Inverse Manipulator Kinematics (1/3)

Direct Versus Inverse Kinematics

Direct (Forward) Kinematics

Given: *Joint angles and links geometry*

Compute: *Position and orientation of the end effector relative to the base frame*

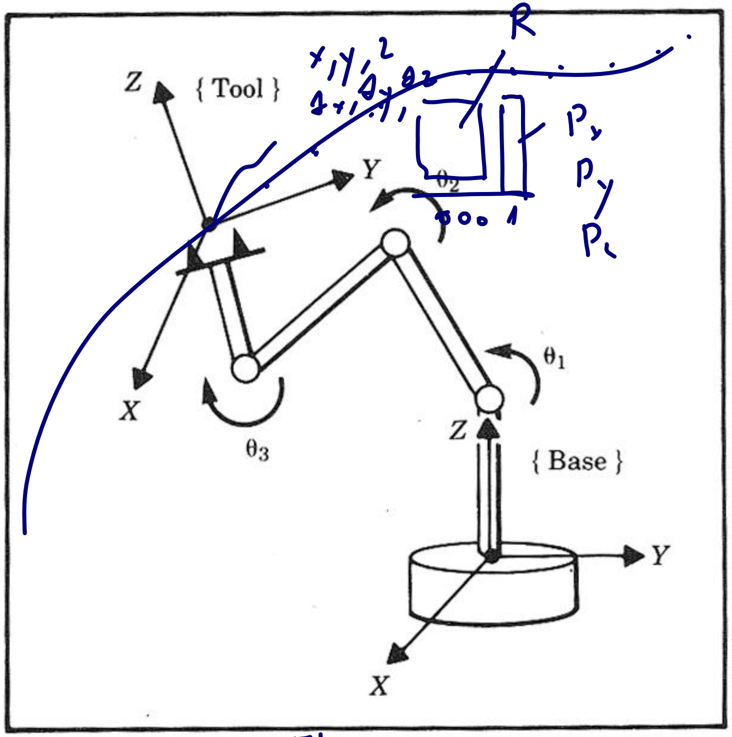
$$f(\theta)={}^{B}_{T}T={}^{0}_{N}T$$

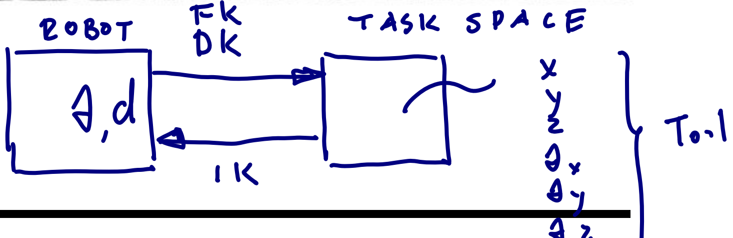
Inverse Kinematics

Given: *Position and orientation of the end effector relative to the base frame*

Compute: *All possible sets of joint angles and links geometry which could be used to attain the given position and orientation of the end effetor*

$$\theta = f^{-1}({}^{B}_{T}T) = f^{-1}({}^{0}_{N}T)$$



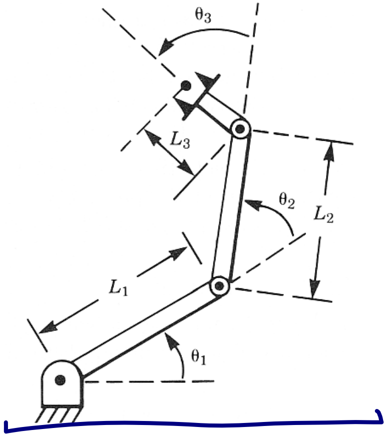


Central Topic - Inverse Manipulator Kinematics - Examples

Geometric Solution - Concept

Decompose spatial geometry into several plane geometry

Examples - *Planar RRR (3R) manipulators -* Geometric Solution



Algebraic Solution - Concept

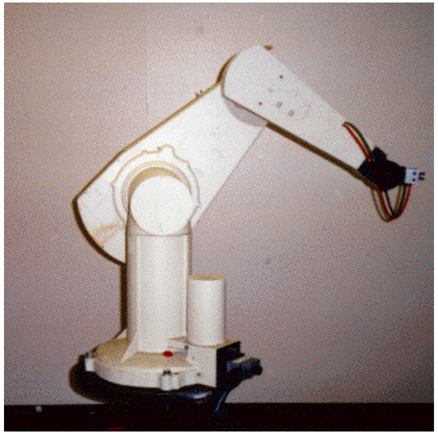
✓

$\theta_1 \dots \theta_6$

$$ {}^{0}_{N}T = {}^{0}_{1}T \dots {}^{N-1}_{N}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Direct Kinematics

Goal (Numeric values)



Examples *- PUMA 560 - Algebraic Solution*

Solvability - PUMA 560

Given : PUMA 560 - 6 DOF, ${}^0_6T$

Solve: $\theta_1 \cdots \theta_6$

$$ {}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T = \begin{bmatrix} \overbrace{r_{11} \quad r_{12} \quad r_{13} \quad p_x}^{\checkmark} \\ r_{21} \quad r_{22} \quad r_{23} \quad p_y \\ r_{31} \quad r_{32} \quad r_{33} \quad p_z \\ 0 \quad\;\; 0 \quad\;\; 0 \quad\;\; 1 \end{bmatrix} $$

✓ $\theta_1, \cdots \theta_6$

$$ \downarrow \left\{ \begin{aligned} r_{11} &= c_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] + s_1(s_4c_5c_6 + c_4s_6), \\ r_{21} &= s_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] - c_1(s_4c_5c_6 + c_4s_6), \\ r_{31} &= -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6, \\ r_{12} &= c_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] + s_1(c_4c_6 - s_4c_5s_6), \\ r_{22} &= s_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] - c_1(c_4c_6 - s_4c_5s_6), \\ r_{32} &= -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6, \\ r_{13} &= -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5, \\ r_{23} &= -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5, \\ r_{33} &= s_{23}c_4s_5 - c_{23}c_5, \\ p_x &= c_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] - d_3s_1, \\ p_y &= s_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] + d_3c_1, \\ p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}. \end{aligned} \right. $$

Total Number of Equations: 12

Independent Equations: 3 - Rotation Matrix
3 - Position Vector

Type of Equations: Non-linear

Solvability

- → **Existence of Solutions**

- **Multiple Solutions**

- **Method of solutions**

  – Close form solution
    - Algebraic solution
    - Geometric solution

  – Numerical solutions

Solvability - Existence of Solution

- For a solution to exist, ${}^{0}_{N}T$ must be in the ***workspace*** of the manipulator

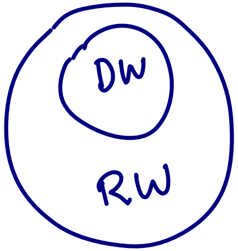
- ***Workspace*** - Definitions

  - ***Dexterous Workspace (DW)***: The subset of space in which the robot end effector can reach *all orientation*.

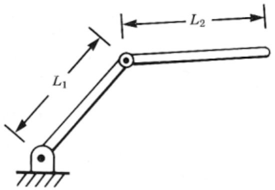
  - ***Reachable Workspace (RW):*** The subset of space in which the robot end effector can reach in *at least 1 orientation*

- The Dexterous Workspace is a subset of the Reachable Workspace

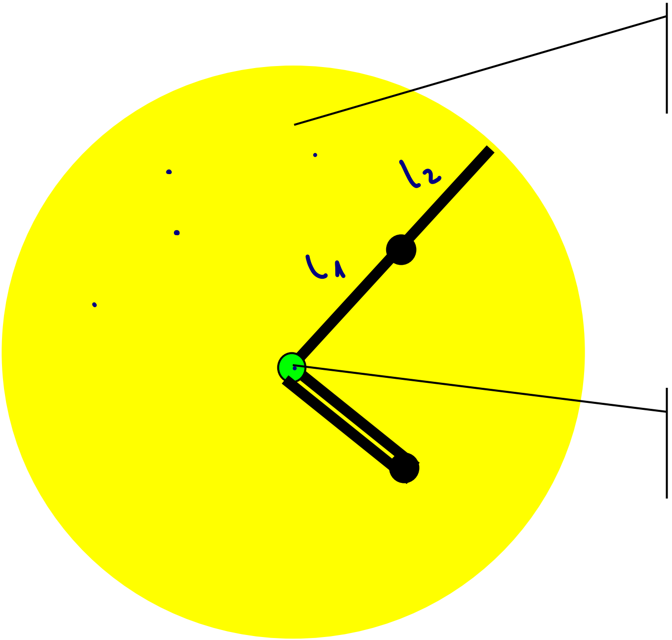
$$DW \subset RW$$



Solvability - Existence of Solution - Workspace - 2R
Example 1 - $L_1 = L_2$



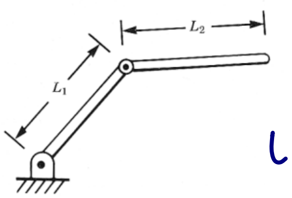
$l_1 = l_2$



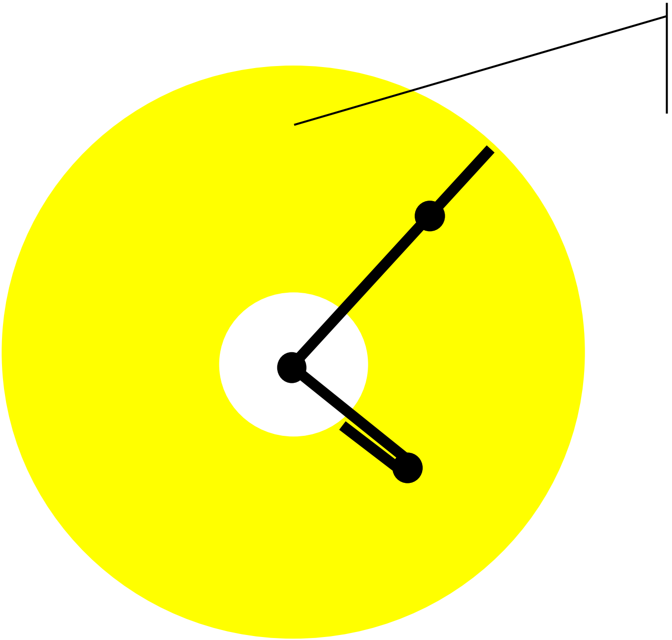
Reachable Workspace RW

Dexterous Workspace DW

Solvability - Existence of Solution - Workspace - 2R
Example 2 - $L_1 \neq L_2$



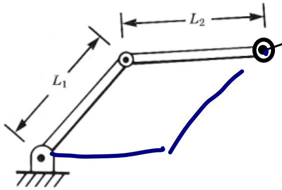
$l_1 \neq l_2$



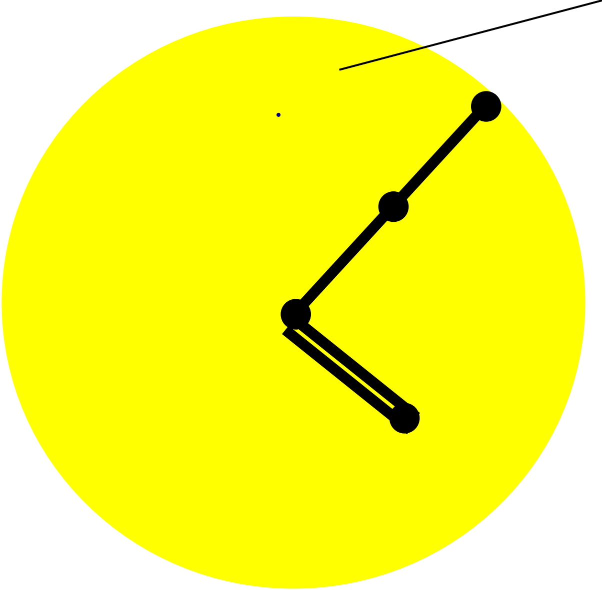
Reachable
Workspace

NO Dexterous
Workspace

Solvability - Existence of Solution - Workspace - 3R
Example 3 - $L_1 = L_2$



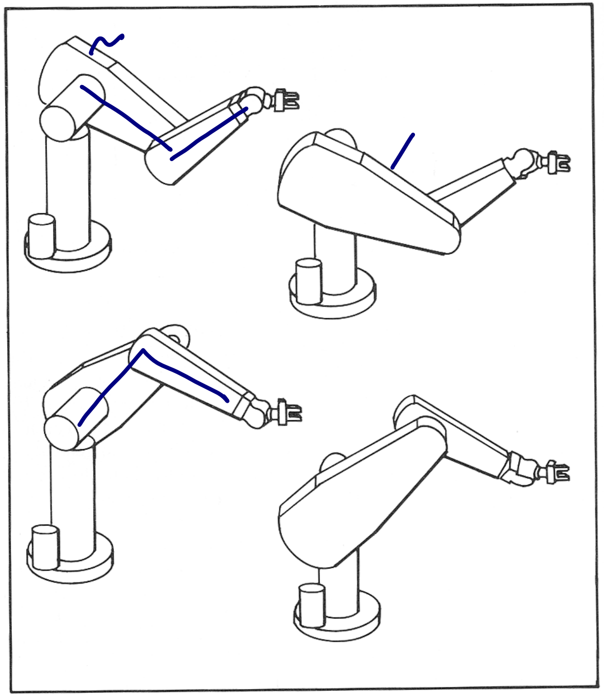
End Effector
Rotation



Reachable
Workspace
&
Dexterous
Workspace

Solvability - Multiple Solutions

- Multiple solutions are a common problem that can occur when solving inverse kinematics because the system has to be able to chose one
- The number of solutions depends on the number of joints in the manipulator but is also a function of the links parameters $a_i \ \alpha_i \ d_i \ \theta_i$
- Example: The PUMA 560 can reach certain goals with 8 different (solutions) arm configurations
  - Four solutions are depicted
  - Four solutions are related to a "flipped" wrist



