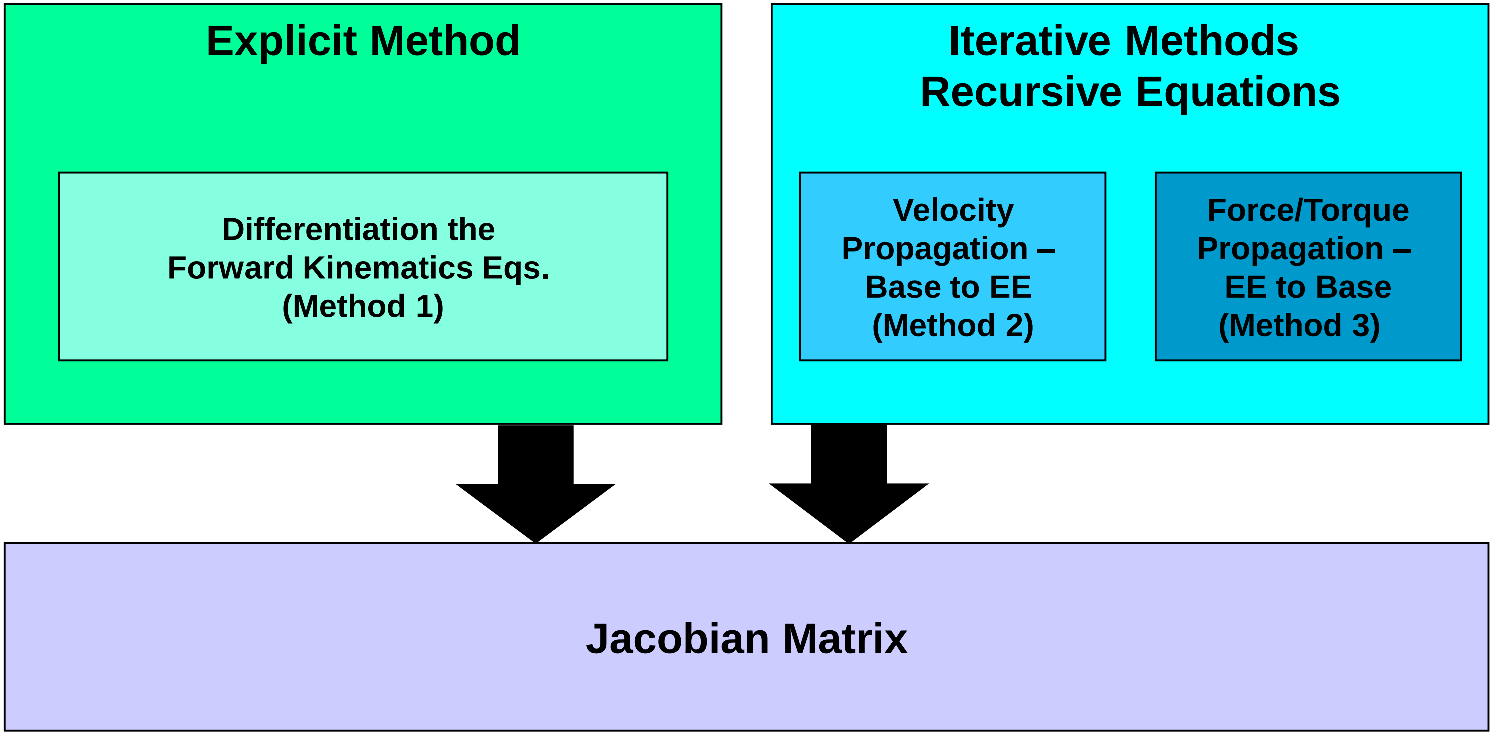
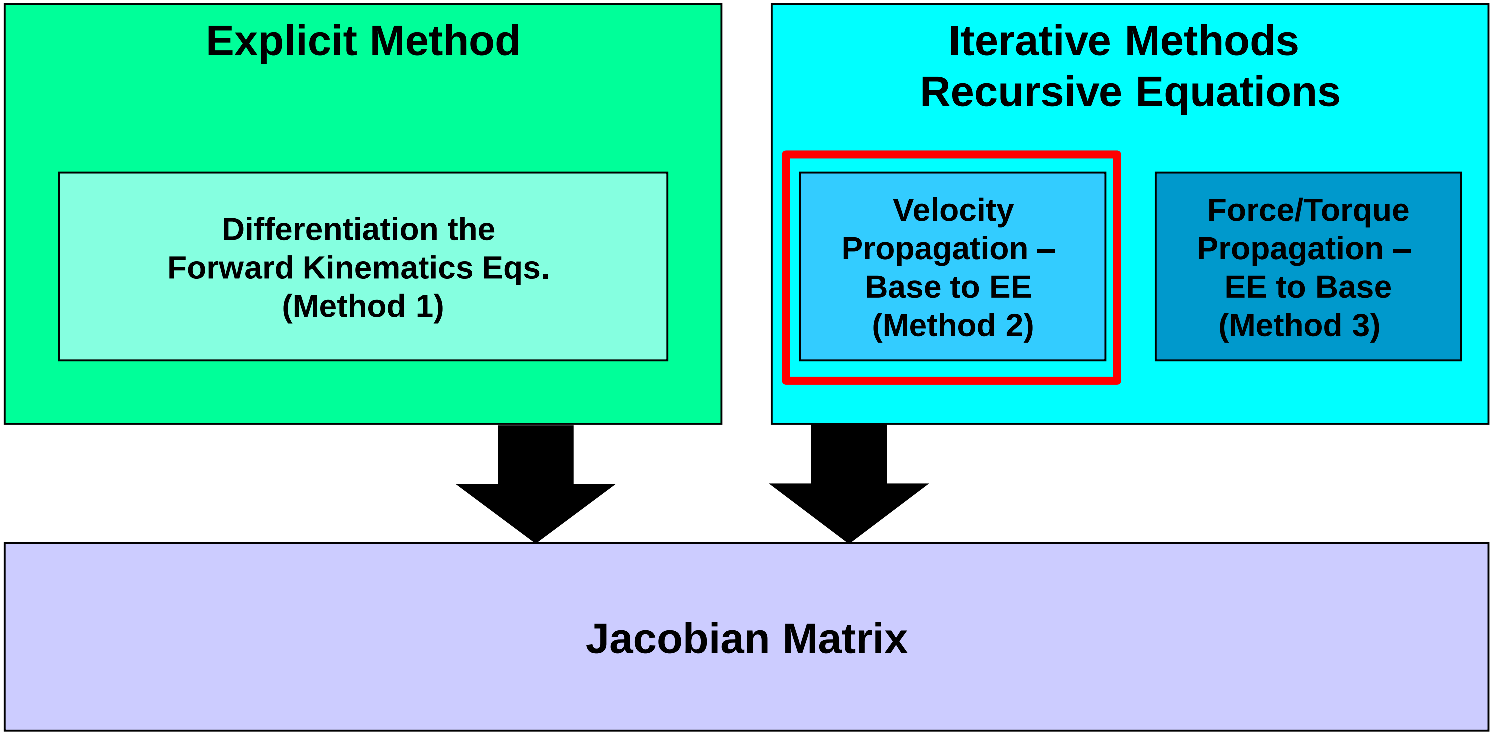
Jacobian: Velocity propagation 4/4

Jacobian Matrix - Derivation Methods



Jacobian Iterative Method -
Velocity propagation (Method No.2)

Jacobian Matrix - Derivation Methods



Jacobian: Velocity propagation

- The recursive expressions for the adjacent joint linear and angular velocities describe a relationship between the joint angle rates ($\dot{\theta}$) and the transnational and rotational velocities of the end effector ($\dot{X}$):

$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega_{i} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_{i+1} \end{bmatrix}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R\left({}^{i}\omega \times {}^{i}P_{i+1} + {}^{i}v_{i}\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_{i+1} \end{bmatrix}$$

Jacobian: Velocity propagation

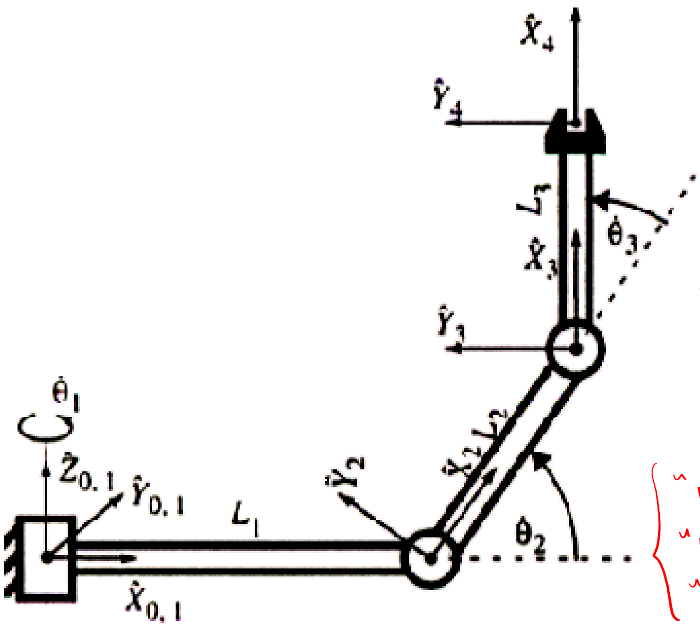
- Therefore the recursive expressions for the adjacent joint linear and angular velocities can be used to determine the Jacobian in the end effector frame

$$^{N}\dot{X} = {}^{N}J(\theta)\dot{\theta}$$

- This equation can be expanded to:

$$^{N}\dot{X} = {}^{N}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = \begin{bmatrix} {}^{N}v_{N} \\ {}^{N}\omega_{N} \end{bmatrix} = {}^{N}\begin{bmatrix} & J(\theta) & \end{bmatrix} \begin{bmatrix} \dot{\theta}1 \\ \dot{\theta}2 \\ \\ \dots \\ \\ \dot{\theta}_{N} \end{bmatrix}$$

- The linear angular velocities of the end effector (N=4)



$\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$

$$\begin{Bmatrix} {}^4v_{4x} \\ {}^4v_{4y} \\ {}^4v_{4z} \end{Bmatrix} \quad {}^4v_4 = \begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3 + L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1 - L2c2 - L3c23)\dot{\theta}_1 \end{bmatrix}$$

$$\begin{Bmatrix} {}^4w_{4x} \\ {}^4w_{4y} \\ {}^4w_{4z} \end{Bmatrix} \quad {}^4\omega_4 = \begin{bmatrix} s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

Jacobian - 3R - Example

- Re-arranged to previous two terms gives an expression that encapsulates

$$ {}^4\dot{X} = {}^4J(\theta)\dot{\theta} $$

$$ \begin{Bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{d}_3 \end{Bmatrix} $$

$$ {}^4\dot{X} = {}^4\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \begin{bmatrix} {}^4v_4 \\ {}^4\omega_4 \end{bmatrix} = \overbrace{\begin{bmatrix} L2s3\dot{\theta}_2 \\ (L2c3+L3)\dot{\theta}_2 + L3\dot{\theta}_3 \\ (-L1-L2c2-L3c23)\dot{\theta}_1 \\ s23\dot{\theta}_1 \\ c23\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}}^{{}^4J(\theta)\dot{\theta}} $$

- We can now factor out the joint velocities vector $\dot{\theta} = [\dot{\theta}_1 \dot{\theta}_2 \dot{\theta}_3]^T$ from the above vector to formulate the Jacobian matrix ${}^4J(\theta)$

Jacobian - 3R - Example

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}^{{}^{4}J(\theta)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

$$
{}^{4}\dot{X} = {}^{4}J(\theta)\dot{\theta}
$$

- The equations for ${}^{N}v_N$ and ${}^{N}\omega_N$ are always a linear combination of the joint velocities, so they can always be used to find the 6xN Jacobian matrix (${}^{N}J(\theta)$) for any robot manipulator.

- Note that the Jacobian matrix is expressed in frame {4}

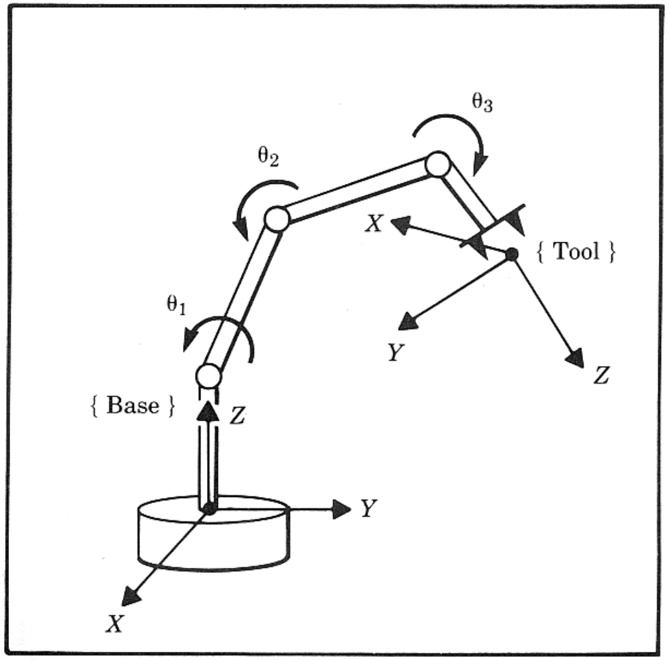
Jacobian: Frame of Representation

- Using the velocity propagation method we expressed the relationship between the velocity of the robot end effector measured relative to the robot base frame {0} and expressed in the end effector frame {N}.

$$^{N}\dot{X} = {}^{N}J(\theta)\dot{\theta}$$

- Occasionally, it may be desirable to express (represent) the end effector velocities in another frame (e.g. frame {0}, in which case we will need a method to provide the transformation.

$$^{0}\dot{X} = {}^{0}J(\theta)\dot{\theta}$$



Jacobian: Frame of Representation

- There are two methods to change the references frame (frame of representation) of the Jacobian Matrix
  - Method 1: Transforming the linear and angular velocities to the new frame prior to formulating the Jabobian matrix.
  - Method 2: Transforming the Jacobian matrix from it existing frame to the new frame after it was formulated.

