Jacobian: Velocities and Static Forces 1/4

Kinematics Relations - Joint & Cartesian Spaces

- A robot is often used to manipulate object attached to its tip (end effector).

- The location of the robot tip may be specified using one of the following descriptions:

- ***Joint Space***

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

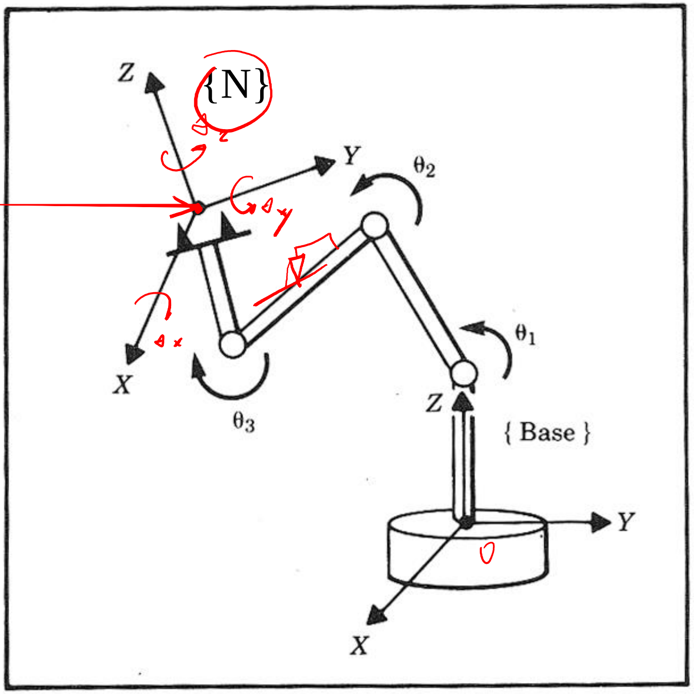
- ***Cartesian Space***

$${}^0_N T = \begin{bmatrix} {}^0_N R & {}^0 P_N \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} [{}^0 P_N] \\ [{}^0 r_N] \end{bmatrix}$$

Euler Angles

θ_x
θ_y
θ_z



Kinematics Relations - Forward & Inverse

- The robot kinematic equations relate the two description of the robot tip location

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

$$X = FK(\theta)$$

$$\theta = IK(X)$$

$$X = \begin{bmatrix} \begin{bmatrix} {}^0P_N \end{bmatrix} \\ \begin{bmatrix} {}^0r_N \end{bmatrix} \end{bmatrix}$$

Tip Location in Joint Space

Tip Location in Cartesian Space

Kinematics Relations - Forward & Inverse

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}}$$

$$\dot{\theta} = \frac{d}{dt}[\theta] = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_N \end{bmatrix}$$

$$\dot{X} = \frac{d}{dt}[X] = \begin{bmatrix} [\, v_N \,] \\ [\, \omega_N \,] \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Tip Velocity in Joint Space

$$\dot{\underline{\theta}} = J^{-1}(\underline{\theta})\dot{\underline{x}}$$

Tip velocity in Cartesian/EE/Task Space

Jacobian Matrix - Introduction

- **The Jacobian is a multi dimensional form of the derivative.**
- Suppose that for example we have 6 functions, each of which is a function of 6 independent variables

$$
\left\{
\begin{array}{l}
y_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6) \\
y_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6) \\
\vdots \\
y_6 = f_6(x_1, x_2, x_3, x_4, x_5, x_6)
\end{array}
\right.
$$

- We may also use a vector notation to write these equations as

$$Y = F(X)$$

Jacobian Matrix - Introduction

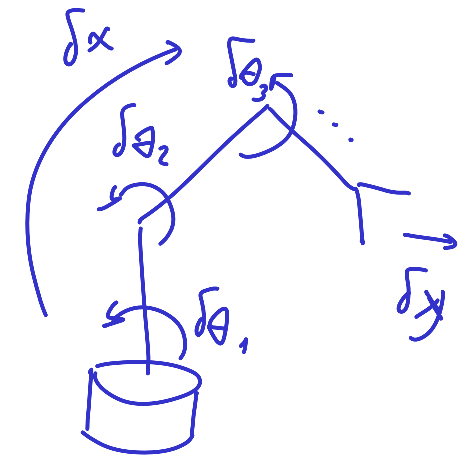
- If we wish to calculate the differential of y_i as a function of the differential x_i we use the chain rule to get

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_1}{\partial x_6}\delta x_6$$

$$\delta y_2 = \frac{\partial f_2}{\partial x_1}\delta x_1 + \frac{\partial f_2}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_2}{\partial x_6}\delta x_6$$

$$\vdots$$

$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \ldots + \frac{\partial f_6}{\partial x_6}\delta x_6$$



- Which again might be written more simply using a vector notation as

$$\rightarrow \quad \delta Y = \frac{\partial F}{\partial X}\delta X$$

Jacobian Matrix - Introduction

- The 6x6 matrix of partial derivative is defined as the Jacobian matrix

$\dot{Y}$ $\quad J \quad$ $\dot{X}$

$$\frac{\delta Y}{dt}=\frac{\partial F}{\partial X}\frac{\delta X}{dt} = J(X)\frac{\delta X}{dt}$$

- By dividing both sides by the differential time element, we can think of the Jacobian as mapping velocities in X to those in Y

$$\dot{Y} = J(X)\dot{X}$$

- Note that the Jacobian is time varying linear transformation

$$\rightarrow \frac{\delta x}{\cancel{dt}} = J\frac{\delta \theta}{\cancel{\delta t}}$$

Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\underline{\dot{\theta}}_N$) and the translation and rotation velocities of the end effector ($\underline{\dot{x}}$). This relationship is given by:

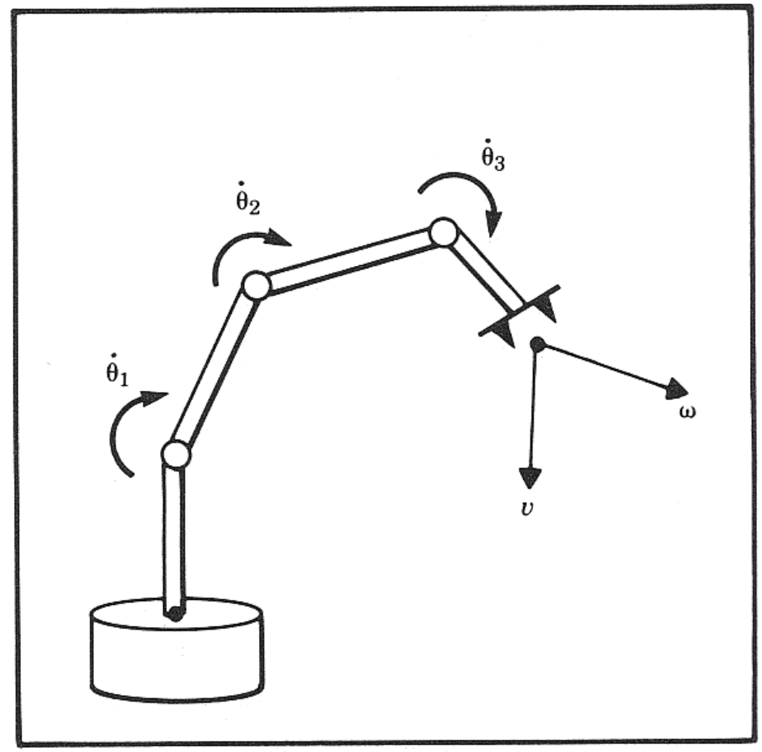
$$\begin{Bmatrix} v_x \\ v_y \\ v_z \\ w_x \\ w_y \\ w_z \end{Bmatrix} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\theta}_x \\ \dot{\theta}_y \\ \dot{\theta}_z \end{Bmatrix}$$

$$\begin{Bmatrix} v \\ w \end{Bmatrix}$$

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{Bmatrix}$$

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$



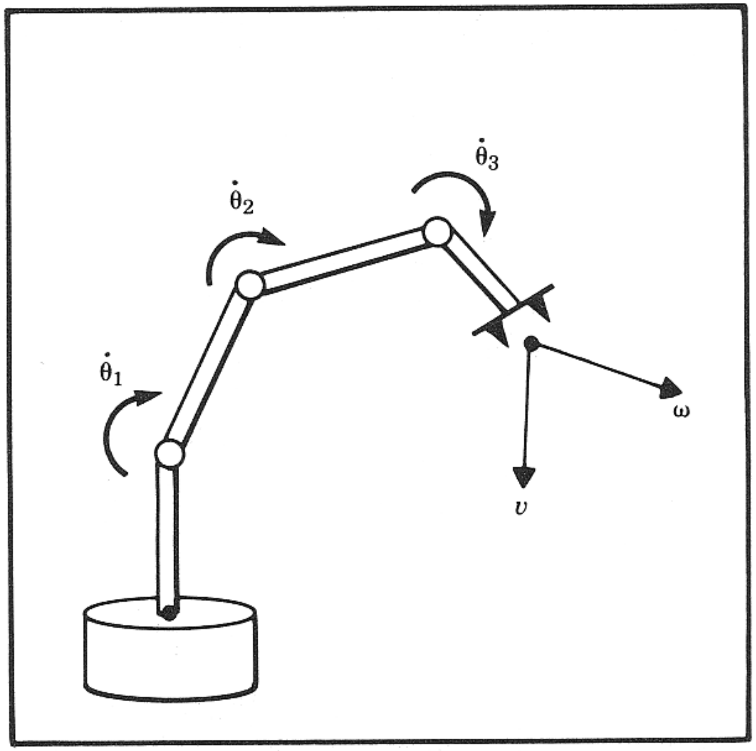
Jacobian Matrix - Introduction

- In the field of robotics the Jacobian matrix describe the relationship between the joint angle rates ($\dot{\underline{\theta}}_N$) and the translation and rotation velocities of the end effector ($\dot{\underline{x}}$). This relationship is given by:

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \\ \\ & J(\underline{\theta}) & \\ \\ \\ \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \\ \cdots \\ \\ \dot{\theta}_N \end{bmatrix}$$

$$\dot{\underline{\theta}} = J(\underline{\theta})^{-1}\dot{\underline{x}}$$



- Note: The Jacobian is a function of joint angle (θ) meaning that the Jacobian varies as the configuration of the arm changes

Jacobian Matrix - Introduction

- This expression can be expanded to:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} \boxed{J_{\upsilon}(\underline{\theta})} \\ \boxed{J_{\cancel{\phi}\,\omega}(\underline{\theta})}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \cdots\\ \dot{\theta}_N\end{bmatrix}$$

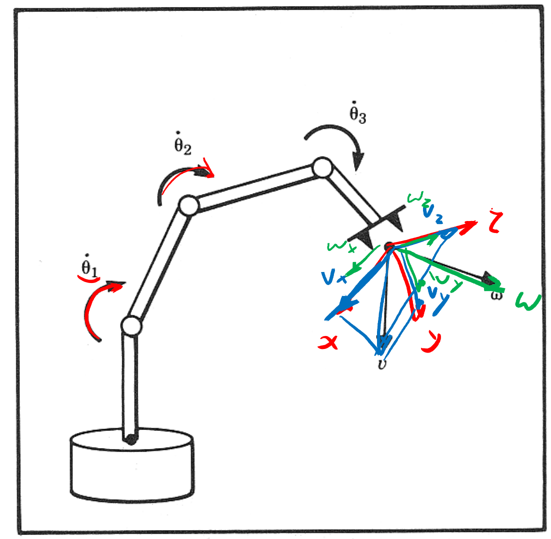
6x1 **6xN** **Nx1**

- Where:
  - $\underline{\dot{x}}$ is a 6x1 vector of the end effector linear and angular velocities
  - $J(\underline{\theta})$ is a 6xN Jacobian matrix
  - $\underline{\dot{\theta}}_N$ is a Nx1 vector of the manipulator joint velocities
  - N is the number of joints

Jacobian Matrix - Introduction

- The meaning of **each line** (e.g. the first line) of the Jacobian matrix:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ \multicolumn{6}{c}{J_v(\underline{\theta})} \\ \multicolumn{6}{c}{J_\omega(\underline{\theta})} \end{bmatrix} \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \cdots\\ \dot{\theta}_N\end{bmatrix}$$



- The first line maps the contribution of the angular velocity of each joint to the linear velocity of the end effector along the x-axis

Jacobian Matrix - Introduction

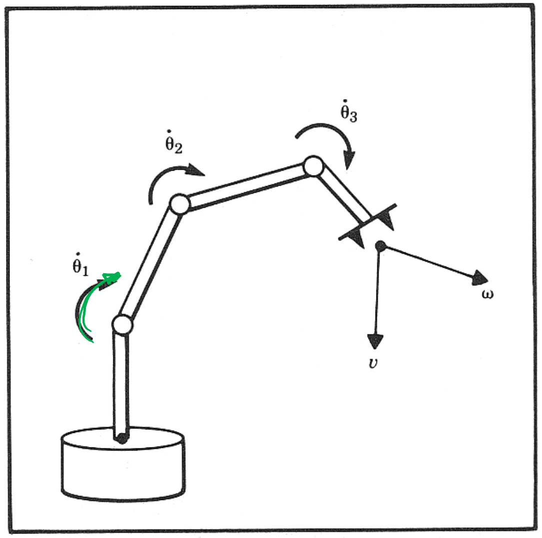
- The meaning of **<u>each column</u>** (e.g. the first column) of the Jacobian matrix:

m/s

RAD/s

RAD/sec

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix} = \begin{bmatrix} J_{11} & \\ J_{21} & J_{\upsilon}(\underline{\theta}) \\ J_{31} & \\ J_{41} & \\ J_{51} & J_{\omega}(\underline{\theta}) \\ J_{61} & \end{bmatrix} \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \cdots\\ \dot{\theta}_N\end{bmatrix}$$

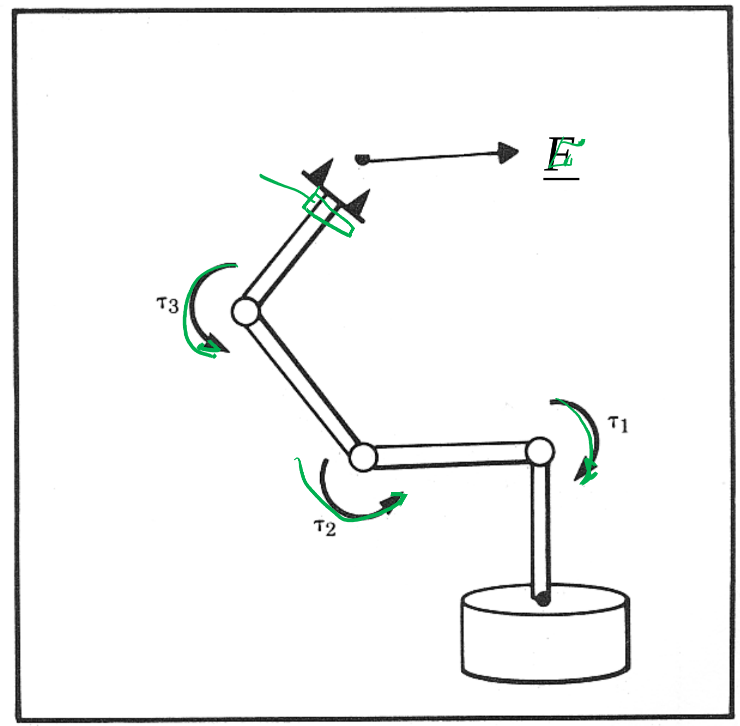


- The first column maps the contribution of the angular velocity of the first joint to the linear and angular velocities of the end effector along all the axis (x,y,z)

Jacobian Matrix - Introduction

- In addition to the velocity relationship, we are also interested in developing a relationship between the robot joint torques ($\underline{\tau}$) and the forces and moments ($\underline{F}$) at the robot end effector (**Static Conditions**). This relationship is given by:

$$\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix} \qquad \underline{\tau} = \underbrace{J(\underline{\theta})}^{T}\underline{F} \qquad \begin{Bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{Bmatrix}$$



Jacobian Matrix - Introduction

- This expression can be expanded to:

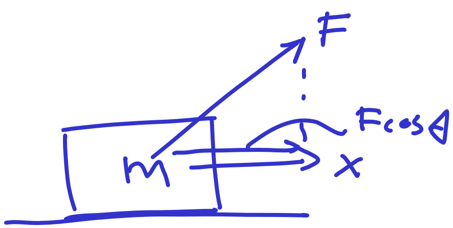
$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \\ \cdots \\ \\ \tau_N \end{bmatrix} = \begin{bmatrix} \boxed{J_f(\underline{\theta})} & \boxed{J_\tau(\underline{\theta})} \end{bmatrix}^T \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix}$$

Nx1 **6xN** **6x1**

- Where:
  - $\underline{\tau}$ is a 6x1 vector of the robot joint torques
  - $J(\underline{\theta})^T$ is a 6xN Transposed Jacobian matrix
  - $\underline{F}$ is a Nx1 vector of the forces and moments at the robot end effector
  - N is the number of joints

Jacobian Matrix - Introduction

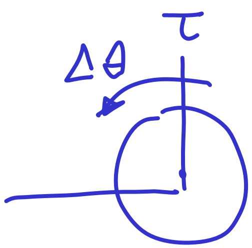
Work



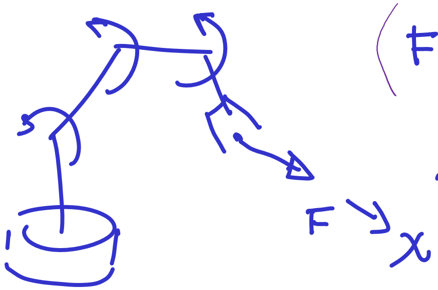
Dot product

$$W = F \cdot \Delta x$$

$$= F\cos\theta \; x$$



$$W = \tau \cdot \Delta\theta$$



$$(F \cdot \delta x = \tau \cdot \delta\theta)^T$$

EE space

Joint space

$$F_x \, x = \tau_1 \, \theta_1$$

$$\vdots$$

$$M_x \theta_x = \tau_4 \, \theta_4$$

$$\vdots$$

$$F^T \delta x = \tau^T \delta\theta$$

$$\delta x = J \, \delta\theta$$

Dot

$$F^T \overbrace{J \cancel{\delta\theta}}^{\delta x} = \tau^T \cancel{\delta\theta}$$

Jacobian Matrix - Introduction

$$\left[\tau^T = F^T J\right]^T \qquad \left[\tau^T\right]^T = \left[F^T J\right]^T$$

$$\boxed{\tau = J^T F}$$

Jacobian Matrix - Calculation Methods

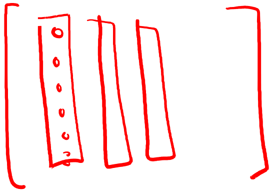
- There are three methods to derive the Jacobian matrix

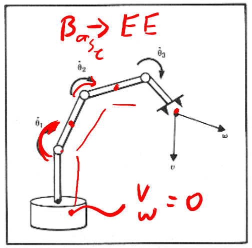
Explicit Approach
Diff. of the FK Eqs
(One Method)

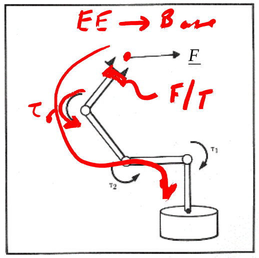
Iterative Propagation Approach
(Velocities or Forces / Torques)
(Two Methods)