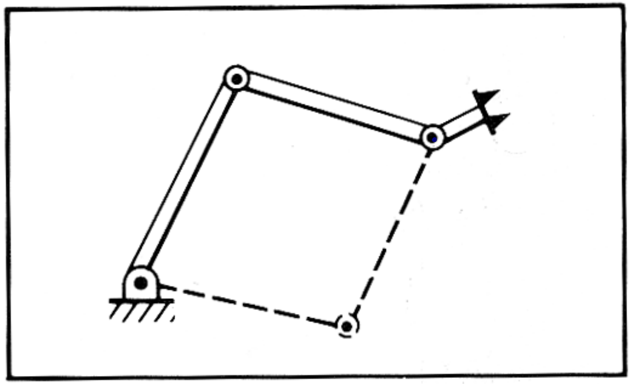
$$\theta'_4 = \theta_4 + 180^o$$

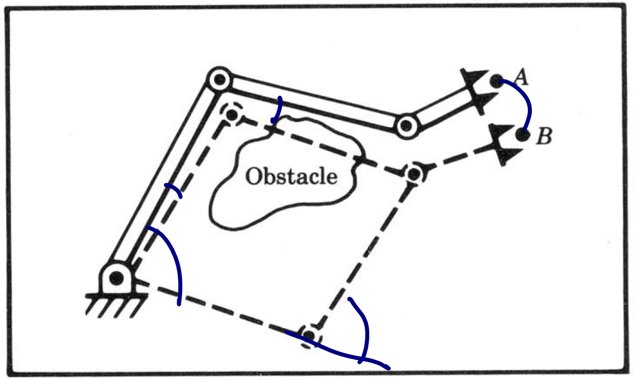
$$\theta'_5 = -\theta_5$$

$$\theta'_6 = \theta_6 + 180^o$$

Solvability - Multiple Solutions

- **Problem:** The fact that a manipulator has multiple multiple solutions may cause problems because the system has to be able to choose one
- **Solution:** Decision criteria
  - The closest (geometrically) - minimizing the amount that each joint is required to move
    - Note 1: input argument - present position of the manipulator
    - Note 2: Joint Weight - Moving small joints (wrist) instead of moving large joints (Shoulder & Elbow)
  - Obstacles exist in the workspace - avoiding collision





Solvability - Multiple Solutions - Number of Solutions

- Task Definition - Position the end effector in a specific point in the plane (2D)

- No. of DOF = No. of DOF of the task

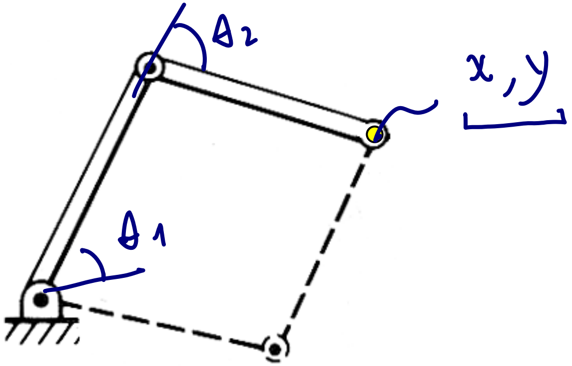
  Number of solution: 2

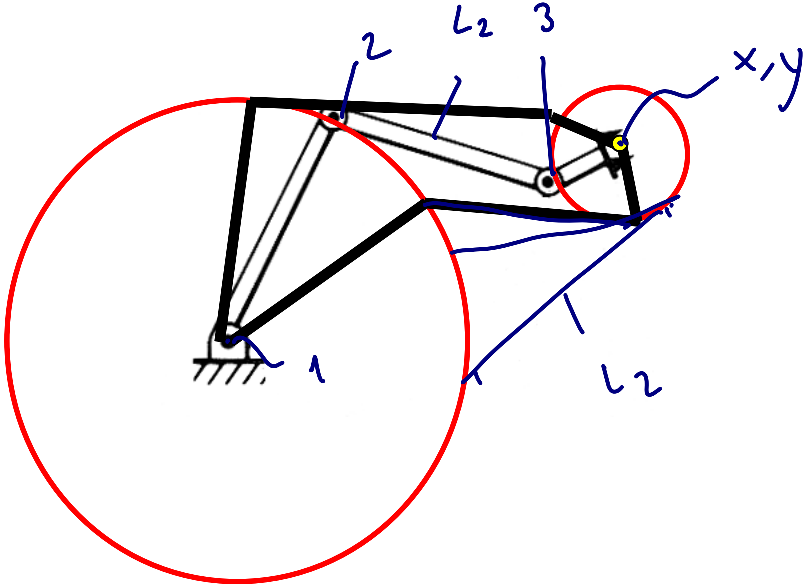
  (elbow up/down)

- No. of DOF > No. of DOF of the task

  Number of solution: ∞

  Self Motion - The robot can be moved without moving the the end effector from the goal





Solvability - Methods of Solutions

- **Solution** (Inverse Kinematics)- A “solution” is the set of joint variables associated with an end effector’s desired position and orientation.

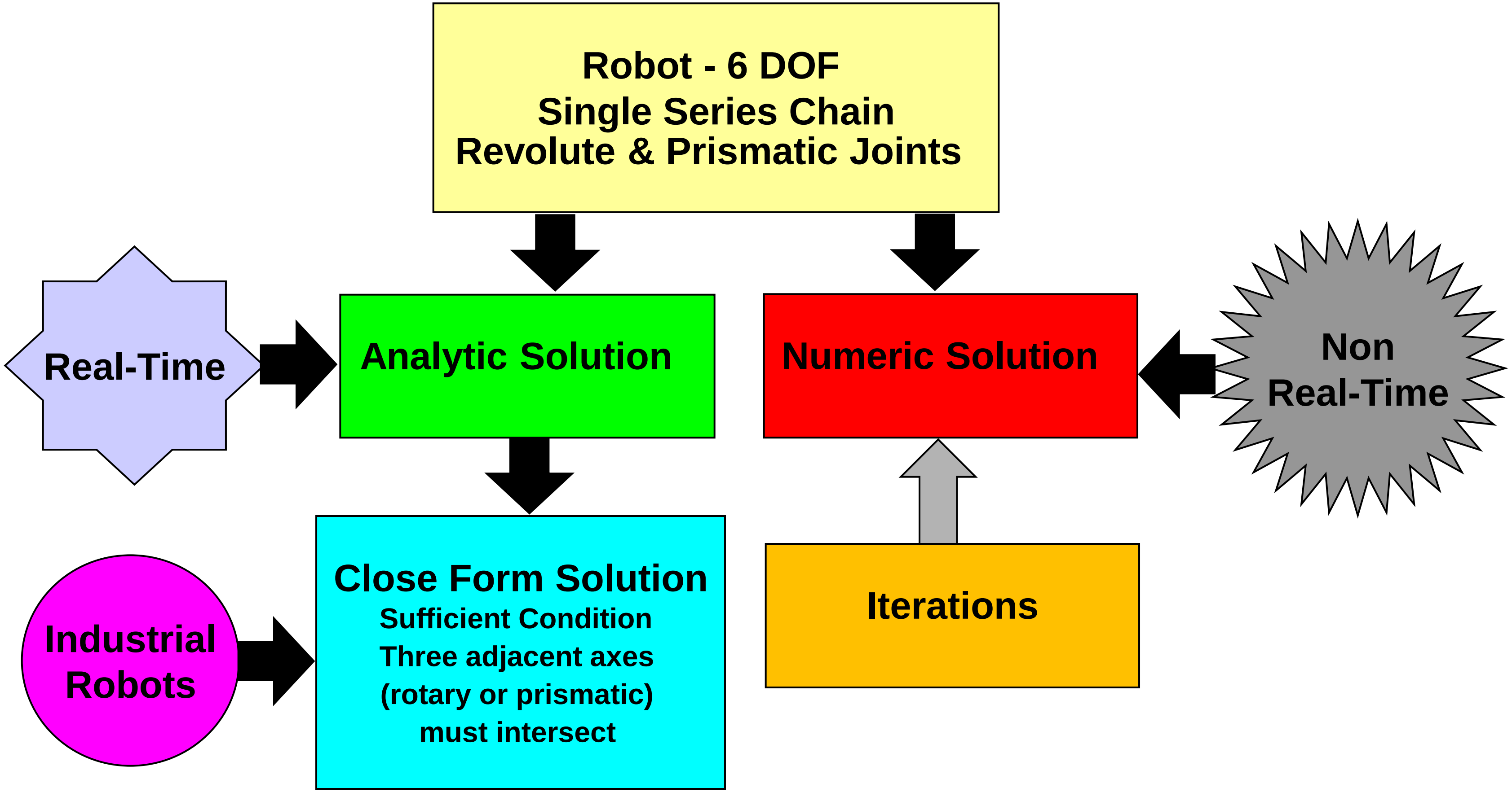
- **No general algorithms** that lead to the solution of inverse kinematic equations.

- **Solution Strategies**

  - **Closed form Solutions** - An analytic expression includes all solution sets.
    - **Algebraic Solution** - Trigonometric (Nonlinear) equations
    - **Geometric Solution** - Reduces the larger problem to a series of plane geometry problems.

  - **Numerical Solutions** - Iterative solutions will not be considered in this course.

Solvability



Robot - 6 DOF
Single Series Chain
Revolute & Prismatic Joints

Real-Time

Analytic Solution

Numeric Solution

Non
Real-Time

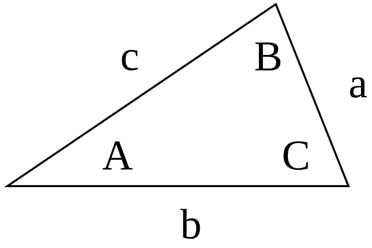
Industrial
Robots

Close Form Solution
Sufficient Condition
Three adjacent axes
(rotary or prismatic)
must intersect

Iterations

Mathematical Equations

- Law of Sinus / Cosines - For a general triangle



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

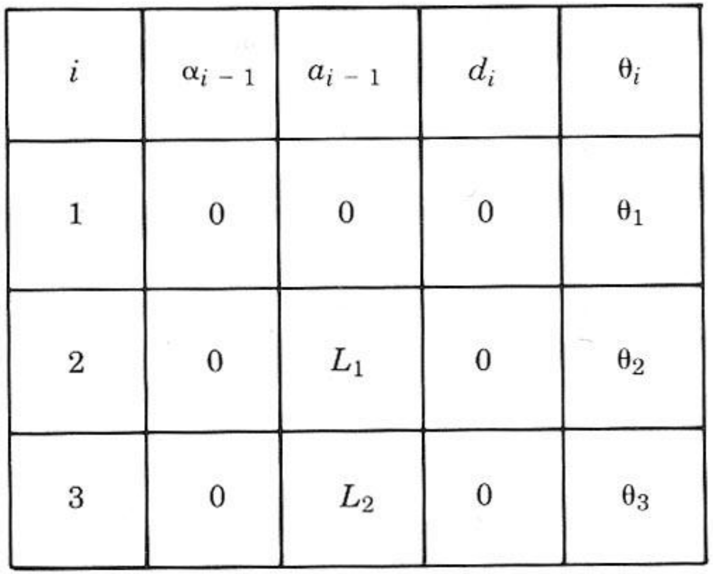
$$a^2 = b^2 + c^2 - 2bc\cos A$$

- Sum of Angles

$$\sin(\theta_1 \pm \theta_2) = s_{12} = c_1 s_2 \pm s_1 c_2$$

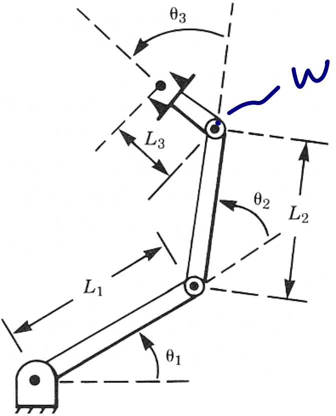
$$\cos(\theta_1 \pm \theta_2) = c_{12} = c_1 c_2 \mp s_1 s_2$$

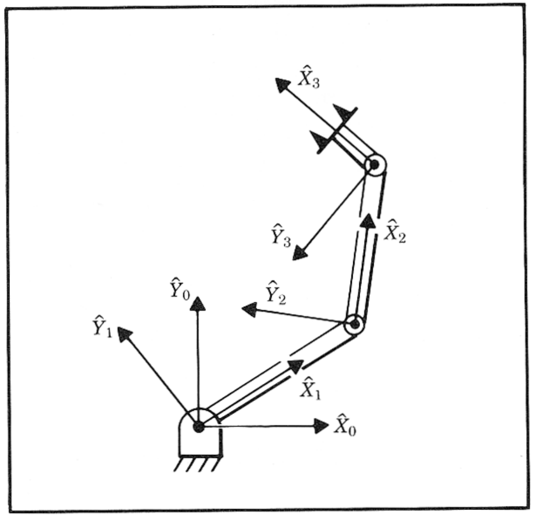
Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 1/12



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

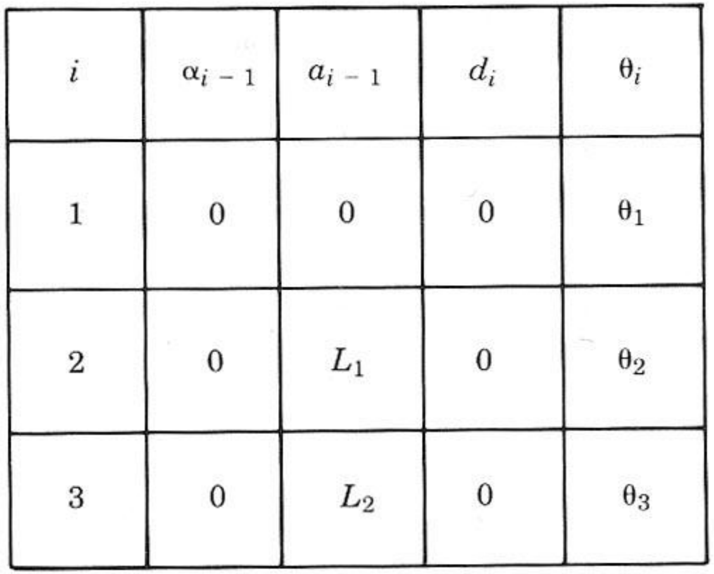
$\underbrace{L_3 = 0}$





Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 2/12

$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$ {}^{0}_{1}T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 3/12

$$ {}^B_W T = {}^0_3 T = {}^0_1 T\, {}^1_2 T\, {}^2_3 T = \begin{bmatrix} c1c2c3 - c1s2s3 - s1s2c3 - s1c2s3 & -c1c2s3 - c1s2c3 + s1s2s3 - s1c2c3 & 0 & c1(L2c2 + L1) - s1s2L2 \\ s1cc3 - s1s2s3 + c1s2c3 + c1c2s3 & -s1c2s3 - s1s2c3 - c1s2s3 + c1c2c3 & 0 & s1(L2c2 + L1) + c1s2L2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