Jacobian: Frame of Representation – Method 1

- Consider the velocities in a different frame {B}

$$ {}^{B}\dot{X} = \begin{bmatrix} {}^{B}v_N \\ {}^{B}\omega_N \end{bmatrix} = {}^{B}J(\theta)\dot{\theta} $$

- We may use the rotation matrix to find the velocities in frame {A}:

$$ {}^{A}\dot{X} = \begin{bmatrix} {}^{A}v_N \\ {}^{A}\omega_N \end{bmatrix} = \begin{bmatrix} {}^{A}_{\cancel{B}}R\,{}^{\cancel{B}}v_N \\ {}^{A}_{\cancel{B}}R\,{}^{\cancel{B}}\omega_N \end{bmatrix} $$

Jacobian: Frame of Representation – Method 1

- Example: Analyzing a 6 DOF manipulator while utilizing velocity propagation method results in an expressing the end effector (frame 6) linear and angular velocities.

$$^{6}\dot{X} = \begin{bmatrix} ^{6}v_6 \\ ^{6}\omega_6 \end{bmatrix}$$

- Using the forward kinematics formulation the rotation matrix from frame 0 to frame 6 can be defined as

$$^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T = \begin{bmatrix} & {}^{0}_{6}R & & {}^{0}P_{6ORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The linear and angular velocities can than be expressed in frame 0 prior to extracting the Jacobian in frame 0

$$^{0}\dot{X} = \begin{bmatrix} ^{0}v_6 \\ ^{0}\omega_6 \end{bmatrix} = \begin{bmatrix} {}^{0}_{6}R\,{}^{6}v_6 \\ {}^{0}_{6}R\,{}^{6}\omega_6 \end{bmatrix} = {}^{0}J\dot{\theta}$$

Jacobian: Frame of Representation – Method 2

- It is possible to define a Jacobian transformation matrix $^{A}_{B}R_J$ that can transform the Jacobian from frame A to frame B

$$^{A}\dot{X} = {^{A}J}(\theta)\dot{\theta} = \underbrace{^{A}_{B}R_J}\,{^{B}J}(\theta)\dot{\theta}$$

- The Jacobian rotation matrix $^{A}_{B}R_J$ is given by

$$\overline{^{A}_{B}R_J} = \begin{bmatrix} \begin{bmatrix} ^{A}_{B}R \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} ^{A}_{B}R \end{bmatrix} \end{bmatrix}$$

Jacobian: Frame of Representation

- or equivalently,

$$
{}^{A}J(\theta)=\begin{bmatrix} \begin{bmatrix} {}^{A}_{B}R \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} {}^{A}_{B}R \end{bmatrix} \end{bmatrix} {}^{B}J(\theta)
$$

Jacobian: Frame of Representation - 3R Example

$$
{}^{0}J(\theta)=\overset{{}^{0}_{4}R_{J}}{\begin{bmatrix} \begin{bmatrix}{}^{0}_{4}R\end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix}{}^{0}_{4}R\end{bmatrix}\end{bmatrix}}\underline{{}^{4}J(\theta)}=\overbrace{\begin{bmatrix} \begin{bmatrix} c1c23 & -c1s23 & s1 \\ s1c23 & -s1s23 & -c1 \\ s23 & c23 & 0 \end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} \\ \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix} c1c23 & -c1s23 & s1 \\ s1c23 & -s1s23 & -c1 \\ s23 & c23 & 0 \end{bmatrix}\end{bmatrix}}^{{}^{0}_{4}R_{J}}\;{}^{4}\overbrace{\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}^{{}^{4}J(\theta)}
$$

- The rotation matrix (${}^{0}_{4}R$) can be calculated base on the direct kinematics given by

$$
{}^{0}_{4}T={}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T=\begin{bmatrix} & {}^{0}_{4}R & & {}^{0}P_{4ORG} \\ & & & \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

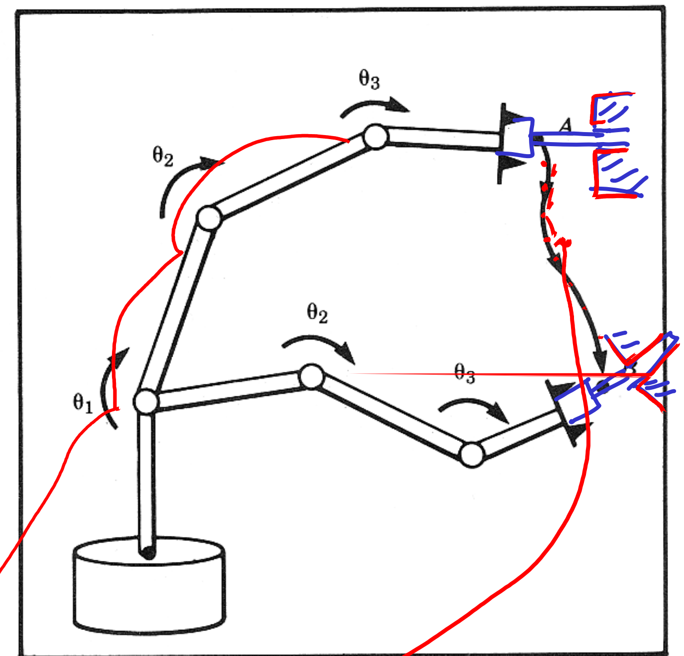
Inverse Jacobian

- ***Given***
  - Tool tip path (defined mathematically)
  - Tool tip position/orientation
  - Tool tip velocity
  - Jacobian Matrix

$$\underline{\dot{x}} = J(\theta)\ \underline{\dot{\theta}}$$

- ***Problem:*** Calculate the joint velocities

- ***Solution:***
  - Compute the inverse Jacobian matrix
  - Use the following equation to compute the joint velocity



$$\underline{\dot{\theta}} = J(\theta)^{-1}\ \underline{\dot{x}}$$

Inverse Jacobian

- Cases in which the Jacobian matrix $J(\theta)$ is not inevitable ($J(\theta)^{-1}$ does not exists). Non invertible matrix is called singular matrix

  – **Case 1** - The Jacobian matrix is not squared

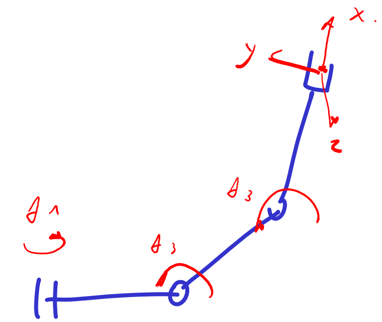
    In general the 6xN Jacobian matrix may be non-square in which case the inverse is not defined

  – **Case 2** - The determinant ($\det(J(\theta))$) is equal to zero

Inverse Jacobian - Reduced Jacobian

- ***Problem***
  - When the number of joints (N) is less than 6, the manipulator does not have the necessary degrees of freedom to achieve independent control of all six velocities components.

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix}$$



- ***Solution***
  - We can reduce the number of rows in the original Jacobian to describe a reduced Cartesian vector. For example, the full Cartesian velocity vector is given by

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 1

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Column of zeroes
- The determinate is equal to zero
- Only two out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 2

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

- Two columns of zeroes
- The determinate is equal to zero
- Only one out of the three variables can be independently specified

Jacobian: Reduced Jacobian - 3R Example

- Matrix Reduction - Option 3

$$
{}^{4}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \Omega x \\ \Omega y \\ \Omega z \end{bmatrix} = {}^{4}\begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \\ s23 & 0 & 0 \\ c23 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}
$$

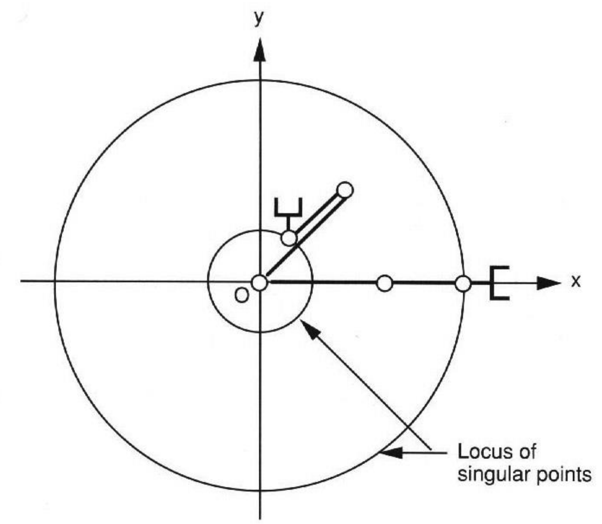
- The resulting reduced Jacobian will be square (the number of independent rows in the Jacobian are equal to the number of unknown variables) and can be inverted unless in a singular configuration.

Jacobian: Singularities

- To avoid singular configurations, the determinant of the Jacobian is often computed symbolically to find the set of joint values for which singularities will occur. Singularities often occur under two situations:

1. Workspace Boundary: the manipulator is fully extended or folded back upon itself.

2. Workspace Interior: generally caused by two or more axes intersecting.



Jacobian: Singular Configuration - 3R Example

- If we want to use the inverse Jacobian to compute the joint angular velocities we need to first find out at what points the inverse exists.

$$\underline{\dot{\theta}} = J(\theta)^{-1}\underline{\dot{x}}$$

- Considering the 3R robot

$$^4J_r(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

- The determinate of the Jacobian is defined as follows

$$\left|{}^4J_r(\theta)\right| = -(L1 + L2c2 + L3c23)(L2s3)L3$$

Jacobian: Singular Configuration - 3R Example

$$\left|{}^{4}J_{r}(\theta)\right| = -(L1+L2c2+L3c23)(L2s3)L3$$

- The reduced Jacbian matrix is singular when it determinate is equal to zero

$$-\underbrace{(L1+L2c2+L3c23)}_{=0}\ \text{OR}\ \underbrace{(L2s3)}_{=0}\boxed{L3}=0$$

- The singular condition occur when either of the following are true

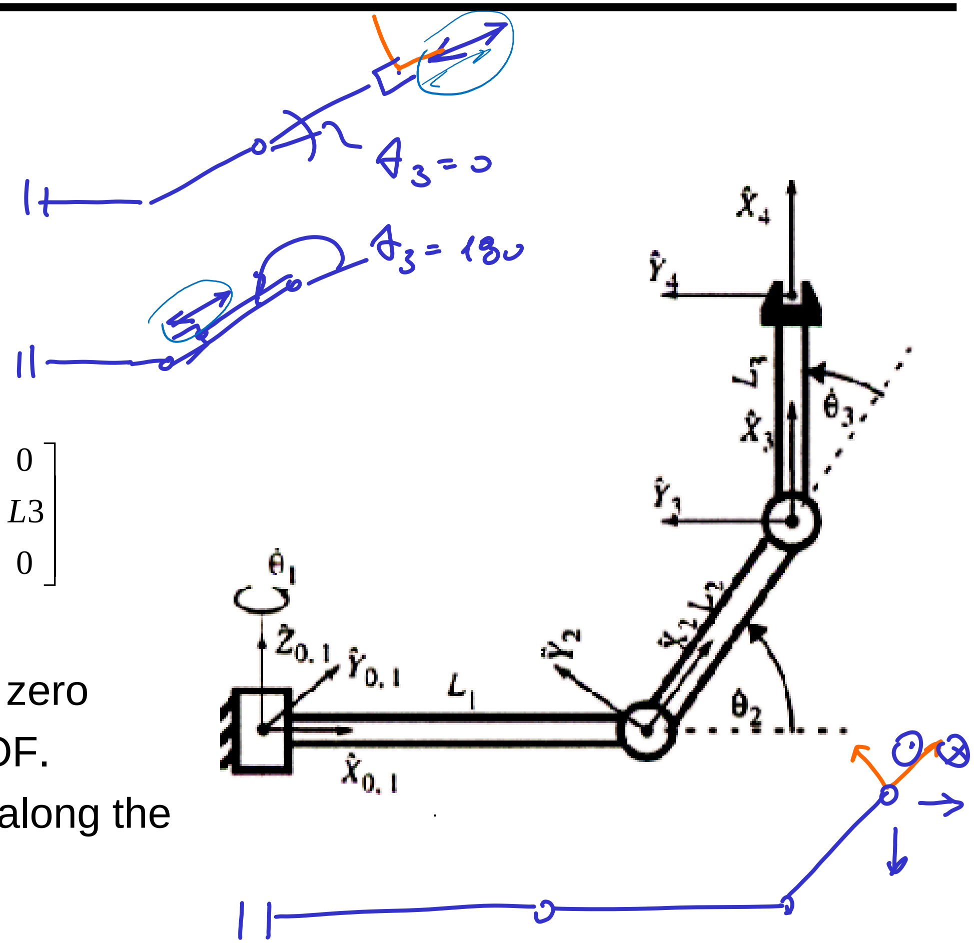
$$\begin{cases} s3=0 \\ -L1-L2c2-L3c23=0 \end{cases}$$

Jacobian: Singular Configuration - 3R Example

- Case 1: $s3 = 0$

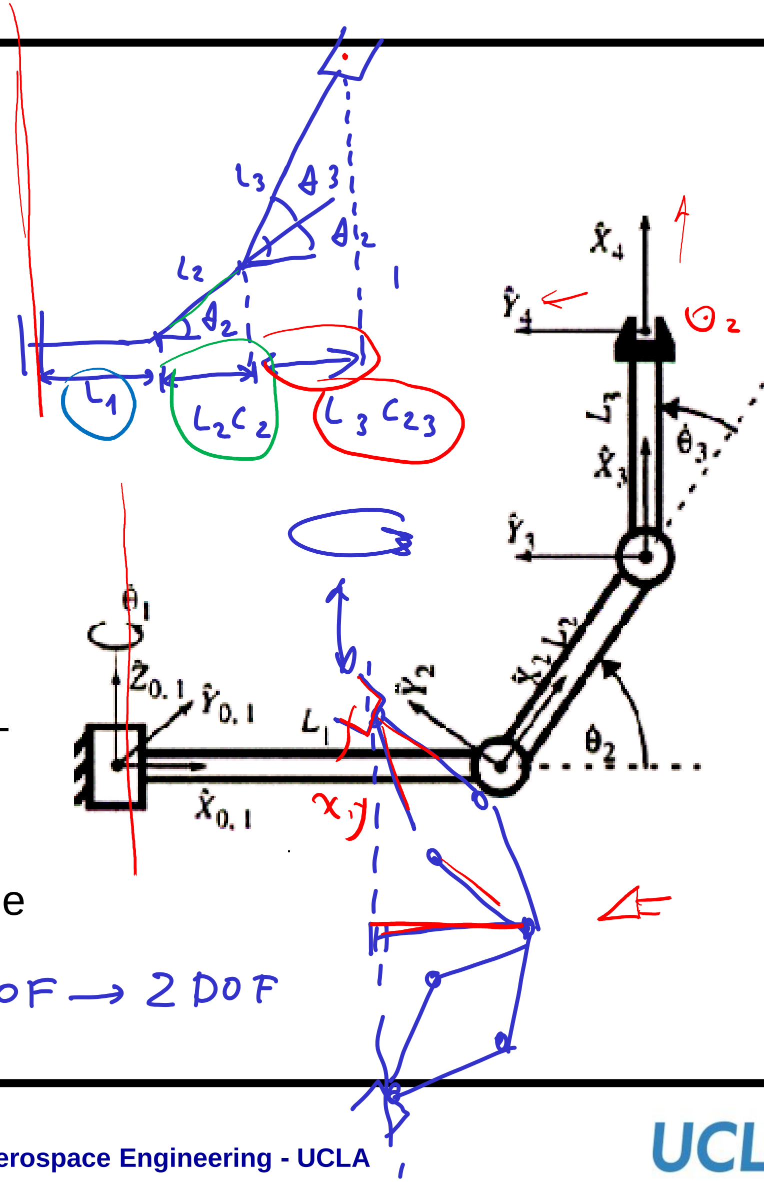
$$s3 = 0 \Rightarrow \begin{cases} \theta_3 = 0^o \\ \theta_3 = 180^o \end{cases}$$

$$^4J_r(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$



- The first row of the Jacobian is zero
- The 3R robot is loosing one DOF.
- The robot can no longer move along the X-axis of frame {4}

Jacobian: Singular Configuration - 3R Example



- Case 2: $-L1 - L2c2 - L3c23 = 0$

$$L1 = -L2c2 - L3c23$$

- Occur only if $L2 + L3 \geq L1$

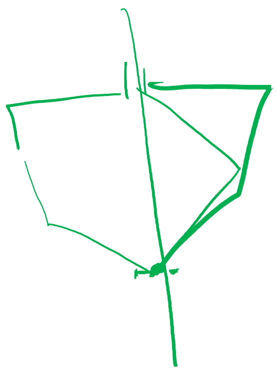
$$^4J_r(\theta) = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3 + L3 & L3 \\ -L1 - L2c2 - L3c23 & 0 & 0 \end{bmatrix}$$

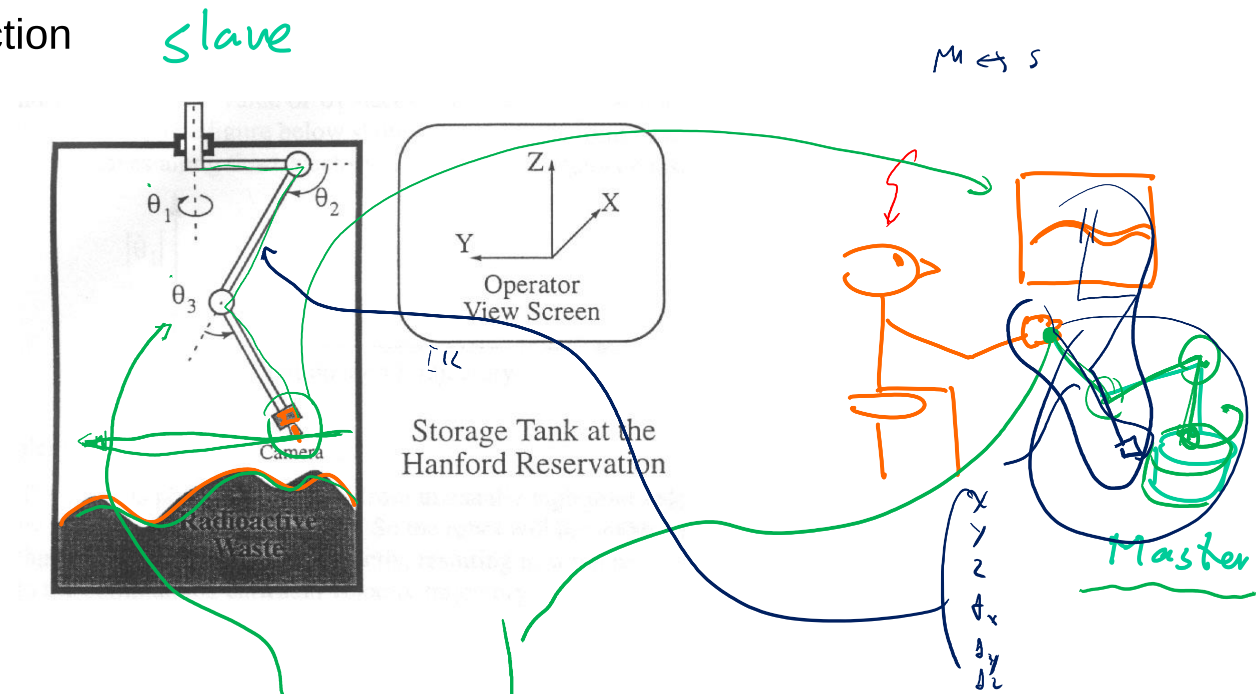
- The third row of the Jacobian is zero
- The origin of frame {4} intersects the Z-axis of frame {1}
- The 3R robot is loosing one DOF.
- The robot can no longer move along the Z-axis of frame {4}