↓

Jacobian Matrix

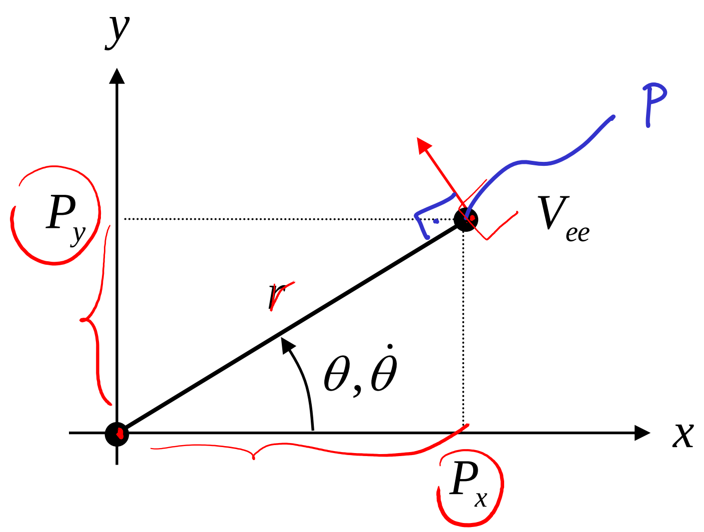






Jacobian Matrix by Differentiation - 1R - 1/4

- Consider a simple planar 1R robot



- The end effector position is given by

$$\begin{cases} {}^0P_x = x = r\cos\theta & /dt \\ {}^0P_y = y = r\sin\theta & /dt \end{cases}$$

Jacobian Matrix by Differentiation - 1R - 2/4

- The velocity of the end effector is defined by

$$
\begin{aligned}
{}^{0}V_x &= {}^{0}\dot{P}_x = \dot{x} = -\dot{\theta}\, r \sin\theta = -\omega\, r \sin\theta \\
{}^{0}V_y &= {}^{0}\dot{P}_y = \dot{y} = \dot{\theta}\, r \cos\theta = \omega\, r \cos\theta
\end{aligned}
$$

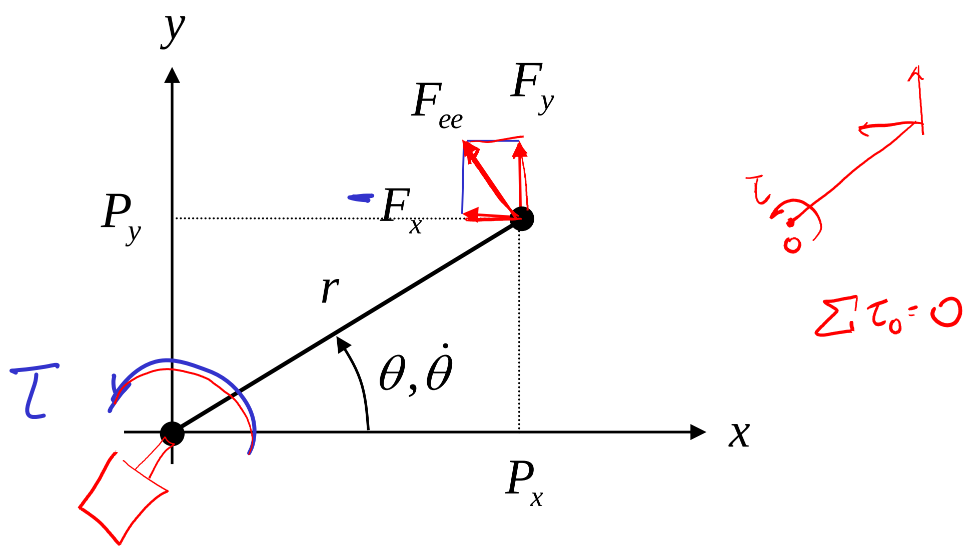
- Expressed in matrix form we have

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \end{bmatrix}$$

2x1 **2x1** **1x1**

Jacobian Matrix by Differentiation - 1R - 3/4



- The moment about the joint generated by the force acting on the end effector is given by

$$\tau = -rF_x \sin\theta + rF_y \cos\theta$$

Jacobian Matrix by Differentiation - 1R - 4/4

- Expressed in matrix form we have

$$\underline{\tau} = J(\underline{\theta})^T \underline{F}$$

$$[\tau] = \begin{bmatrix} -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

1x1 **1x2** **2x1**

J^T

T

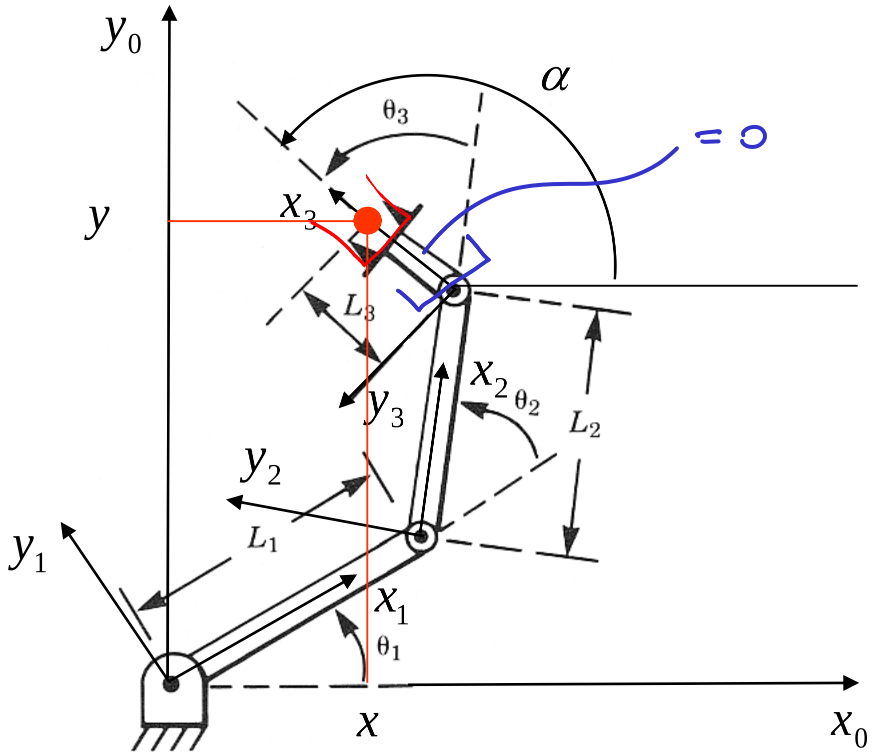
$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}}$$

J

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} [\dot{\theta}]$$

Jacobian Matrix by Differanciation - 3R - 1/4

- Consider the following 3 DOF Planar manipulator



Jacobian Matrix by Differanciation - 3R - 2/4

- **Problem:** Compute the Jacobian matrix that describes the relationship

$$\dot{\underline{x}} = J(\underline{\theta})\dot{\underline{\theta}} \qquad\qquad \underline{\tau} = J(\underline{\theta})^T \underline{F}$$

- **Solution:**

- The end effector position and orientation is defined in the base frame by

$$\underline{x} = \begin{bmatrix} x \\ y \\ \alpha \end{bmatrix}$$

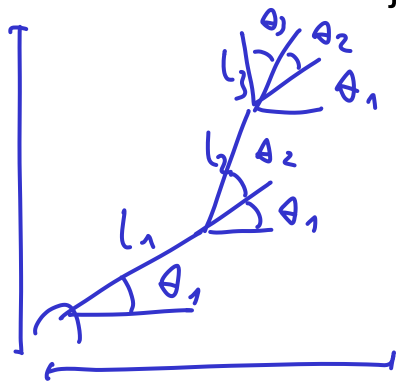
Jacobian Matrix by Differanciation - 3R - 3/4

- The forward kinematics gives us relationship of the end effector to the joint angles:

$$ {}^{0}P_{3org,x} = x = L_1 c_1 + L_2 c_{12} + L_3 c_{123} $$

$$ {}^{0}P_{3org,y} = y = L_1 s_1 + L_2 s_{12} + L_3 s_{123} $$

$$ {}^{0}P_{3org,\alpha} = \alpha = \theta_1 + \theta_2 + \theta_3 $$



- Differentiating the three expressions gives

J

$\dot\theta_1 \;\; \dot\theta_2 \;\; \dot\theta_3$

$$ \dot{x} = -L_1 s_1 \dot\theta_1 - L_2 s_{12}\left(\dot\theta_1 + \dot\theta_2\right) - L_3 s_{123}\left(\dot\theta_1 + \dot\theta_2 + \dot\theta_3\right) $$

$$ = -\left(L_1 s_1 + L_2 s_{12} + L_3 s_{123}\right)\dot\theta_1 - \left(L_2 s_{12} + L_3 s_{123}\right)\dot\theta_2 - \left(L_3 s_{123}\right)\dot\theta_3 $$

$$ \dot{y} = L_1 c_1 \dot\theta_1 + L_2 c_{12}\left(\dot\theta_1 + \dot\theta_2\right) + L_3 c_{123}\left(\dot\theta_1 + \dot\theta_2 + \dot\theta_3\right) $$

$$ = \left(L_1 c_1 + L_2 c_{12} + L_3 c_{123}\right)\dot\theta_1 + \left(L_2 c_{12} + L_3 c_{123}\right)\dot\theta_2 + \left(L_3 c_{123}\right)\dot\theta_3 $$

$$ \dot\alpha = \dot\theta_1 + \dot\theta_2 + \dot\theta_3 $$

UCLA

Jacobian Matrix by Differanciation - 3R - 4/4

- Using a matrix form we get

FRAME 0

$$\underline{\dot{x}} = {}^{0}J(\underline{\theta})\underline{\dot{\theta}}$$

J

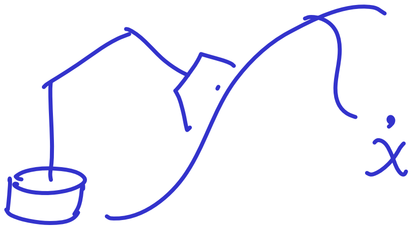
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & -L_2 s_{12} - L_3 s_{123} & -L_3 s_{123} \\ L_1 c_1 + L_2 c_{12} + L_3 c_{123} & L_2 c_{12} + L_3 c_{123} & L_3 c_{123} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$

- The Jacobian provides a linear transformation, giving a velocity map and a force map for a robot manipulator. For the simple example above, the equations are trivial, but can easily become more complicated with robots that have additional degrees a freedom. Before tackling these problems, consider this brief review of linear algebra.