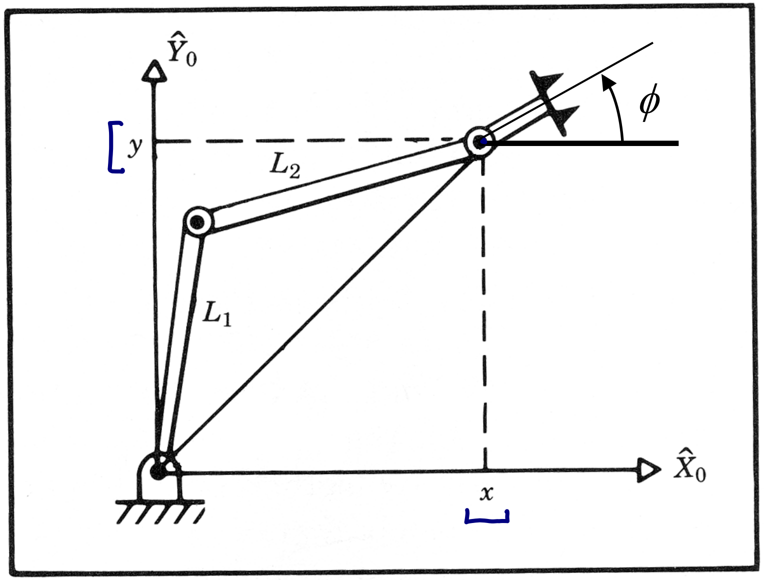
- Using trigonometric identities to simplify ${}^B_W T$, the solution to the forward kinematics is:

$$ ({}^B_W T = {}^0_3 T = \begin{bmatrix} \underline{c_{123}} & -s_{123} & 0 & \overbrace{L_1 c_1 + L_2 c_{12}}^{P_x} \\ \underline{s_{123}} & c_{123} & 0 & \underbrace{L_1 s_1 + L_2 s_{12}}_{P_y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- were $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$ $s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 4/12

- ***Given:***
  - ***Direct Kinematics:*** *The homogenous transformation from the base to the wrist* ${}^{B}_{W}T$
  - ***Goal Point Definition*****:** For a planar manipulator, specifying the goal can be accomplished by specifying three parameters: The position of the wrist in space (x, y) and the orientation of link 3 in the plane relative to the $\hat{X}$ axis (ϕ)



Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 5/12

- ***Problem:***

What are the joint angles $(\theta_1, \theta_2, \theta_3)$ as a function of the wrist position and orientation (x, y, ϕ)

- ***Solution:***
- The goal in terms of position and orientation of the wrist expressed in terms of the homogeneous transformation is defined as follows

√

$\theta_1, \theta_2, \theta_3 = ?$

$$\underline{{}^B_W T_{Goal}} = \begin{bmatrix} \boxed{c_\phi} & -s_\phi & 0 & x \\ \boxed{s_\phi} & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^B_W T = {}^0_3 T = \begin{bmatrix} \boxed{c_{123}} & -s_{123} & 0 & \boxed{L_1 c_1 + L_2 c_{12}} \\ \boxed{s_{123}} & c_{123} & 0 & \boxed{L_1 s_1 + L_2 s_{12}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

=

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 6/12

$${}^{B}_{W}T_{Goal} = {}^{0}_{3}T$$

- A set of four nonlinear equations which must be solved for $\theta_1, \theta_2, \theta_3$

$$\left\{ \begin{array}{l} c_\phi = c_{123} \\ s_\phi = s_{123} \\ \left\{ \begin{array}{l} x = l_1 c_1 + l_2 c_{12} \\ y = l_1 s_1 + l_2 s_{12} \end{array} \right. \end{array} \right.$$

- Solving for θ_2
- If we square x and y add them while making use of $c_{12} = c_1 c_2 - s_1 s_2$;$s_{12} = c_1 s_2 + s_1 c_2$ we obtain

$$\underline{x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2}$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 6/12 (Continue)

$$x = l_1 c_1 + l_2 c_{12} \qquad c_{12} = c_1 c_2 - s_1 s_2$$

$$y = l_1 s_1 + l_2 s_{12} \qquad s_{12} = c_1 s_2 + s_1 c_2$$

$$x^2 = l_1^2 c_1^2 + l_2^2 (c_{12})^2 + 2 l_1 l_2 c_1 c_{12}$$

$$y^2 = l_1^2 s_1^2 + l_2 (s_{12})^2 + 2 l_1 l_2 s_1 s_{12}$$

$$x^2 + y^2 = l_1^2 \left(\cancelto{1}{s_1^2 + c_1^2} \right) + l_2 \left(\cancelto{1}{s_{12}^2 + c_{12}^2} \right) + 2 l_1 l_2 \left(\underbrace{c_1^2 c_2} - \cancel{c_1 s_1 s_2} + \cancel{s_1 c_1 s_2} + \underbrace{s_1^2 c_2} \right)$$

$$c_2 \left(\cancelto{1}{c_1^2 + s_1^2} \right)$$

$$\rightarrow \quad x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \underbrace{c_2}_{\uparrow \; \theta_2}$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 7/12

- Solving for c_2 we obtain

$$\longrightarrow\quad c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\overset{x}{\boxed{\theta_2 = \cos^{-1}\left[\quad\right]}}$$

- Note: In order for a solution to exist, the right hand side must have a value between -1 and 1. Physically if this constraints is not satisfied, then the goal point is too far away for the manipulator to reach.

- Assuming the goal is in the workspace, and making use of we write an expression for s_2 as $\underline{c_2^2 + s_2^2 = 1} \longleftarrow$

$$\longrightarrow\quad s_2 = \enclose{circle}{\pm}\sqrt{1 - c_2{}^2}$$

- Note: The chose of the sign corresponds to the multiple solutions in which we can choose the “elbow-up” or the “elbow-down” solution

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 8/12

- Finally, we compute θ_2 using the two argument arctangent function

$$\theta_2 = \underbrace{\mathrm{Atan2}}(\mathrm{s}_2, \mathrm{c}_2) = A\tan 2(\underbrace{\overset{\downarrow}{\pm}\sqrt{1-c_2^2}}, \underbrace{\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}})$$

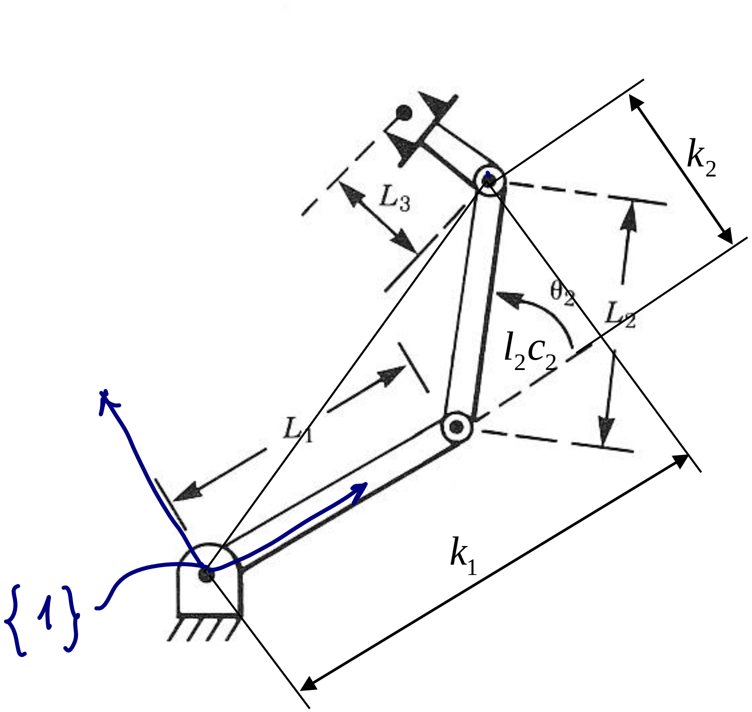
Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 8/12

- Solving for $\boxed{\theta_1}$
- For solving θ_1 we rewrite the original nonlinear equations using a **change of variables** as follows

$$\begin{cases} x = l_1 c_1 + l_2 c_{12} \\ y = l_1 s_1 + l_2 s_{12} \end{cases}$$

$$\rightarrow \begin{cases} x = k_1 c_1 - k_2 s_1 \\ y = k_1 s_1 + k_2 c_1 \end{cases}$$

k_1, k_2



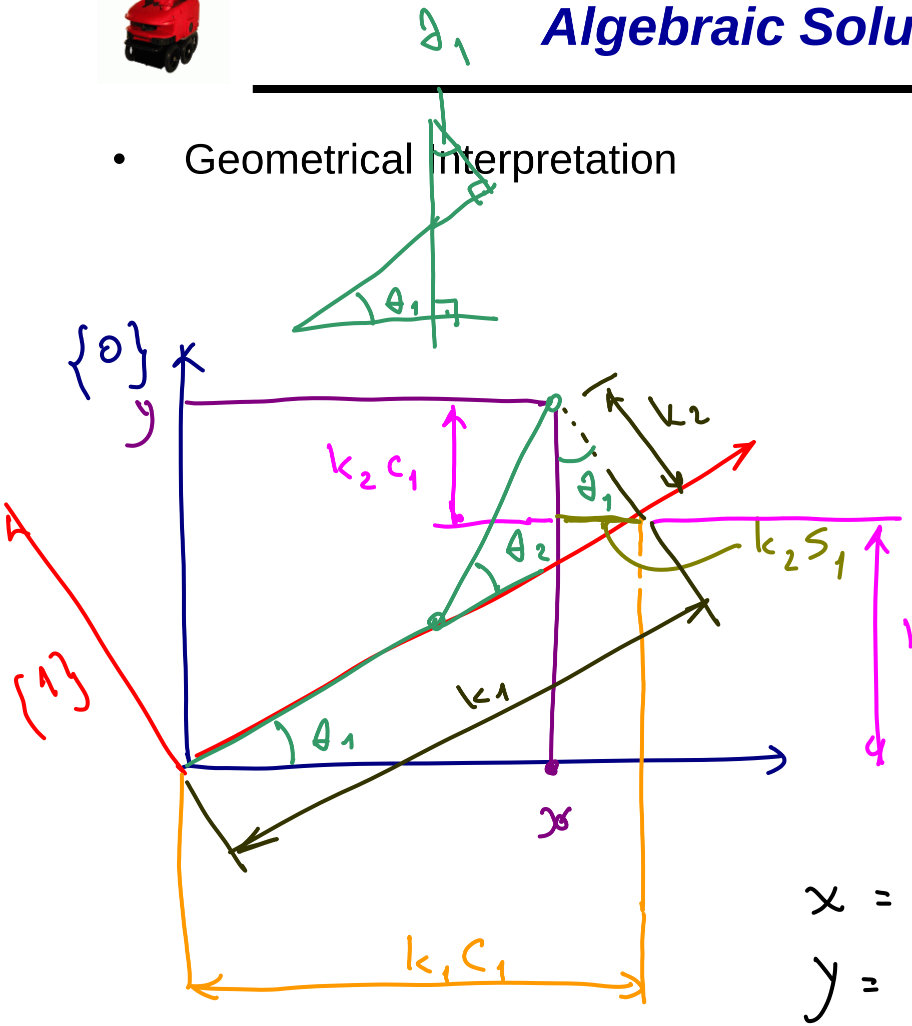
- where

$$\rightarrow \quad k_1 = l_1 + l_2 \underset{\uparrow}{c_2}$$

$$\rightarrow \quad k_2 = l_2 \underset{\uparrow}{s_2}$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 8/12 (continue)

- Geometrical Interpretation



$$x = c_1 k_1 - s_1 k_2$$

$$y = c_1 k_2 + s_1 k_1$$

$$x = c_1 \overbrace{(l_1 + l_2 c_2)}^{k_1} - s_1 \overbrace{(l_1 s_2)}^{k_2}$$

$$y = c_1 \underbrace{(l_2 s_1)}_{k_2} + s_1 \underbrace{(l_1 + l_2 c_2)}_{k_2}$$

$$x = l_1 c_1 + l_1 c_{12} = l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2$$

$$y = l_1 s_1 + l_2 s_{12} = l_1 s_1 + l_2 c_1 s_1 + l_2 s_1 s_2$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 9/12

- Changing the way in which we write the constants k_1 and k_2

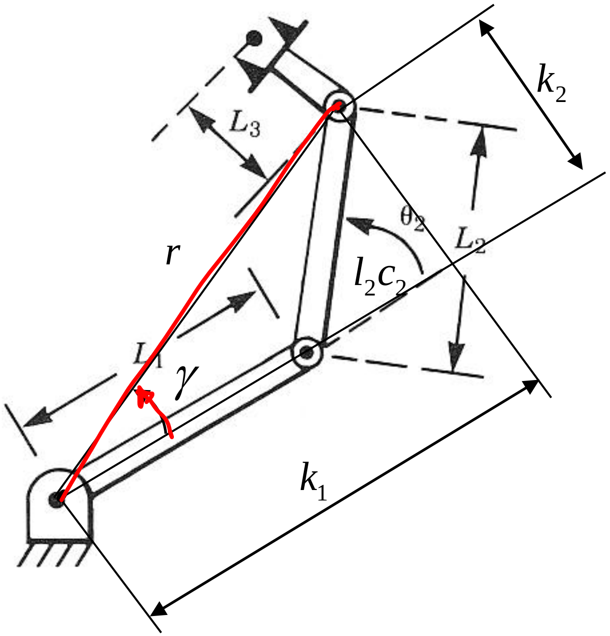
$$r = +\sqrt{k_1^2 + k_2^2}$$

$$\gamma = A\tan 2(k_2, k_1)$$

- Then

$$k_1 = r\cos\gamma$$

$$k_2 = r\sin\gamma$$



Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 10/12

- Base on the previous two transformations the equations can be rewritten as

$$\left\{\begin{aligned} x &= k_1 c_1 - k_2 s_1 \\ y &= k_1 s_1 + k_2 c_1 \end{aligned}\right. \qquad \left\{\begin{aligned} k_1 &= r\cos\gamma \\ k_2 &= r\sin\gamma \end{aligned}\right. \qquad \begin{aligned} x &= r\cos\gamma\cos\underbrace{\theta_1} - r\sin\gamma\sin\underbrace{\theta_1} \\ y &= r\cos\gamma\sin\underbrace{\theta_1} + r\sin\gamma\cos\underbrace{\theta_1} \end{aligned}$$

