Linear and Angular Velocities 2/4

Jacobian Matrix - Calculation Methods

Explicit Derivation

Differentiation the Forward Kinematics Eqs.

Iterative Propagation (Velocities or Forces / Torques)

Velocities Propagation

Forces / Torques Propagation

Jacobian Matrix

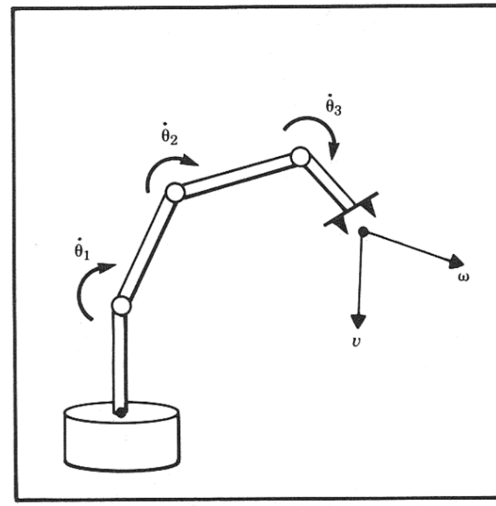
Jacobian Matrix - Introduction

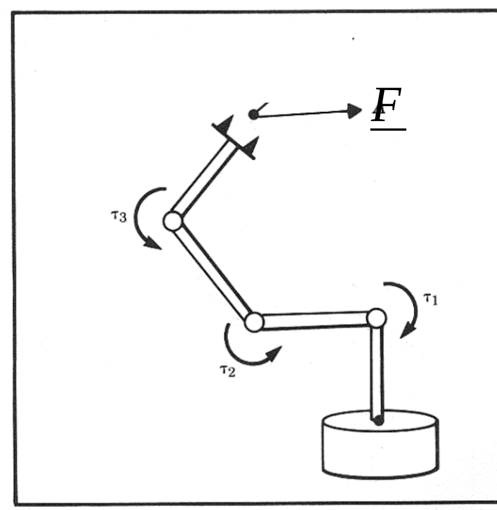
- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\dot{\underline{\theta}}_N$) and the translation and rotation velocities of the end effector ($\dot{\underline{x}}$). This relationship is given by:

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}}$$

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

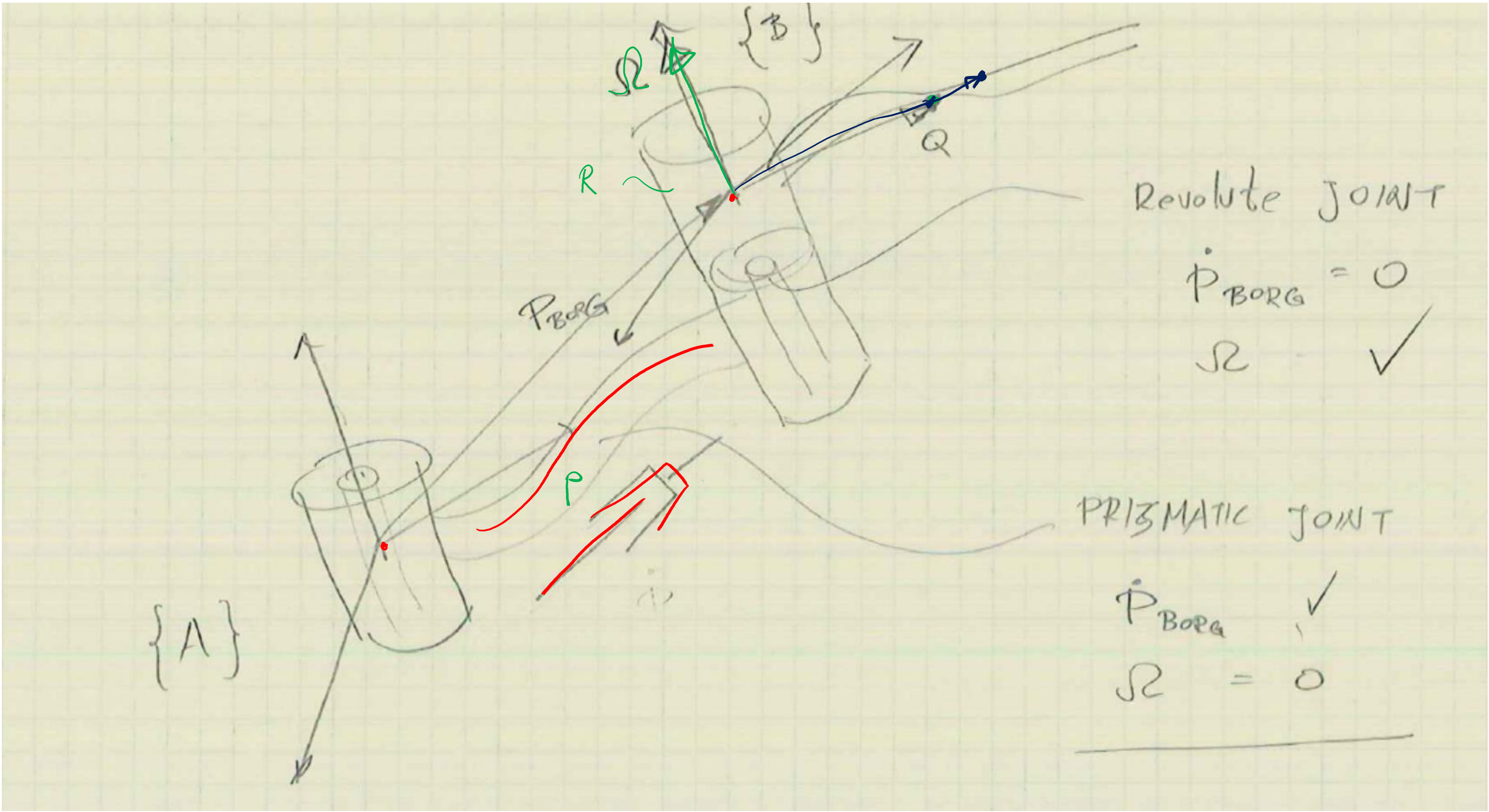
$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$





UCLA

Velocity Propagation – Link / Joint Abstraction

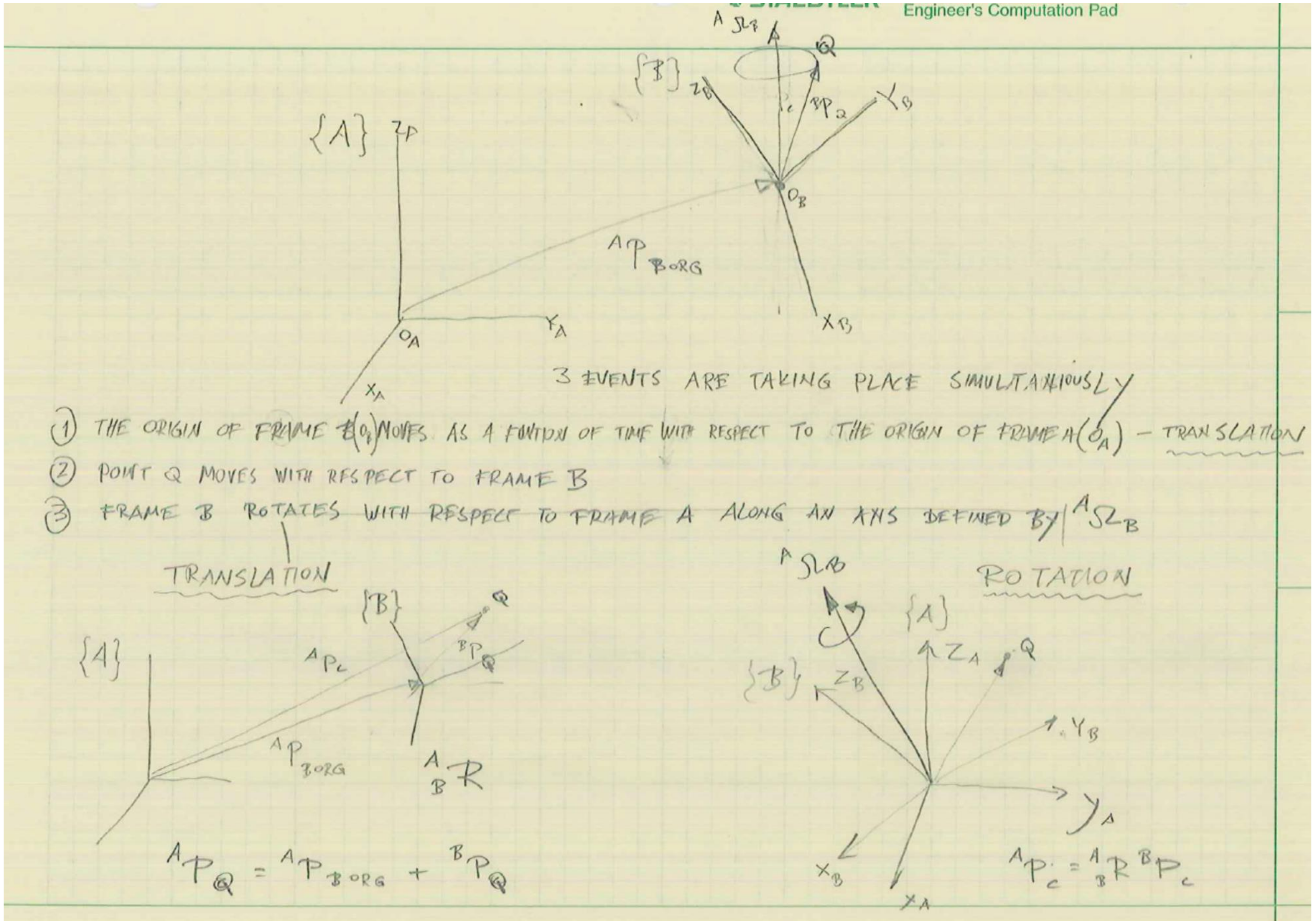


Velocity Propagation – Intuitive Explanation

- Show a demo with the stick like frames

- Three Actions
  - The origin of frame B moves as a function of time with respect to the origin of frame A
  - Point Q moves with respect to frame B
  - Frame B rotates with respect to frame A along an axis defined by $^{A}\Omega_{B}$

Velocity Propagation – Intuitive Explanation



Central Topic - Simultaneous Linear and Rotational Velocity

$$^{A}V_Q = f\left(^{B}P_Q,\ \ ^{B}V_Q,\ \ ^{A}V_{BORG},\ \ ^{A}\Omega_B,\ \ ^{A}_{B}R\right)$$

- Vector Form (Method No. 1)