Singularity - The Concept

- ***Motivation:*** We would like the hand of a robot (end effecror) to move with a certain velocity vector in Cartesian space. Using linear transformation relating the joint velocity to the Cartesian velocity we could calculate the necessary joint rates at each instance along the path.

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1}\underline{\dot{x}}$$



- ***Given:*** a linear transformation relating the joint velocity to the Cartesian velocity (usually the end effector)

- ***Question:*** Is the Jacobian matrix invertable? (Or) Is it nonsingular?
  Is the Jacobian invertable for all values of θ ?
  If not, where is it not invertable?

Singularity - The Concept

- ***Answer (Conceptual):*** Most manipulator have values of θ where the Jacobian becomes singular . Such locations are called ***singularities of the mechanism*** or ***singularities*** for short

Singularities of the mechanism

Workspace boundary singularities

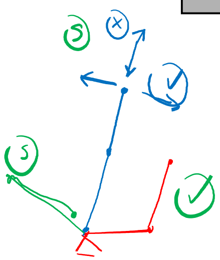
Workspace interior Singularities

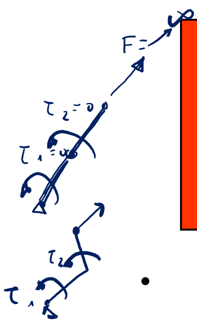
- Stretched out
- Folded back

End Effector → **Workspace Boundary**

- Two or more joints are lining up

Singularity - The Concept





Manipulator

Singular Configuration

General Configuration

Losing One or More DOF

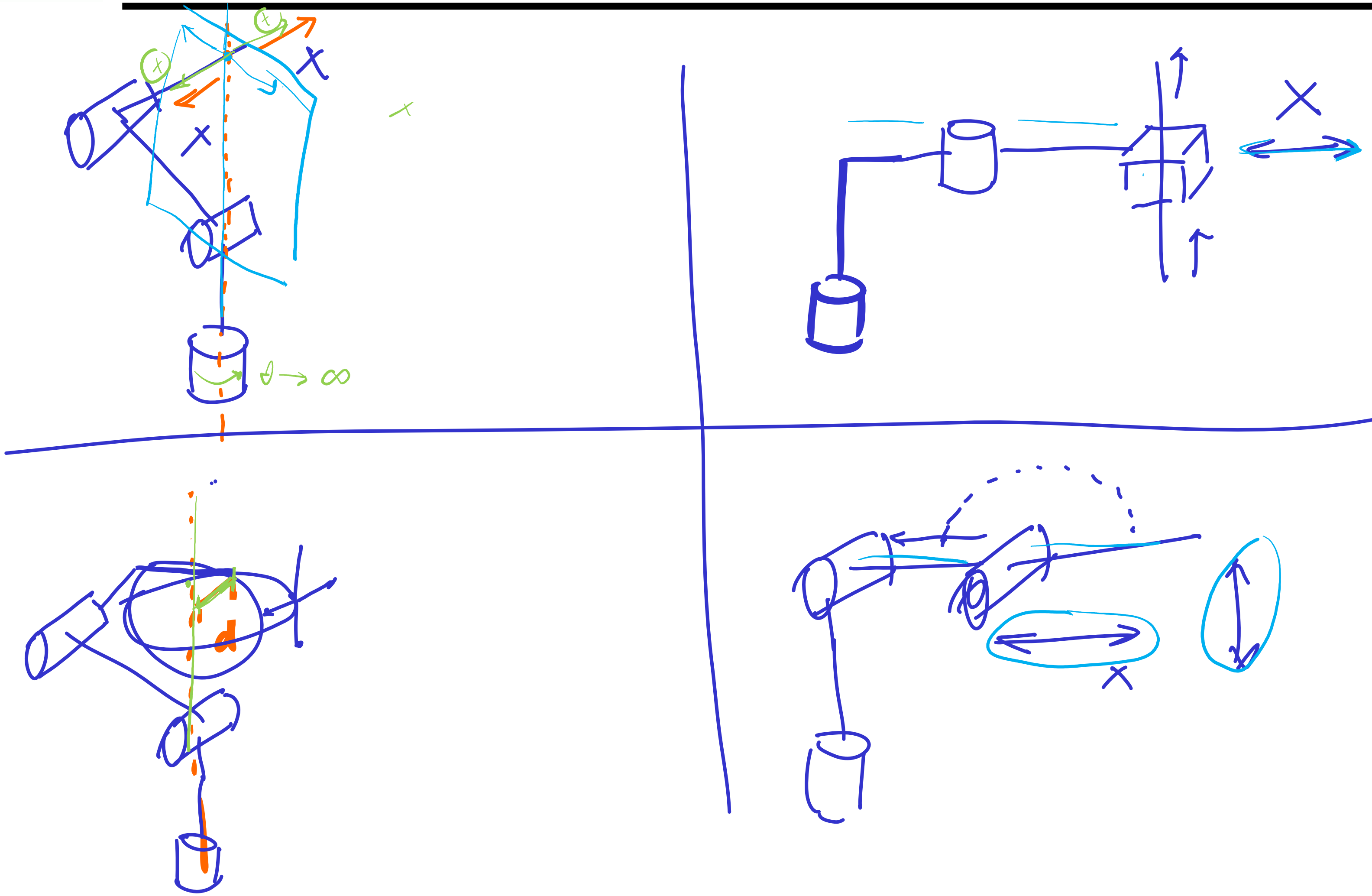
Problematic Load Balance

Problematic Joint Velocity

All DOF Are Available

- **Lost of DOF** - Losing one or more DOF means that there is a some direction (or subspace) in Cartesian space along which it is impossible to move the hand of the robot (end effector) no matter which joint rate are selected
- **Load Balance** – A finite force can be applied to the end effector that produces no torque at the robot’s joints
- **Joint Velocity** – A zero end effector velocity will cause high joint velocity

Singularity – Physical Interpretation - Examples



Singularities

YouTube


ROBOT
SINGULARITY
roboticsbook.com
A3
A2
A1

Singularities

YouTube

Brief Linear Algebra Review - 1/

- Inverse of Matrix A exists ***if and only if*** the determinant of A is non-zero.

A^{-1} Exists ***if and only if***

$$Det(A) = |A| \neq 0$$

- If the determinant of A is equal to zero, then the matrix A is a singular matrix

$$Det(A) = |A| = 0$$

A Singular

Brief Linear Algebra Review - 2/

- The rank of the matrix A is the size of the largest squared Matrix S for which

$$Det(S) \neq 0$$

- Example 1 - $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $A = S = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ $|A| = |S| = 3$ $Rank(A) = 2$

- Example 2 - $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $S = [1]$ $|S| = 1$ $Rank(A) = 1$

UCLA

Brief Linear Algebra Review - 3/

- If two rows or columns of matrix A are equal or related by a constant, then

$$Det(A) = 0$$

- Example

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 6 & -3 & -3 \\ 10 & -6 & -5 \end{bmatrix}$$

$$\det(A) = |A| = 2\begin{vmatrix} -3 & -3 \\ -6 & -5 \end{vmatrix} - 0\begin{vmatrix} 6 & -3 \\ 10 & -5 \end{vmatrix} - 1\begin{vmatrix} 6 & -3 \\ 10 & -6 \end{vmatrix} = 6 + 0 - 6 = 0$$

- ***Eigenvalues***

$$\boxed{AX = \lambda X}$$

$$(A - \lambda I)X = 0$$

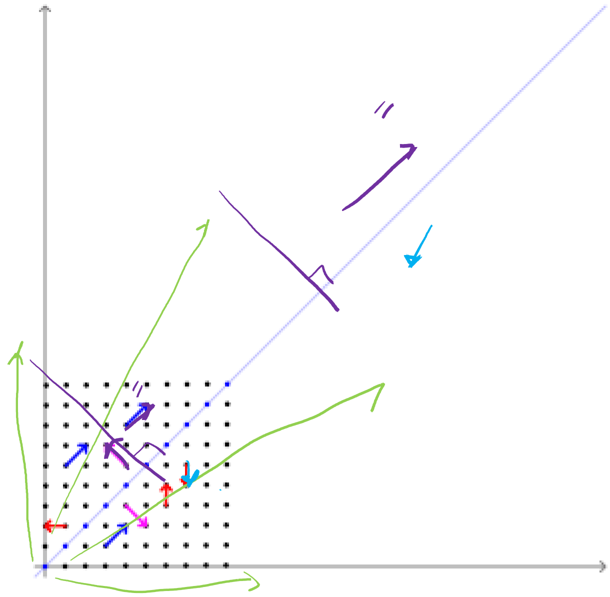
- Eigenvalues are the roots of the polynomial

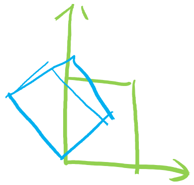
$$Det(A - \lambda I)$$

- If $X \neq 0$ each solution to the characteristic equation λ (Eigenvalue) has a corresponding Eigenvector