- or

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$$

- or

$$\frac{x}{r} = \cos(\gamma + \theta_1)$$

$$\frac{y}{r} = \sin(\gamma + \theta_1)$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 11/12

- Using the two argument arctangent we finally get a solution for θ_1

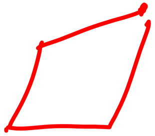
$$\gamma+\theta_1 = \text{Atan}2\left(\frac{y}{\cancel{r}},\frac{x}{\cancel{r}}\right) = \underline{\text{Atan}2(y,x)}$$

By Definition $\rightarrow$ $\gamma = \text{Atan}2(k_2,k_1)$

$$\longrightarrow \theta_1 = \underline{\text{Atan}2(y,x)} - \overbrace{\text{Atan}2(\underset{\theta_2}{\underset{\uparrow}{k_2}},\underset{\theta_2}{\underset{\uparrow}{k_1}})}^{\gamma}$$

$$k_1 = l_1 + l_2 c_2$$

$$k_2 = l_2 s_2$$

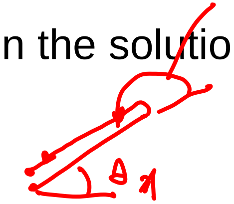


- Note:

(1) When a choice of a sign is made in the solution of θ_2 $\left(\overbrace{\theta_2 = \text{Atan}2(s_2,c_2) = \text{Atan}2\left(\underset{\uparrow}{\pm}\sqrt{1-c_2^2},\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}\right)}^{\theta_2 \downarrow}\right)$, it will cause a sign change in k_2 thus affecting θ_1

$\theta_2 = 180$

(2) If $x = y = 0$ then the solution becomes undefined - in this case θ_1 is arbitrary



$\theta_2 = 180$

$\theta_1 = $ Arbitrary

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution - 12/12

- Solving for θ_3

- Base on the original equations

$$\left\{ \begin{aligned} c_\phi &= c_{123} \\ s_\phi &= s_{123} \end{aligned} \right.$$

- We can solve for the sum of $\theta_1, \theta_2, \theta_3$

$$\overset{\checkmark}{\theta_1} + \overset{\checkmark}{\theta_2} + \theta_3 = A\tan 2(s_\phi, c_\phi) = \phi$$

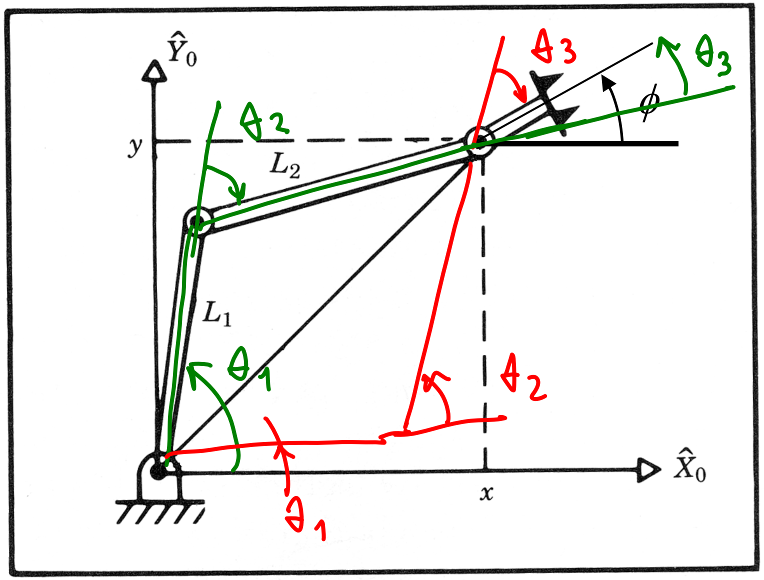
$$\longrightarrow \quad \theta_3 = \overset{\checkmark}{\phi} - \overset{\downarrow}{\theta_1} + \overset{\downarrow}{\theta_2}$$

- Note: It is typical with manipulators that have two or more links moving in a plane that in the course of a solution, expressions for sum of joint angles arise

Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 1/5

- ***Given:***
  - ***Manipulator Geometry***
  - ***Goal Point Definition***: The position x, y and orientation ϕ of the wrist in space
- ***Problem:***

What are the joint angles ($\theta_1, \theta_2, \theta_3$) as a function of the goal (wrist position and orientation)



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 2/5

- ***Solution:***

- We can apply the law of cosines to solve for θ_2

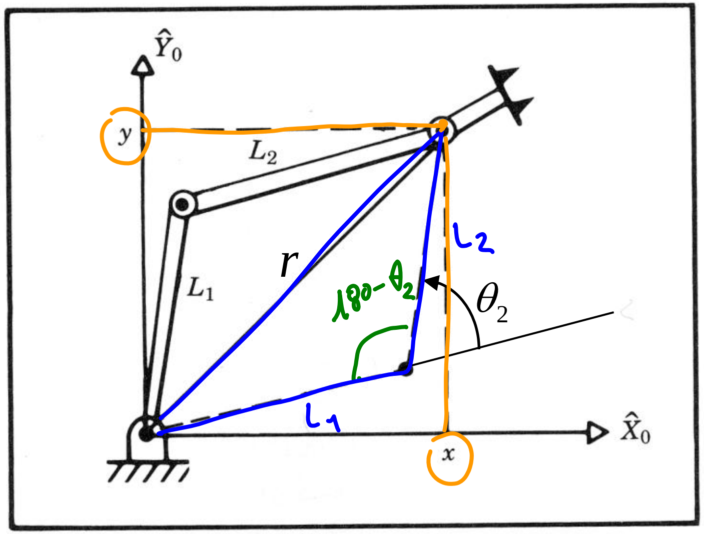
$$r^2 = x^2 + y^2 = \overbrace{l_1^2 + l_2^2 - 2l_1l_2\cos(180-\theta_2)}^{\triangle}$$

- Since

$$\cos(180-\theta_2) = \underbrace{-\cos\theta_2}$$

- We have

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$$



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 3/5

$$\rightarrow \quad c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$$\rightarrow \quad s_2 = \pm\sqrt{1 - c_2^2}$$

$$\rightarrow \quad \theta_2 = A\tan 2(s_2, c_2)$$

Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 4/5

- Note : Condition - Should be checked by the computational algorithm to verify existence of solutions.

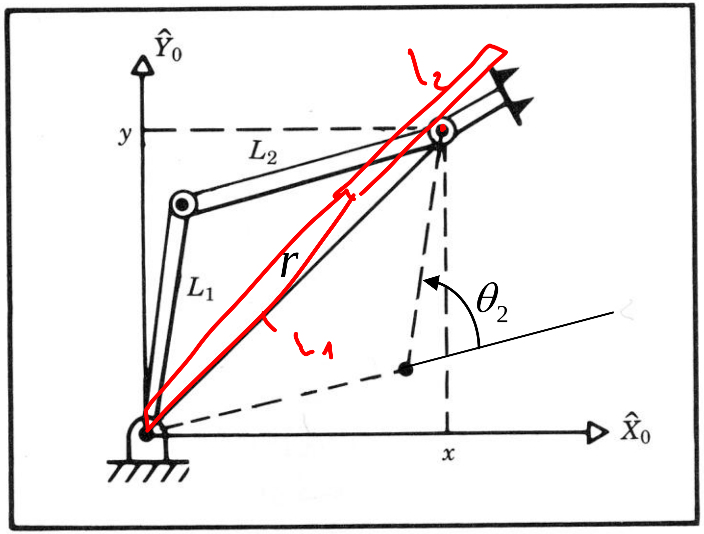
$$\underline{l_1 + l_2 \geq \sqrt{x^2 + y^2}}$$

- Assuming that the solution exist it lies in the range of

$$0^o \leq \theta_2 \leq 180^o$$

- The other possible solution may found by symmetry to be

$$\underline{\theta_2' = -\theta_2}$$



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 5/5

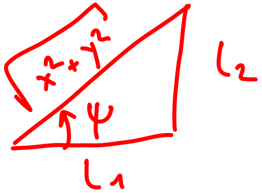
- By definition

$$\theta_1 = \beta \overset{\boxed{\theta_2 < 0}}{\underset{\boxed{\theta_2 > 0}}{\pm}} \psi$$

- Defining β as a function of x,y

$$\beta = A\tan 2(y,x)$$

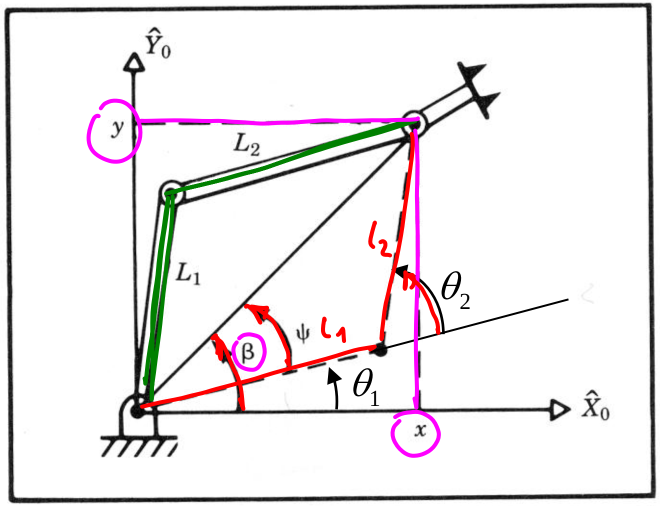
- Applying the law of cosine to find

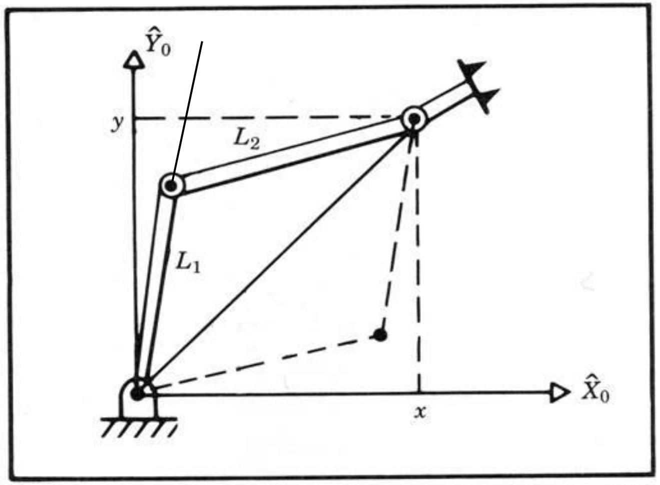


$$\cos\psi = \frac{x^2+y^2+l_1^2-l_2^2}{2l_1\sqrt{x^2+y^2}}$$

- Note: $0^0 \le \psi \le 180^o$

$$\theta_1 = \beta \pm \psi = A\tan 2(y,x) \pm A\tan 2(\sqrt{1-\cos^2\psi}, \cos\psi)$$



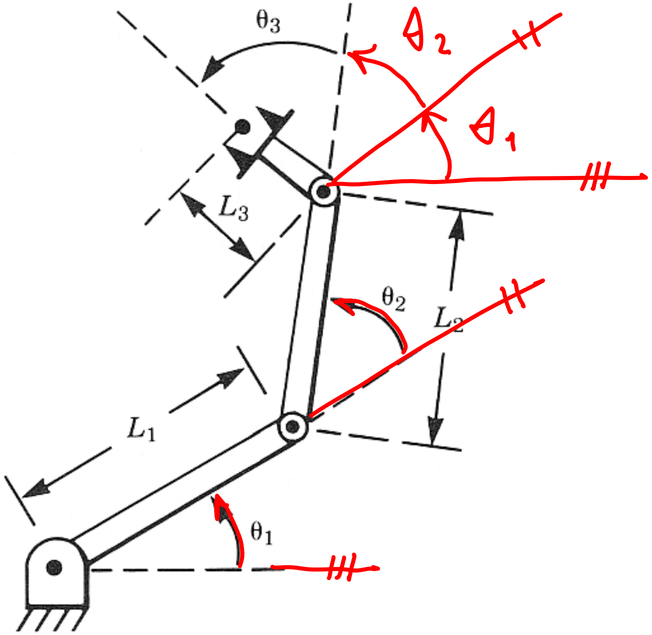


Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 6/5

- Angle in the plane add up to define the orientation of the last link

$$\underbrace{\phi}_{} = \theta_1 + \theta_2 + \theta_3$$

$$\theta_3 = \overset{\checkmark}{\phi} - \left(\overset{\downarrow}{\theta_1} + \overset{\downarrow}{\theta_2}\right)$$



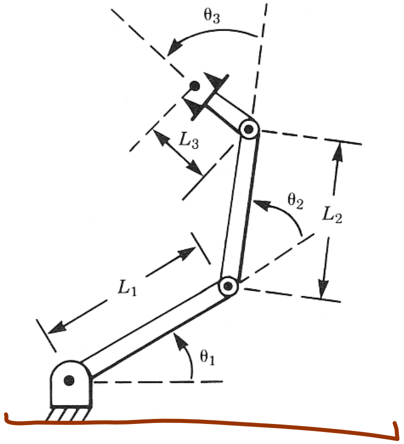
Inverse Manipulator Kinematics (2/4)

