thin disk $I_x = \frac{1}{2} m r^2$ $I_y = I_z = \frac{1}{4} m r^2$		
Circular cylinder $I_x = \frac{1}{2} m a^2$ $I_y = I_z = \frac{1}{12} m (3a^2 + L^2)$		
Circular cone $I_x = \frac{3}{10} m a^2$ $I_y = I_z = \frac{3}{5} m (\frac{1}{4} a^2 + h^2)$		
Sphere $I_x = I_y = I_z = \frac{2}{5} m a^2$		

Inertia Tensor 2/

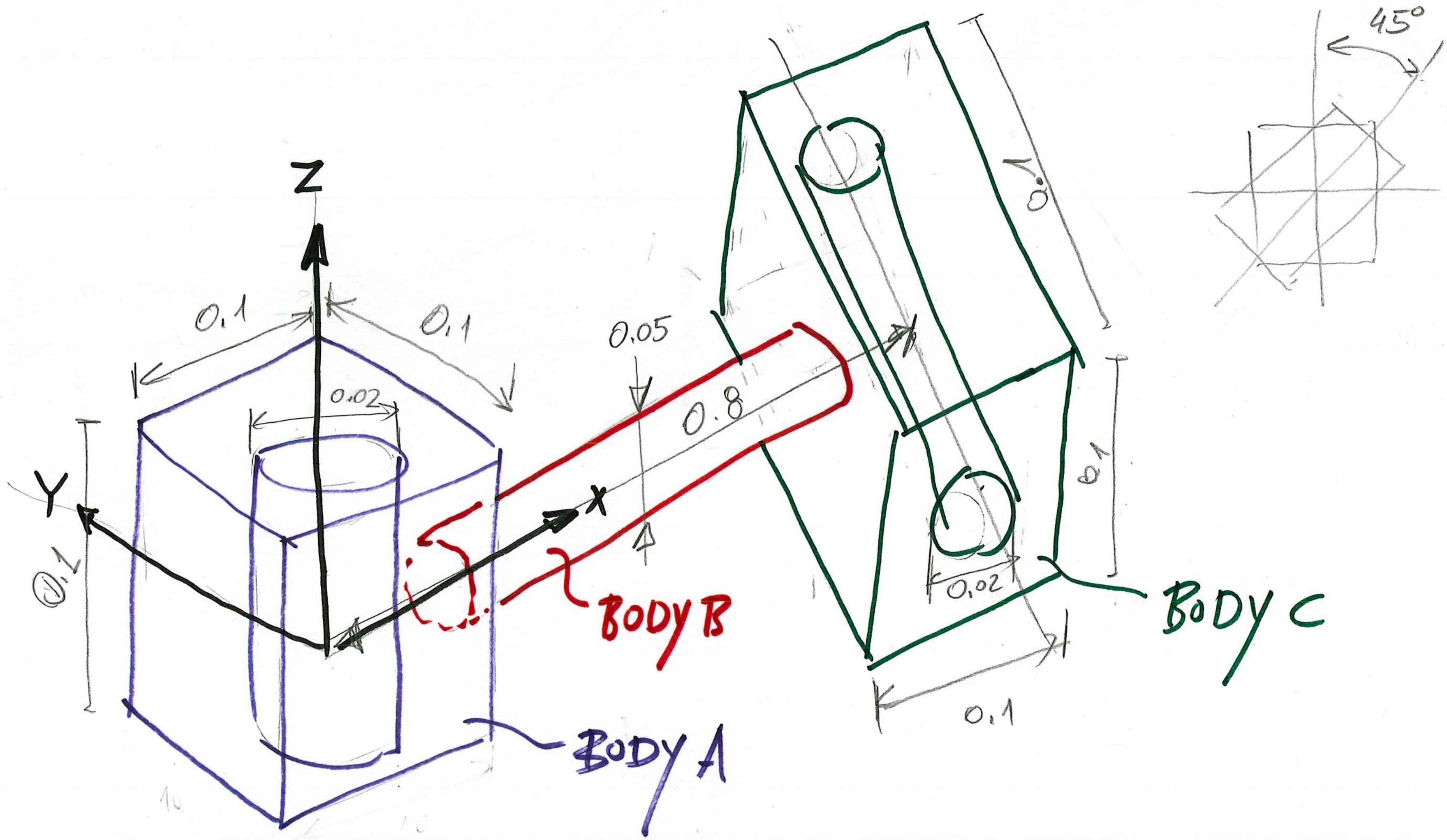
Slender rod $I_y = I_z = \frac{1}{12} m L^2$	
Thin rectangular plate $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m c^2$ $I_z = \frac{1}{12} m b^2$	
Rectangular prism $I_x = \frac{1}{12} m (b^2 + c^2)$ $I_y = \frac{1}{12} m (c^2 + a^2)$ $I_z = \frac{1}{12} m (a^2 + b^2)$	
Thin disk $I_x = \frac{1}{2} m r^2$ $I_y = I_z = \frac{1}{4} m r^2$	