Brief Linear Algebra Review - 4/

- Wikipedai - https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors





Brief Linear Algebra Review - 4/

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$(A - \lambda I)X = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$$Det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

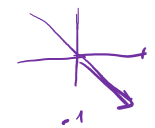
$$\lambda_1 = 1$$

$$\lambda_2 = 3$$

Brief Linear Algebra Review - 4/

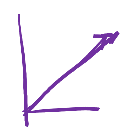
$$\lambda_1 = 1$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



$$\lambda_2 = 3$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \qquad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Brief Linear Algebra Review - 5/

- Any singular matrix ($Det(A)=0$) has at least one Eigenvalue equal to zero

Brief Linear Algebra Review - 6/

- If A is non-singular ($Det(A) \neq 0$), and λ is an eigenvalue of A with corresponding to eigenvector X, then

$$A^{-1}X = \lambda^{-1}X$$

Brief Linear Algebra Review - 7/

- If the $n \times n$ matrix A is of full rank (that is, $\boldsymbol{Rank\ (A) = n}$), then the only solution to

$$AX = 0$$

is the trivial one

$$X = 0$$

- If A is of less than full rank (that is $\boldsymbol{Rank\ (A) < n}$), then there are $n\text{-}r$ linearly independent (orthogonal) solutions

$$x_j \qquad 0 \le j \le n - r$$

for which

$$Ax_j = 0$$

Brief Linear Algebra Review - 8/

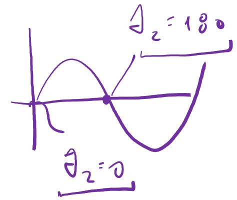
- If A is square, then A and A^T have the same eigenvalues

Properties of the Jacobian - Velocity Mapping and Singularities

- ***Example:*** Planar 3R

$$\det(J(\theta)) = \begin{vmatrix} -L_1s_1 - L_2s_{12} - L_3s_{123} & -L_2s_{12} - L_3s_{123} & -L_3s_{123} \\ L_1c_1 + L_2c_{12} + L_3c_{123} & L_2c_{12} + L_3c_{123} & L_3c_{123} \\ 1 & 1 & 1 \end{vmatrix} = L_1L_2s_2$$

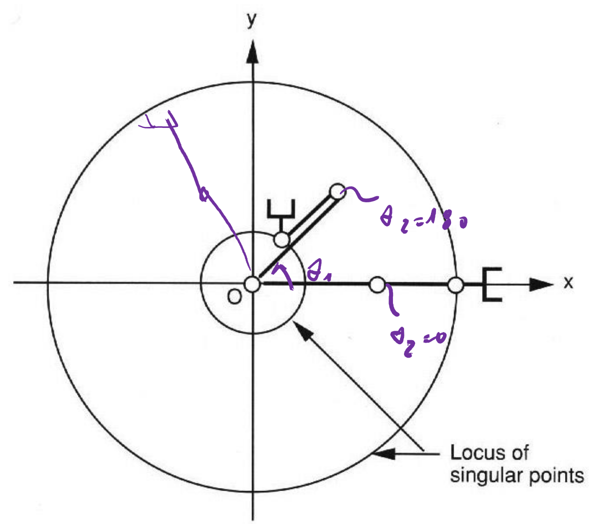
$$\det(J(\theta)) = \underbrace{L_1L_2s_2}_{\theta_1} = 0$$



- Note that $\det(J(\theta))$ is not a function of θ_1, θ_3

Properties of the Jacobian - Velocity Mapping and Singularities

$$\text{singular configuration} \begin{cases} \theta_2 = 0 & \text{Stretched Out} \\ \theta_2 = \pi & \text{Fold Back} \end{cases}$$



- The manipulator loses 1 DEF. The end effector can only move along the tangent direction of the arm. Motion along the radial direction is not possible.

Properties of the Jacobian - Force Mapping and Singularities

- The relationship between joint torque and end effector force and moments is given by:

$$\underline{\tau} = J\left(\underline{\theta}\right)^T \underline{F}$$

- The rank of $J\left(\theta\right)^T$ is equals the rank of $J\left(\theta\right)$.

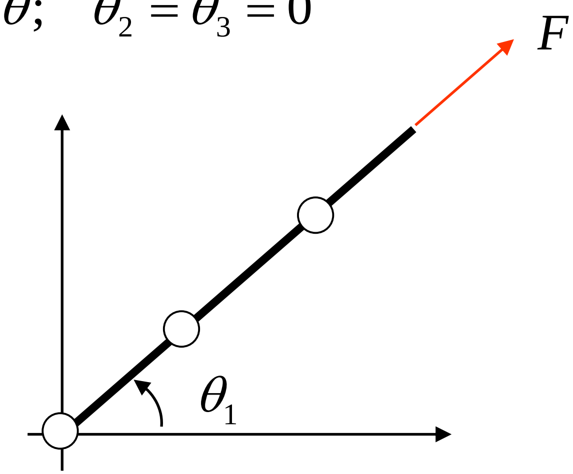
- At a singular configuration there exists a non trivial force $\underline{F}$ such that

$$J\left(\underline{\theta}\right)^T \underline{F} = 0$$

- In other words, a finite force can be applied to the end effector that produces no torque at the robot's joints. In the singular configuration, the manipulator can "lock up."

Properties of the Jacobian - Force Mapping and Singularities

- ***Example:*** Planar 3R $\theta_1 = \theta; \quad \theta_2 = \theta_3 = 0$



- In this case the force acting on the end effector (relative to the {0} frame) is given by

$$ {}^{0}F = \begin{bmatrix} Fc_1 \\ Fs_1 \\ 0 \end{bmatrix} $$

Properties of the Jacobian - Force Mapping and Singularities

$$
{}^{0}\underline{\tau} = {}^{0}J\left(\underline{\theta}\right)^{T}\,{}^{0}\underline{F} = \begin{bmatrix} -L_1 s_1 - L_2 s_{12} - L_3 s_{123} & L_1 c_1 + L_2 c_{12} + L_3 c_{123} & 1 \\ -L_2 s_{12} - L_3 s_{123} & L_2 c_{12} + L_3 c_{123} & 1 \\ -L_3 s_{123} & L_3 c_{123} & 1 \end{bmatrix} \begin{bmatrix} F c_1 \\ F s_1 \\ 0 \end{bmatrix}
$$

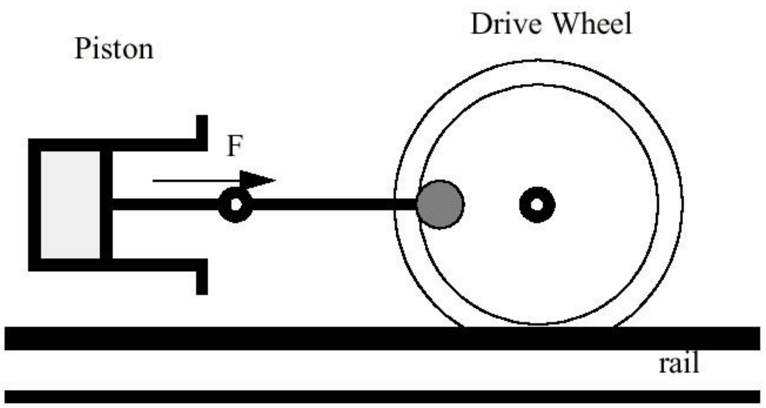
- For $\theta_1 = \theta; \quad \theta_2 = \theta_3 = 0$ we get

$$
{}^{0}\underline{\tau} = {}^{0}J\left(\underline{\theta}\right)^{T}\,{}^{0}\underline{F} = \begin{bmatrix} -L_1 s_1 - L_2 s_1 - L_3 s_1 & L_1 c_1 + L_2 c_1 + L_3 c_1 & 1 \\ -L_2 s_1 - L_3 s_1 & L_2 c_1 + L_3 c_1 & 1 \\ -L_3 s_1 & L_3 c_1 & 1 \end{bmatrix} \begin{bmatrix} F c_1 \\ F s_1 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} -F s_1 c_1 (L_1 + L_2 + L_3) + F s_1 c_1 (L_1 + L_2 + L_3) \\ -F s_1 c_1 (L_2 + L_3) + F s_1 c_1 (L_2 + L_3) \\ -F s_1 c_1 (L_3) + F s_1 c_1 (L_3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Properties of the Jacobian - Force Mapping and Singularities

- This situation is an old and famous one in mechanical engineering.
- For example, in the steam locomotive, “top dead center” refers to the following condition



- The piston force, F, cannot generate any torque around the drive wheel axis because the linkage is singular in the position shown.

Properties of the Jacobian - Velocity Mapping and Singularities

- We have shown the relationship between joint space velocity and end effector velocity, given by

$$\underline{\dot{x}} = J\left(\underline{\theta}\right)\underline{\dot{\theta}}$$

- It is interesting to determine the inverse of this relationship, namely

$$\underline{\dot{\theta}} = J\left(\underline{\theta}\right)^{-1}\underline{\dot{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

- Consider the square 6x6 case for $J(\underline{\theta})$.