Central Topic - Inverse Manipulator Kinematics - Examples

- ***Geometric Solution - Concept***

  Decompose spatial geometry into several plane geometry

  Examples - *Planar RRR (3R) manipulators* - Geometric Solution



- ***Algebraic Solution - Concept***

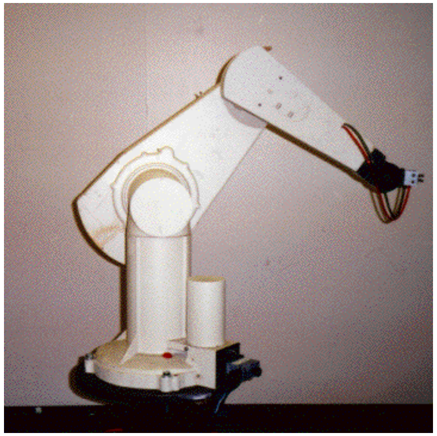
BASE 0

WRIST

$$\underbrace{{}^{0}_{N}T={}^{0}_{1}T\ldots{}^{N-1}_{N}T}_{\theta_1 \ldots \theta_n\ 6,\ \theta_i} = \underbrace{\begin{bmatrix} \boxed{\begin{matrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{matrix}} & \boxed{\begin{matrix} p_x \\ p_y \\ p_z \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}}_{12}$$

Direct Kinematics   Goal (Numeric values)

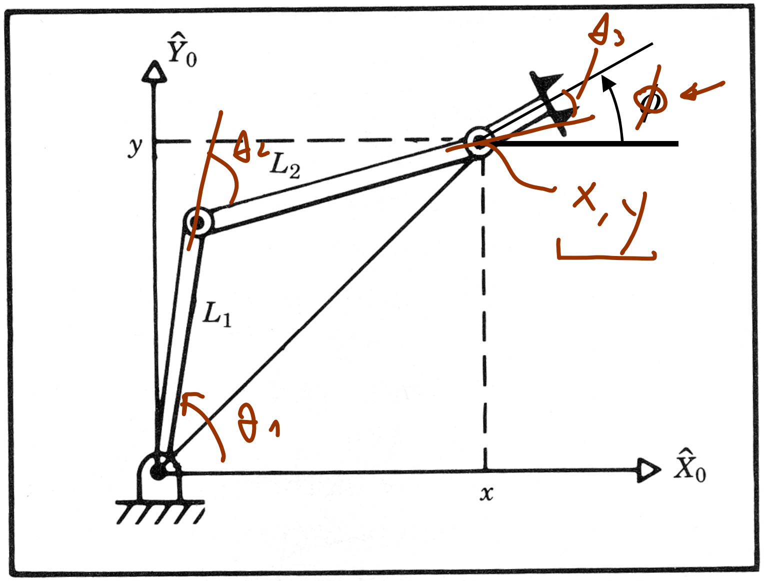
Examples - *PUMA 560 - Algebraic Solution*



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 1/5

- ***Given:***
  - ***Manipulator Geometry***
  - ***Goal Point Definition***: The position x, y and orientation ϕ of the wrist in space
- ***Problem:***

What are the joint angles ($\underbrace{\theta_1, \theta_2, \theta_3}$) as a function of the goal (wrist position and orientation)



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 2/5

- ***Solution:***

- We can apply the law of cosines to solve for θ_2

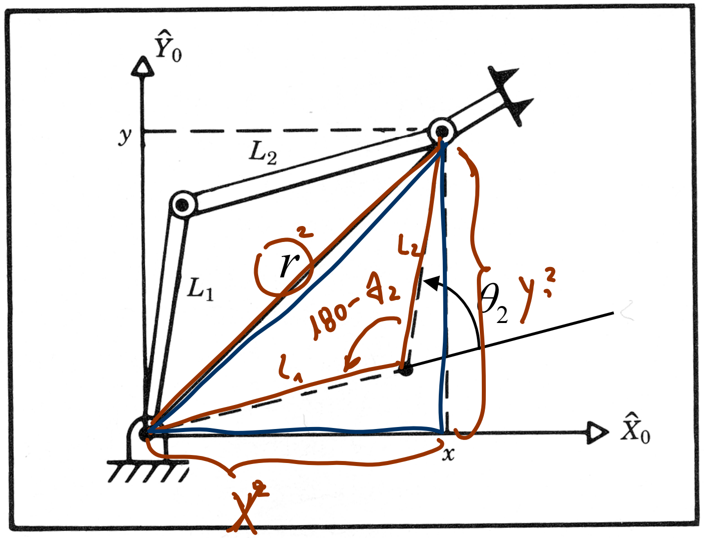
$$\underbrace{r^2}_{} = \underbrace{x^2 + y^2}_{} = \underbrace{l_1^2 + l_2^2 - 2l_1l_2\cos(180 - \theta_2)}_{\uparrow}$$

- Since

$$\cos(180 - \theta_2) = -\cos\theta_2$$

- We have

$$c_2 = \frac{\overset{\downarrow}{x^2} + \overset{\downarrow}{y^2} - l_1^2 - l_2^2}{2l_1l_2}$$



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 3/5

$$\left\{\begin{aligned} c_2 &= \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2} \\[2em] s_2 &= \pm\sqrt{1-c_2^2} \end{aligned}\right.$$

~~$\theta_2 = \cos^{-1}\left(\frac{\sim}{\sim}\right)$~~

$1 = \sin^2\theta_2 + \cos^2\theta_2$

$$\theta_2 = A\tan 2(s_2, c_2)$$

$\uparrow$

Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 4/5

- Note : Condition - Should be checked by the computational algorithm to verify existence of solutions.

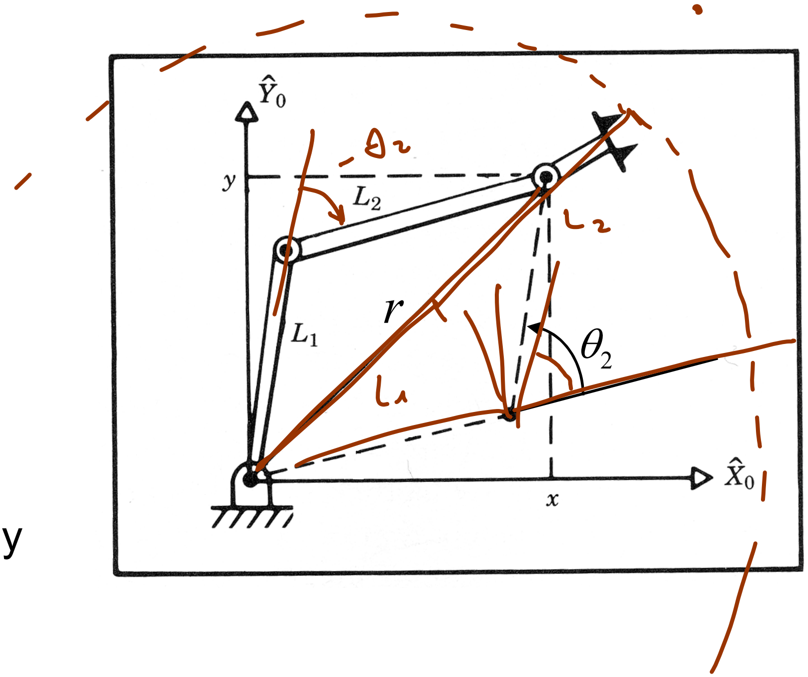
$$l_1 + l_2 \geq \sqrt{x^2 + y^2} \quad \leftarrow$$

- Assuming that the solution exist it lies in the range of

$$\underline{0^o \leq \theta_2 \leq 180^o}$$

- The other possible solution may found by symmetry to be

$$\underline{\theta_2^{'}} = -\theta_2$$



Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 5/5

ϕ

ψ

- By definition

$$\underbrace{\theta_2 < 0}$$

$$\theta_1 = \beta \pm \psi$$

$$\underbrace{\theta_2 > 0}$$

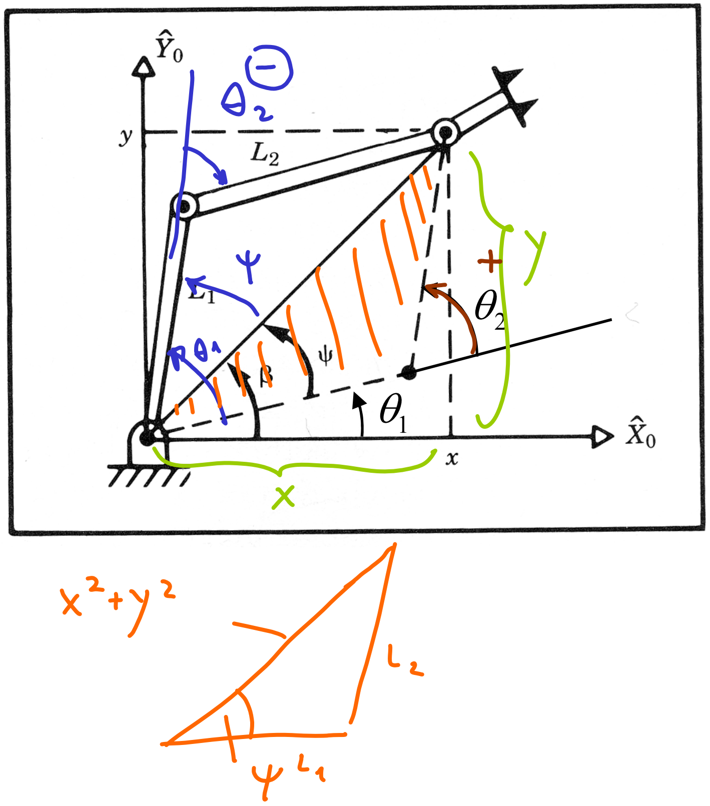
- Defining β as a function of x,y

$$\beta = A\tan 2(y,x)$$

- Applying the law of cosine to find

$$\cos \underbrace{\psi} = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2 l_1 \sqrt{x^2 + y^2}}$$

- Note: $0^0 \le \psi \le 180^o$

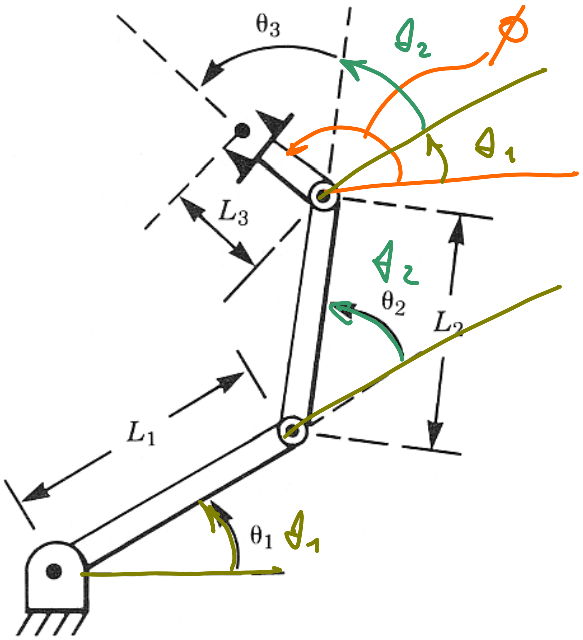


Inverse Kinematics - Planar RRR (3R) - Geometric Solution - 6/5

- Angle in the plane add up to define the orientation of the last link

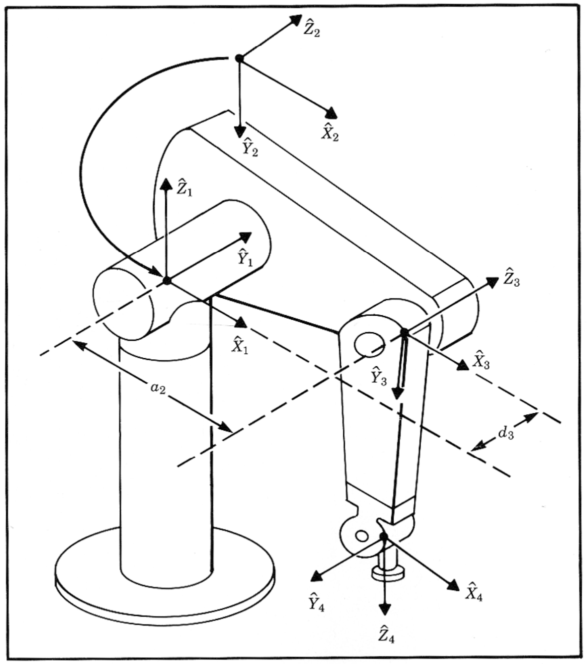
$$\underbrace{\phi}_{} = \theta_1 + \theta_2 + \theta_3$$

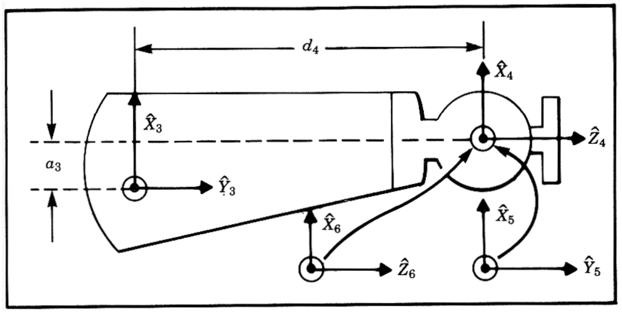
$$\underbrace{\theta_3}_{} = \underset{\uparrow}{\phi} - \underset{\uparrow}{\theta_1} + \underset{\uparrow}{\theta_2}$$



Inverse Kinematics - PUMA 560 - Algebraic Solution - 1/

- *Given:*
  - ***Direct Kinematics:*** *The homogenous transformation from the base to the wrist* ${}^{B}_{W}T$ ${}^{0}_{6}T$
  - ***Goal Point Definition*****:** The position and orientation of the wrist in space





Inverse Kinematics - PUMA 560 - Algebraic Solution - 2/

- ***Problem:***

  What are the joint angles ($\theta_1 \cdots \theta_6$) as a function of the wrist position and orientation (or when ${}^0_6T$ is given as numeric values)

$$
{}^0_6T = \underbrace{\underbrace{{}^0_1T(\theta_1)\;{}^1_2T(\theta_2)\;{}^2_3T(\theta_3)\;{}^3_4T(\theta_4)\;{}^4_5T(\theta_5)\;{}^5_6T(\theta_6)}_{?}}_{\text{Direct Kinematics}} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Goal}}
$$

✓

Inverse Kinematics - PUMA 560 - Algebraic Solution - 3/

- ***Solution (General Technique):*** Multiplying each side of the direct kinematics equation by a an inverse transformation matrix for separating out variables in search of solvable equation