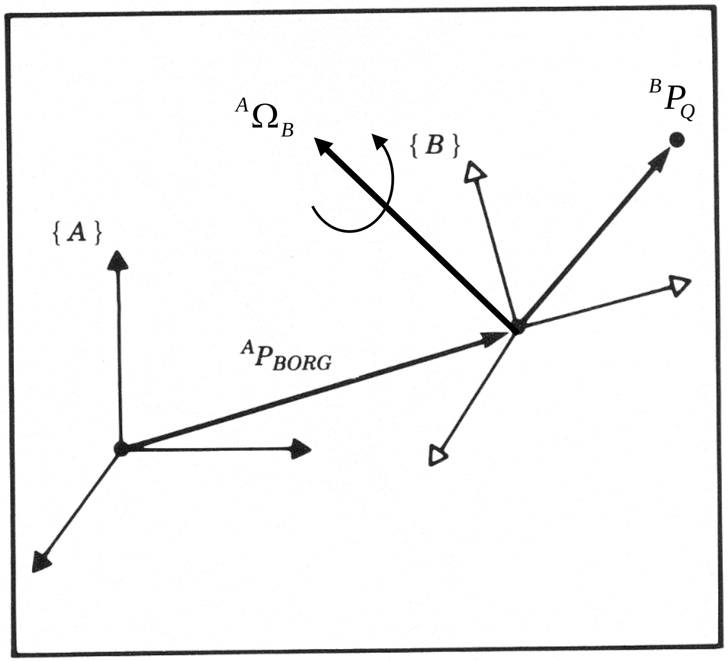
$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

- Matrix Form (Method No. 2)

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

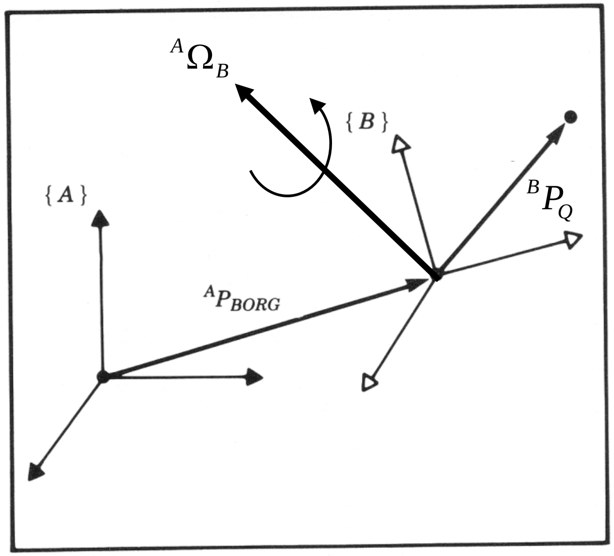
- Matrix Formulation – Homogeneous Transformation Form – Method No. 3

$$\begin{bmatrix} \left[{}^{A}V_Q\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega} \cdot {}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_Q\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\,org}\right] \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_Q\right] \\ 0 \end{bmatrix}$$



Central Topic - Changing Frame of Representation – Angular Velocity

- Angular Velocity Representation in Various Frames
  - Vector Form $$^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C}$$
  - Matrix Form $$^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T}$$



Velocity – Derivation Method No. 1 & 2

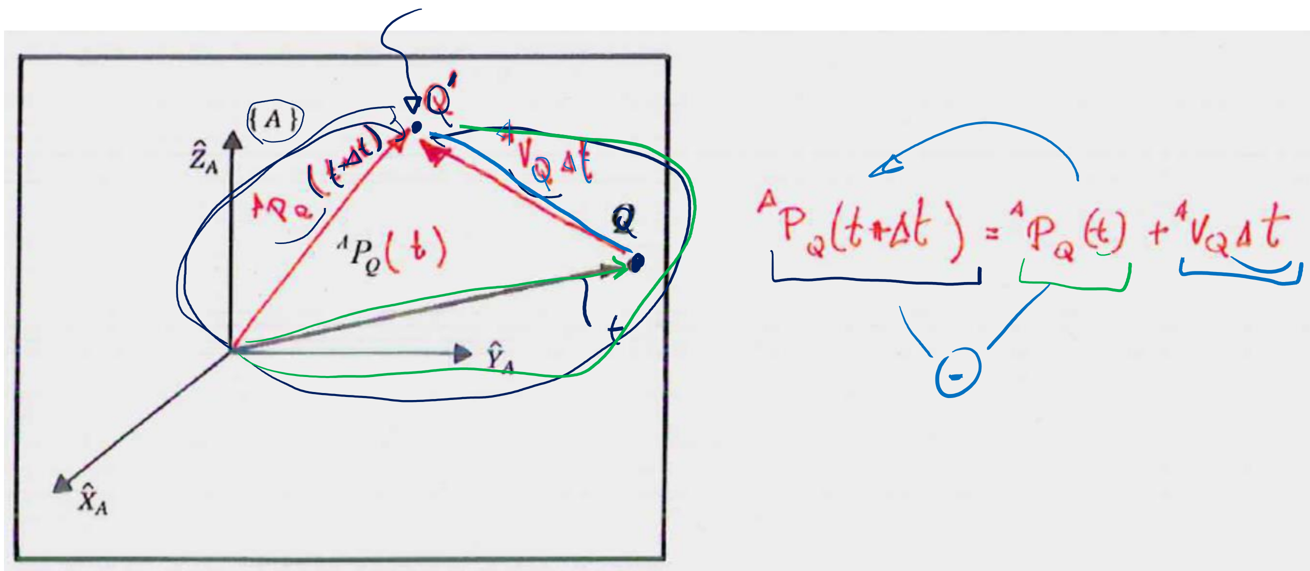
Vector Form

Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Linear Velocity

- ***Linear velocity*** - The instantaneous rate of change in linear position of a point relative to some frame.



$$ {}^{A}V_{Q} = \frac{d}{dt}{}^{A}P_{Q} \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_{Q}(t+\Delta t) - {}^{A}P_{Q}(t)}{\Delta t} \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_{Q}(t) - {}^{A}P_{Q}(t+\Delta t)}{\Delta t} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

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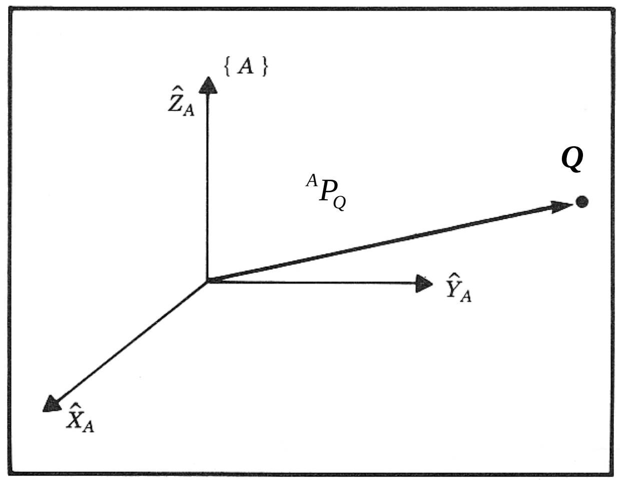


$$ {}^{A}V_{Q} = \frac{d}{dt}\,{}^{A}P_{Q} \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_{Q}(t+\Delta t) - {}^{A}P_{Q}(t)}{\Delta t} \approx \lim_{\Delta t \to 0} \frac{{}^{A}P_{Q}(t) - {}^{A}P_{Q}(t-\Delta t)}{\Delta t} $$

$$ {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A\Omega_B \times {}^A_BR\,{}^BP_Q \qquad {}^AV_Q = {}^AV_{BORG} + {}^A_BR\,{}^BV_Q + {}^A_B\dot{R}_\Omega\left({}^A_BR\,{}^BP_Q\right) $$

Definitions - Linear Velocity

- The position of point Q in frame {A} is represented by the ***linear position vector***

$$ {}^AP_Q = \begin{bmatrix} {}^AP_{Qx} \\ {}^AP_{Qy} \\ {}^AP_{Qz} \end{bmatrix} $$

- The velocity of a point Q relative to frame {A} is represented by the ***linear velocity vector***

$$ {}^AV_Q = \frac{{}^Ad}{dt} \begin{bmatrix} {}^AP_{Qx} \\ {}^AP_{Qy} \\ {}^AP_{Qz} \end{bmatrix} = \begin{bmatrix} {}^A\dot{P}_{Qx} \\ {}^A\dot{P}_{Qy} \\ {}^A\dot{P}_{Qz} \end{bmatrix} $$

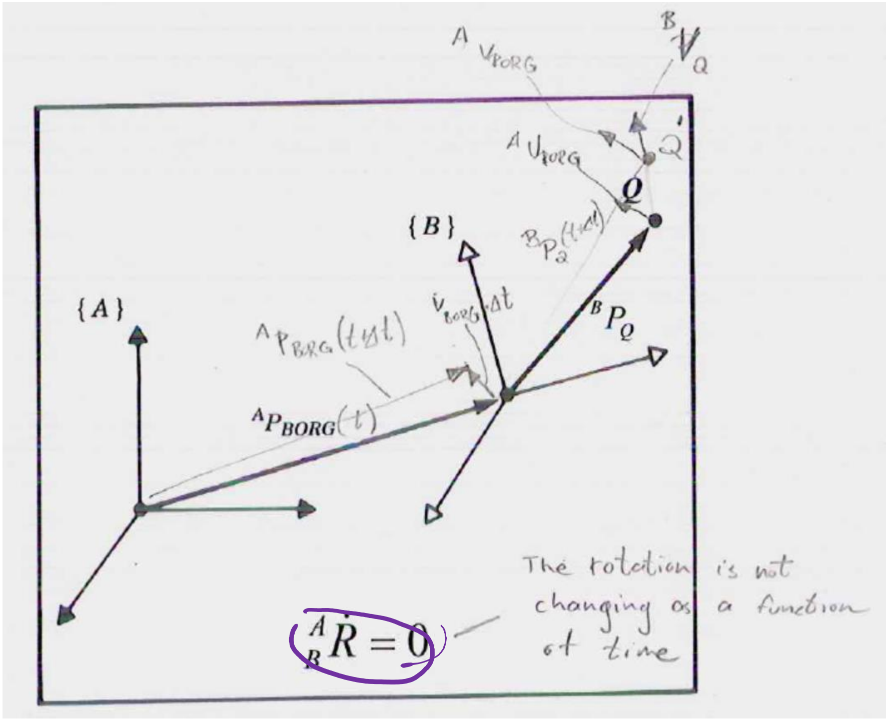
$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Linear Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The orientation of frame {A} with respect to frame {B} is not changing as a function of time $^{A}_{B}\dot{R} = 0$

- ***Problem:*** describe the motion of of the vector $^{B}P_{Q}$ relative to frame {A}

- ***Solution:*** Frame {B} is located relative to frame {A} by a position vector $^{A}P_{BORG}$ and the rotation matrix $^{A}_{B}R$ (assume that the orientation is not changing in time $^{A}_{B}\dot{R} = 0$) expressing both components of the velocity in terms of frame {A} gives