3 DOF → 2 DOF

Joint Velocity Near Singular Position - 3R Example

- Robot : 3R robot
- Task: Visual inspection





- Control

Operator → $\dot{\underline{\theta}} = J(\theta)^{-1}\dot{\underline{x}}$ **Control** → **Robot**

Joint Velocity Near Singular Position - 3R Example

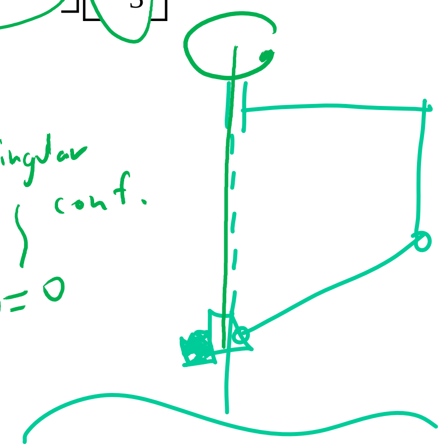
- ***Singularity (Case 2)-*** The origin of frame {4} intersects the Z-axis of frame {1}

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & L2s3 & 0 \\ 0 & L2c3+L3 & L3 \\ -L1-L2c2-L3c23 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- Solve for $\dot{\theta}_1$ in terms of $\dot{z}$ we find

$$\dot{\theta}_1 = \frac{\dot{z}}{\underbrace{-L1-L2c2-L3c23}_{}} = 0$$

singular conf.



$$-L1-L2c2-L3c23=0$$

$$\dot{\theta}_1 \to \infty$$

Joint Velocity Near Singular Position - 3R Example

- Singularity - $$\det\left(J\left(\theta\right)\right)=0$$

- Problems:

  – **Motor Constrains** - The robot is physically limited from moving in unusual high joint velocities by motor power constrains. Therefore, the robot will be unable to track the required joint velocity trajectory exactly resulting in some perturbation to the commanded Cartesian velocity trajectory.

  – **Gears and Shafts** - The derivative of the angular velocity is the angular acceleration. The high acceleration of the joint resulting form approaching too close to a singularity may cause damage to the gear/shafts.

  – **DOF** - At a singular configuration (specific point in space) the manipulator loses one or more DOF.

- Consequences – Certain tasks can not be performed at a singular configuration

Designing Well Conditioned Workspace

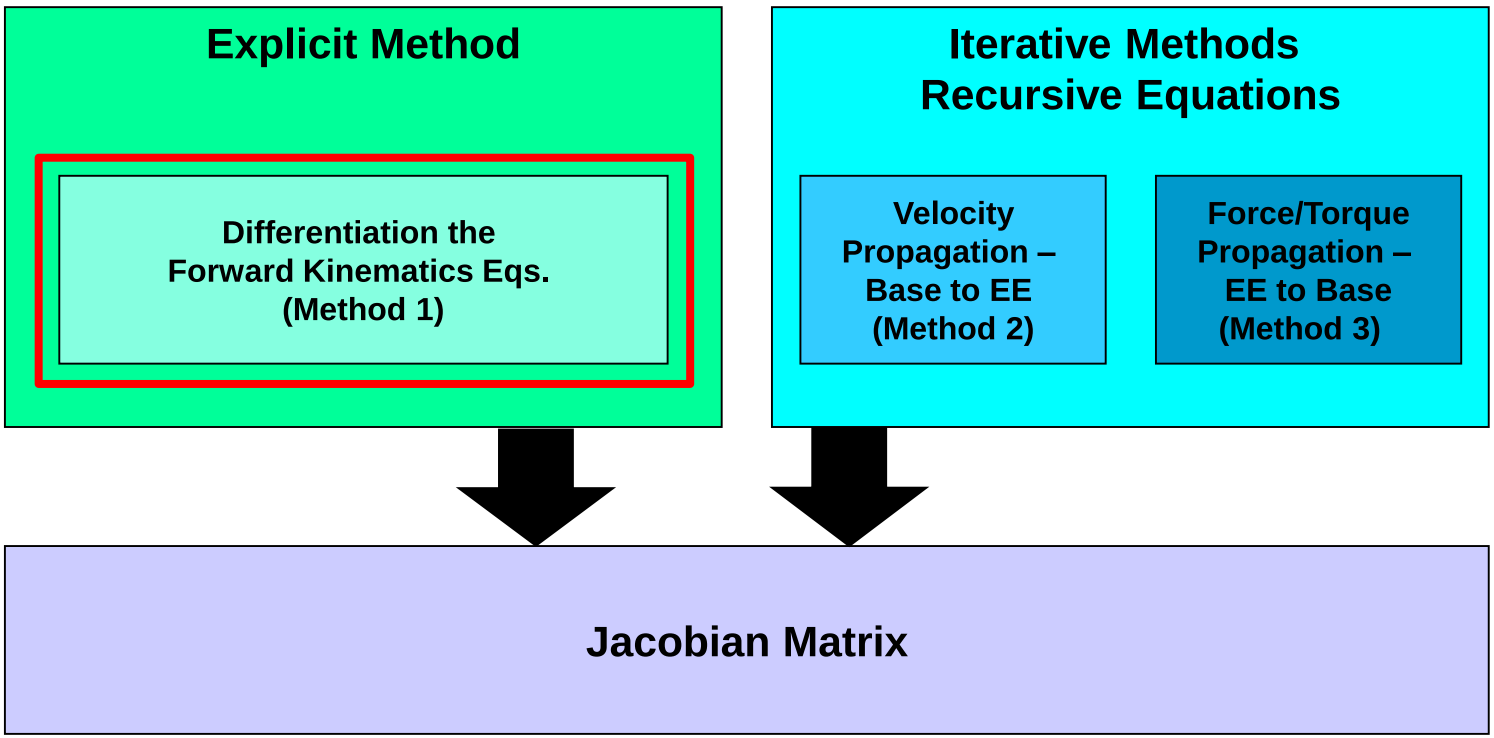
- Difficulty in operating at
  - Workspace Boundaries
  - Neighborhood of singular point inside the workspace

- The further the manipulator is away from singularities the better it moves uniformly and apply forces in all directions

- **Manipulability Measure** - How far / close the manipulator is from singularity
  - Range $0 \le w < \infty$
  - For Non redundant manipulators $w = \det\left(J(\theta)\right)$
  - For redundant manipulators $w = \sqrt{\det\left(J(\theta)J^T(\theta)\right)}$
  - A good manipulator design has large area of characterized by high value of the manipulability (w)

Jacobian Explicit Method -
Differentiation the Forward Kinematics Eqs. (Method No. 1)

Jacobian Matrix - Derivation Methods



Jacobian – Explicit Form – Overview

Frame 0

$$v = {}^{0}J\,\dot{\theta}$$

$$\begin{bmatrix} {}^{0}v_n \\ \hline {}^{0}\omega_n \end{bmatrix} \qquad \begin{bmatrix} {}^{0}v_{nx} \\ {}^{0}v_{ny} \\ {}^{0}v_{ny} \\ \hline {}^{0}\omega_{nx} \\ {}^{0}\omega_{ny} \\ {}^{0}\omega_{nz} \end{bmatrix}$$

6×6

$$\begin{bmatrix} J_v \\ \hline J_\omega \end{bmatrix}$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 / \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{Bmatrix}$$

OR

UCLA

Jacobian – Explicit Form

$${}^0_iT = \left[\begin{array}{c|c} {}^0_iR & \\ \hline & \end{array}\right] \qquad {}^0_iZ \qquad {}^0P_i$$

i

J_v

Prismatic Joint

$$\frac{\partial\, {}^0P_6}{\partial d_i}$$

$${}^0_iZ$$

Revolute Joint

$$\frac{\partial\, {}^0P_6}{\partial \theta_i}$$

$${}^0_iZ \times \left({}^0P_6 - {}^0P_i\right)$$

J_w

$\rho = 1$ Revolute

$\rho = 0$ Prismatic

$$\rho\, {}^0_iZ$$

Jacobian – Explicit Form – Angular Velocity J_ω

$$ {}^{i+1}\omega_{i+1} = {}^{i+1}_{\ \ i}R\; {}^{i}\omega_{i} + \begin{bmatrix}0\\0\\\dot\theta_{i+1}\end{bmatrix} \rho_{i+1} $$

$\rho = 0$ prismatic

$\rho = 1$ Revolute

$i = 0$

$$ {}^{1}\omega_{1} = {}^{1}_{0}R\; \cancel{{}^{0}\omega_{0}}^{\,0} + \begin{bmatrix}0\\0\\\dot\theta_{1}\end{bmatrix} \rho_{1} $$

${}^{1}\omega_{1}$

$i = 1$

$$ {}^{2}\omega_{2} = {}^{2}_{1}R\; {}^{1}\omega_{1} + \begin{bmatrix}0\\0\\\dot\theta_{2}\end{bmatrix} \rho_{2} = {}^{2}_{1}R \left[\begin{bmatrix}0\\0\\\dot\theta_{1}\end{bmatrix} \rho_{1} \right] + \begin{bmatrix}0\\0\\\dot\theta_{2}\end{bmatrix} \rho_{2} $$

$i = 2$

$$ {}^{3}\omega_{3} = {}^{3}_{2}R\; {}^{2}\omega_{2} \quad \begin{bmatrix}0\\0\\\dot\theta_{3}\end{bmatrix} \rho_{3} = {}^{3}_{2}R \left[{}^{2}_{1}R \begin{bmatrix}0\\0\\\dot\theta_{1}\end{bmatrix} \rho_{1} + \begin{bmatrix}0\\0\\\dot\theta_{2}\end{bmatrix} \rho_{2} \right] + \begin{bmatrix}0\\0\\\dot\theta_{3}\end{bmatrix} \rho_{3} $$

${}^{0}_{n}R$ *

$$ \rightarrow \quad {}^{n}\omega_{n} = \qquad = {}^{n}_{1}R \begin{bmatrix}0\\0\\\dot\theta_{1}\end{bmatrix} \rho_{1} + {}^{n}_{2}R \begin{bmatrix}0\\0\\\dot\theta_{2}\end{bmatrix} \rho_{2} + \cdots \cancel{{}^{n}_{n}R}^{\,1} \begin{bmatrix}0\\0\\\dot\theta_{n}\end{bmatrix} \rho_{n} $$

Jacobian – Explicit Form – Angular Velocity J_ω

$$\overbrace{{}^0_nR\,{}^n\omega_n}^{{}^0\omega_n} = \overbrace{{}^0_nR\,{}^n_1R}^{{}^0_1R}\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} f_1 + \overbrace{{}^0_nR\,{}^n_2R}^{{}^0_2R}\begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix} f_2 + \cdots + \overbrace{{}^0_nR\,\cancelto{1}{{}^n_nR}}^{{}^n_nR}\begin{bmatrix}0\\0\\\dot\theta_n\end{bmatrix} f$$

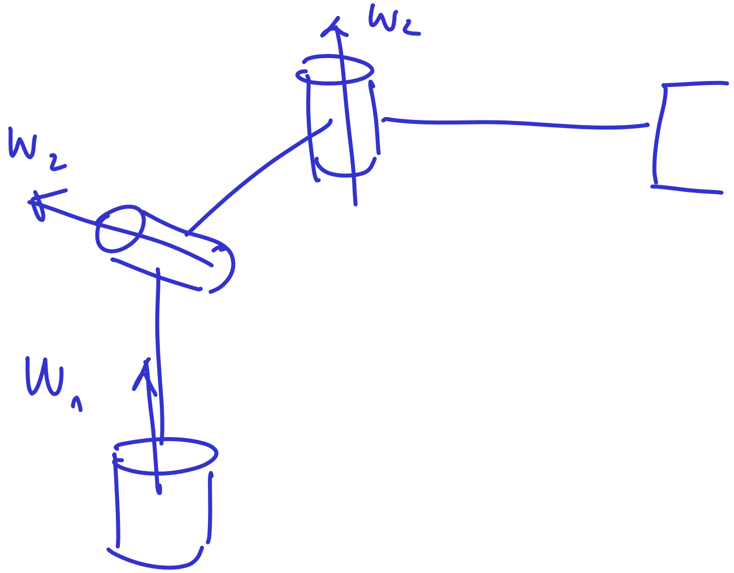
$${}^0\omega_n = {}^0_1R\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} f_1 + {}^0_2R\begin{bmatrix}0\\0\\\dot\theta_2\end{bmatrix} f_2 + \cdots + {}^0_nR\begin{bmatrix}0\\0\\\dot\theta_n\end{bmatrix} f_n$$

$${}^0\omega_n = \sum_{i=1}^{n} {}^0_iR\begin{bmatrix}0\\0\\\dot\theta_i\end{bmatrix} f_i = \sum_{i=1}^{n} {}^0Z_i\,\dot\theta_i\, f_i$$

$${}^0_nR = \begin{bmatrix}[\;] & [\;] & \underbrace{[\;\bullet\;]}_{{}^0Z_n}\end{bmatrix}$$

Instructor: Jacob Rosen
Advanced Robotic - MAE 263B - Department of Mechanical & Aerospace Engineering - UCLA