- Put the dependence on θ_1 on the left hand side of the equation by multiplying the direct kinematics eq. with $[{}^0_1T(\theta_1)]^{-1}$ gives

$$[{}^0_1T(\overset{\downarrow}{\theta_1})]^{-1}\ {}^0_6T = \overset{I}{\boxed{[{}^0_1T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)}}\ {}^1_2T(\theta_2)\ {}^2_3T(\overset{\downarrow}{\theta_3})\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6) \leftarrow$$

$$[{}^0_3T(\theta_1,\overset{\downarrow}{\theta_2},\theta_3)]^{-1}\ {}^0_6T = \boxed{[{}^0_3T(\overset{\theta_2,\theta_3}{\theta_1})]^{-1}\ \overset{I}{\ }\underbrace{{}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)}_{{}^0_3T}}\ {}^3_4T(\overset{\downarrow}{\theta_4})\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6) \swarrow$$

$$[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_6T = \boxed{[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)}\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_6T = \boxed{[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)}\ {}^5_6T(\theta_6)$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 4/

- Put the dependence on θ_1 on the left hand side of the equation by multiplying the direct kinematics eq. with $[{}^0_1T(\theta_1)]^{-1}$ gives

$$[{}^0_1T(\theta_1)]^{-1}\ {}^0_6T = \underbrace{[{}^0_1T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)}_{I}\ {}^1_2T(\theta_2)\ {}^2_3T(\overset{\downarrow}{\theta_3})\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1_0T = [{}^0_1T]^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[{}^A_BT]^{-1} = {}^B_AT = \begin{bmatrix} & {}^A_BR^T & & -{}^A_BR^T\ {}^AP_{BORG} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 5/

$$\underbrace{\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{}\underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{} = {}^1_6T$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 6/

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_6 & -c\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{6}T = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}r_{11} = c_{23}\left[c_4c_5c_6 - s_4s_6\right] - s_{23}s_5s_6, $$
$$ {}^{1}r_{21} = -s_4c_5c_6 - c_4s_6, $$
$$ {}^{1}r_{31} = -s_{23}\left[c_4c_5c_6 - s_4s_6\right] - c_{23}s_5c_6, $$

$$ {}^{1}r_{12} = -c_{23}\left[c_4c_5s_6 + s_4c_6\right] + s_{23}s_5s_6, $$
$$ {}^{1}r_{22} = s_4c_5s_6 - c_4c_6, $$
$$ {}^{1}r_{32} = s_{23}\left[c_4c_5s_6 + s_4c_6\right] + c_{23}s_5s_6, $$

$$ {}^{1}r_{13} = -c_{23}c_4s_5 - s_{23}c_5, $$
$$ {}^{1}r_{23} = s_4s_5, $$
$$ {}^{1}r_{33} = s_{23}c_4s_5 - c_{23}c_5, $$

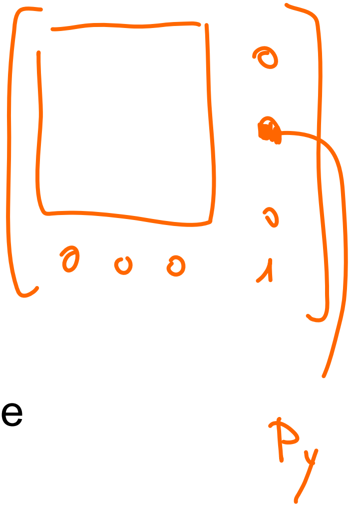
➡ $ {}^{1}p_x = a_2c_2 + a_3c_{23} - d_4s_{23}, $

→ ➡ $ {}^{1}p_y = d_3, $

➡ $ {}^{1}p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}. $

Inverse Kinematics - PUMA 560 - Algebraic Solution - 7/

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T =$$

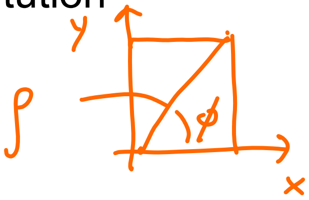


- Equating the (2,4) elements from both sides of the equation we have

$$\boxed{-s_1 p_x + c_1 p_y = d_3} \quad \theta_1$$

- To solve the equation of this form we make the trigonometric substitution

$$\left\{\begin{array}{l} p_x = \overset{\downarrow}{\rho}\cos\phi \\ p_y = \rho\sin\phi \end{array}\right.$$



Inverse Kinematics - PUMA 560 - Algebraic Solution - 8/

$$\rho = \sqrt{p_x^2 + p_y^2}$$

$$\phi = A\tan 2(p_x, p_y)$$

- Substituting p_x, p_y with ρ, ϕ we obtain

$$\underbrace{c_1 s_\phi - s_1 c_\phi} = \frac{d_3}{\rho}$$

- Using the difference of angles formula

$$\rightarrow \quad \sin(\phi - \theta_1) = \frac{d_3}{\rho}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 9/

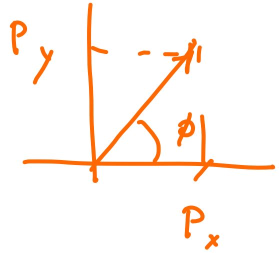
- Based on

$$\sin^2(\phi-\theta_1)+\cos^2(\phi-\theta_1)=1$$

- and so

$$\cos(\phi-\theta_1)=\pm\sqrt{1-\frac{d_3^2}{\rho^2}}$$

$\phi = A\tan 2(p_y, p_x)$



$$\phi-\theta_1=A\tan 2\left(\frac{d_3}{\rho},\pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)$$

- The solution for θ_1 may be written

$$\boxed{\theta_1=A\tan 2(p_y,p_x)-A\tan 2\left(\frac{d_3}{\rho},\pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)}$$

- Note: we have found two possible solutions for θ_1 corresponding to the +/- sign

Inverse Kinematics - PUMA 560 - Algebraic Solution - 10/

- Equating the $\underbrace{(1,4)}$ element and $\underbrace{(3,4)}$ element

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T$$

- We obtain

$$\left(\underbrace{c_1p_x + s_1p_y = a_3c_{23} - d_4s_{23} + a_2c_2}\right)^2$$

$$+ \left(\underbrace{-p_z = a_3c_{23} + d_4s_{23} + a_2c_2}\right)^2$$

$$\overset{\checkmark}{\theta_1}, \boxed{\theta_2}, \overset{?}{\theta_3}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 11/

- If we square the following equations and add the resulting equations

(1) $(\quad)^2$ ⇨ $-s_1p_x + c_1p_y = d_3$

$(\quad)^2$ ⇨ $c_1p_x + s_1p_y = a_3c_{23} - d_4s_{23} + a_2c_2$

$(\quad)^2$ ⇨ $-p_z = a_3s_{23} + d_4c_{23} + a_2s_2$

(1) $-2s_1\cancel{c_1}p_xp_y + \boxed{s_1^2p_x^2} + \boxed{c_1p_y^2} = d_3^2$

(2) $+2s_1\cancel{c_1}p_xp_y + \boxed{c_1^2p_x^2} + \boxed{s_1^2p_y^2} = \boxed{a_3^2c_{23}^2} + \boxed{d_4^2s_{23}^2} + \boxed{a_2^2c_2^2} + \boxed{2a_3c_{23}a_2c_2} - 2a_3c_{23}\cancel{d_4}s_{23}\boxed{-2d_4s_{23}a_2c_2}$

(3) $\boxed{p_z^2} = \boxed{a_3^2s_{23}^2} + \boxed{d_4^2c_{23}^2} + \boxed{a_2^2s_2^2} + \boxed{2a_3s_{23}a_2s_2} + 2a_3\cancel{s_{23}}d_4c_{23} + \boxed{2d_4c_{23}a_2s_2}$

$$p_x^2 + p_y^2 + p_z^2 = a_3^2 + d_4^2 + a_2^2 + d_3^2 + 2a_3a_2c_3 - 2a_2d_4s_3$$

$$\longrightarrow p_x^2 + p_y^2 + p_z^2 - a_3^2 - d_4^2 - a_2^2 - d_3^2 = 2a_2(a_3c_3 - d_4s_3)$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 11/ (Continue)

$$\underbrace{\frac{P_x^2 + P_y^2 + P_z^2 - a_3^2 - d_4^2 - a_2^2 - d_3^2}{2a_2}}_{K} = a_3 c_3 - d_4 s_3$$

$$\overset{\checkmark}{a_3}\,\underbrace{c_3} - \overset{\checkmark}{d_4}\,\underbrace{s_3} = k$$

$\rightarrow d_4 = \rho c\phi$

$\rightarrow a_3 = \rho s\phi$

$\phi = \mathrm{Atan2}(a_3, d_4)$

$\rho^2 = a_3^2 + d_4^2$

$$\frac{k}{\rho} = \underbrace{c_3 s\phi - s_3 \phi}$$

$$\left.\begin{array}{l} \dfrac{k}{\rho} = \sin(\phi - \theta_3) \\ \pm\sqrt{1-\left(\dfrac{k}{\rho}\right)^2} = \cos(\phi - \theta_3) \end{array}\right]$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 11/ (Continue)

$$\phi - \theta_3 = \text{Atan}\,2\left(\underbrace{\frac{k}{\rho}\ ,\ \pm\sqrt{1-\frac{k^2}{\rho^2}}}_{\times\rho}\right)$$

$$\rho > 1$$

$$\phi - \theta_3 = \text{Atan}\,2\left(k \pm \sqrt{\rho^2 - k^2}\right)$$

$$\underbrace{\uparrow}$$

$$\phi - \theta_3 = \text{Aant}\,2\left(k \pm \sqrt{a_3^2 + d_4^2 - k^2}\right)$$

$$\frac{k}{\pm\sqrt{a_3^2 + d_4^2 - k^2}}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 12/

- we obtain

$$a_3c_3 - d_4s_3 = K$$

- where

$$K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - a_4^2}{2a_2}$$

- Note that the dependence on θ_1 has be removed. Moreover the eq. for θ_3 is of the same form as the eq. for θ_1 and so may be solved by the same kind of trigonometric substitution to yield a solution for θ_3

Inverse Kinematics - PUMA 560 - Algebraic Solution - 13/

$$\boxed{\theta_3 = A\tan 2(a_3, d_4) - A\tan 2\left(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}\right)}$$

$\bar{\theta}_3$

$\bar{\bar{\theta}}_3$

- Note that the +/- sign leads to two different solution for θ_3

Inverse Kinematics - PUMA 560 - Algebraic Solution - 14/

$$[{}^0_3T(\theta_1,\theta_2,\theta_3)]^{-1}\ {}^0_6T = \overset{I}{\boxed{[{}^0_3T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)}}\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & \underline{a_3} \\ s_5c_6 & -s_5s_6 & -c_5 & \underline{d_4} \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the $\underline{(1,4)}$ element and $\underline{(2,4)}$ element we obtain

$$c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3$$

$$-c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4$$

? ✓

θ_2 , θ_3

- These equations may be solved simultaneously for s_{23} and c_{23} resulting in

Inverse Kinematics - PUMA 560 - Algebraic Solution - 14/ (Continue)

$$c_{23}, s_{23} \begin{cases} c_{23}\left[\underbrace{c_1 p_x + s_1 p_y}_{}\right] - s_{23}\left[\underbrace{p_z}_{}\right] = a_3 + a_2 c_3 \\ c_{23}\left[\underbrace{p_z}_{}\right] + s_{23}\left[\underbrace{c_1 p_x + s_1 p_y}_{}\right] = -d_4 + a_2 a_3 \end{cases}$$

$$\Delta = (c_1 p_x + s_1 p_y)^2 + p_z^2$$

$$c_{23} = \frac{\begin{vmatrix} a_3 + a_2 c_3 & -p_z \\ a_2 s_3 - d_4 & c_1 p_x + s_1 p_y \end{vmatrix}}{\Delta} = \frac{(a_3 + a_2 c_3)(c_1 p_x + s_1 p_x) + p_z (a_3 s_3 - d_4)}{\Delta}$$

$$s_{12} = \frac{\begin{vmatrix} c_1 p_x + s_1 p_y & a_3 + a_2 c_3 \\ p_z & a_2 s_3 - d_4 \end{vmatrix}}{\Delta} = \frac{(c_1 p_x + s_1 p_y)(a_3 s_3 - d_4) - p_z (a_3 + a_2 c_3)}{\Delta}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 15/

$$s_{23} = \frac{(-a_3 - a_2c_2)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4)}{p_z^2 + (c_1p_x + s_1p_y)^2}$$

$$c_{23} = \frac{(a_2s_3 - d_4)p_z - (-a_3 - a_2c_3)(c_1p_x + s_1p_y)}{p_z^2 + (c_1p_x + s_1p_y)^2}$$

- Since the denominator are equal and positive, we solve for the sum of θ_2 and θ_3 as

$\theta_2 + \theta_3$

$$\theta_{23} = A\tan 2[(-a_3 - a_2c_2)p_z + (c_1p_x + s_1p_y)(a_2s_3 - d_4), (a_2s_3 - d_4)p_z - (-a_3 - a_2c_3)(c_1p_x + s_1p_y)]$$

- The equation computes four values of θ_{23} according to the four possible combination of solutions for θ_1 and θ_3

$\uparrow$ +/- $\quad$ $\uparrow$ +/-

	θ_1	θ_3
θ_2	+	+
	+	−
	−	+
	−	−

Inverse Kinematics - PUMA 560 - Algebraic Solution - 16/

- Then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3$$

- Equating the (1,3) and the (3,3) elements

$\theta_{\cancel{5}4}$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & \boxed{-c_4s_5} & a_3 \\ s_5c_6 & -s_5s_6 & -c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & \boxed{s_4s_5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- we get

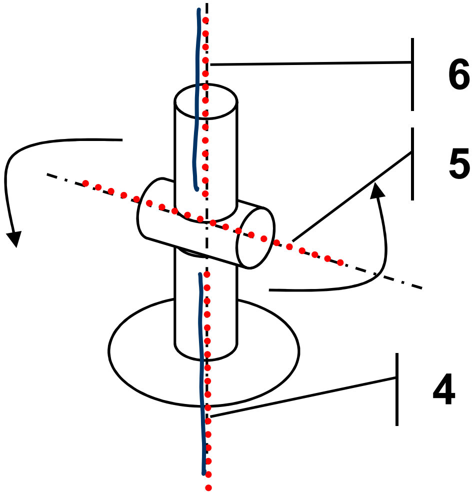
$$\left.\begin{aligned} r_{13}c_1c_{23} + r_{23}s_1c_{23} - s_{23}r_{33} &= -\overset{\downarrow}{c_4}s_5 \\ -r_{13}s_1 + r_{23}c_1 &= \underset{\uparrow}{s_4}s_5 \end{aligned}\right\}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 17/