$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}({}^{B}V_{Q}) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

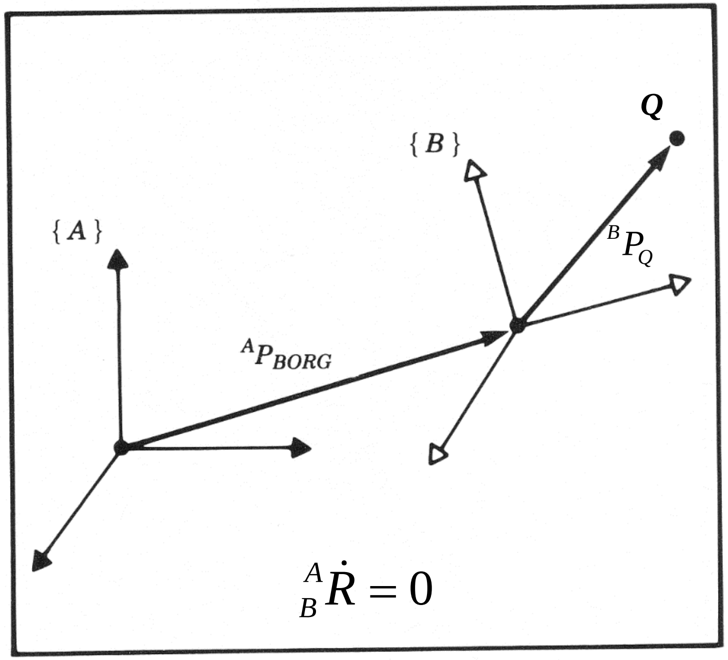
$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Linear Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The orientation of frame {A} with respect to frame {B} is not changing as a function of time ${}^{A}_{B}\dot{R} = 0$

- ***Problem:*** describe the motion of of the vector ${}^{B}P_{Q}$ relative to frame {A}

- ***Solution:*** Frame {B} is located relative to frame {A} by a position vector ${}^{A}P_{BORG}$ and the rotation matrix ${}^{A}_{B}R$ (assume that the orientation is not changing in time ${}^{A}_{B}\dot{R} = 0$) expressing both components of the velocity in terms of frame {A} gives

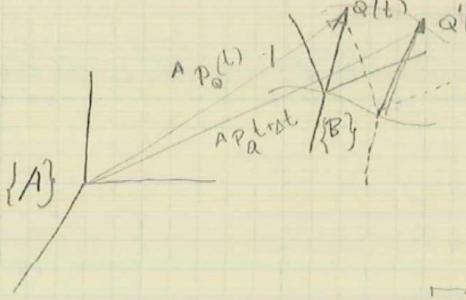


$${}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}({}^{B}V_{Q}) = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

Linear Velocity – Translation (No Rotation)

① (I) POINT Q IS FIXED IN B

(II) B TRANSLATE WITH RESPECT TO A



SAME VECTOR

$= 0$

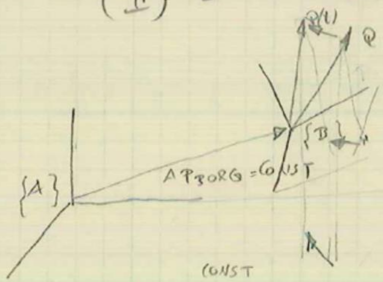
$$ {}^B\frac{d}{dt}\left({}^BP_Q\right) = {}^B\left(\frac{{}^BP_Q(t+\Delta t) - P_Q(t)}{\Delta t}\right) = {}^B\left({}^Bv_Q\right) = 0 $$

$$ {}^A\frac{d}{dt}\left({}^AP_Q\right) = {}^A\left(\frac{{}^AP_Q(t+\Delta t) - {}^AP(t)}{\Delta t}\right) = {}^A\left({}^Av_Q\right) = {}^Av_Q = {}^Av_{BORG} $$

ALL THE POINTS IN FRAME {B} TRASLATE WITH THE SAME VEVOCITY WITH RESPECT TO A

② (I) POINT Q MOVES WITH RESPECT TO B

(II) B IS FIXED WITH RESPECT TO A



CONST

SAME VECTOR

$= 0$

$$ {}^A\frac{d}{dt}\left({}^AP_{BORG}\right) = {}^A\left(\frac{{}^AP_{BORG}(t+\Delta) - {}^AP_{BORG}(t)}{dt}\right) = {}^A\left({}^Av_{BORG}\right) = {}^Av_{BORG} = 0 $$

$$ {}^A\frac{d}{dt}\left({}^BP_Q\right) = {}^A\left(\frac{{}^BP_Q(t+\Delta t) - {}^BP_Q(t)}{\Delta t}\right) = {}^A\left({}^Bv_Q\right) $$

$$ {}^Av_Q = {}^A_BR\,{}^Bv $$

$$ {}^Av_Q = {}^Av_{BORG} + {}^A\left({}^Bv_Q\right) = {}^Av_{BORG} + {}^A_BR\,{}^Bv $$

Linear Velocity – Translation

1

$$^{A}P_{Q} = {}^{A}P_{BORG} + {}^{B}P_{Q}$$

DIFF WITH RESPECT TO COORDINATE SYSTEM A

$$\frac{^{A}d}{dt}\left(^{A}P_{Q}\right) = \frac{^{A}d}{dt}\left(^{A}P_{BORG}\right) + \frac{^{A}d}{dt}\left(^{B}P_{Q}\right)$$

$$^{A}\left(^{A}\dot{P}_{Q}\right) = {}^{A}\left(^{A}\dot{P}_{BORG}\right) + {}^{A}\left(^{B}\dot{P}_{Q}\right)$$

$$^{A}\left(^{A}V_{Q}\right) = {}^{A}\left(^{A}V_{BORG}\right) + {}^{A}\left(^{B}V_{Q}\right)$$

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q}$$

MATRIX FORM

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}\left(^{B}V_{Q}\right)$$

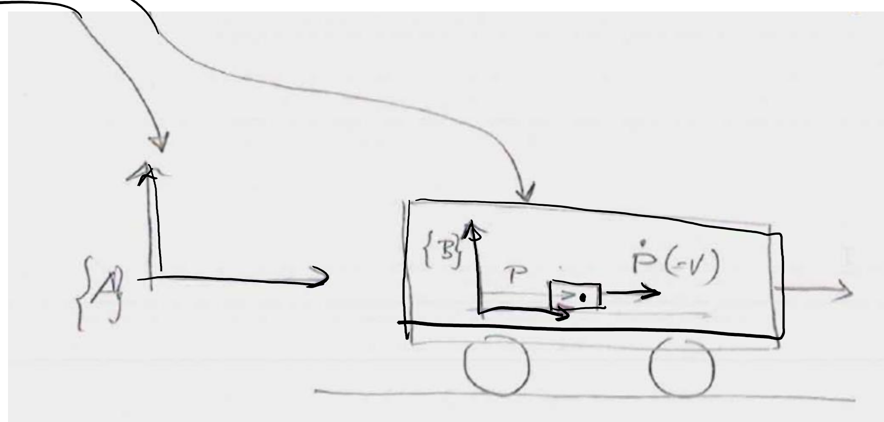
VECTOR FORM

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Linear & Angular Velocities - Frames

- When describing the velocity (linear or angular) of an object, there are two important frames that are being used:
  - ***Represented Frame (Reference Frame)*** : e.g. [A]
    This is the frame used to **represent (express)** the object's velocity.
  - ***Computed Frame***: e.g. [B]
    This is the frame in which the velocity is **measured** (differentiate the position).



$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Frame - Velocity

- As with any vector, a velocity vector may be described in terms of any frame, and this frame of reference is noted with a leading superscript.

- A velocity vector **computed** in frame {B} and **represented** in frame {A} would be written

Represented
(Reference Frame) -– Projected on

$$ {}^{A}\left({}^{B}V_Q\right) = \frac{{}^{A}d}{dt}\,{}^{B}P_Q $$

Computed
(Measured) - Differentiate with respect to

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frame - Linear Velocity

- We can always remove the outer, leading superscript by explicitly including the rotation matrix which accomplishes the change in the reference frame

$$ {}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} $$

- Note that in the general case $ {}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} \neq {}^{A}V_{Q} $ because ${}^{A}_{B}R$ may be time-verging ${}^{A}_{B}\dot{R} \neq 0$

- If the calculated velocity is written in terms of of the frame of differentiation the result could be indicated by a single leading superscript.

$$ {}^{A}({}^{A}V_{Q}) = {}^{A}V_{Q} $$

- In a similar fashion when the angular velocity is expresses and measured as a vector

$$ {}^{A}({}^{B}\Omega_{C}) = {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

Frame - Linear Velocity

Note: If frame B doesn't move with respect to frame A $\underset{\sim}{\text{and}}$ frame B is not rotated with respect to frame A $({}^{B}_{A}R = I)$ then ${}^{A}V_Q = {}^{B}V_Q$

The velocity is a $\underset{\sim}{\text{free}}$ $\underset{\sim}{\text{vector}}$

■ Projected on frame A

■ Projected on frame A

Represented (Reference Frame) – Projected on

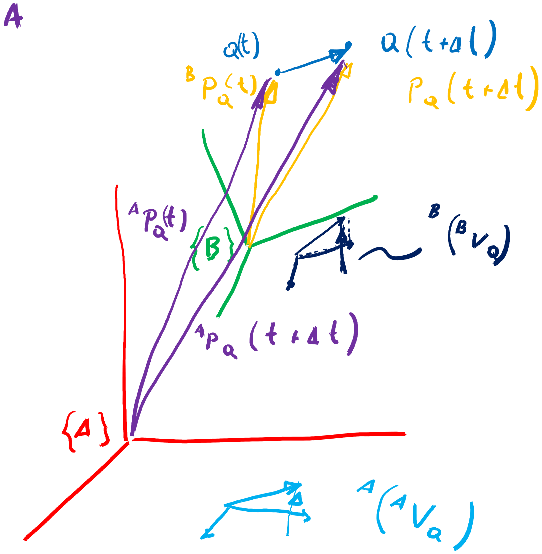
■ - computed with respect to frame B

■ - computed with respect to frame A

Computed (Measured / Differentiate with respect to)