Jacobian – Explicit Form – Angular Velocity J_ω



Jacobian – Explicit Form – Linear Velocity J_v

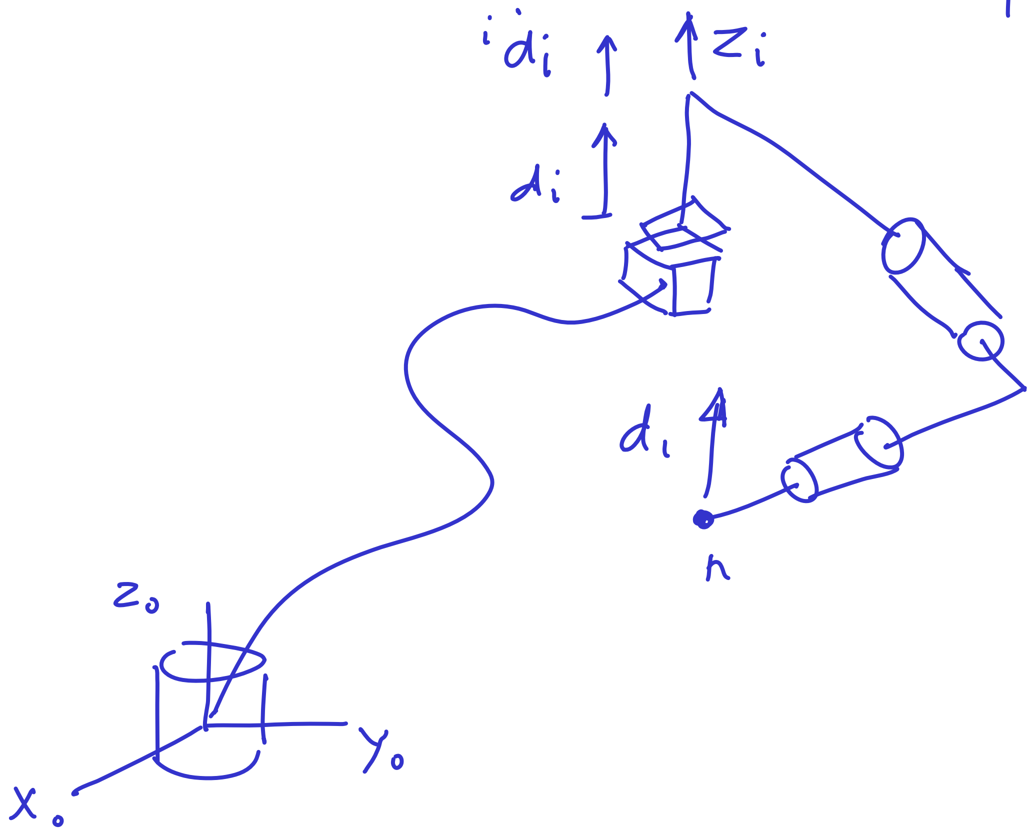
$${}^0\dot{P}_n = \sum \underbrace{\frac{\partial\, {}^0P_n}{\partial q_i}} \cdot \dot{q}_i \sim \begin{cases} \dot{\theta}_i \\ \dot{d}_i \end{cases}$$

$$J_{v_i} = \frac{\partial\, {}^0P_n}{\partial q_i}$$

Jacobian – Explicit Form – Linear Velocity
Case 1- Prismatic Joint

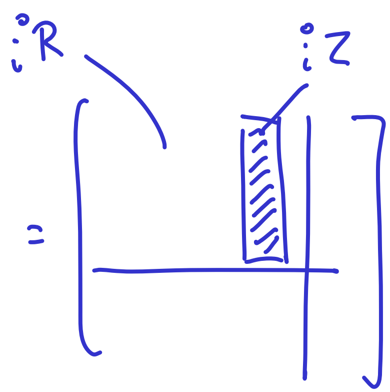
J_υ

$${}^0\dot{P}_n = \dot{d}_i \, {}^0_iR \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i \, {}^0_iZ$$



$$J_{v_i} = {}^0_iZ$$

${}^0_iT =$

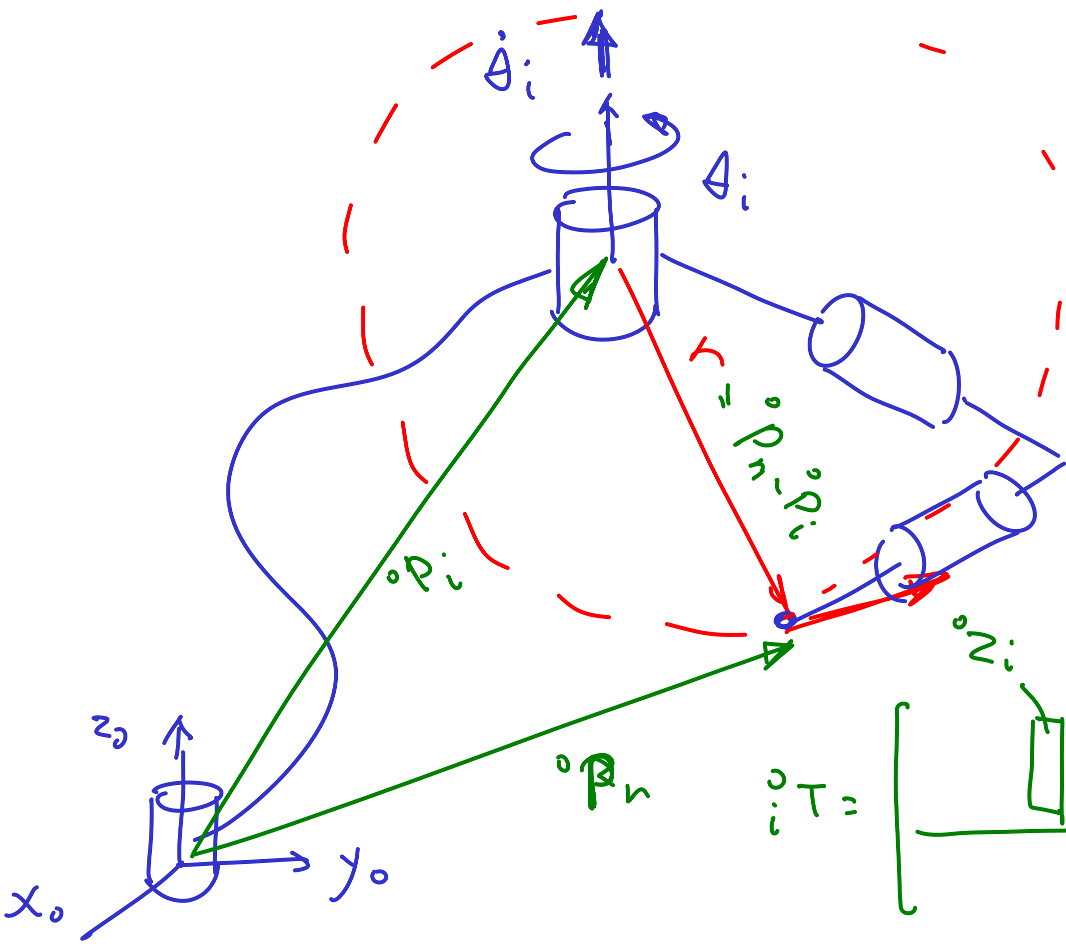


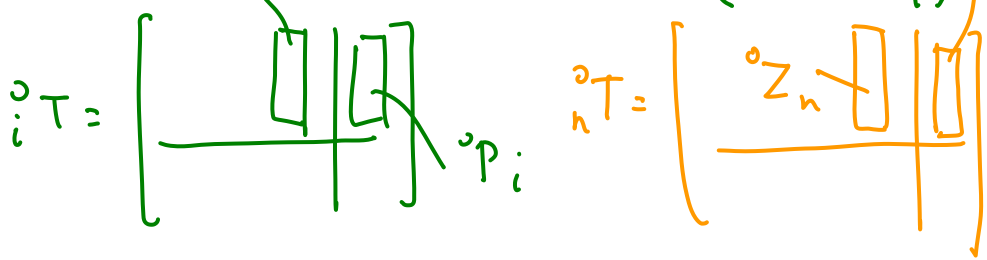
$w \times r$

$\dot{\theta}_i \times r$

$$\underbrace{\dot{\theta}_i}_{\omega} \times \underbrace{({}^0p_n - {}^0p_i)}_{r}$$

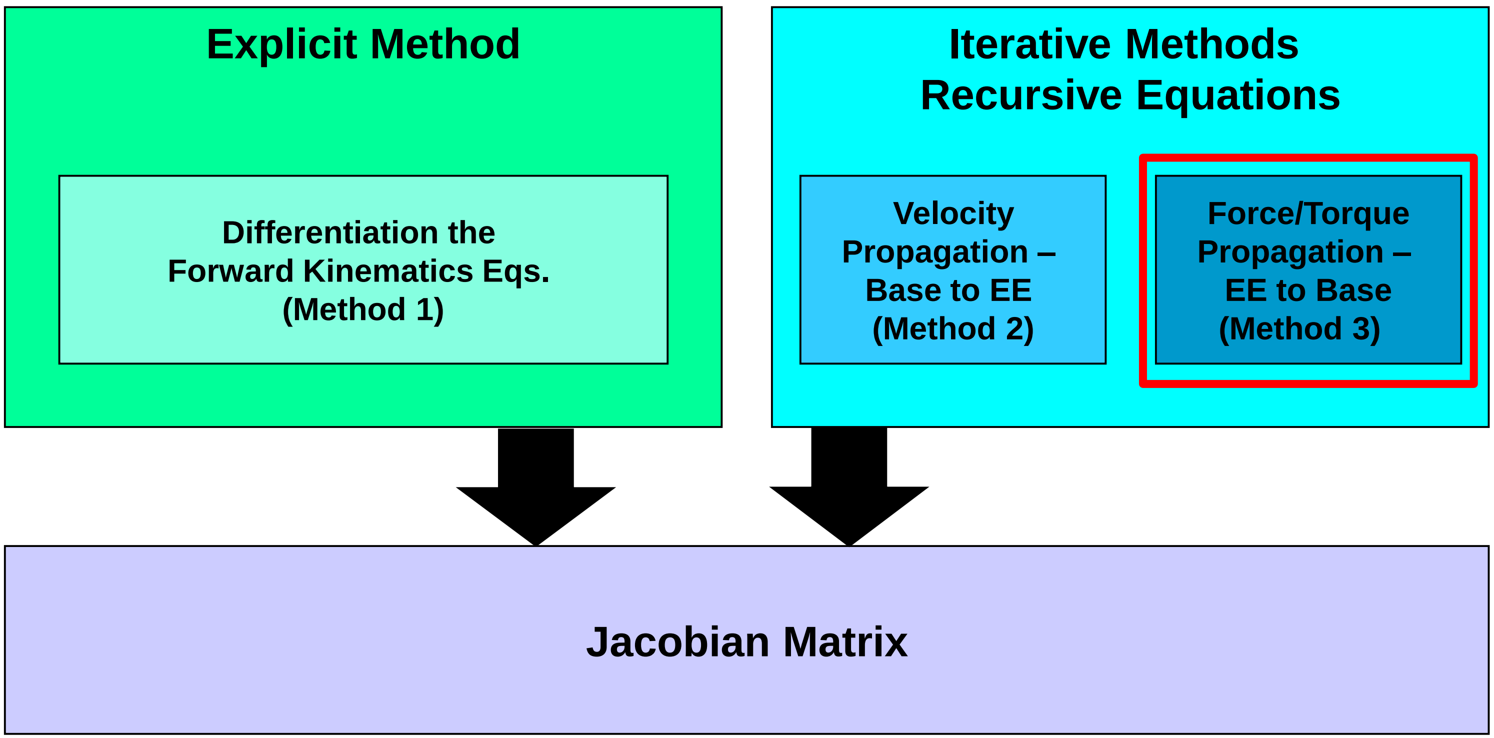
$$J_{vi} = {}^0Z_i \times ({}^0p_n - {}^0p_i)$$





Jacobian Iterative Method - Force/Torque Propagation (Method No. 3)

Jacobian Matrix - Derivation Methods



Statics - Forces & Torques

Problem

Given: Typically the robot's end effector is applying forces and torques on an object in the environment or carrying an object (gravitational load).

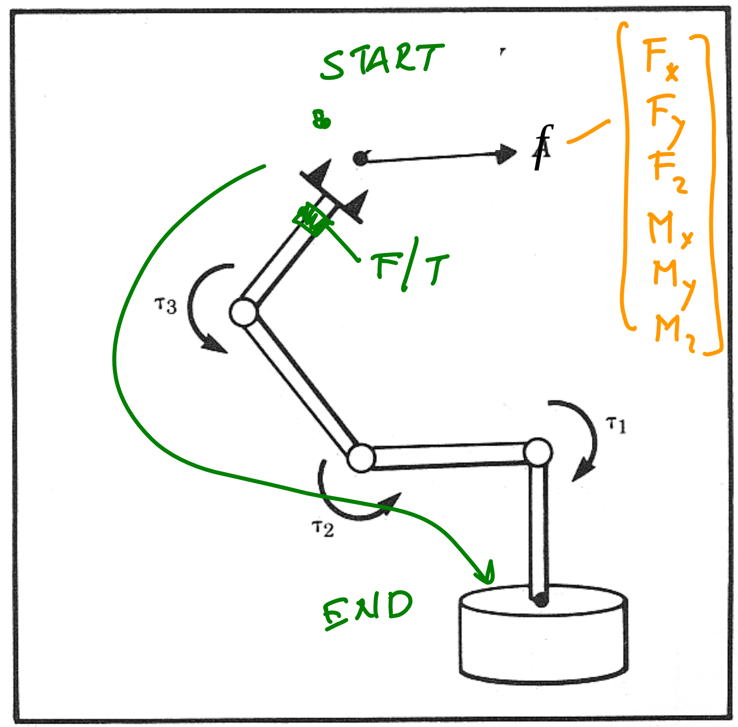
Compute: The joint torques which must be acting to keep the system in static equilibrium.

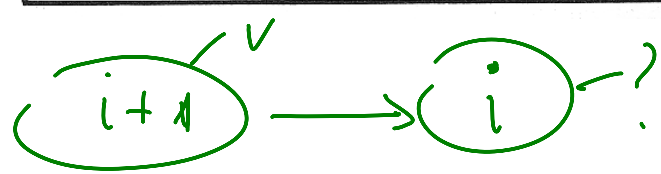
Solution

Jacobian - Mapping from the joint force/torques - τ to forces/torque in the Cartesian space applied on the end effector) - f .

$$\tau = \mathbf{J}^T f$$

Free Body Diagram - The chain like nature of a manipulator leads to decompose the chain into individual links and calculate how forces and moments propagate from one link to the next.





Static Analysis Protocol - Free Body Diagram 1/

Step 1

Lock all the joints - Converting the manipulator (mechanism) to a structure

Step 2

Consider each link in the structure as a free body and write the force / moment equilibrium equations

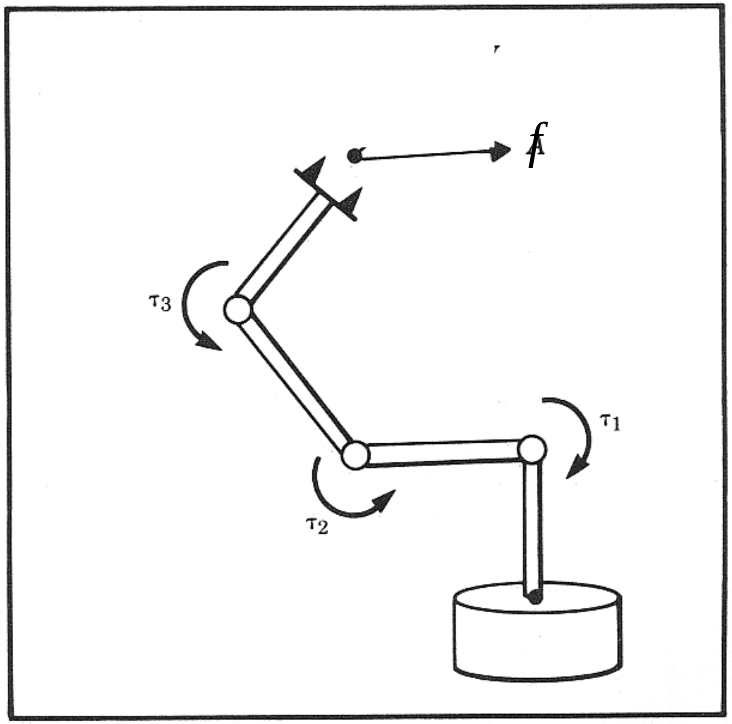
(3 Eqs.) $$\sum F = 0$$

(3 Eqs.) $$\sum M = 0$$

Step 3

Solve the equations - 6 Eq. for each link.