- As long as $s_5 \neq 0$ we can solve for θ_4

$$\boxed{\theta_4 = A\tan 2(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + s_{23}r_{33})}$$

- When $\theta_5 = 0$ the manipulator is in a ***singular configuration*** in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot. In this case all that can be solved for is the sum or difference of θ_4 and θ_6 . This situation is detected by checking whether both arguments of Atan2 are near zero. If so θ_4 is chosen arbitrary (usually chosen to be equal to the present value of joint 4), and θ_6 is computed later, it will be computed accordingly



Inverse Kinematics - PUMA 560 - Algebraic Solution - 18/

$$[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_6T = \boxed{[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)}\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$\begin{bmatrix} c_1c_{23}c_4+s_1s_4 & s_1c_{23}c_4-c_1s_4 & -s_{23}c_4 & -a_2c_3c_4+d_3s_4-a_3c_4 \\ -c_1c_{23}s_4+s_1c_4 & -s_1c_{23}s_4-c_1c_4 & s_{23}s_4 & a_2c_3s_4+d_3c_4-a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & c_{23} & a_2s_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the (1,3) and the (3,3) elements we get

$$\left.\begin{aligned} r_{13}(c_1c_{23}c_4+s_1s_4)+r_{23}(s_1c_{23}c_4-c_1s_4)-r_{33}(s_{23}c_4) &= -s_5 \\ r_{13}(-c_1s_{23})+r_{23}(-s_1s_{23})+r_{33}(-c_{23}) &= c_5 \end{aligned}\right\}$$

- We can solve for θ_5

$$\boxed{\theta_5 = A\tan 2(s_5, c_5)}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 19/

$$[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_6T = \boxed{[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)}\ {}^5_6T(\theta_6)$$

$$\begin{bmatrix} c_1c_{23}c_4c_5+s_1s_4c_5-s_1c_{23}c_4s_5+c_1s_4s_5 & s_{23}c_4 & c_1c_{23}c_4s_5+s_1s_4s_5+s_1c_{23}c_4c_5-c_1s_4c_5 & -a_2c_3c_4+d_3s_4-a_3c_4 \\ -c_1c_{23}s_4c_5+s_1c_4c_5+s_1c_{23}s_4s_5+c_1c_4s_5 & -s_{23}s_4 & -c_1c_{23}s_4s_5+s_1c_4s_5-s_1c_{23}s_4c_5-c_1c_4c_5 & a_2c_3s_4+d_3c_4-a_3s_4 \\ -c_1s_{23}c_5+s_1s_{23}s_5 & -c_{23} & -c_1s_{23}s_5-s_1s_{23}c_5 & a_2s_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_6 & -c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the (3,1) and the (1,1) elements we get

$$r_{11}(c_1s_{23}c_5-s_1s_{23}s_5)+r_{21}(c_{23})+r_{31}(+c_1s_{23}s_5+s_1s_{23}c_5)=s_6$$

$$r_{11}(c_1c_{23}c_4c_5+s_1s_4c_5-s_1c_{23}c_4s_5+c_1s_4s_5)+r_{21}(s_{23}c_4)+r_{31}(c_1c_{23}c_4s_5+s_1s_4s_5+s_1c_{23}c_4c_5-c_1s_4c_5)=c_6$$

- We can solve for θ_6

$$\boxed{\theta_6 = A\tan 2(s_6, c_6)}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 21/

- Summary - Number of Solutions

- Four solution

$$\theta_1 = A\tan 2(p_y, p_x) - A\tan 2\left(\frac{d_3}{\rho}, \pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)$$

$$\theta_3 = A\tan 2(a_3, d_4) - A\tan 2\left(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}\right)$$

- For each of the four solutions the wrist can be flipped

$$\theta_4^{'} = \theta_4 + 180^o$$
$$\theta_5^{'} = -\theta_5$$
$$\theta_6^{'} = \theta_6 + 180^o$$

Inverse Kinematics - PUMA 560 - Algebraic Solution - 22/

- After all eight solutions have been computed, some or all of them may have to be discarded because of joint limit violations.

- Of the remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.

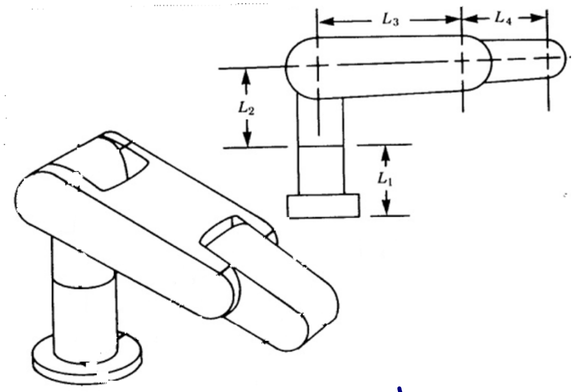
Inverse Manipulator Kinematics (3/4)

Central Topic - Inverse Manipulator Kinematics - Examples

- ***Geometric Solution - Concept***

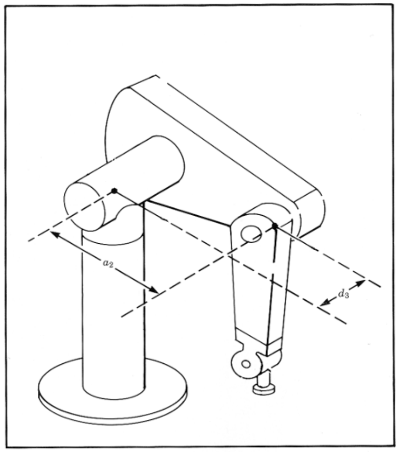
  Decompose spatial geometry into several plane geometry

  Example - *3D - RRR (3R) manipulators -* Geometric Solution



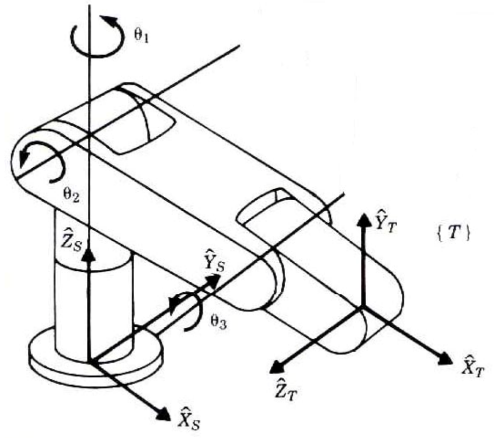
- ***Algebraic Solution*** *(closed form) - Piepers Method - Last three consecutive axes intersect at one point*

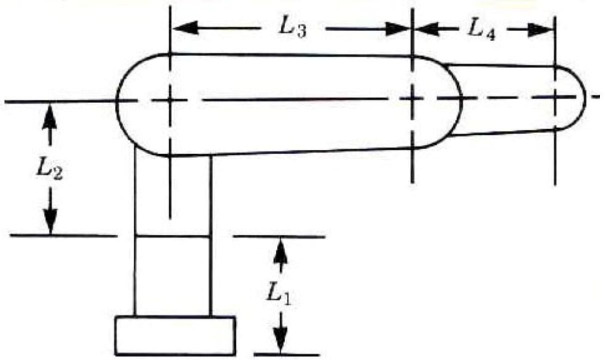
  Example - *Puma 560*



Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 1/

- ***Given:***
  - ***Manipulator Geometry***
  - ***Goal Point Definition***: The position $\underline{x_d, y_d, z_d}$ of the wrist in space

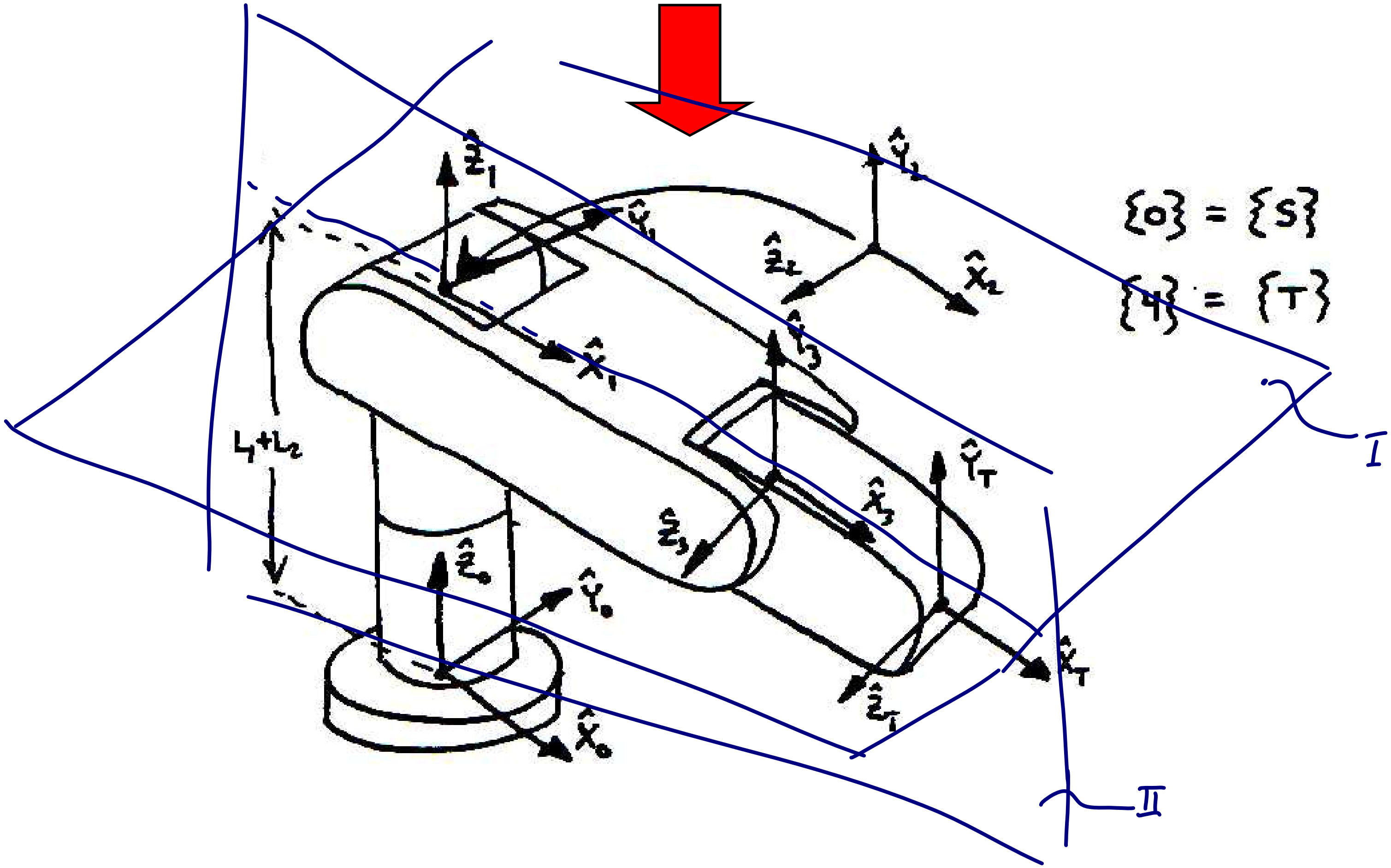




- ***Problem:***

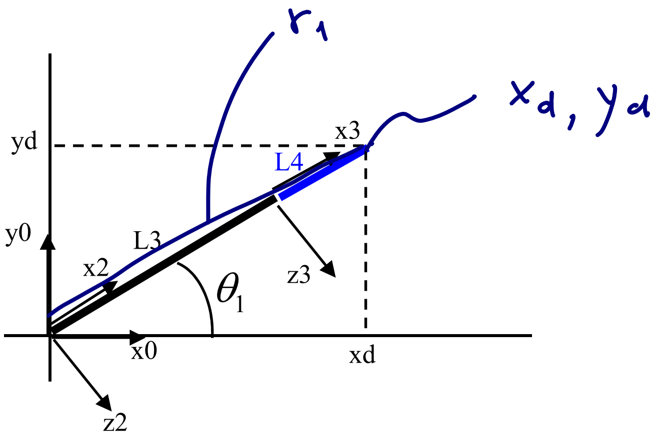
What are the joint angles ($\underline{\theta_1, \theta_2, \theta_3}$) as a function of the goal (wrist position and orientation)