$${}^{\gamma}[{}^{\beta}V_Q]$$

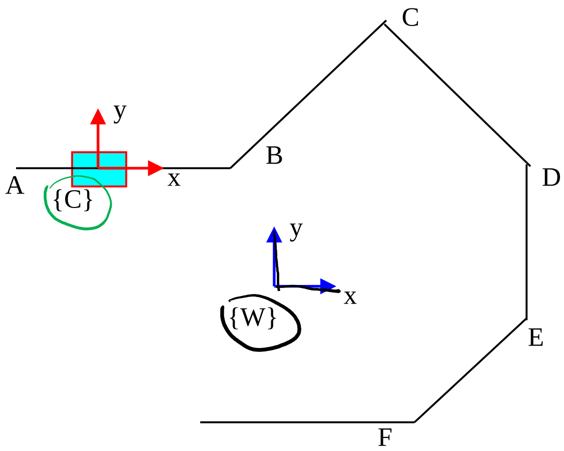
Object Frame

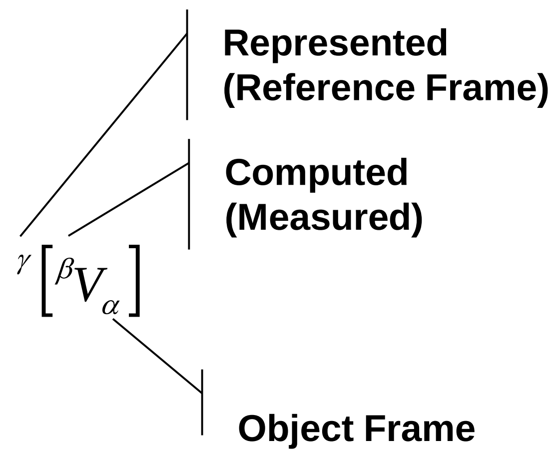


$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

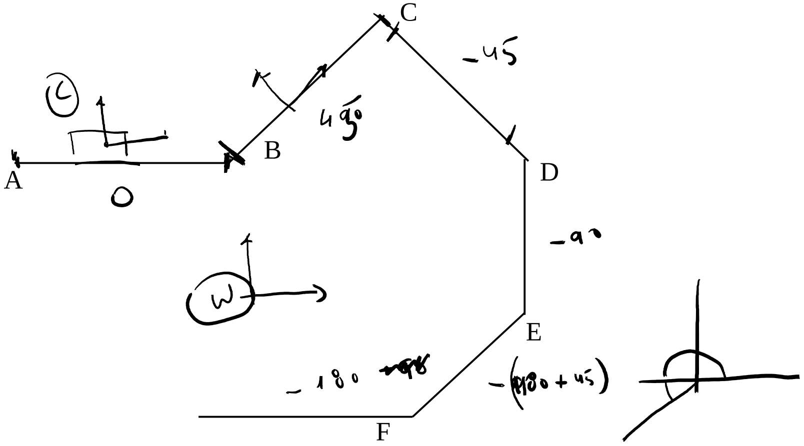
Frames - Linear Velocity - Example

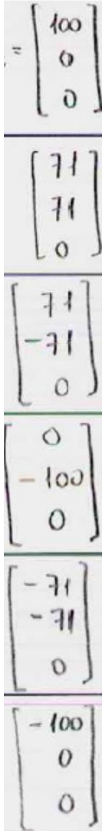
- ***Given:*** The driver of the car maintains a ***speed*** of 100 km/h (as shown to the driver by the car's speedometer).
- ***Problem:*** Express the velocities ${}^{C}\left[{}^{C}V_{C}\right]$ ${}^{W}\left[{}^{W}V_{C}\right]$ ${}^{W}\left[{}^{C}V_{C}\right]$ ${}^{C}\left[{}^{W}V_{C}\right]$ in each section of the road A, B, C, D, E, F where {C} - Car frame, and {W} - World frame





Frames - Linear Velocity - Example





Frames - Linear Velocity - Example

$$ {}^{A}_{B}R = Rot(\hat{z},\theta) = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ Rot(\hat{z},+45^o) = \begin{bmatrix} 0.707 & -0.707 & 0.000 \\ 0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},-45^o) = \begin{bmatrix} 0.707 & 0.707 & 0.000 \\ -0.707 & 0.707 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix} $$

$$ Rot(\hat{z},+90^o) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Rot(\hat{z},-90^o) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Frames - Linear Velocity - Example

COMPUTED (MEASURED)

$$ {}^{A}({}^{B}V_Q) = {}^{A}_{B}R\,{}^{B}V_Q $$

REPRESENTED (Ref FRAME)

- $ {}^{A}_{B}\dot{R} = 0 $ is not time-varying (in this example)

$$ {}^{C}({}^{C}V_C) = {}^{C}_{C}R\,{}^{C}V_C = I[0] = [0] $$

$$ {}^{W}({}^{W}V_C) = {}^{W}_{W}R\,{}^{W}V_C = I\,{}^{W}V_C $$

$$ {}^{W}({}^{C}V_C) = {}^{W}_{C}R\,{}^{C}V_C = {}^{W}_{C}R[0] = [0] $$

$$ {}^{C}({}^{W}V_C) = {}^{C}_{W}R\,{}^{W}V_C $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + \boxed{{}^{A}_{B}R\,{}^{B}V_{Q}} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Frames - Linear Velocity - Example

$$ {}^{A}({}^{B}V_{Q}) = {}^{A}_{B}R\,{}^{B}V_{Q} $$

- $ {}^{A}_{B}\dot{R} = 0 $ is not time-varying (in this example)

$$ {}^{C}({}^{C}V_{C}) = {}^{C}_{C}R\,{}^{C}V_{C} = I[0] = [0] $$

$$ {}^{W}({}^{W}V_{C}) = {}^{W}_{W}R\,{}^{W}V_{C} = I\,{}^{W}V_{C} $$

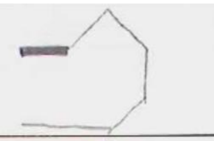
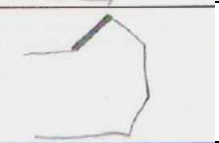
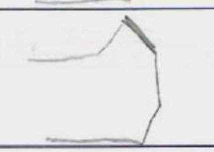
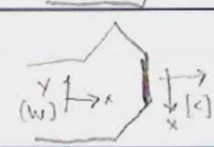
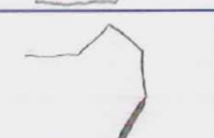
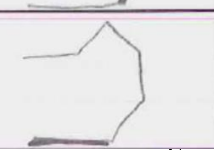
$$ {}^{W}({}^{C}V_{C}) = {}^{W}_{C}R\,{}^{C}V_{C} = {}^{W}_{C}R[0] = [0] $$

$$ {}^{C}({}^{W}V_{C}) = {}^{C}_{W}R\,{}^{W}V_{C} $$

W - WORLD
C - CAR

Reference frame
Represented / Described / expressed $\square(\square V_{\square})$ — computed / measured (or differentiation) — Object

$$^A(^BV_Q) = {}^A_BR\,^BV_Q$$

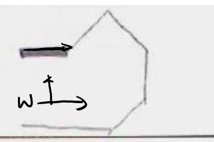
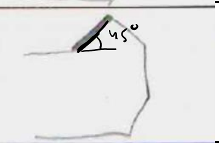
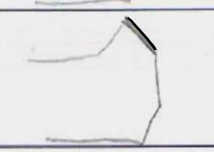
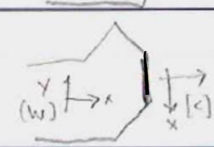
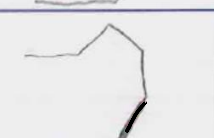
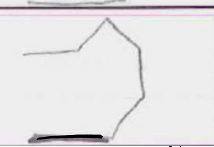
Road Section	Velocity			
	${}^C[{}^CV_C]$	${}^W[{}^WV_C]$	${}^W[{}^CV_C]$	${}^C[{}^WV_C]$
A 	${}^C_CR\,{}^CV_c = I\,{}^CV_c = \begin{bmatrix}0\\0\\0\end{bmatrix}$			
B 				
C 				
D 				
E 				
F 				

W - WORLD

C - CAR

Reference frame | Represented / Described / expressed $\square(\square V_\square)$ — computed / measured (or differentiation); $\square$ — Object

$${}^A({}^B V_Q) = {}^A_B R\, {}^B V_Q$$

Road Section	Velocity			
	${}^C[{}^C V_C]$	${}^W[{}^W V_C]$	${}^W[{}^C V_C]$	${}^C[{}^W V_C]$
A 	${}^C_C R\, {}^C V_c = I\, {}^C V_c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	${}^W_W R\, {}^W V_c = I\, {}^W V_c = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$		
B 	↓	$\begin{bmatrix} 71 \\ 71 \\ 0 \end{bmatrix}$		
C 	↓	$\begin{bmatrix} 71 \\ -71 \\ 0 \end{bmatrix}$		
D 	↓	$\begin{bmatrix} 0 \\ -100 \\ 0 \end{bmatrix}$		
E 	↓	$\begin{bmatrix} -71 \\ -71 \\ 0 \end{bmatrix}$		
F 	↓	$\begin{bmatrix} -100 \\ 0 \\ 0 \end{bmatrix}$		

W – WORLD

C – CAR

Refereance frame | Represented / Described / expressed

computed / measured (or differentiation)

Object

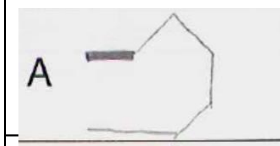
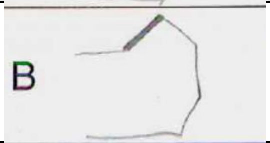
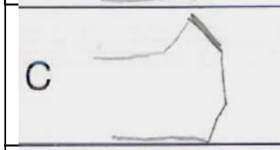
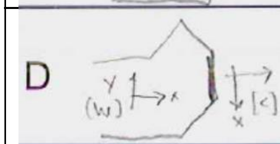
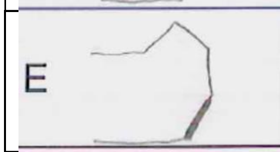
$$^{A}\left(^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

Road Section	Velocity			
	${}^{C}\left[{}^{C}V_{C}\right]$	${}^{W}\left[{}^{W}V_{C}\right]$	${}^{W}\left[{}^{C}V_{C}\right]$	${}^{C}\left[{}^{W}V_{C}\right]$
A	${}^{C}_{C}R\,{}^{C}V_{c} = I\,{}^{C}V_{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_{c} = I\,{}^{W}V_{c} = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	
B	↓	$\begin{bmatrix}71\\71\\0\end{bmatrix}$	↓	
C	↓	$\begin{bmatrix}71\\-71\\0\end{bmatrix}$	↓	
D	↓	$\begin{bmatrix}0\\-100\\0\end{bmatrix}$	↓	
E	↓	$\begin{bmatrix}-71\\-71\\0\end{bmatrix}$	↓	
F	↓	$\begin{bmatrix}-100\\0\\0\end{bmatrix}$	↓	