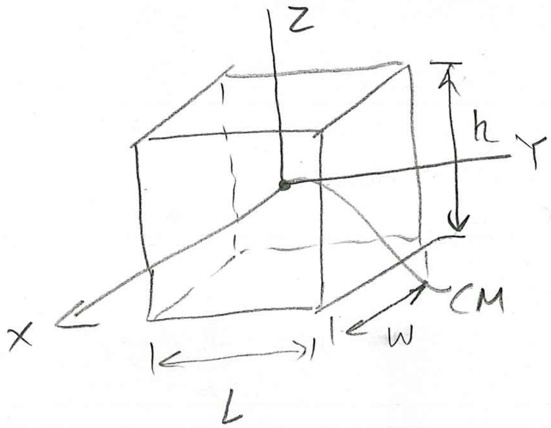
Inertia Tensor 2/

Circular cylinder $I_x = \frac{1}{2} m a^2$ $I_y = I_z = \frac{1}{12} m (3a^2 + L^2)$	
Circular cone $I_x = \frac{3}{10} m a^2$ $I_y = I_z = \frac{3}{5} m (\frac{1}{4} a^2 + h^2)$	
Sphere $I_x = I_y = I_z = \frac{2}{5} m a^2$	



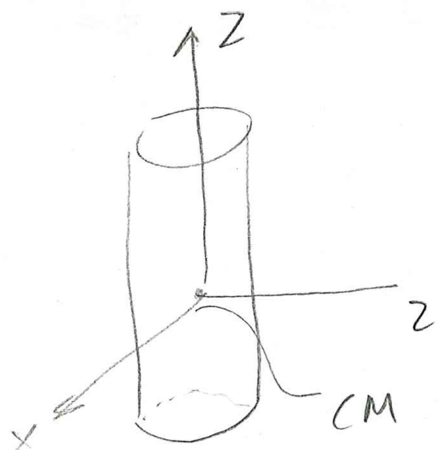
LINK - Moment of Inertia





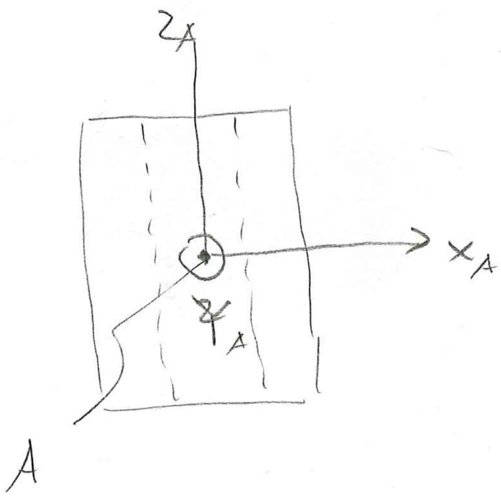
$$ {}^{CM}I = \frac{m}{12}\begin{bmatrix} h^2+l^2 & 0 & 0 \\ 0 & w^2+h^2 & 0 \\ 0 & 0 & l^2+h^2 \end{bmatrix} $$

Note : $I_{xx} = I_{yy}$

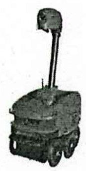


$$ {}^{CM}I = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{mr^2}{2} \end{bmatrix} $$