- If rank < 6 ($Det(J(\underline{\theta})) = 0$) , then there is no solution to the inverse equation (see Brief Linear Algebra Review - 1,7).

$$Rank(J(\underline{\theta})) < 6$$

$$\underline{\dot{\theta}} = J(\underline{\theta})^{-1} \underline{\dot{x}}$$

- However, if the rank = 5, then there is at least one non-trivial solution to the forward equation (see Brief Linear Algebra Review - 7). That is, for

$$\underline{\dot{x}} = J(\underline{\theta})\underline{\dot{\theta}} = 0$$

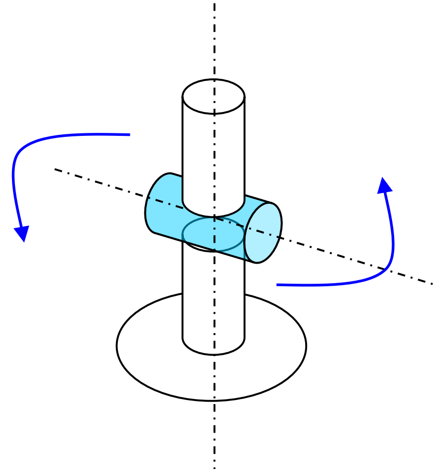
Properties of the Jacobian - Velocity Mapping and Singularities

- The solution is a direction $\left(\underline{\theta}\right)$ in the in joint velocity space for which joint motion produces no end effector motion.

- We call any joint configuration $\underline{\theta} = Q$ for which

$$Rank\left(J\left(\underline{\theta}\right)\right) < 6$$

a ***singular configuration***.



Properties of the Jacobian - Velocity Mapping and Singularities

- For certain directions of end effector motion , $\underline{\dot{x}}_i \quad 1 \le i \le 6$

$$\underline{\dot{x}} = J(\theta)\underline{\dot{\theta}} = \lambda_i(\underline{\theta})\underline{\omega}_i$$

where:

- λ_i are the eigenvalues of $J(\theta)$
- $\underline{\omega}_i$ are the eigenvectors of $J(\theta)$

- If $J(\theta)$ is fully ranked (see Brief Linear Algebra Review - 6/), we have

$$\underline{\omega}_i = J(\theta)^{-1}\underline{\dot{x}} = \lambda_i(\underline{\theta})^{-1}\underline{\dot{x}}$$

Properties of the Jacobian - Velocity Mapping and Singularities

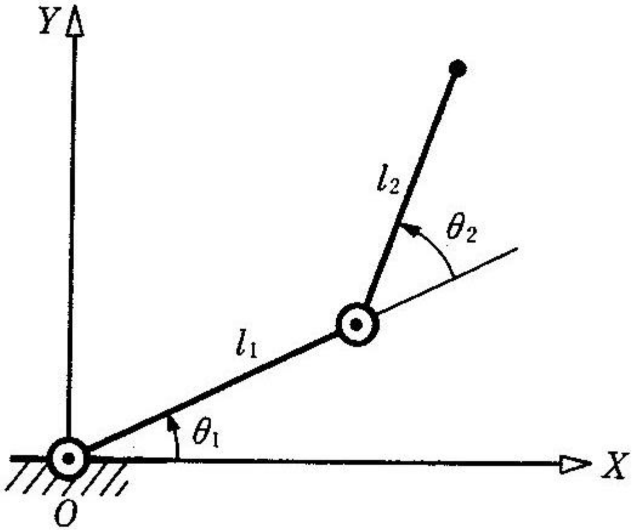
- As the joint approach a singular configuration $\underline{\theta} = Q$ there is at least one eigenvalue for which $\lambda_i \to 0$. This results in

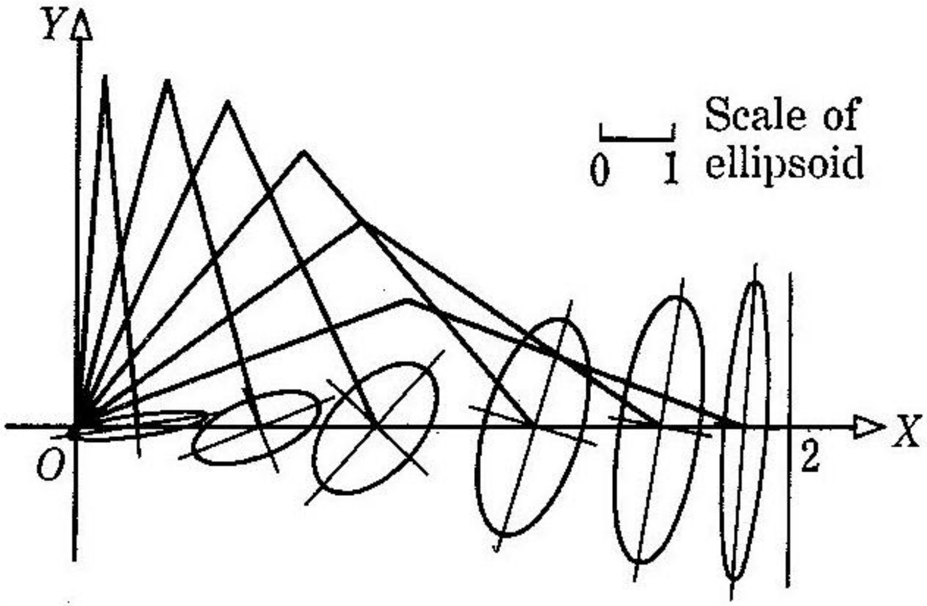
$$\underline{\omega}_i = \frac{\underline{\dot{x}}}{\lambda_i\left(\underline{\theta}\right)} \to \frac{\underline{\dot{x}}}{0} \to \infty$$

- In other word, as the joints approach the singular configuration, the end effector motion in a particular task direction $\underline{\dot{x}}_j$ causes the joint velocities to approach infinity. However, there are task velocities that can have solutions.

- If $J\left(\underline{\theta}\right)$ loses rank by only one, then there are n-1 eigenvectors in the task velocity space ($\underline{\dot{x}}_j$) for which solutions do exist. However, there can be multiple solutions.

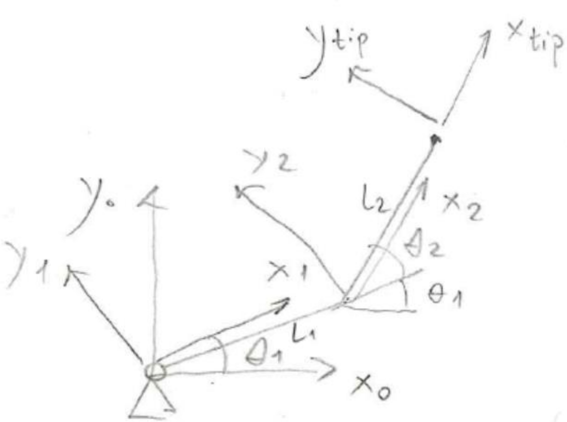
Properties of the Jacobian - Velocity Mapping and Singularities





- Note: See Mathematica Simulations
  - Two Link: https://demonstrations.wolfram.com/ForwardAndInverseKinematicsForTwoLinkArm/
  - Three links : https://demonstrations.wolfram.com/ManipulabilityEllipsoidOfARobotArm/

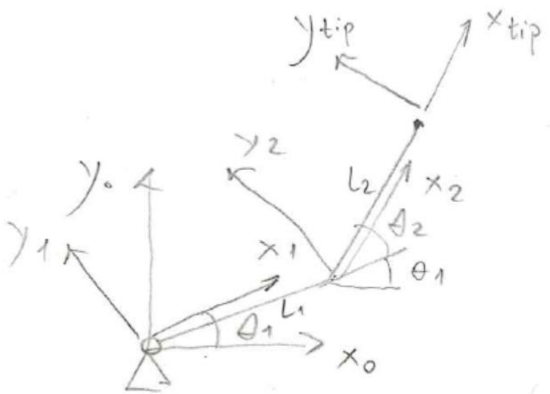
Jacobian – 2R



$$\begin{cases} x_{tip} = L_1\, c_1 + l_2\, c_{12} \\ y_{tip} = L\, s_1 + l_2\, s_{12} \end{cases}$$

$$V_{x\,tip} = \frac{dx_{tip}}{dt} = -L_1 \dot{\theta}_1 s_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2) s_{12}$$

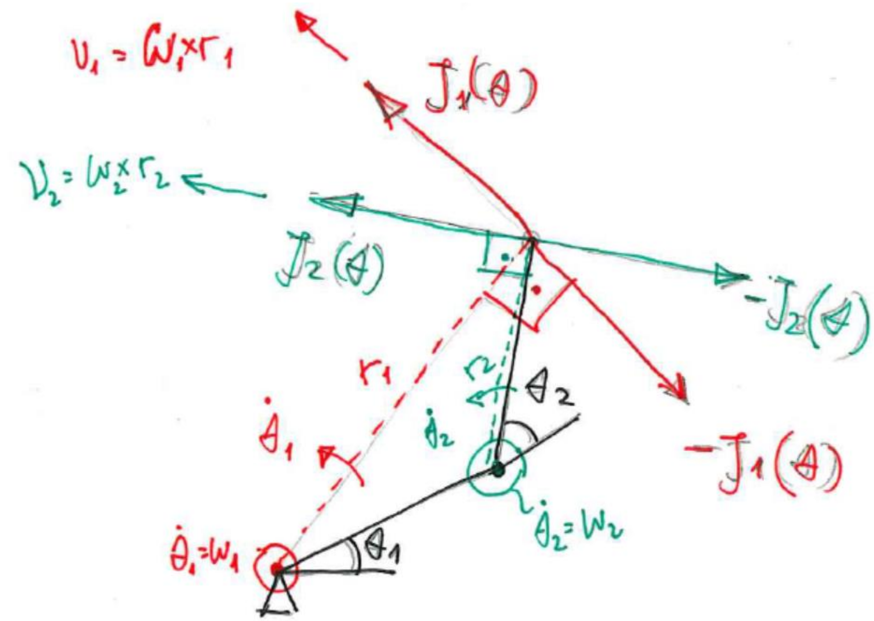
$$V_{y\,tip} = \frac{dy_{tip}}{dt} = L_1 \dot{\theta}_1 c_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) c_{12}$$



$$
\begin{cases}
x_{tip} = l_1 c_1 + l_2 c_{12} \\
y_{tip} = l \; s_1 + l_2 \; s_{12}
\end{cases}
$$

$$V_{x\,tip} = \frac{dx_{tip}}{dt} = -L_1 \dot{\theta}_1 s_1 - l_2(\dot{\theta}_1 + \dot{\theta}_2) s_{12}$$

$$V_{y\,tip} = \frac{dy_{tip}}{dt} = L_1 \dot{\theta}_1 c_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) c_{12}$$