W – WORLD
C – CAR

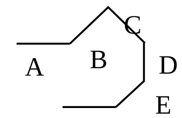
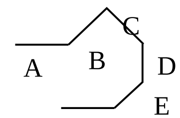
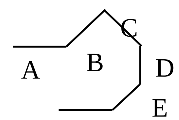
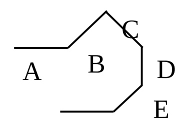
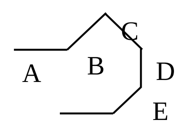
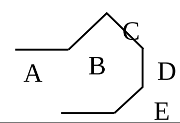
Reference frame | Represented / Described / expressed $\square \left({}^{\square}V_{\square} \right)$ — computed / measured (or differentiation) — Object

$${}^{A}\left({}^{B}V_{Q}\right) = {}^{A}_{B}R\,{}^{B}V_{Q}$$

Road Section	Velocity			
	${}^{C}\left[{}^{C}V_{C}\right]$	${}^{W}\left[{}^{W}V_{C}\right]$	${}^{W}\left[{}^{C}V_{C}\right]$	${}^{C}\left[{}^{W}V_{C}\right]$
A 	${}^{C}_{C}R\,{}^{C}V_{c} = I\,{}^{C}V_{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	${}^{W}_{W}R\,{}^{W}V_{c} = I\,{}^{W}V_{c} = \begin{bmatrix}100\\0\\0\end{bmatrix}$	${}^{W}_{C}R\,{}^{C}V_{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}$	$R(\hat{z},0)\,[I]\begin{bmatrix}100\\0\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
B 	↓	$\begin{bmatrix}71\\71\\0\end{bmatrix}$	↓	$R(\hat{z},-45)\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}.707 & .707 & 0\\-.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
C 		$\begin{bmatrix}71\\-71\\0\end{bmatrix}$		$R(\hat{z},+45)\begin{bmatrix}71\\-71\\0\end{bmatrix} = \begin{bmatrix}.707 & -.707 & 0\\.707 & .707 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}71\\-71\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
D 		$\begin{bmatrix}0\\-100\\0\end{bmatrix}$		$R(\hat{z},+90)\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\-100\\0\end{bmatrix} = \begin{bmatrix}100\\0\\0\end{bmatrix}$
E 		$\begin{bmatrix}-71\\-71\\0\end{bmatrix}$		↓
F 	↓	$\begin{bmatrix}-100\\0\\0\end{bmatrix}$	↓	↓

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

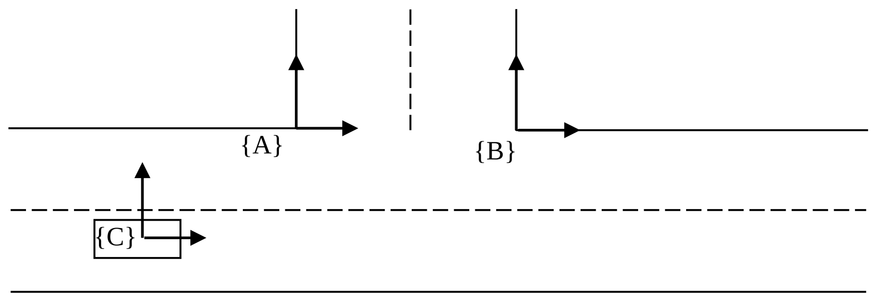
Frames - Linear Velocity - Example

Road Section	Velocity			
	${}^{C}\left[{}^{C}V_C\right]$	${}^{W}\left[{}^{W}V_C\right]$	${}^{W}\left[{}^{C}V_C\right]$	${}^{C}\left[{}^{W}V_C\right]$
A 				
B 				
C 				
D 				
E 				
F 				

$${}^A V_Q = {}^A V_{BORG} + {}^A_B R \, \boxed{{}^B V_Q} + {}^A\Omega_B \times {}^A_B R \, {}^B P_Q \qquad {}^A V_Q = {}^A V_{BORG} + {}^A_B R \, \boxed{{}^B V_Q} + {}^A_B \dot{R}_\Omega \left({}^A_B R \, {}^B P_Q \right)$$

Linear Velocity - Free Vector

- Linear velocity vectors are insensitive to shifts in origin.
- Consider the following example:



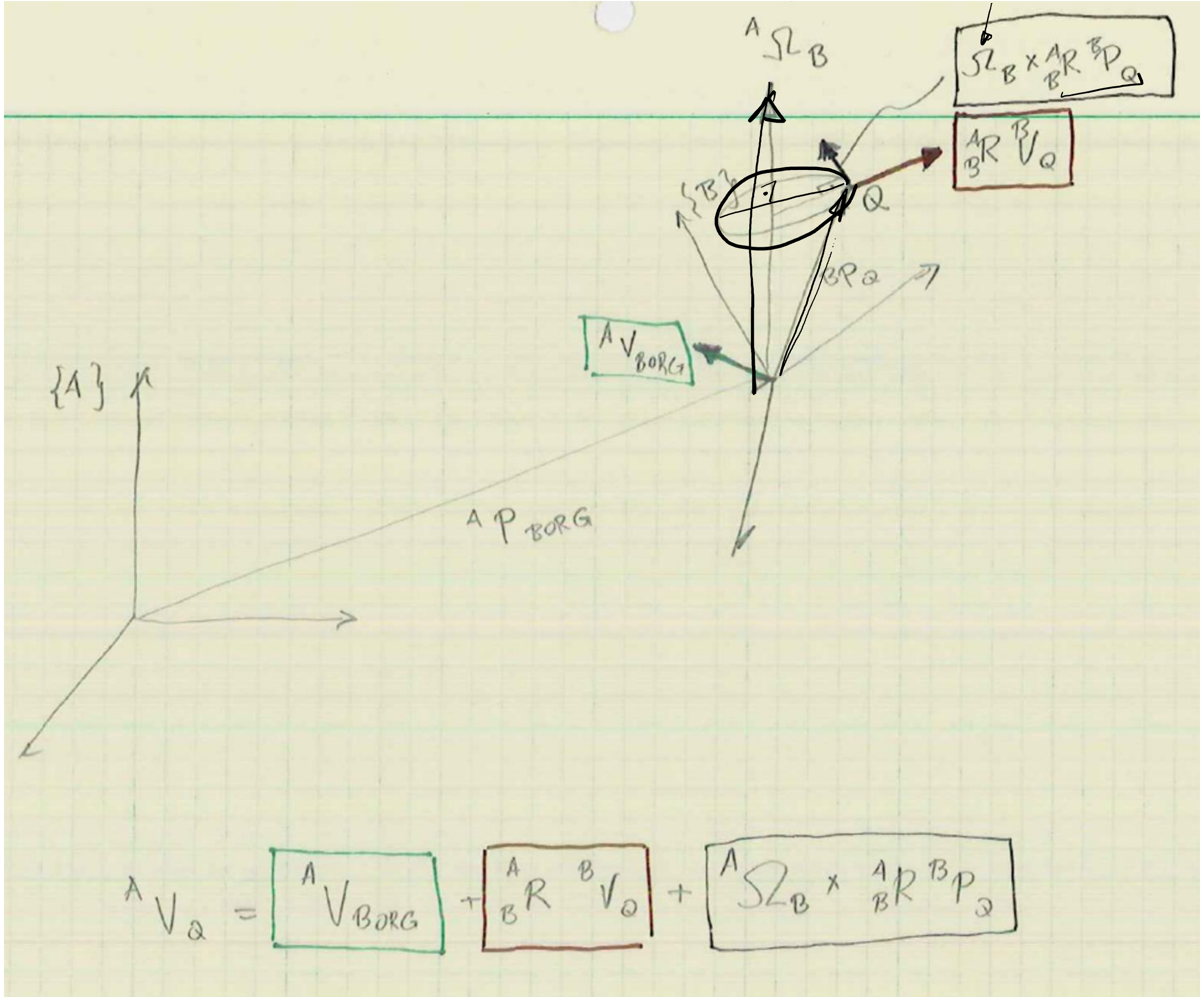
- The velocity of the object in {C} relative to both {A} and {B} is the same, that is

$${}^A V_C = {}^B V_C$$

- As long as {A} and {B} remain fixed relative to each other (translational but not rotational), then the velocity vector remains unchanged (that is, a ***free vector***).

$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach



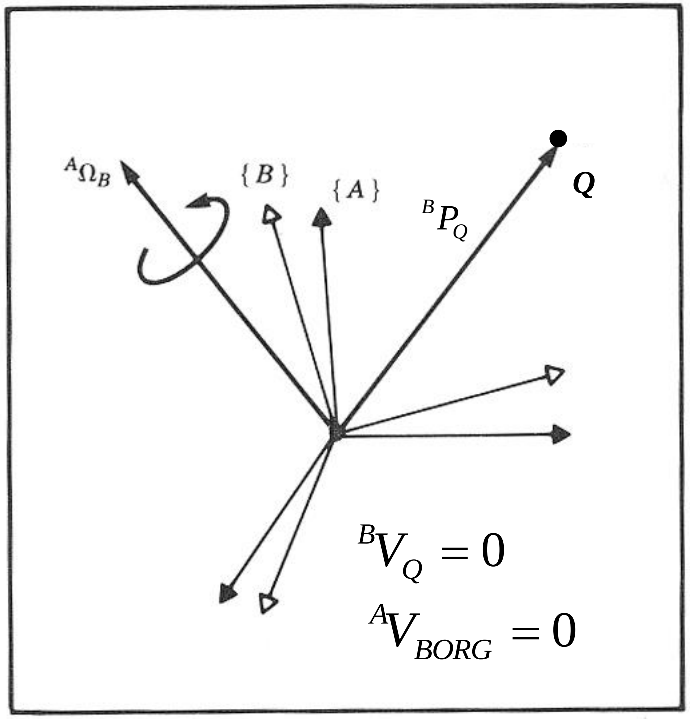
$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body

- ***Given:*** Consider a frame {B} attached to a rigid body whereas frame {A} is fixed. The vector $^{B}P_Q$ is constant as view from frame {B} $^{B}V_Q = 0$