Apply backward solution starting from the last link (end effector) and end up at the first link (base)

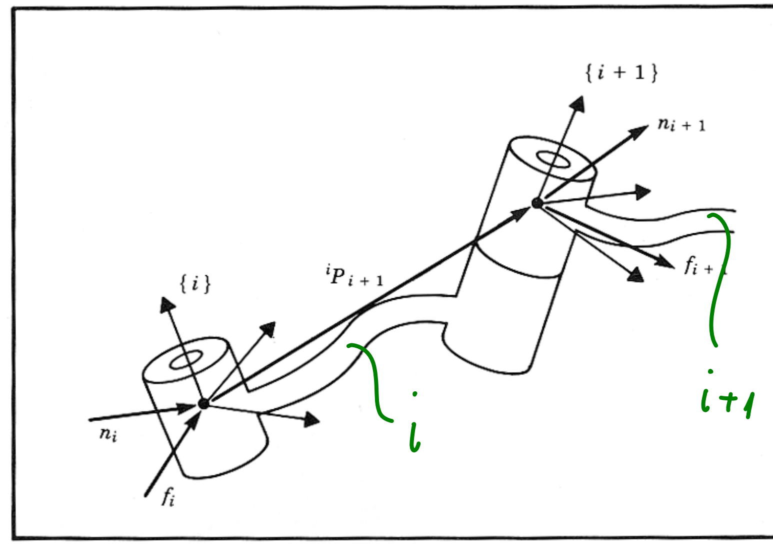


Static Analysis Protocol - Free Body Diagram 2/

- Special Symbols are defined for the force and torque exerted by the neighbor link

f_i - Force exerted on link ***i*** by link ***i-1***

n_i - Torque exerted on link ***i*** by link ***i-1***



Reference coordinate system {B}

$$^{B}C_{A}$$

Force ***f*** or torque ***n***

Exerted on link A by link A-1

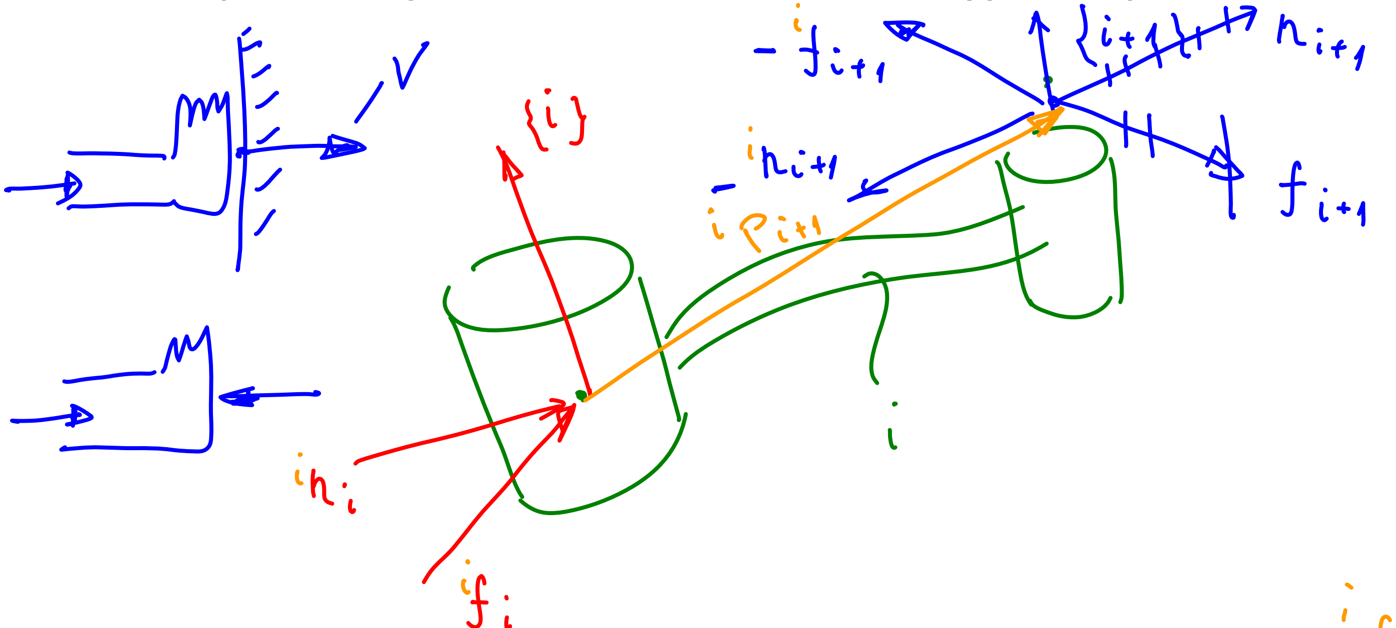
$^{B}f_{A}$

$^{B}n_{A}$

- For easy solution superscript index (A) should the same as the subscript (B)

Static Analysis Protocol - Free Body Diagram 3/

- For serial manipulator in static equilibrium (joints locked), the sum the forces and torques acting on link $\boldsymbol{i}$ in the link frame $\{\boldsymbol{i}\}$ are equal to zero.



$$\sum {}^{i}F = 0 \quad \Rightarrow \quad \sum {}^{i}F = {}^{i}f_i - {}^{i}f_{i+1} = 0 \longrightarrow \underbrace{{}^{i}f_i = {}^{i}f_{i+1}}$$

$$\sum {}^{i}M = 0 \quad \Rightarrow \quad \sum {}^{i}M = {}^{i}n_i - {}^{i}n_{i+1} - {}^{i}P_{i+1} \times {}^{i}f_{i+1} = 0$$

ORIGIN of i

Static Analysis Protocol - Free Body Diagram 4/

- **Procedural Note**: The solution starts at the end effector and ends at the base
- Re-writing these equations in order such that the known forces (or torques) are on the right-hand side and the unknown forces (or torques) are on the left, we find

$$ {}^{i}f_{i} = {}^{i}f_{i+1} $$

$$ {}^{i}n_{i} = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i} $$

Static Analysis Protocol - Free Body Diagram 5/

- Changing the reference frame such that each force (and torque) is expressed upon their link's frame, we find the static force (and torque) propagation from link ***i+1*** to link ***i***

$$ {}^{i}f_i = {}^{i}f_{i+1} = {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} $$

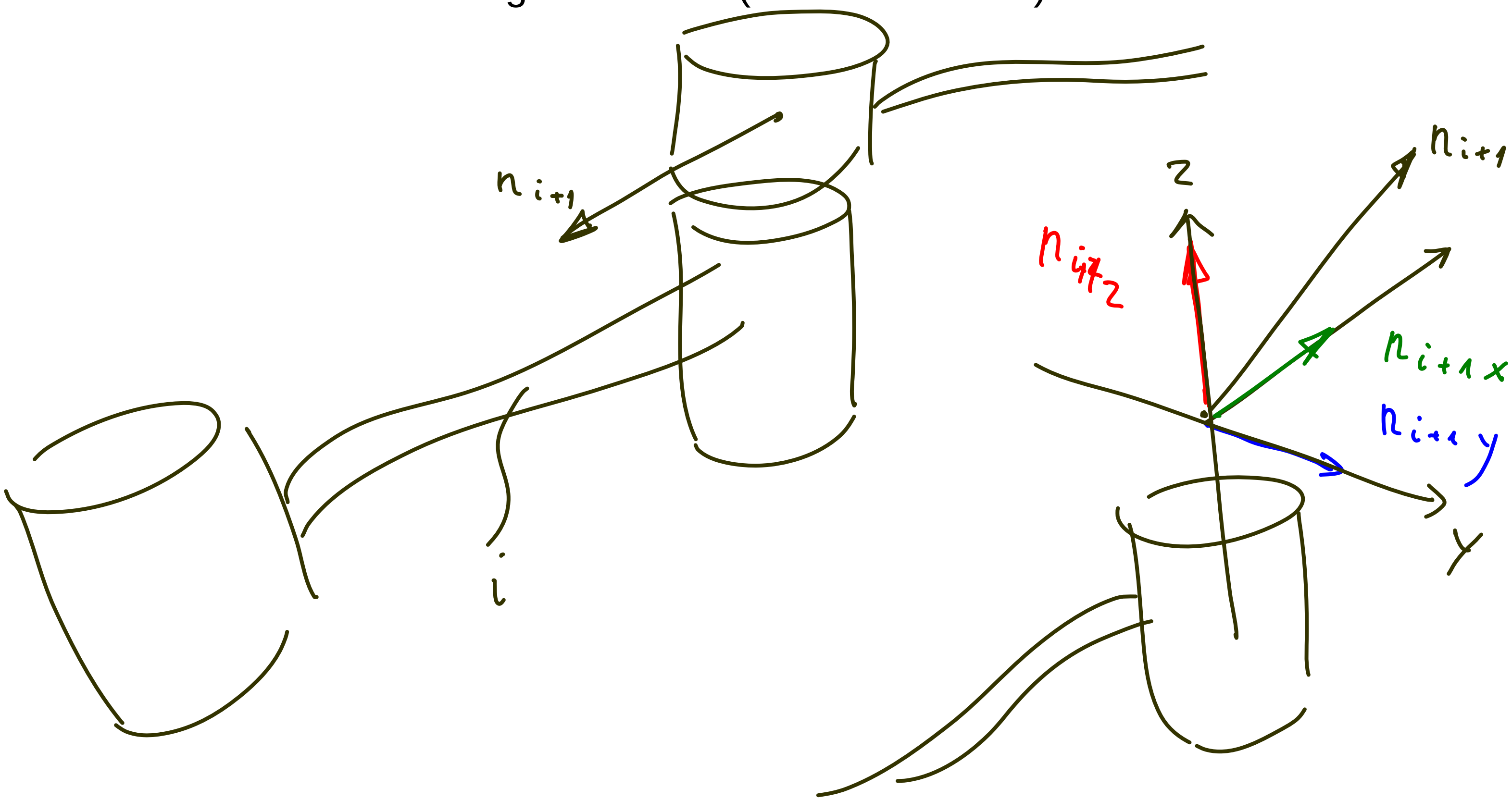
$$ {}^{i}n_i = {}^{i}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_{i+1} = {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i $$

$$ \boxed{\begin{aligned} {}^{i}f_i &= {}^{i}_{i+1}R \; {}^{i+1}f_{i+1} \\ {}^{i}n_i &= {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i \end{aligned}} $$

- These equations provide the static force (and torque) propagation from link to link. They allow us to start with the force and torque applied at the end effector, and calculate the force and torque at each joint all the way back to the robot base frame.

Static Analysis Protocol - Free Body Diagram 6/

- **Question:** What torques are needed at the joints in order to balance the reaction moments acting on the link (**Revolute Joint**).



Static Analysis Protocol - Free Body Diagram 7/

- **Question:** What forces are needed at the joints in order to balance the reaction forces acting on the link (**Prismatic Joint**).

Static Analysis Protocol - Free Body Diagram 8/

- **Answer:** All the components of the force and moment vectors are resisted by the structure of mechanism itself, except for the torque about the joint axis (revolute joint) or the force along the joint (prismatic joint).

- Therefore, to find the joint the torque or force required to maintain the static equilibrium, the dot product of the joint axis vector with the moment vector or force vector acting on the link is computed

Revolute Joint

$$\tau_i = {}^i n_i^T \hat{z}_i = \begin{bmatrix} {}^i n_{ix} & {}^i n_{iy} & {}^i n_{iz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Prismatic Joint

$$f_i = {}^i f_i^T \hat{z}_i = \begin{bmatrix} {}^i f_{ix} & {}^i f_{iy} & {}^i f_{iz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

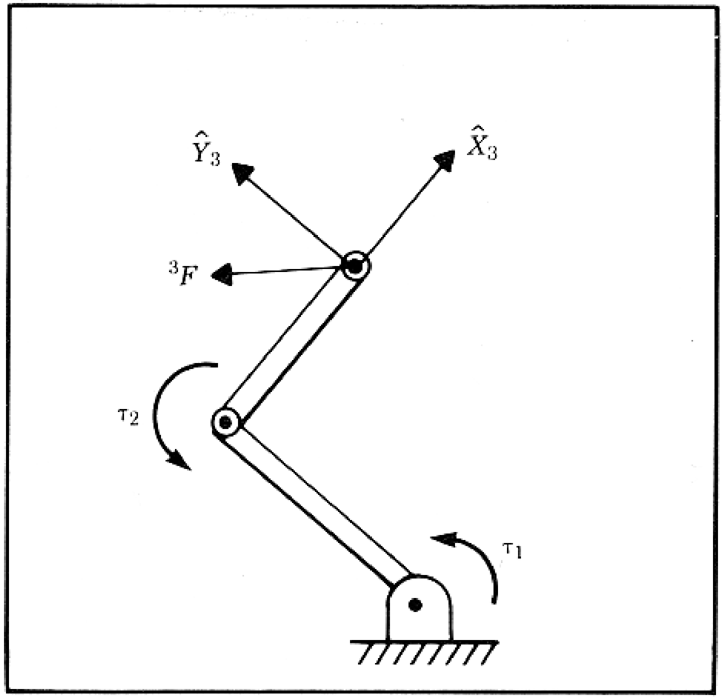
Problem

Given:

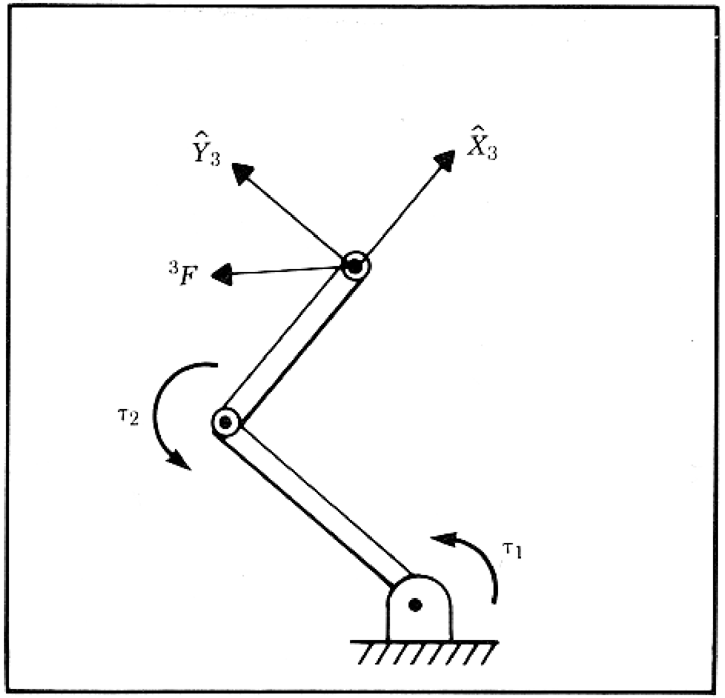
- 2R Robot
- A Force vector ${}^{3}f_3$ is applied **by** the end effector
- A torque vector ${}^{3}n_3 = 0$

Compute:

The required joint torque as a function of the robot configuration and the applied force



Example - 2R Robot - Static Analysis



Example - 2R Robot - Static Analysis

Solution

- Lock the revolute joints
- Apply the static equilibrium equations starting from the end effector and going toward the base

$$^{i}f_i = {}^{i}_{i+1}R \; {}^{i+1}f_{i+1}$$

$$^{i}n_i = {}^{i}_{i+1}R \; {}^{i+1}n_{i+1} + {}^{i}P_{i+1} \times {}^{i}f_i$$

Example - 2R Robot - Static Analysis

- For i=2

$$ {}^{2}f_{2} = {}^{2}_{3}R \; {}^{3}f_{3} $$

$$ {}^{2}f_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} $$

$$ {}^{2}n_{2} = {}^{2}_{3}R \; {}^{3}n_{3} + {}^{2}P_{3} \times {}^{2}f_{2} $$

$$ {}^{2}n_{2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{X} & \hat{Y} & \hat{Z} \\ l_2 & 0 & 0 \\ f_x & f_y & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} $$

Example - 2R Robot - Static Analysis

- For i=1

$$ {}^{1}f_1 = {}^{1}_{2}R \; {}^{2}f_2 $$

$$ {}^{1}f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x - c_2 f_y \\ 0 \end{bmatrix} $$

$$ {}^{1}n_1 = {}^{1}_{2}R \; {}^{2}n_2 + {}^{1}P_2 \times {}^{1}f_1 $$

$$ {}^{1}n_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_2 f_x - s_2 f_y \\ s_2 f_x + c_2 f_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + \begin{bmatrix} \hat{X} & \hat{Y} & \hat{Z} \\ l_1 & 0 & 0 \\ c_2 f_x - s_2 f_y & s_2 f_x + c_2 f_y & 0 \end{bmatrix} = $$

$$ \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} $$

Example - 2R Robot - Static Analysis

$$
{}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \end{bmatrix} \qquad {}^{2}n_2 = \begin{bmatrix} 0 \\ 0 \\ l_2 f_y \end{bmatrix}
$$

$$
\tau_i = {}^{i}n_i^T \hat{z}_i = \begin{bmatrix} {}^{i}n_{ix} & {}^{i}n_{iy} & {}^{i}n_{iz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

$$
\tau_1 = l_1 s_2 f_x + l_1 c_2 f_y + l_2 f_y \qquad \tau_2 = l_2 f_y
$$

Example - 2R Robot - Static Analysis

- Re-writing the equations in a matrix form

$$\tau = \begin{bmatrix} l_1 s_2 & l_1 c_2 + l_2 \\ 0 & l_2 \end{bmatrix} \begin{bmatrix} f_x \\ f_y \end{bmatrix} = [{}^N J]^T f$$