Column 1 of $J(\theta) \rightarrow J_1(\theta)$ when $\dot{\theta}_1 = 1$ $\dot{\theta}_2 = 0$

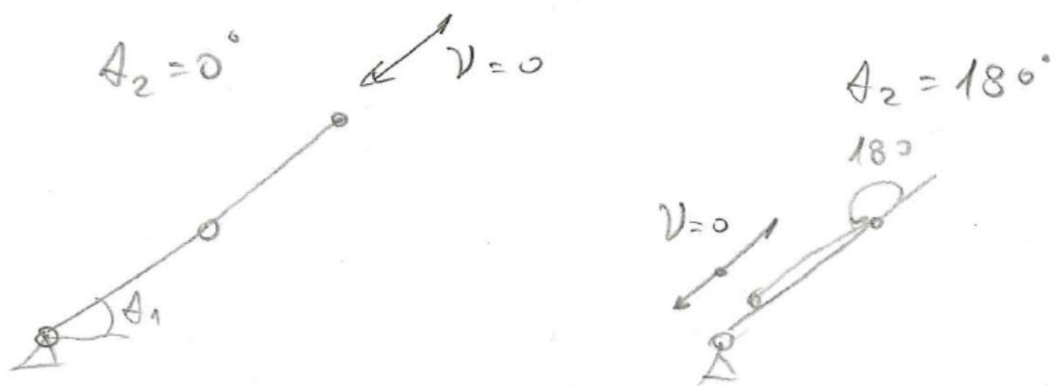
Column 2 of $J(\theta) \rightarrow J_2(\theta)$ when $\dot{\theta}_1 = 0$ $\dot{\theta}_2 = 1$

- As long as $J_1(\theta)$ and $J_2(\theta)$ are not collinear, it is possible to generate an endeffector velocity V_{tip} in any arbitrary direction in the $x_0 - y_0$ plane by choosing appropriate joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$
- Since $J_1(\theta)$, and $J_2(\theta)$ depend on the joint values θ_1 and θ_2 there are some configurations where $J_1(\theta)$, $J_2(\theta)$ become collinear

- If $\left\{\begin{array}{l}\theta_2 = 0 \\ \theta_2 = 180\end{array}\right\}$ regardless of the value of θ_1

  $J_1(\theta)$ and $J_2(\theta)$ will be collinear and the Jacobian $J(\theta)$ become a singular matrix

- Such configurations are called singularities, and they are characterized by a situation where the robot's endeffector is unable to generate velocities in certain directions for any θ_1 $\left\{ \begin{array}{ll} \theta_2 = 0 & J_1 \parallel J_2 \\ \theta_2 = 180 & J_1 \parallel J_2 \end{array} \right]$ $\longrightarrow$ singularities

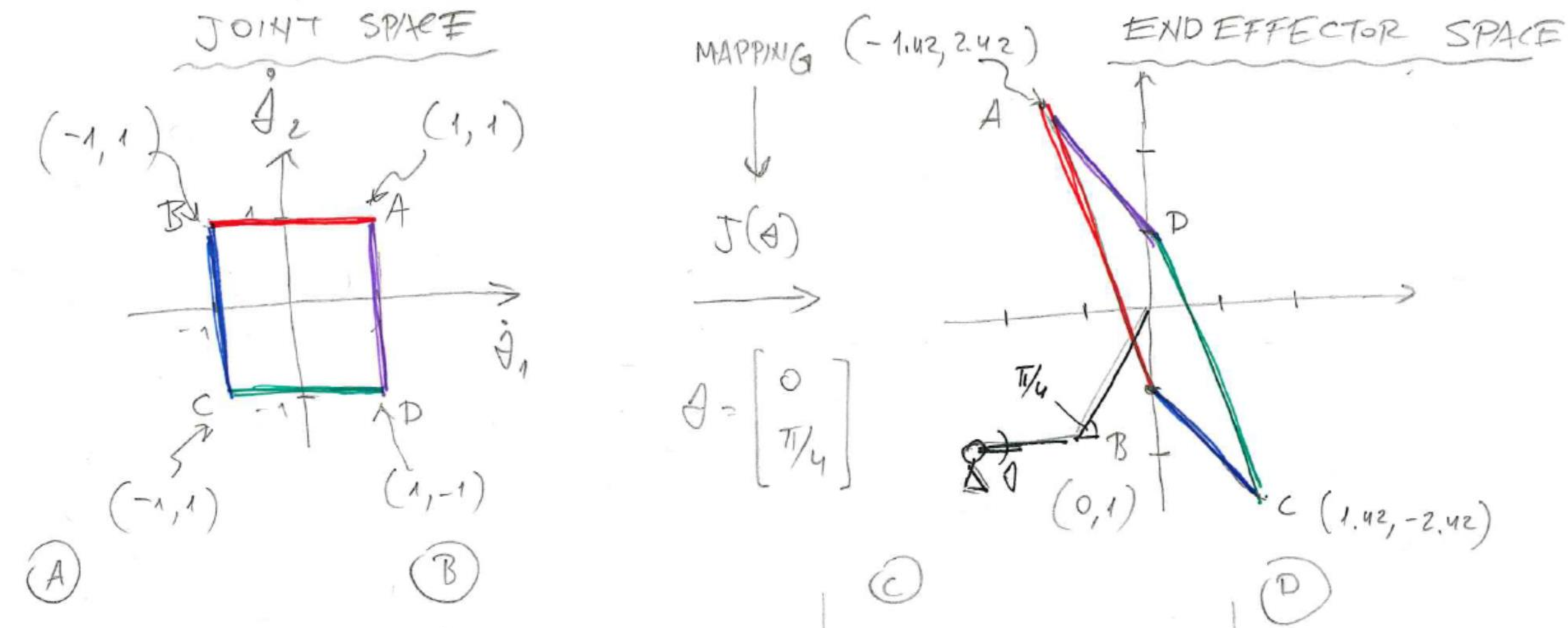


- Substitute $l_1 = 1$; $l_2 = 1$
- Consider the robot at two different non singular postures

$$\theta = \begin{bmatrix} 0 \\ \pi/4 \end{bmatrix} \qquad \theta = \begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \quad ; \quad J\left(\begin{bmatrix} 0 \\ 3\pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 0.29 & -0.71 \end{bmatrix}$$

- The Jacobian can be used to map bounds on rotational speed of the joints ($\dot{\theta}$) to bounds on the endeffector velocity (v_{tip})



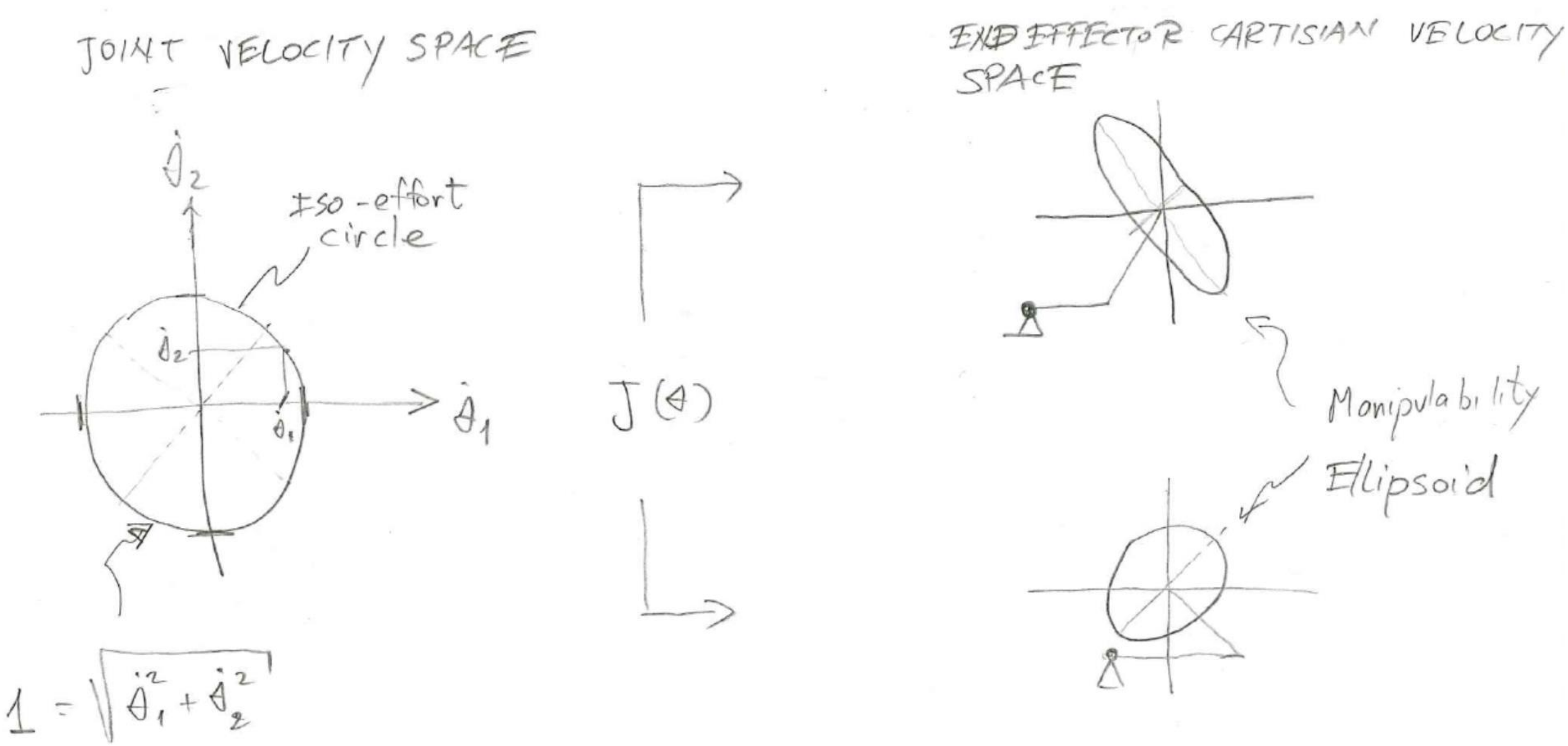
$$V_{tip} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.42 \\ 2.42 \end{bmatrix}$$

$$V_{tip} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$V_{tip} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.42 \\ -2.42 \end{bmatrix}$$

$$V_{tip} = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Rather than mapping a polygon of joint velocities through the jacobian we could insead map a unite circle of joint velocities into the endeffector velocities in the x_0, y_0 plane
- The circle represents an iso-effort contour in the joint velocity space, where total actuator effort is consider to be the sum of sequars of the joint velocities