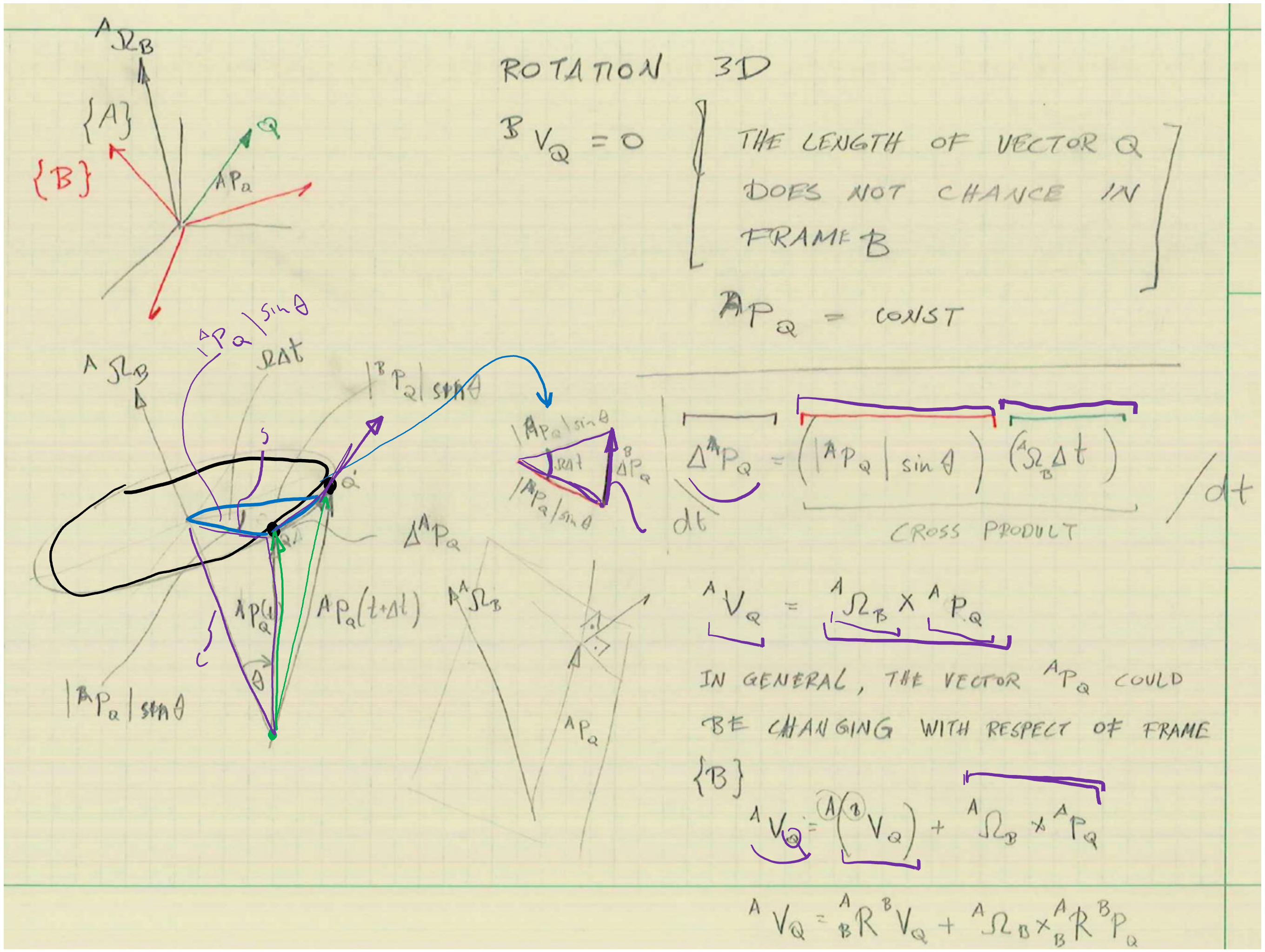
- ***Problem:*** describe the velocity of the vector $^{B}P_Q$ representing the the point **Q** relative to frame {A}

- ***Solution:*** Even though the vector $^{B}P_Q$ is constant as view from frame {B} it is clear that point **Q** will have a velocity as seen from frame {A} due to the rotational velocity $^{A}\Omega_B$



$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach



$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Angular Velocity - Rigid Body - Intuitive Approach

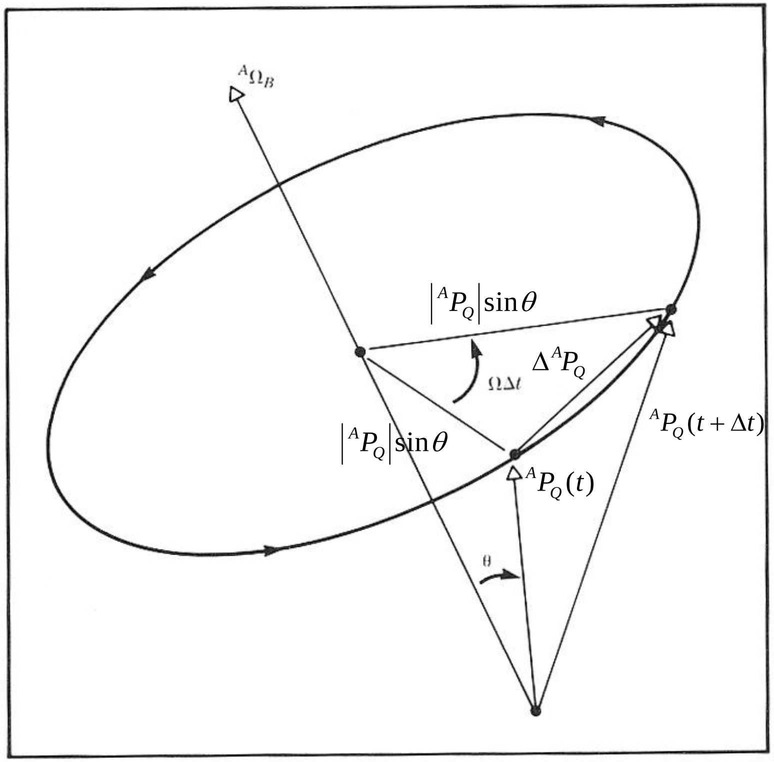
- The figure shows to instants of time as the vector ${}^{A}P_{Q}$ rotates around ${}^{A}\Omega_{B}$ This is what an observer in frame {A} would observe.

- The Magnitude of the differential change is

$$ \left|\Delta\,{}^{A}P_{Q}\right| = \left(\left|{}^{A}\Omega_{B}\right|\Delta t\right)\left(\left|{}^{A}P_{Q}\right|\sin\theta\right) $$

- Using a vector cross product we get

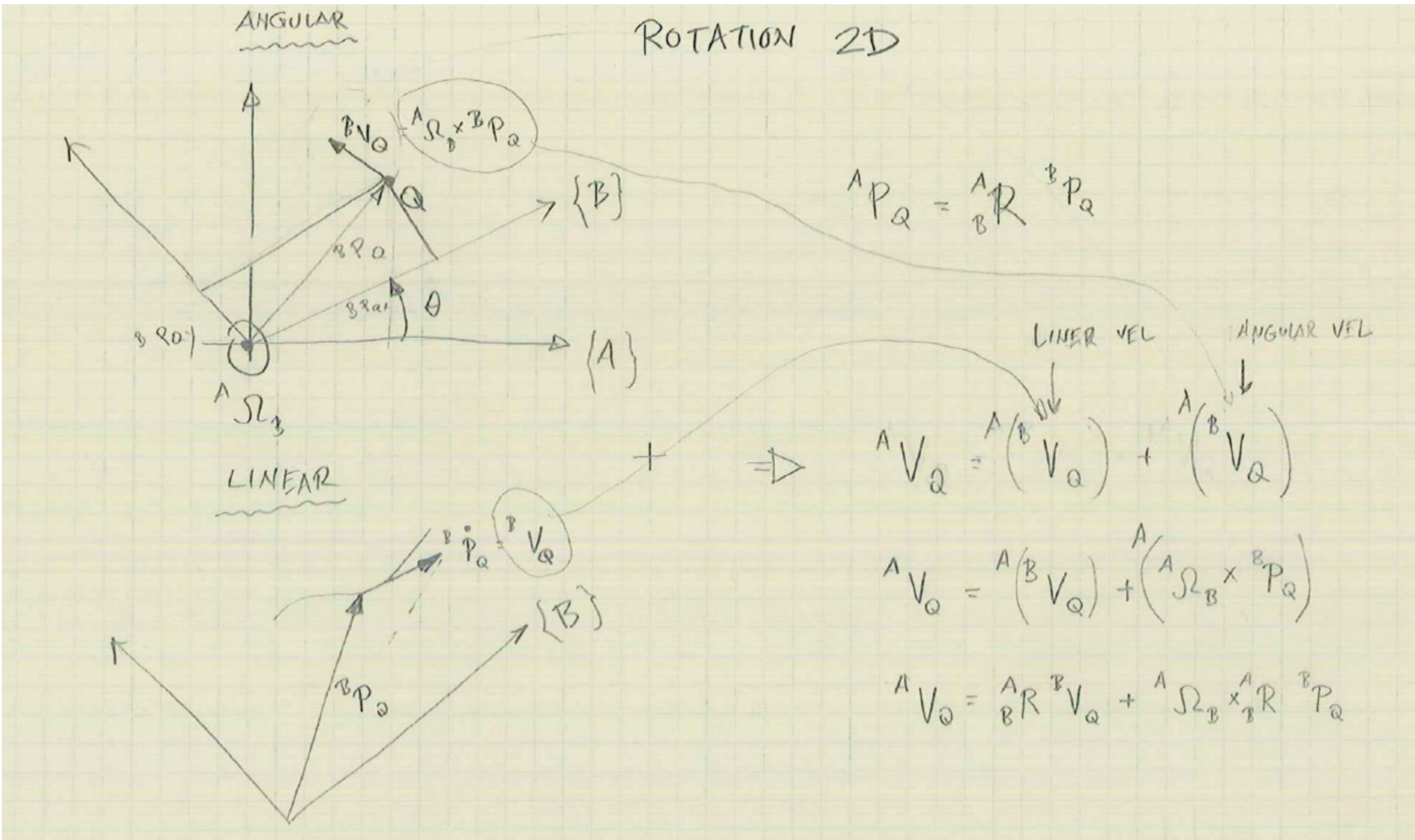
$$ \frac{\Delta\,{}^{A}P_{Q}}{\Delta t} = {}^{A}V_{Q} = {}^{A}\Omega_{B} \times {}^{A}P_{Q} $$



$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach

- Rotation in 2D



$${}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Rigid Body - Intuitive Approach

- In the general case, the vector Q may also be changing with respect to the frame {B}. Adding this component we get.