Jacobian Propagation to the Tip of the Tool

Jacobian Propagation to the Tip of the Tool

Goal

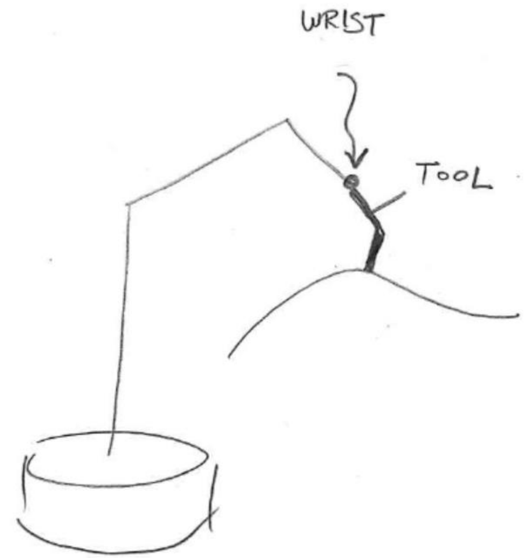
- $$\underset{\downarrow}{T_{path}} = \underset{\theta_1}{{}^0_1T}\;\underset{\theta_2}{{}^1_2T}\;\underset{\theta_3}{{}^2_3T}\;\underset{\theta_4}{{}^3_4T}\;\underset{\theta_5}{{}^4_5T}\;\underset{\theta_6}{{}^5_6T}\;\underset{\text{KNOWN}}{{}^6_TT}$$

- The position of the tool does not change as a function of time in frame 6

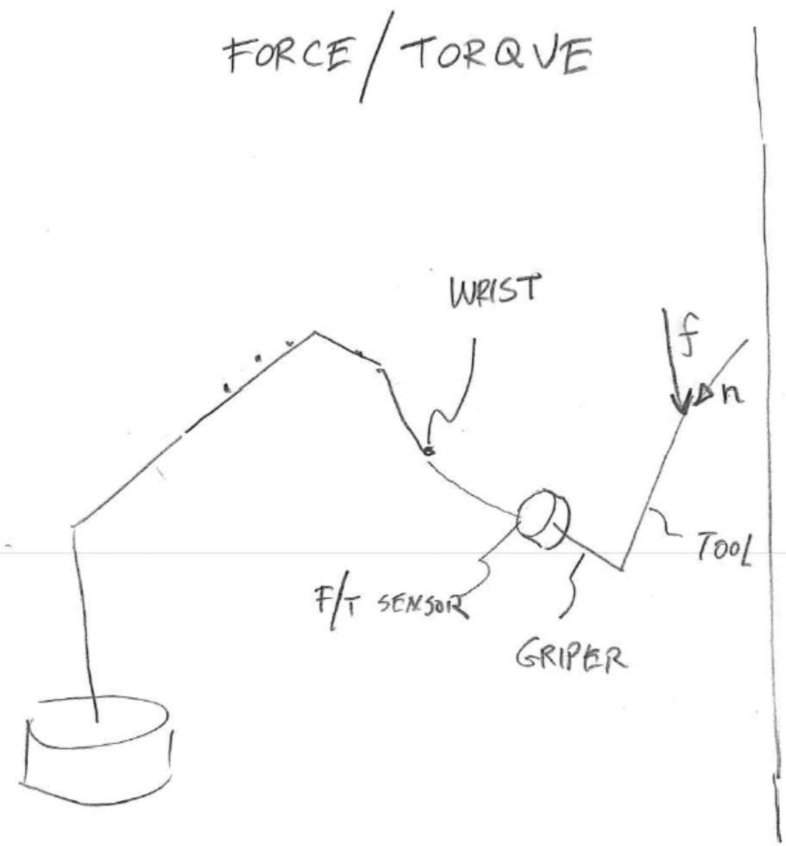
- Multiply both sides by $({}^6_TT)^{-1} = ({}^6_TT)^T = {}^T_6T$

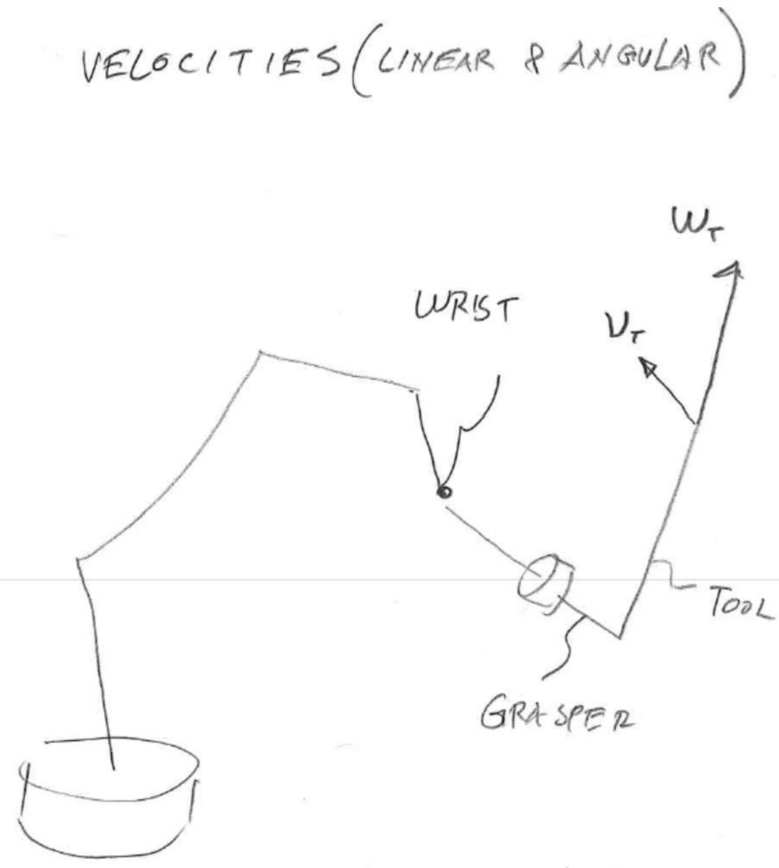
$$T_{path}({}^6_TT)^{-1} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T\,\underbrace{{}^6_TT({}^6_TT)^{-1}}_{I}$$

$$T_{path}\,{}^T_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T \longrightarrow \text{solve the inverse kin (IK)}$$



Jacobian Propagation to the Tip of the Tool





Jacobian Propagation to the Tip of the Tool

POSITION

VECTOR FORM

$$^{A}P_{Q} = {}^{A}P_{BORG} + {}^{A}_{B}R\,{}^{B}P_{Q}$$

MATRIX FORM

$$^{A}P_{Q} = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}P_{ORG} \\ \hline & \end{array}\right] {}^{B}P_{Q}$$

Jacobian Propagation to the Tip of the Tool

VELOCITY - FOR TWO FRAMES THAT ARE RIGIDILY CONNECTED

VECTOR FORM

$$\begin{cases} \boxed{{}^{i+1}\omega_{i+1}} = \boxed{{}^{i+1}_{i}R}\,\boxed{{}^{i}\omega_{i}} + \cancel{\dot{\theta}_{i+1}}\,{}^{i+1}Z_{i+1} \quad (\dot{\theta}_{i+1} = 0) \\ \boxed{{}^{i+1}v_{i+1}} = \boxed{{}^{i+1}_{i}R\left(\boxed{{}^{i}v_{i}} + \boxed{{}^{i}\omega_{i}} \times \boxed{{}^{i}P_{i+1}}\right)} \end{cases}$$

$$\begin{cases} i+1 \rightarrow A \\ i \rightarrow B \end{cases}$$

MATRIX FORM

$${}^{B}V_{B} = {}^{B}_{A}T_{v}\,{}^{A}v_{A}$$

$${}^{A}V_{A} = {}^{B}_{A}T_{v}\,{}^{B}v_{B}$$

$$\begin{Bmatrix} \boxed{{}^{A}v_{A}} \\ \boxed{{}^{A}\omega_{A}} \end{Bmatrix} = \left[\begin{array}{c|c} \boxed{{}^{A}_{B}R} & \boxed{{}^{A}P_{BORG} \times {}^{A}_{B}R} \\ \hline 0 & \boxed{{}^{A}_{B}R} \end{array}\right] \begin{Bmatrix} \boxed{{}^{B}v_{B}} \\ \boxed{{}^{B}\omega_{B}} \end{Bmatrix}$$

$$\begin{Bmatrix} {}^{B}v_{B} \\ {}^{B}\omega_{B} \end{Bmatrix} = \left[\begin{array}{c|c} {}^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}P_{BORG} \times \\ 0 & \end{array}\right] \begin{Bmatrix} {}^{A}v_{A} \\ {}^{A}\omega_{A} \end{Bmatrix}$$

Jacobian Propagation to the Tip of the Tool

FORCE / TORQUE

VECTOR FORM

$$\left\{\begin{array}{l} \boxed{{}^{i}f_{i}} = \boxed{{}^{i}_{i+1}R}\,\boxed{{}^{i+1}f_{i+1}} \\ \boxed{{}^{i}n_{i}} = \boxed{{}^{i}_{i+1}R}\,\boxed{{}^{i+1}n_{i+1}} + \boxed{{}^{i}P_{i+1}} \times \boxed{{}^{i}f_{i+1}} \end{array}\right.$$

$$\left\{\begin{array}{l} i \rightarrow A \\ i+1 \rightarrow B \end{array}\right.$$

MATRIX FORM

$${}^{A}F_{A} = {}^{A}_{B}T_{f}\,{}^{B}F_{B}$$

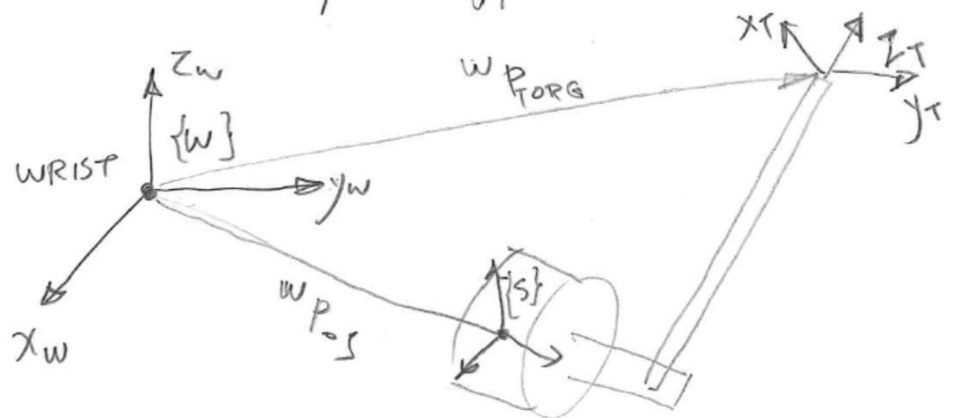
$$\left\{\begin{array}{c} {}^{A}F_{A} \\ {}^{A}N_{A} \end{array}\right\} = \left[\begin{array}{c|c} \boxed{{}^{A}_{B}R} & 0 \\ \hline \boxed{{}^{A}P_{BORG} \times \boxed{{}^{A}_{B}R}} & \boxed{{}^{A}_{B}R} \end{array}\right] \left\{\begin{array}{c} \boxed{{}^{B}F_{B}} \\ \boxed{{}^{B}N_{B}} \end{array}\right\}$$

NOTE:

$${}^{A}_{B}T_{f} = {}^{A}_{B}T_{v}^{\,T}$$

Jacobian Propagation to the Tip of the Tool

- The end effector holds a tool
- Located at the point where the end effector attached to the manipulator is a force sensing wrist. This device can measure the forces and torque applied to it



Jacobian Propagation to the Tip of the Tool

$$ {}^{W}_{T}T = {}^{W}_{S}T \; {}^{S}_{T}T $$

- Multiply both sides by $\left({}^{W}_{S}T\right)^{-1} \longrightarrow {}^{S}_{W}T$

$$ {}^{S}_{W}T \, {}^{W}_{T}T = \underbrace{{}^{S}_{W}T \; {}^{W}_{S}T}_{I} \; {}^{S}_{T}T $$

- Inverting ${}^{S}_{T}T \longrightarrow {}^{T}_{S}T$

- The forces & torques applied on the tool tip based on the measurment at the sensor can be calculated as

Jacobian Propagation to the Tip of the Tool

$$^{T}\mathcal{F}_{T} = \underbrace{{}^{T}_{S}T_{f}}\,{}^{S}\mathcal{F}_{S}$$

$$^{T}_{S}T_{f} = \left[\begin{array}{c|c} {}^{T}_{S}R & 0 \\ \hline {}^{T}P_{SORG} \times {}^{T}_{S}R & {}^{T}_{S}R \end{array}\right]$$

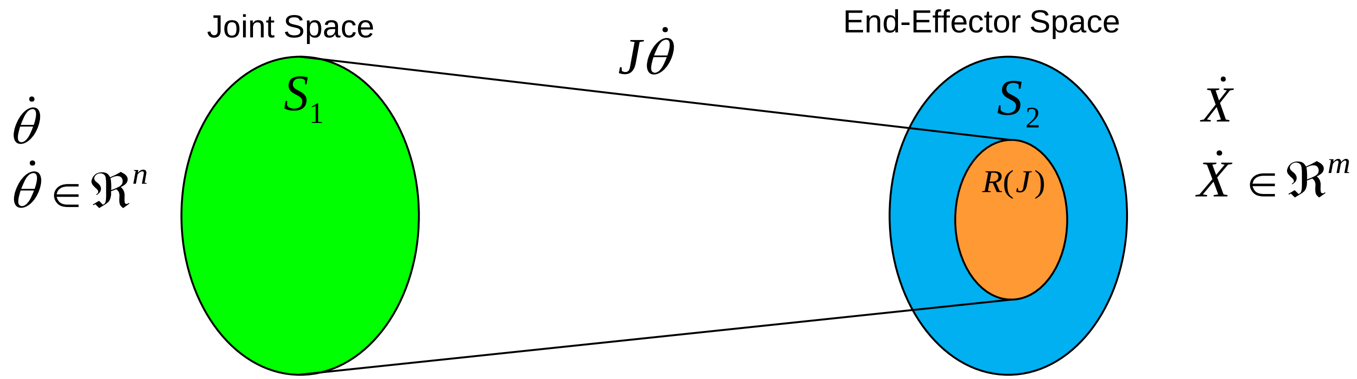
Jacobian Methods – Reference Frame - Summary

Jacobian Methods of Derivation & the Corresponding Reference Frame – Summary

Method	Jacobian Matrix Reference Frame	Transformation to Base Frame (Frame 0)
Explicit (Diff. the Forward Kinematic Eq.)	$^{0}J_{N}$	None
Iterative Velocity Eq.	$^{N}J_{N}$	Transform Method 1: $^{0}v_{N} = {}^{0}_{N}R\,{}^{N}v_{N}$ $^{0}\omega_{N} = {}^{0}_{N}R\,{}^{N}\omega_{N}$ Transform Method 2: $^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$
Iterative Force Eq.	$^{N}J_{N}^{T}$	Transpose $^{N}J_{N} = [{}^{N}J_{N}^{T}]^{T}$ Transform $^{0}J_{N}(\theta) = \begin{bmatrix} {}^{0}_{N}R & 0 \\ 0 & {}^{0}_{N}R \end{bmatrix} {}^{N}J_{N}(\theta)$