JOINT VELOCITY SPACE
$\dot{\theta}_2$
Iso-effort circle
$\dot{\theta}_1$
$J(\theta)$
END EFFECTOR CARTISIAN VELOCITY SPACE
Manipulability Ellipsoid
$1 = \sqrt{\dot{\theta}_1^2 + \dot{\theta}_2^2}$

MANIPULABILITY ELLIPSOID & MANIPULABILITY MEASURES - GENERALIZATION

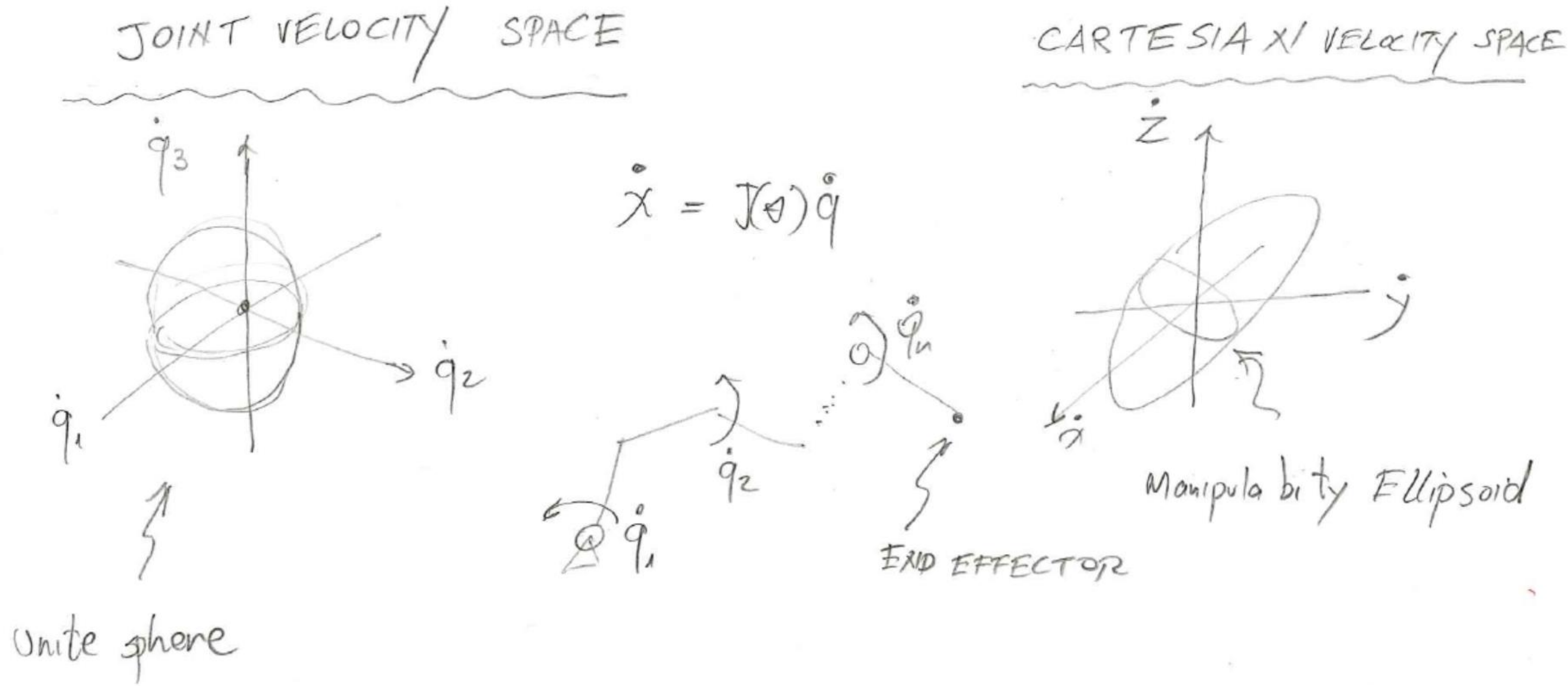
- TASK REQUIRMENTS

  + DESIGN - MECHANISM SIZE

  + POSTURE OF THE ROBOTIC ARM WITHIN THE WORKSPACE FOR PERFORMING A GIVEN TASK

  + EASE OF ARBITRARILY CHANGING THE POSITION AND ORINTATION OF THE END EFFECTOR

MANIPULABILITY ELLIPSOID - DEFENITION



PROBLEM 2

INVERSE ORI. KIM.

$$^0_6R = {}^0_3R \; {}^3_4R \; {}^4_6R$$

$$^0_3R \left(R_{x_3}(\alpha_3) \;\; \underset{\substack{\uparrow \\ D_{x_3}(a_3)}}{I} \;\; R_{z_4}(\theta_4) \;\; \underset{\substack{\uparrow \\ D_{z_4}(d_i)}}{I} \right) {}^4_6R$$

PROBLEM 1

$$^0_6R = \left[{}^0_3R \; R_{x_3}\alpha_3 \right] \left[R_{z_4}(\theta_4) \; {}^4_6R \right]$$

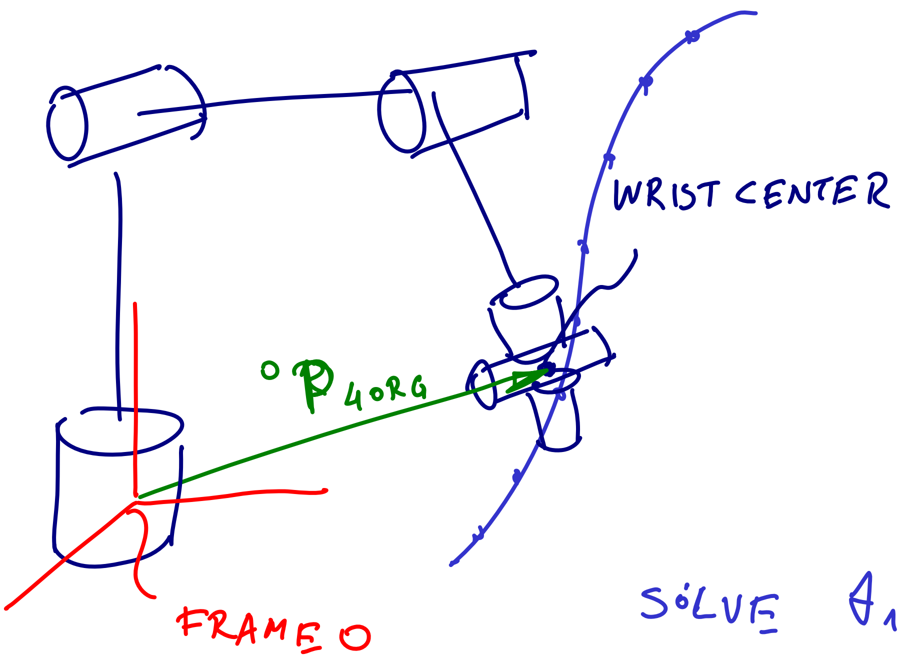
$$R_{z_4}(\theta_4) \; {}^4_6R = \left[{}^0_3R \; R_{x_3}(\alpha_3) \right]^{-1} \; \underbrace{{}^0_6R}_{\text{GIVE}}$$

GIVEN FOR EVERY POINT ALONG THE TRAJECTORY

PROBLEM 1

INVERSE POSITION KIN.

$$^0P_4 = {}^0_1T\,{}^1_2T\,{}^2_3T\;{}^3P_{4ORG}$$



$$^3_4T = \left[\begin{array}{c|c} {}^3_4R & {}^3P_{4ORG} \\ \hline & \end{array}\right]$$

SOLVE $\theta_1, \theta_2, \theta_3$

$${}^0_6T = {}^0_1T\ {}^1_2T\ {}^2_3T \ \Big|\ {}^3_4T\ \Big|\ {}^4_5T\ {}^5_6T$$

PROBLEM 1

PROBLEM 2

$$\{\text{FRAME } 3\} \xrightarrow{\alpha_3} \{\text{FRAME}(R)\} \xrightarrow{a_3} \{\text{FRAME}(Q)\} \xrightarrow{\theta_4} \{\text{FRAME}(P)\} \xrightarrow{d_4} \{\text{FRAME}(4)\}$$

$$R_{x_3}(\alpha_3)\, D_{x_3}(a_3) \ \Big|\ R_{z_4}(\theta_4)\, D_{z_4}(d_4)$$

PROBLEM 1

$${}^3_4T \ \Big|\ \theta_4 = 0$$

PROBLEM 2

$${}^3_4T \ \Big|\ \alpha_3 = 0$$