$${}^{A}V_Q = {}^{A}\left({}^{B}V_Q\right) + {}^{A}\Omega_B \times {}^{A}P_Q$$

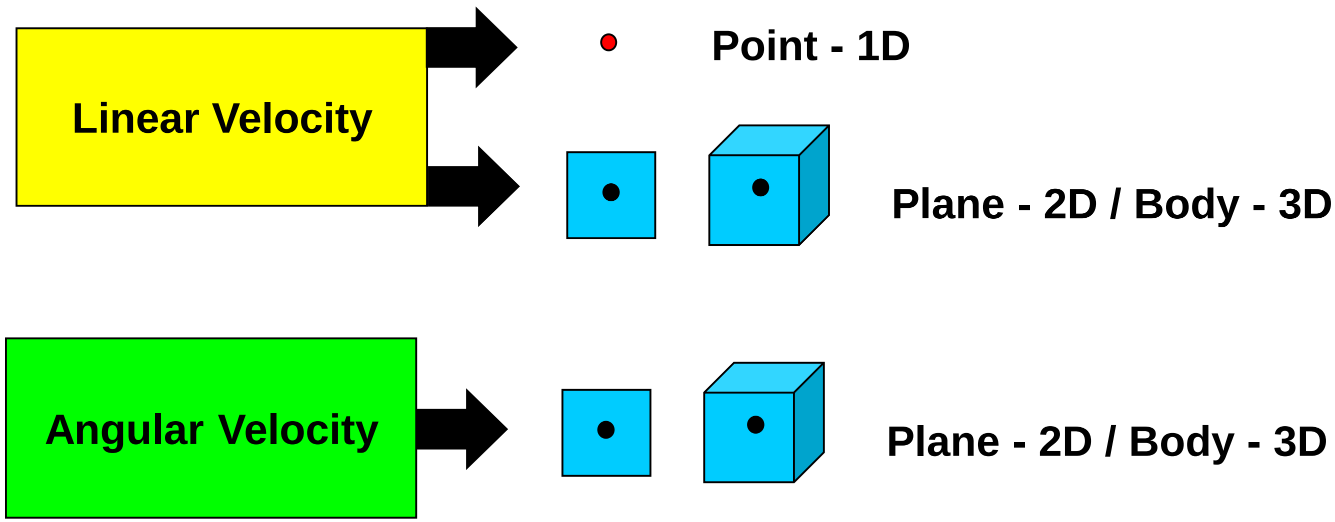
- Using the rotation matrix to remove the dual-superscript, and since the description of ${}^{A}P_Q$ at any instance is ${}^{A}_{B}R\,{}^{B}P_Q$ we get

$${}^{A}V_Q = {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q$$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Angular Velocity

- ***Angular Velocity:*** The instantaneous rate of change in the orientation of one frame relative to another.

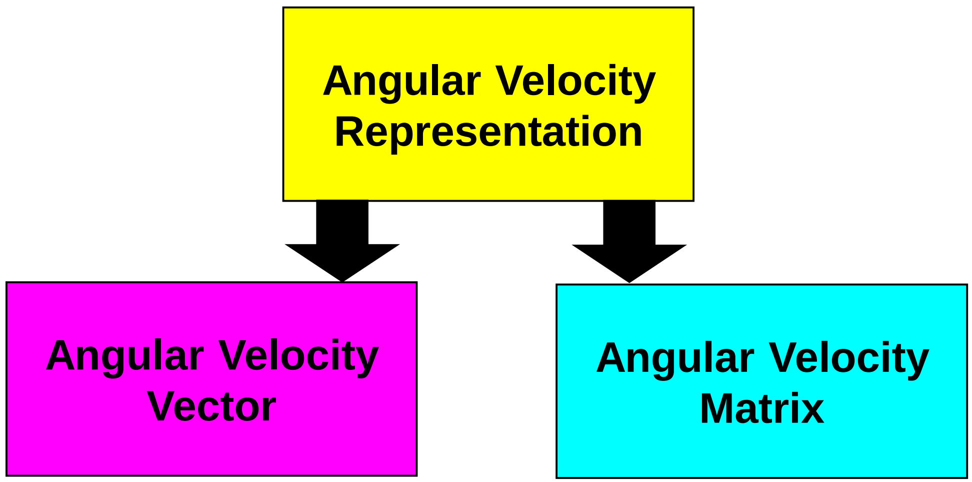


$$^{A}V_Q = {^{A}V_{BORG}} + {^{A}_{B}R}\,{^{B}V_Q} + {^{A}\Omega_B} \times {^{A}_{B}R}\,{^{B}P_Q}$$

$$^{A}V_Q = {^{A}V_{BORG}} + {^{A}_{B}R}\,{^{B}V_Q} + {^{A}_{B}\dot{R}_{\Omega}}\left({^{A}_{B}R}\,{^{B}P_Q}\right)$$

Definitions - Angular Velocity

- Just as there are many ways to represent orientation (Euler Angles, Roll-Pitch-Yaw Angles, Rotation Matrices, etc.) there are also many ways to represent the rate of change in orientation.



- The angular velocity vector is convenient to use because it has an easy to grasp physical meaning. However, the matrix form is useful when performing algebraic manipulations.

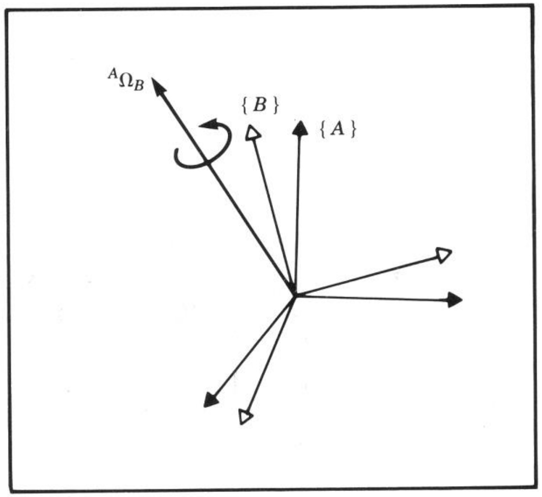
$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q}$$

$$^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$

Definitions - Angular Velocity - Vector

- ***Angular Velocity Vector:*** A vector whose direction is the instantaneous axis of rotation of one frame relative to another and whose magnitude is the rate of rotation about that axis.

$$^{A}\Omega_{B} \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$



- The angular velocity vector $^{A}\Omega_{B}$ describes the instantaneous change of rotation of frame {B} relative to frame {A}

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Definitions - Angular Velocity - Matrix

- ***Angular Velocity Matrix:***

$$ \left[{}^{A}_{B}\dot{R}_{\Omega}\right]\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{matrix} -\Omega_z y + \Omega_y z \\ \Omega_z x - \Omega_x z \\ -\Omega_y x + \Omega_x y \end{matrix} $$

$$ {}^{A}\Omega_B \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{vmatrix} i & j & w \\ \Omega_x & \Omega_y & \Omega_z \\ x & y & z \end{vmatrix} = \begin{Bmatrix} \Omega_y z - \Omega_z y \\ -\Omega_x z + \Omega_z x \\ \Omega_x y - \Omega_y x \end{Bmatrix} $$

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} \qquad {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$

Definitions - Angular Velocity - Matrix

- The rotation matrix (${}^{A}_{B}R$) defines the orientation of frame {B} relative to frame {A}. Specifically, the columns of ${}^{A}_{B}R$ are the unit vectors of {B} represented in {A}.

$$ {}^{A}_{B}R = {}^{A}\begin{bmatrix} \left[{}^{B}P_{x}\right] & \left[{}^{B}P_{y}\right] & \left[{}^{B}P_{z}\right] \\ & & \end{bmatrix} $$

- If we look at the derivative of the rotation matrix, the columns will be the velocity of each unit vector of {B} relative to {A}:

$$ {}^{A}_{B}\dot{R} = \frac{d}{dt}\left[{}^{A}_{B}R\right] = {}^{A}\begin{bmatrix} \left[{}^{B}V_{x}\right] & \left[{}^{B}V_{y}\right] & \left[{}^{B}V_{z}\right] \\ & & \end{bmatrix} $$

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Definitions - Angular Velocity - Matrix

- The relationship between the rotation matrix ${}^{A}_{B}R$ and the derivative of the rotation matrix ${}^{A}_{B}\dot{R}$ can be expressed as follows:

$$^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R$$

$$^{A}\begin{bmatrix} \left[{}^{B}V_x\right] & \left[{}^{B}V_y\right] & \left[{}^{B}V_z\right] \end{bmatrix} = {}^{A}_{B}\dot{R}_{\Omega}\;{}^{A}\begin{bmatrix} \left[{}^{B}P_x\right] & \left[{}^{B}P_y\right] & \left[{}^{B}P_z\right] \end{bmatrix}$$

- where ${}^{A}_{B}\dot{R}_{\Omega}$ is defined as the ***angular velocity matrix***

$$^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \qquad {}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$$

$$^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}\Omega_B} \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + \boxed{{}^{A}_{B}\dot{R}_{\Omega}}\left({}^{A}_{B}R\,{}^{B}P_Q\right)$$

Angular Velocity - Matrix & Vector Forms

	Matrix Form	Vector Form
Definition	${}^{A}_{B}\dot{R}_{\Omega} \equiv \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$	${}^{A}\Omega_B \equiv \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$
Multiply by Constant	$k\left[{}^{A}_{B}\dot{R}_{\Omega}\right]$	$k\left[{}^{A}\Omega_B\right]$
Multiply by Vector	$\left[{}^{A}_{B}\dot{R}_{\Omega}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	${}^{A}\Omega_B \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\left[w \times r \right]$
Multiply by Matrix	$\left[{}^{s}_{t}R\right]\left[{}^{A}_{B}\dot{R}_{\Omega}\right]\left[{}^{s}_{t}R\right]^T$	$\left[{}^{s}_{t}R\right]\left[{}^{A}\Omega_B\right]$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q \qquad {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

Simultaneous Linear and Rotational Velocity - Vector Versus Matrix Representation

Vector Form

Matrix Form

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}P_Q $$

$$ {}^{A}V_Q = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_Q + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_Q\right) $$

$$ \Omega \times P = \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ P_x & P_y & P_z \end{vmatrix} = i\,(\Omega_y P_z - \Omega_z P_y) - j\,(\Omega_x P_z - \Omega_z P_x) + k\,(\Omega_x P_y - \Omega_y P_x) $$

$$ \dot{R}_{\Omega} P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} -\Omega_z P_y + \Omega_y P_z \\ \Omega_z P_x - \Omega_x P_z \\ -\Omega_y P_x + \Omega_x P_y \end{bmatrix} $$

Simultaneous Linear and Rotational Velocity

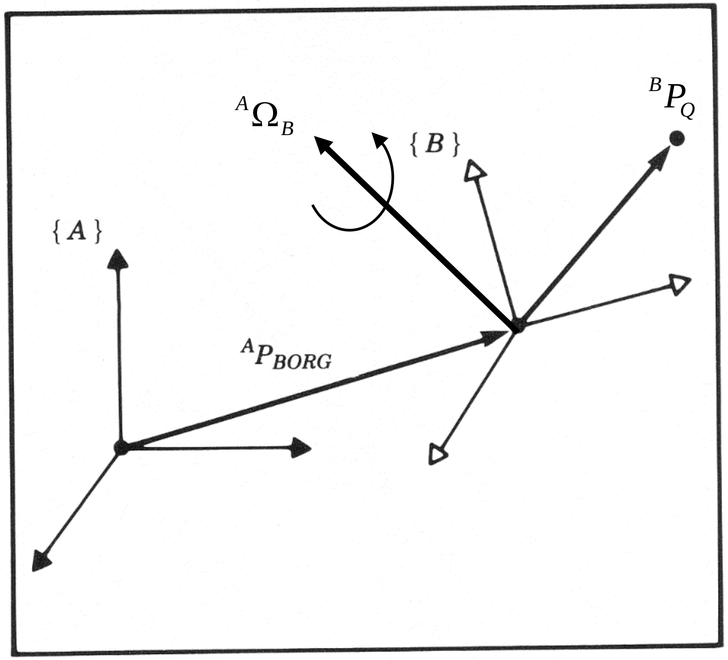
- The final results for the derivative of a vector in a moving frame (linear and rotation velocities) as seen from a stationary frame