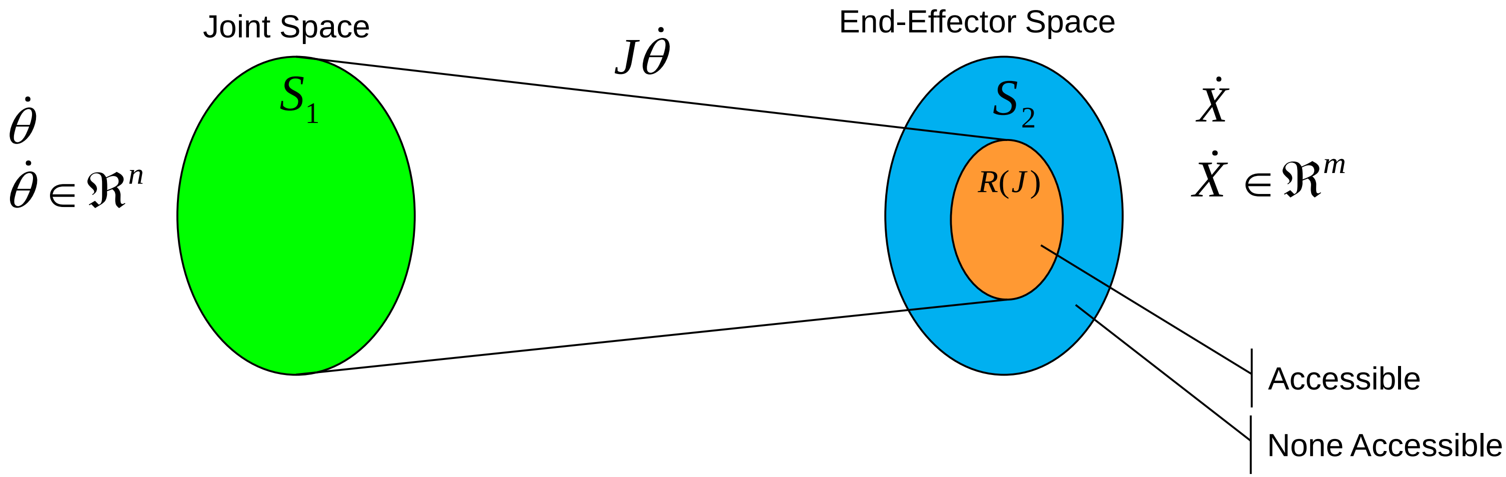
Jacobian – High Level Overview

Jacobian – Duality



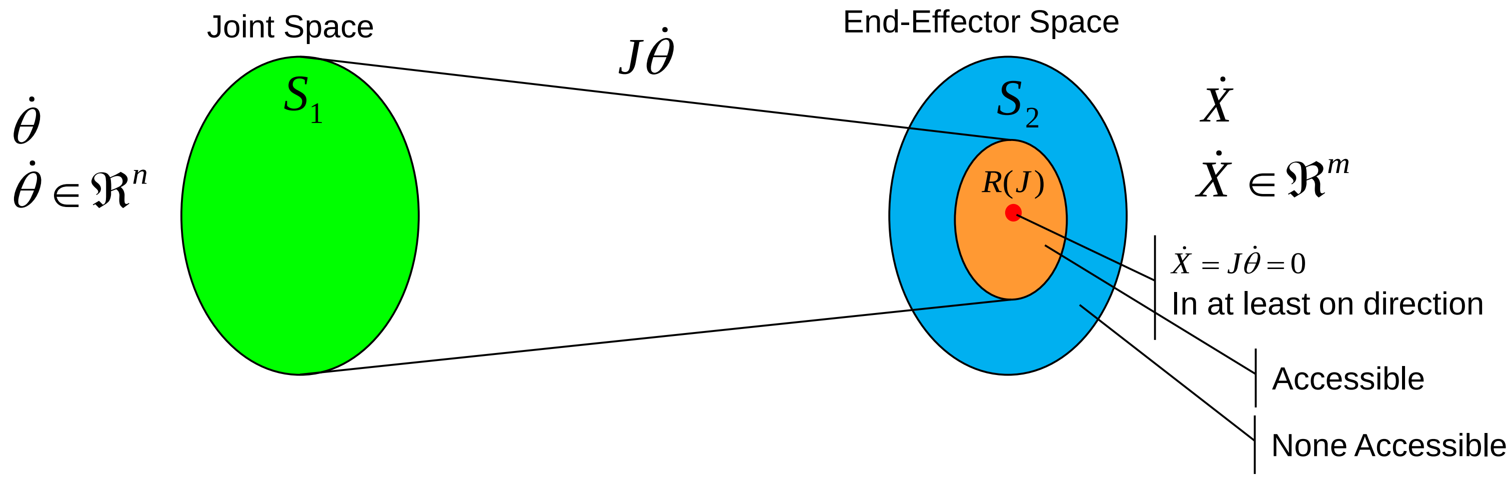
$\dot{X} = J\dot{\theta}$ is a linear mapping of the joint space velocities $\dot{\theta}$ which is a n - dimensional vector space $\dot{\theta} \in \Re^n$ to the end effector velocities $\dot{X}$ which is a m – dimensional vector space $\dot{X} \in \Re^m$

Jacobian – Duality



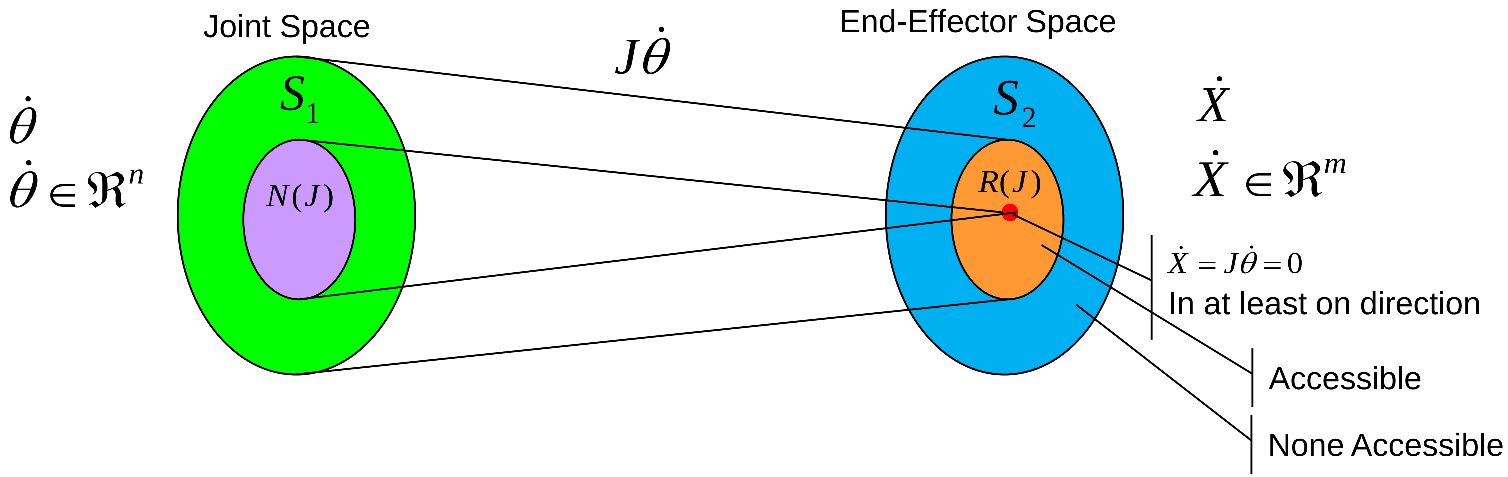
The subset of all the end effector velocities $\boxed{\dot{X}}$ resulting from the mapping $\dot{X} = J\dot{\theta}$ represents all the possible end effector velocities that can be generated by the n joints given the arm configuration

Jacobian – Duality



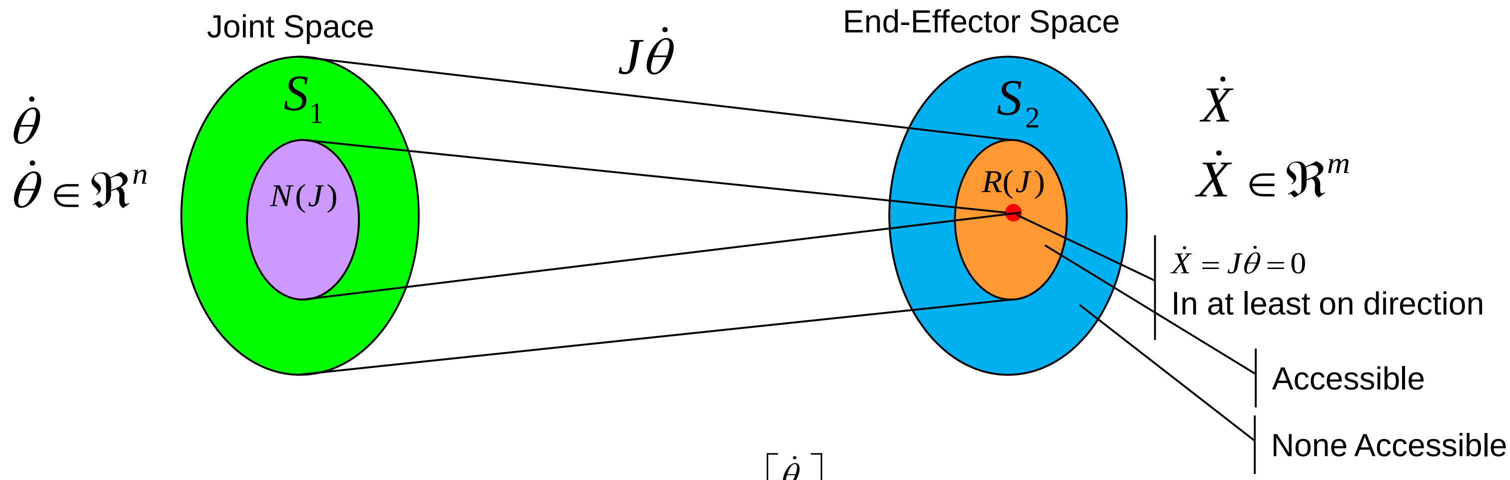
If the rank of the Jacobian matrix J is at full of row rank (square matrix) the joint space $\boxed{\dot{\theta}}$ covers the entire end effector vector $\boxed{\dot{X}}$ otherwise there is at least one direction in which the end effector can not be moved ●

Jacobian – Duality



The subset $\boxed{N(J)}$ is the null space of the linear mapping. Any element in this subspace is mapped into a zero vector in $\Re^m$ such that $\dot{X} = J\dot{\theta} = 0$ therefore any joint velocity vector $\dot{\theta}$ that belongs to the null space does not produce any velocity at the end effector

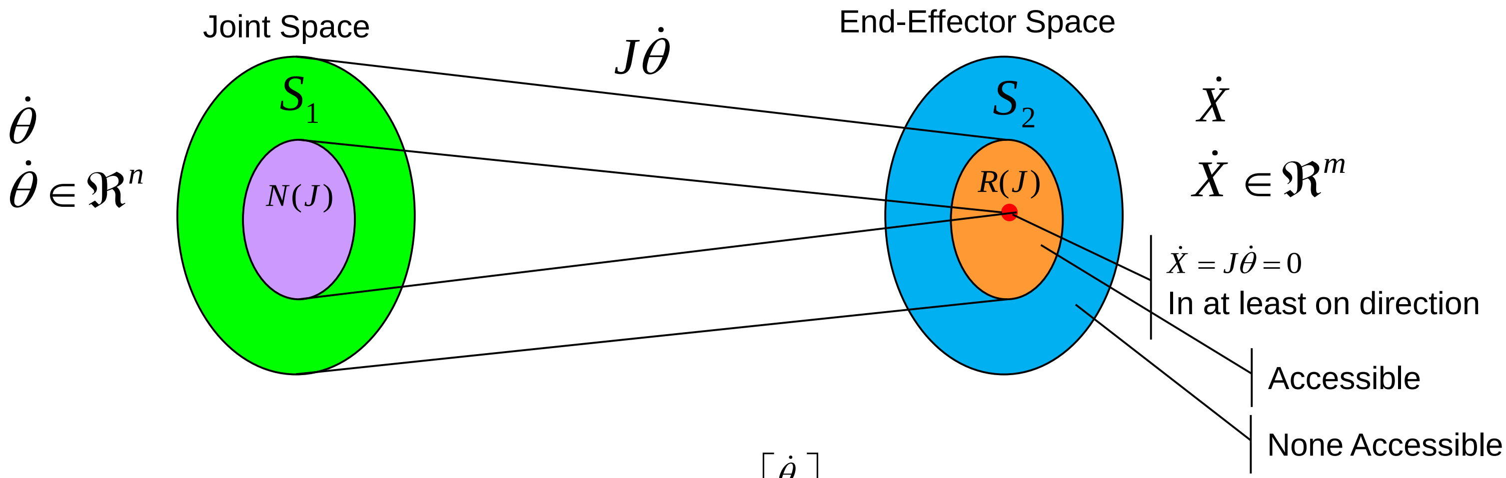
Jacobian – Duality



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & & \\ & 6X7 & \\ & J(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \\ \dot{\theta}_7 \end{bmatrix}$$

If the Jacobian of a manipulator is full rank the dimension of the null space $\boxed{\dim(N(J))}$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the hand may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

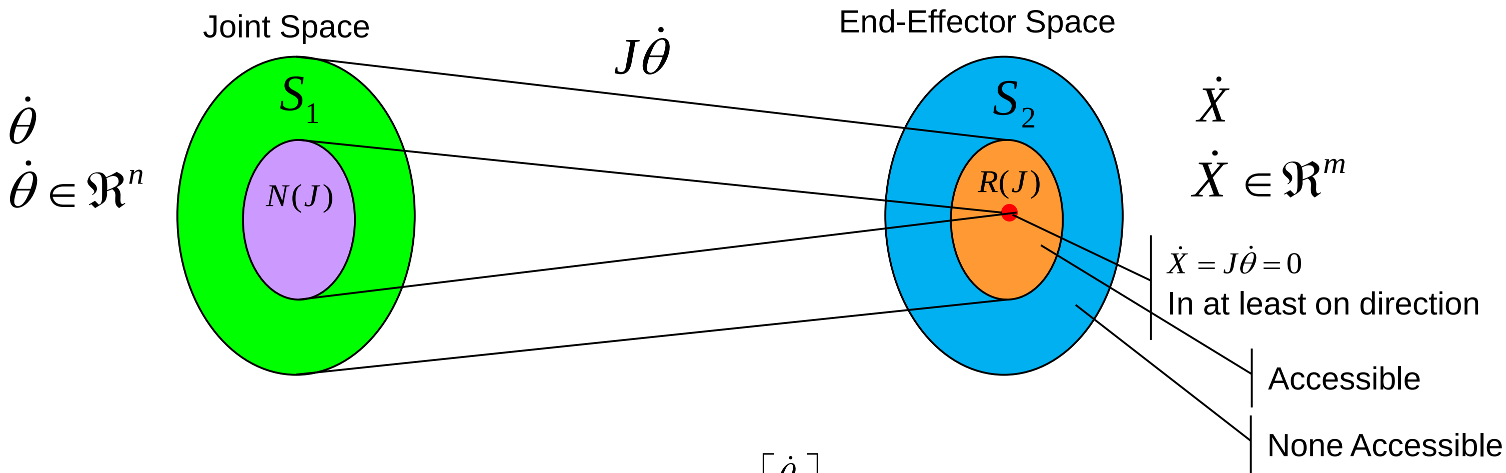
Jacobian – Duality



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & & \\ & 6X7 & \\ & J(\theta) & \\ & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \\ \dot{\theta}_7 \end{bmatrix}$$

If the Jacobian of a manipulator is full rank (i.e. n>m full row rank where the rows are linearly independent) the dimension of the null space $\boxed{\dim(N(J))}$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the hand may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

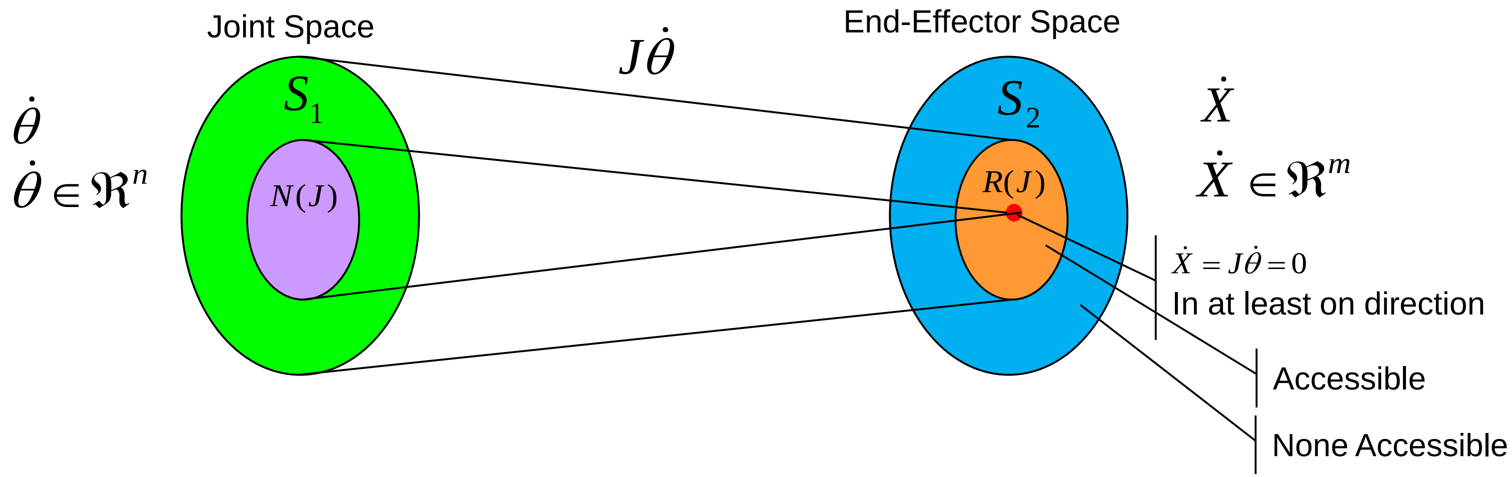
Jacobian – Duality



$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} & & \\ & 6X7 & \\ & J(\theta) & \\ & & \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \\ \dot{\theta}_7 \end{bmatrix}$$