BODY A



$$ {}^{A}I = I_{\square} - I_{\bigcirc} $$



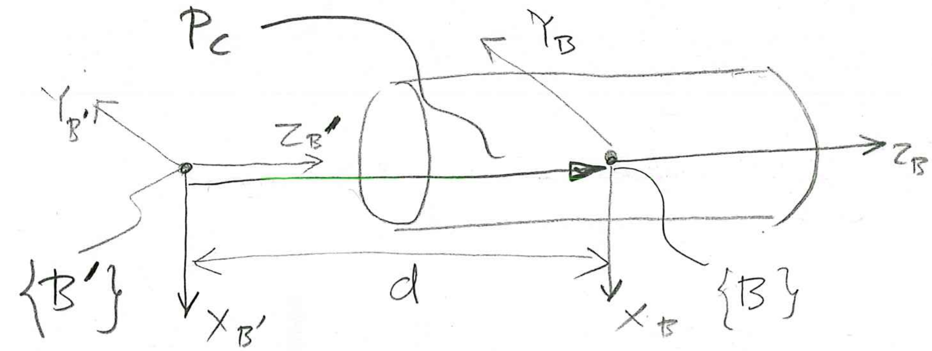
$$
{}^{B'}I = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} + m\left[\begin{bmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^2 \end{bmatrix}\right]
$$

$$
{}^{B'}I = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} + m\begin{bmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
{}^{B'}I = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2)+md^2 & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2)+md^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}
$$

BODY B

same origin as frame {A}



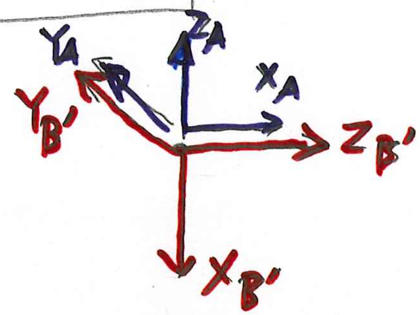
$$P_c = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}$$

$\boxed{\text{step 1}}$ – Translation along z by d

$$^{B'}I = {}^{B}I + m\left[P_c^T P_c I_3 - P_c \otimes P_c\right]$$

$$^{B'}I = \begin{bmatrix} \frac{1}{12}m(3r^2+h^2) & 0 & 0 \\ 0 & \frac{1}{12}m(3r^2+h^2) & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} + m\left[\begin{bmatrix} 0 & 0 & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^2 \end{bmatrix}\right]$$