- Vector Form

$$^{A}V_{Q}={}^{A}V_{BORG}+{}^{A}_{B}R\,{}^{B}V_{Q}+{}^{A}\Omega_{B}\times{}^{A}_{B}R\,{}^{B}P_{Q}$$

- Matrix Form

$$^{A}V_{Q}={}^{A}V_{BORG}+{}^{A}_{B}R\,{}^{B}V_{Q}+{}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right)$$



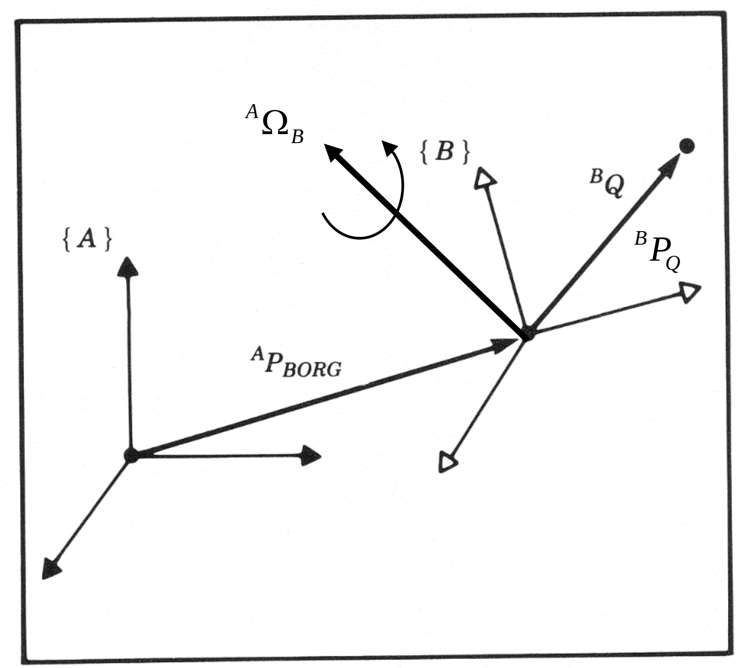
Velocity – Derivation Method No. 3

Homogeneous Transformation Form

Changing Frame of Representation - Linear Velocity

- We have already used the homogeneous transform matrix to compute the location of position vectors in other frames:

$$^{A}P_{Q} = {}^{A}_{B}T\,{}^{B}P_{Q}$$



- To compute the relationship between velocity vectors in different frames, we will take the derivative:

$$\frac{d}{dt}\left[{}^{A}P_{Q}\right] = \frac{d}{dt}\left[{}^{A}_{B}T\,{}^{B}P_{Q}\right]$$

$$^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q}$$

$$ {}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q} $$

Changing Frame of Representation - Linear Velocity

- Recall that

$$ {}^{A}_{B}T = \begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- so that the derivative is

$$ {}^{A}_{B}\dot{T} = \frac{d}{dt}\begin{bmatrix} \left[{}^{A}_{B}R\right] & & & \left[{}^{A}P_{B\,org}\right] \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}\right] & & & \left[{}^{A}\dot{P}_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} == \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\right] & & & \left[{}^{A}V_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

$$ {}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q} $$

Changing Frame of Representation - Linear Velocity

$$ {}^{A}_{B}\dot{T} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\right] & & & \left[{}^{A}V_{B\,org}\right] \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

- Substitute the previous results into the original equation $ {}^{A}\dot{P}_{Q} = {}^{A}_{B}\dot{T}\,{}^{B}P_{Q} + {}^{A}_{B}T\,{}^{B}\dot{P}_{Q} $ we get

$$ \begin{bmatrix} \left[{}^{A}V_{Q}\right] \\ 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{A}_{B}\dot{R}_{\Omega}\cdot{}^{A}_{B}R\right] & \left[{}^{A}V_{B\,org}\right] \\ 0\,0\,0 & 0 \end{bmatrix} \begin{bmatrix} \left[{}^{B}P_{Q}\right] \\ 1 \end{bmatrix} + \begin{bmatrix} \left[{}^{A}_{B}R\right] & \left[{}^{A}P_{B\,org}\right] \\ 0\,0\,0 & 1 \end{bmatrix} \begin{bmatrix} \left[{}^{B}V_{Q}\right] \\ 0 \end{bmatrix} $$

- This expression is equivalent to the following three-part expression:

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

Changing Frame of Representation - Linear Velocity

$$ {}^{A}V_{Q} = {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

- Converting from matrix to vector form yields

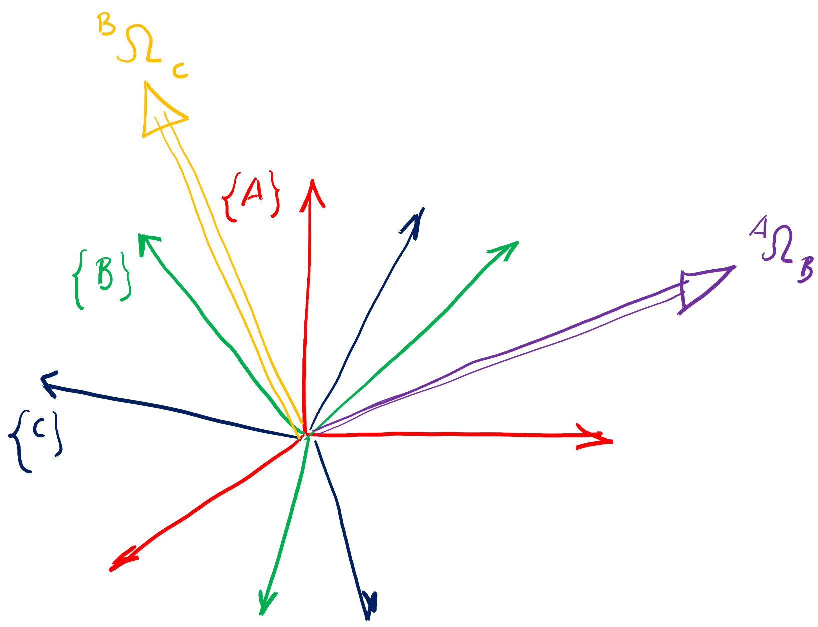
$$ {}^{A}V_{Q} = {}^{A}\Omega_{B} \times \left({}^{A}_{B}R\,{}^{B}P_{Q}\right) + {}^{A}V_{B\,org} + {}^{A}_{B}R\,{}^{B}V_{Q} $$

Angular Velocity –
Changing Frame of Representation

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} \qquad {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$

Angular Velocity

- Frame {C} is rotated around frame {B} by ${}^{B}\Omega_{C}$
- Frame {B} is rotated around frame {A} by ${}^{A}\Omega_{B}$
- Given ${}^{B}\Omega_{C}$ ${}^{A}\Omega_{B}$
- Find ${}^{A}\Omega_{C}$



$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} \qquad {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$

Changing Frame of Representation - Angular Velocity

- We use rotation matrices to represent angular position so that we can compute the angular position of {C} in {A} if we know the angular position of {C} in {B} and {B} in {A} by

$$ {}^{A}_{C}R = {}^{A}_{B}R\,{}^{B}_{C}R $$

- To derive the relationship describing how angular velocity propagates between frames, we will take the derivative

$$ {}^{A}_{C}\dot{R} = {}^{A}_{B}\dot{R}\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R} $$

- Substituting the angular velocity matrixes

$$ {}^{A}_{B}\dot{R} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R \qquad {}^{B}_{C}\dot{R} = {}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R \qquad {}^{A}_{C}\dot{R} = {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R $$

- we find

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{B}R\,{}^{B}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{C}R + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R $$

Changing Frame of Representation - Angular Velocity

- Post-multiplying both sides by ${}^{A}_{C}R^{T}$,which for rotation matrices, is equivalent to ${}^{A}_{C}R^{-1}$

$$ {}^{A}_{C}\dot{R}_{\Omega}\,{}^{A}_{C}R\,{}^{A}_{C}R^{T} = {}^{A}_{B}\dot{R}_{\Omega}\,{}^{A}_{C}R\,{}^{A}_{C}R^{T} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{B}_{C}R\,{}^{A}_{C}R^{T} $$

$$ {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$

- The above equation provides the relationship for changing the frame of representation of angular velocity matrices.

- The vector form is given by

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

- To summarize, the angular velocities of frames may be added as long as they are expressed in the same frame.

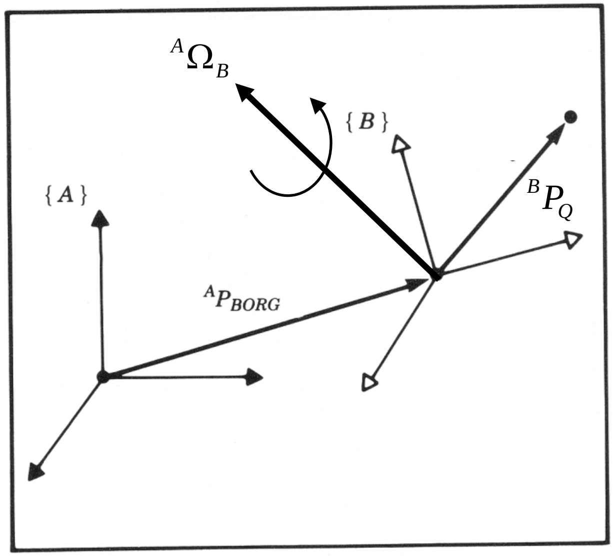
Summary – Changing Frame of Representation

- Linear and Rotational Velocity
  - Vector Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}\Omega_{B} \times {}^{A}_{B}R\,{}^{B}P_{Q} $$

  - Matrix Form

$$ {}^{A}V_{Q} = {}^{A}V_{BORG} + {}^{A}_{B}R\,{}^{B}V_{Q} + {}^{A}_{B}\dot{R}_{\Omega}\left({}^{A}_{B}R\,{}^{B}P_{Q}\right) $$



- Angular Velocity
  - Vector Form

$$ {}^{A}\Omega_{C} = {}^{A}\Omega_{B} + {}^{A}_{B}R\,{}^{B}\Omega_{C} $$

  - Matrix Form

$$ {}^{A}_{C}\dot{R}_{\Omega} = {}^{A}_{B}\dot{R}_{\Omega} + {}^{A}_{B}R\,{}^{B}_{C}\dot{R}_{\Omega}\,{}^{A}_{B}R^{T} $$