If the Jacobian of a manipulator is full rank (i.e. for redundant manipulator n>m full row rank where the rows are linearly independent) the dimension of the null space $\boxed{\dim(N(J))}$ is the same as the redundant degrees of freedom (n-m). For example the human arm has 7 DOF whereas the end effector (hand) may have 6 linear and angular velocities therefore the null dimension is one (n-m=7-1=1)

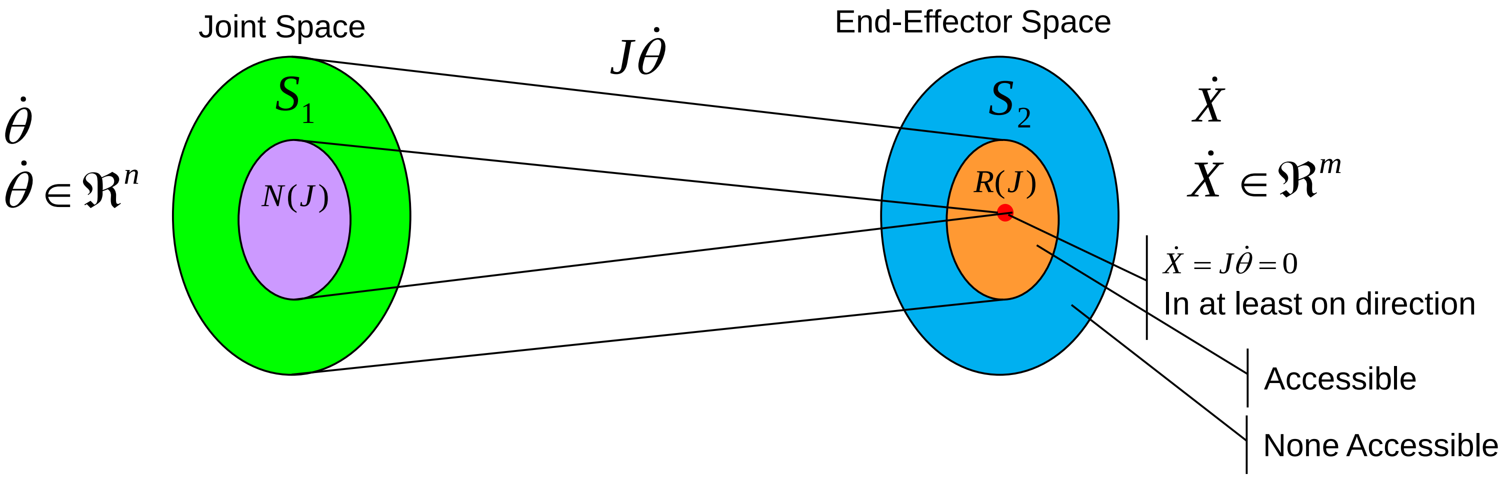
Jacobian – Duality



When the Jacobian matrix degenerates (i.e. not full rank e.g. due to singularity) the dimension of the range space $\boxed{\dim(R(J))}$ decreases at the same time as the dimension of the null space increases $\boxed{\dim(N(J))}$ by the same amount. The sum of the two is always equal to n

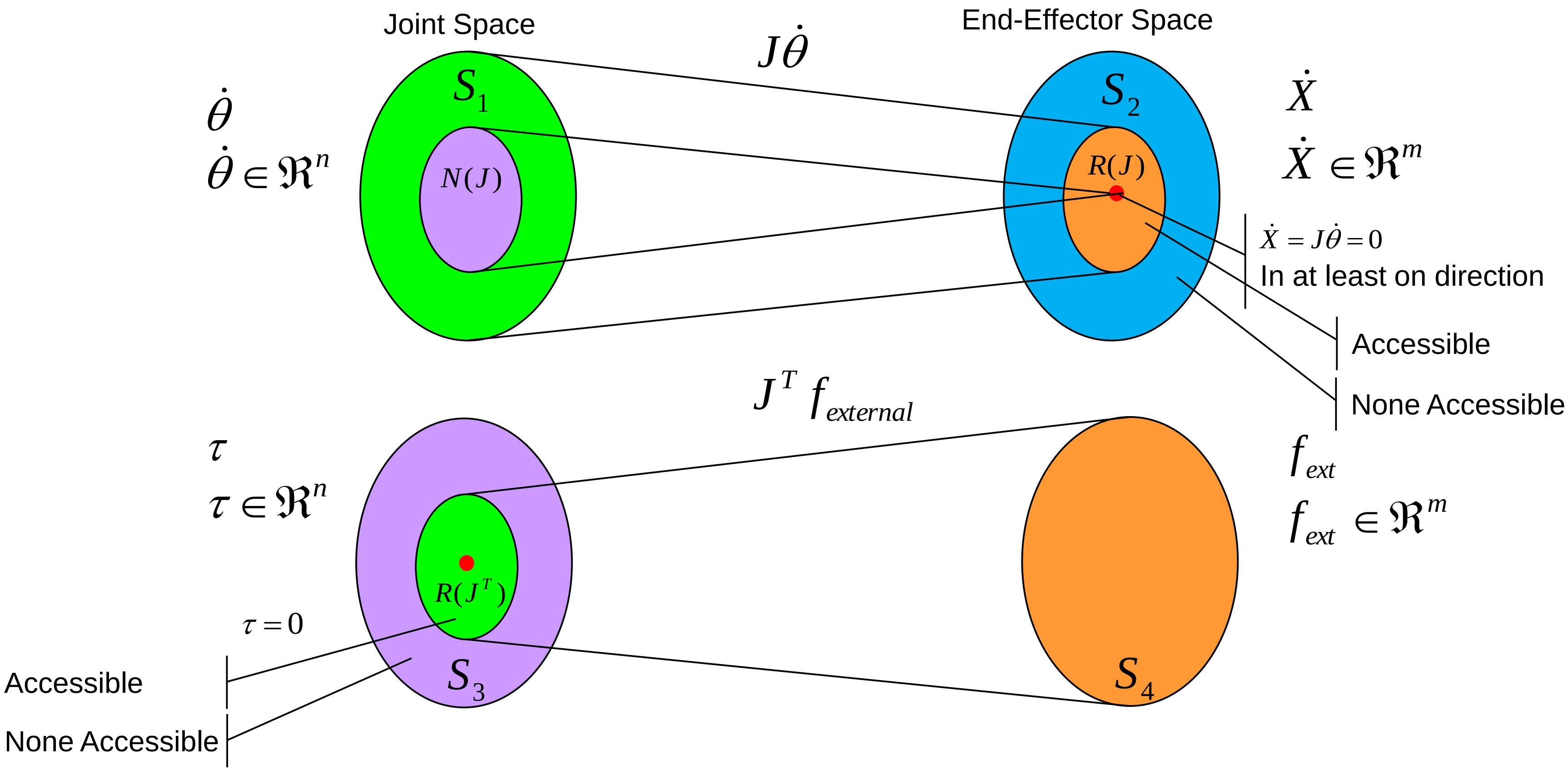
$$\dim(R(J)) + \dim(N(J)) = n$$

Jacobian – Duality



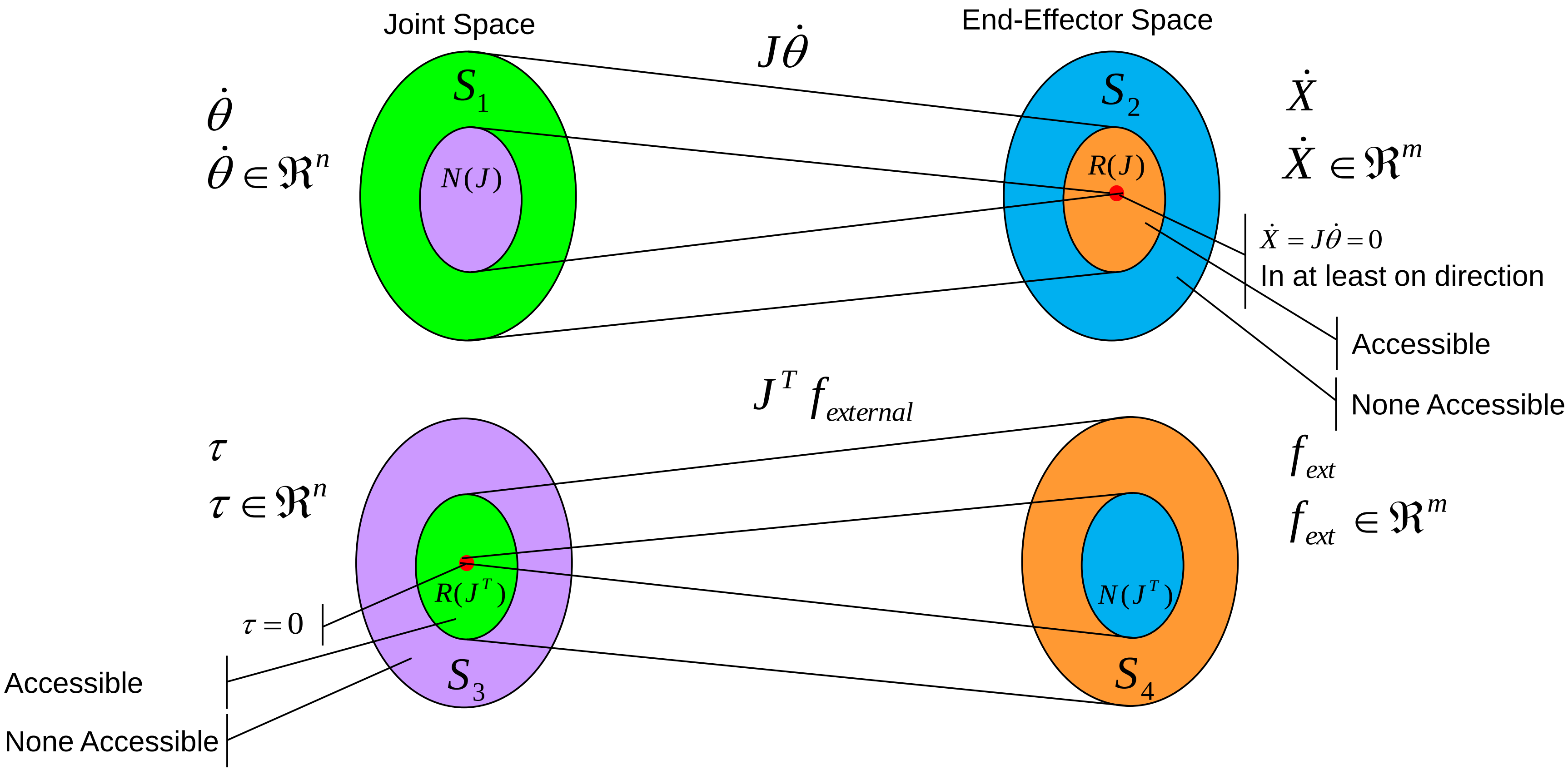
If the null space is not empty set, the instantaneous kinematic equation has an infinite number of solutions that cause the same end effector velocities (recall the 3 axis end effector)

Jacobian – Duality



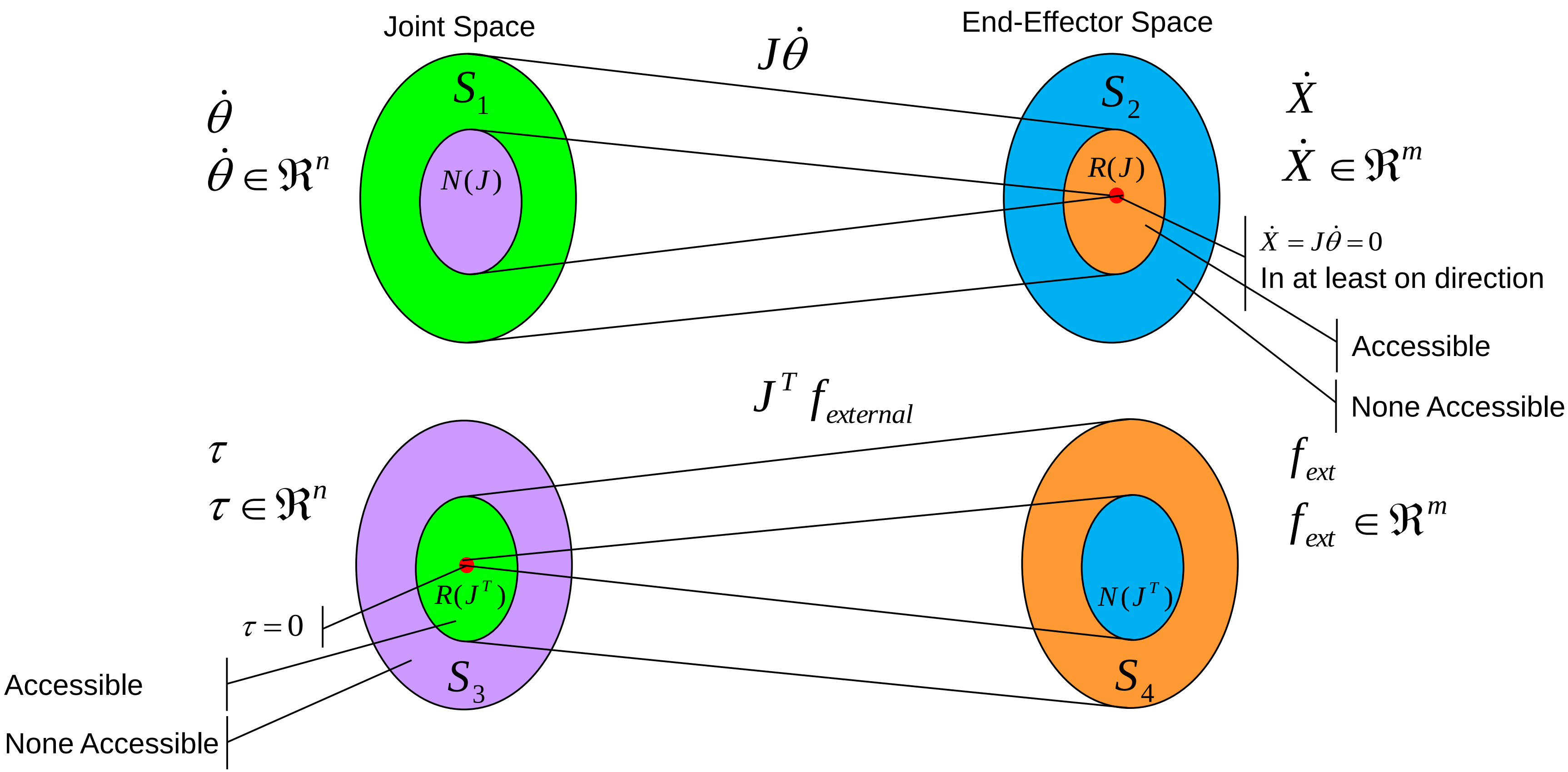
Unlike the mapping of the instantaneous kinematics the mapping of the static external forces is from the m-th vector space $\boxed{f_{ext} \in \Re^m}$ associated with the end effector coordinates to the n-th dimensional vector space $\boxed{\tau \in \Re^n}$ associated with the torques at the joint space. Therefore the joint torque are always determined uniquely from any end effector point force

Jacobian – Duality



The null space $N(J^T)$ represents the set of all end point forces that do not require any torques at the joints to bear the corresponding load ● (e.g. 2R fully stretched or collapsed elbow).

Jacobian – Duality



When the Jacobian matrix is degenerated or the arm is in a singular configuration external endpoint force is borne entirely by the structure and not by the joint torque.