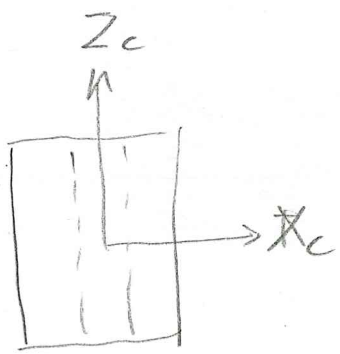
STEP 2 - Rotate

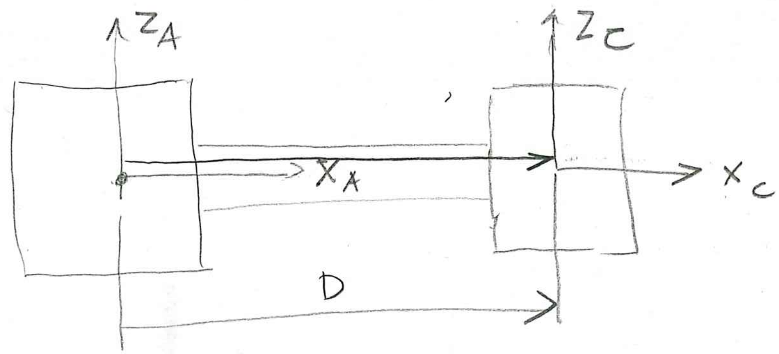


$$ {}^{A}_{B'}R = \mathrm{RoT}(\hat{Y}, +90) $$

$$ {}^{A}I = {}^{A}_{B'}R \; {}^{B'}I \; {}^{A}_{B'}R^{T} $$

Body C





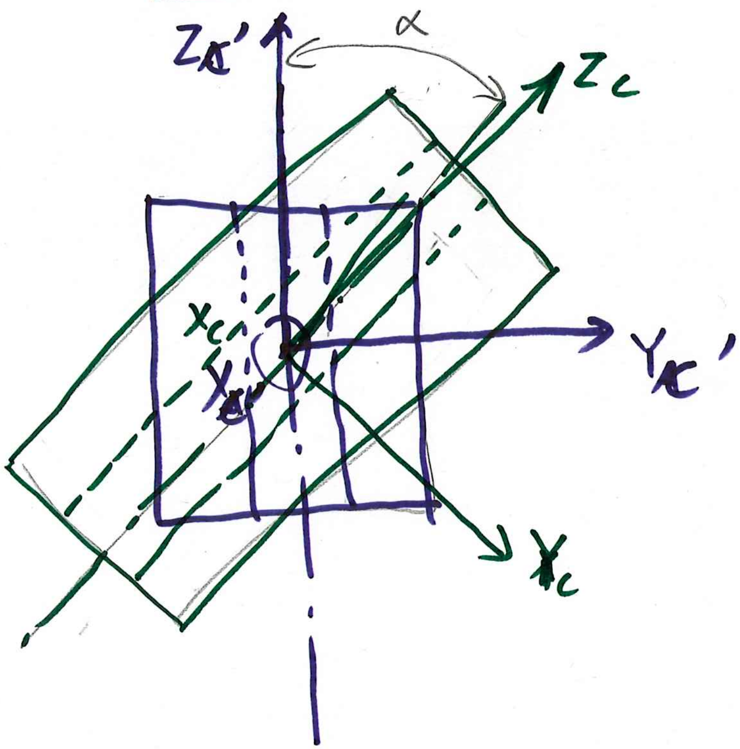
$${}^{C}I = I_{\square} - I_{\bigcirc}$$

$\boxed{\text{STEP 1}}$ - Translate along x by D

$$P_c = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$$

(Note : see Body B step 1)

$\boxed{\text{Body C}}$

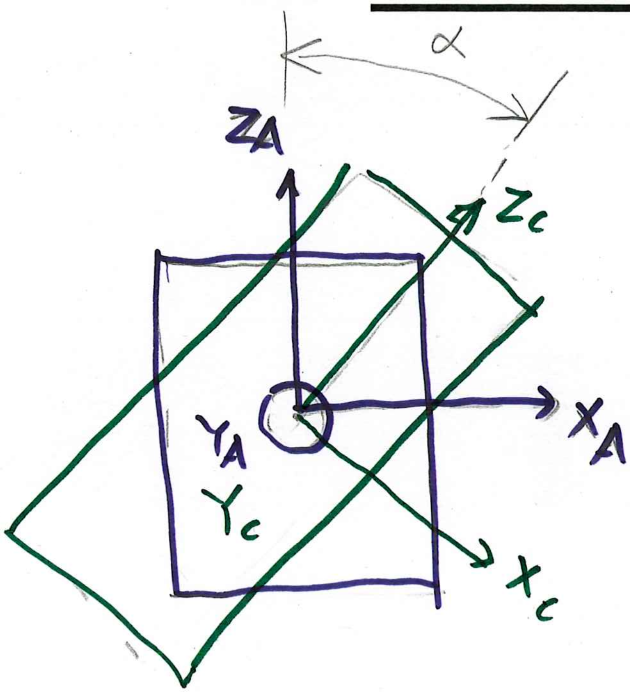


$${}^{C}I = I_{\square} - I_{\text{B}}$$

$\boxed{\text{STEP I}}$ - Rotate

$${}^{C'}_{C}R = Rot(\hat{X}, -\alpha)$$

$${}^{C'}I = {}^{C'}_{C}R \; {}^{C}I \; {}^{C'}_{C}R^{T}$$



$\boxed{\text{STEP 2}}$ - Rotate

$$ {}^{A}_{C}R = R(\hat{X}, -\alpha) $$

$$ {}^{A}I = {}^{A}_{C}R \; {}^{C}I \; {}^{A}_{C}R^{T} $$