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 2/



Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 3/

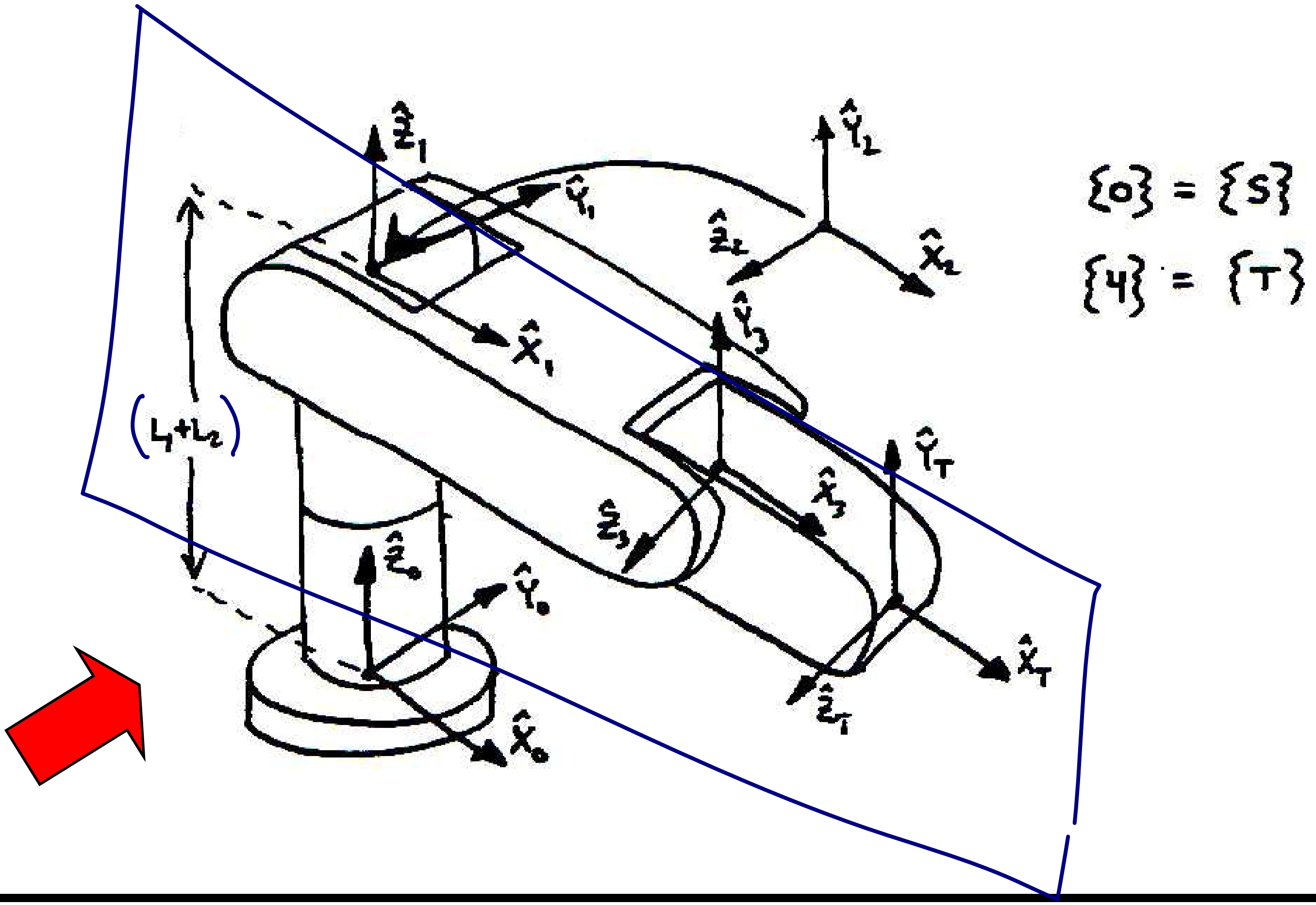
- The planar geometry - top view of the robot



$$\checkmark \quad \longrightarrow \boxed{\theta_1 = A\tan 2(y_d, x_d)}$$

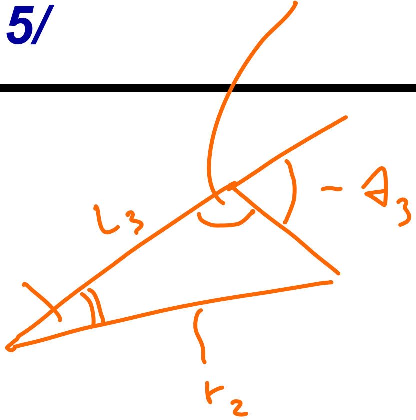
$$\longrightarrow r_1 = \sqrt{x_d^2 + y_d^2}$$

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 4/

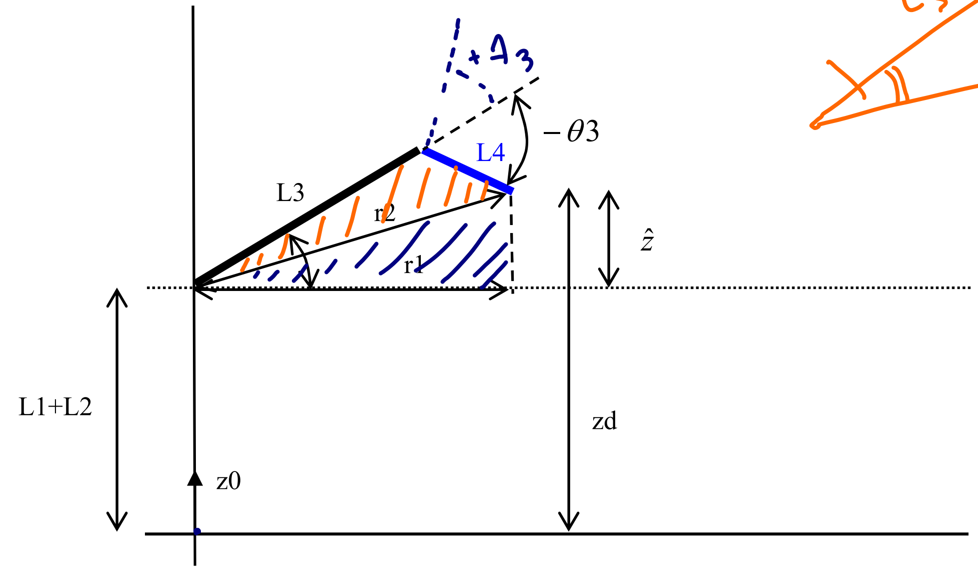


Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 5/

$180-(-\theta_3) = 180+\theta_3$



- The planar geometry - side view of the robot:



$$r_2 = \sqrt{r_1^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2}$$

- where

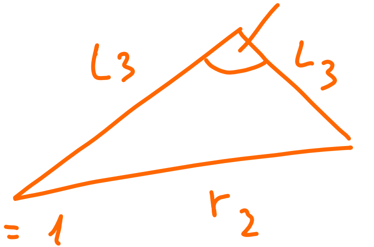
$$\hat{z} = z_d - (L_1 + L_2)$$

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 6/

- By Apply the law of cosines we get

$$\underline{r_2^2} = L_3^2 + L_4^2 - 2L_3L_4\cos(180+\theta_3) = L_3^2 + L_4^2 + 2L_3L_4\underline{\cos(\theta_3)}$$

$180+\theta$



- Rearranging gives

$$c_3 = \frac{r_2^2 - (L_3^2 + L_4^2)}{2L_3L_4}$$

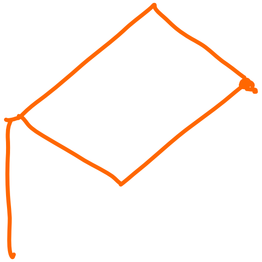
- and

$$s_3 = \sqrt{1-c_3^2}$$

$c_3^2 + s_3^2 = 1$

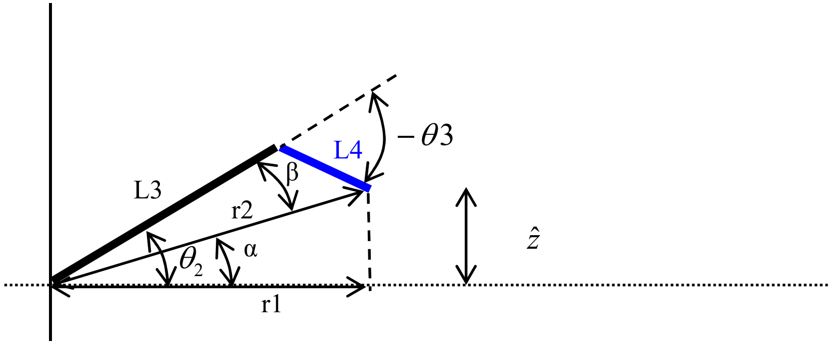
- Solving for θ_3 we get

$$\boxed{\theta_3 = A\tan(\underset{\uparrow}{\pm}\sqrt{1-c_3^2}, c_3)}$$



- Where c_3 is defined above in terms of known parameters $L_{3,}L_{4,}x_d, y_d$, and z_d

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 8/



- Finally we need to solve for θ_2

$$\theta_2 = \alpha + \beta$$

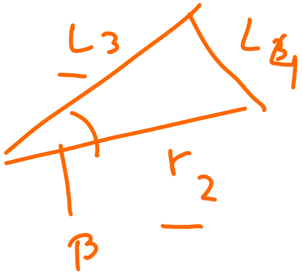
$$\alpha = A\tan 2(\hat{z}, r_1)$$

- where ✓ $r_1 = \sqrt{x_d^2 + y_d^2}$ $\hat{z} = z_d - (L_1 + L_2)$ ✓

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 9/

- Based on the law of cosines we can solve for β

$$L_4^2 = r_2^2 + L_3^2 - 2r_2L_3\cos(\beta)$$



$$c_\beta = \frac{r_2^2 + L_3^2 - L_4^2}{2r_2L_3}$$

$$s_\beta = \sqrt{1 - c_\beta^2}$$

$$\rightarrow \quad \beta = A\tan2(\pm\sqrt{1 - c_\beta^2}, c_\beta)$$

$$\boxed{\theta_2 = \underbrace{A\tan2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2})}_{\alpha} + \underbrace{A\tan2(\pm\sqrt{1 - c_\beta^2}, c_\beta)}_{\beta}}$$

Inverse Kinematics - 3D RRR (3R) - Geometric Solution - 10/

- Summary

x_d
y_d
z_d

$$\left\{\begin{aligned}
&\theta_1 = A\tan 2(y_d, x_d) \\
&\theta_2 = A\tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) + \\
&\qquad A\tan 2(\pm\sqrt{1 - \left(\frac{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2)}L_3)^2}\right)^2}, \frac{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2)}L_3)^2}) \\
&\theta_3 = A\tan(\pm\sqrt{1 - \left[\frac{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2 - (L_3^2 + L_4^2)}{2L_3L_4}\right]^2}, \frac{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2 - (L_3^2 + L_4^2)}{2L_3L_4})
\end{aligned}\right.$$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 1-7

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 1

- Equation

$$\begin{cases} \sin\theta = a & a \in [-1,1] \\ \cos\theta = b & b \in [-1,1] \end{cases}$$

- Solution (Unique)

$$\theta = A\tan \overset{\downarrow}{2}(a,b)$$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 2

- Equation

$$\rightarrow \sin\theta = a \qquad a \in [-1,1]$$

$$\rightarrow \underbrace{\cos\theta = \pm\sqrt{1-a^2}}_{s^2\theta + c^2\theta = 1}$$

$$\rightarrow \cos\theta = b \qquad a \in [-1,1]$$

$$\rightarrow \underbrace{\sin\theta = \pm\sqrt{1-b^2}}$$

- Solution

$$\theta = A\tan 2(a, \overset{\downarrow}{\pm}\sqrt{1-a^2})$$

$$\theta = A\tan 2(\overset{\downarrow}{\pm}\sqrt{1-b^2}, b)$$

  - Two Solutions

$$\theta_1 = \theta$$

$$\theta_2 = 180 - \theta$$

$$\theta_1 = \theta$$

$$\theta_2 = -\theta$$

  - Singularity at the Boundary

$$When \quad \theta = \pm 90^o, \quad |\mathrm{a}| = 1$$

$$When \quad \theta = 0^o, 180^0, \quad |\mathrm{b}| = 1$$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 3

- Equation

$$\longrightarrow \quad a(\cos\theta) + b(\sin\theta) = 0$$

$$\frac{\sin\theta}{\cos\theta} = \boxed{-}\frac{a}{\underline{b}}$$

- Solution
  - Two Solutions 180^0 apart

$$\begin{cases} \theta = A\tan 2(a, -b) \\ \theta = A\tan 2(-a, b) \end{cases}$$

  - Singularity $\underline{b = 0}$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 4

- Equation

$$\longrightarrow\; a(\cos\theta)+b(\sin\theta)=c$$

$$a,b,c\neq 0$$

- Solution
  - Two Solutions

$$\theta = A\tan 2(\underset{\uparrow}{\pm}\sqrt{\underbrace{a^2+b^2-c^2}},c)+A\tan 2(b,a) \quad \longleftarrow$$

  - For a solution to exist $\quad a^2+b^2-c^2>0$
  - No solution (outside of the workspace) $\quad a^2+b^2-c^2<0$
  - One solution (singularity) $\quad a^2+b^2-c^2=0$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 4

$$a\cos\theta + b\sin\theta = c \qquad a, b, c > 0$$

$$\begin{cases} a = \rho\cos\phi \\ b = \rho\sin\phi \end{cases} \Rightarrow \begin{cases} \rho = \sqrt{a^2+b^2} \\ \phi \ \mathrm{Atan2}(b, a) \end{cases}$$

$$\underbrace{\cos\phi\cos\theta + \sin\phi\cos\theta}_{} = \frac{c}{\rho}$$

→ $$\cos(\theta-\phi) = \frac{c}{\rho} = \frac{c}{\sqrt{a^2+b^2}}$$

$$\sin^2(\theta-\phi) + \cos^2(\theta-\phi) = 1$$

→ $$\sin(\theta-\phi) = \pm\sqrt{1-\cos^2(\theta-\phi)}$$

$$= \pm\sqrt{1-\frac{c^2}{a^2+b^2}}$$

$$\underbrace{\theta-\phi}_{} = \mathrm{Atan2}\left(\pm\sqrt{1-\frac{c^2}{a^2+b^2}}, \frac{c}{\sqrt{a^2+b^2}}\right)$$

$$= \mathrm{Atan2}\left(\pm\frac{\sqrt{a^2+b^2-c^2}}{\cancel{\sqrt{a^2+b^2}}}; \frac{c}{\cancel{\sqrt{a^2+b^2}}}\right)$$

$$\theta = \mathrm{Atan2}\left(\pm\sqrt{a^2+b^2-c^2}, c\right) + \underbrace{\mathrm{Atan2}(b, a)}_{\phi}$$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 4

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 5

- Equation

$$\begin{cases} \sin\theta \sin\phi = a \\ \cos\theta \sin\phi = b \end{cases}$$

- Solution

$$\rightarrow \theta = A\tan 2(a,b) \qquad \text{if } \sin\phi \text{ is positive}$$

$$\rightarrow \theta = A\tan 2(-a,-b) \quad \text{if } \sin\phi \text{ is negative}$$

Inverse Kinematics -
Generalized Algebraic (Analytical) Solutions – Case 6

- Equation

$$\begin{cases} a\cos\theta + c\sin\theta = d \\ e\cos\theta + f\sin\theta = g \end{cases}$$

$\cos\theta$, $\sin\theta$

- Solution

$$\theta = A\tan 2(ag - de, df - cg)$$

  - For an exiting solution (the determinant must be positive)

$$\Delta = af - ce > 0$$

Algebraic Solution by Reduction to Polynomial

Case - 7

- Transcendental equations are difficult to solve because they are a function of

$c\theta, s\theta$

$$f(c\theta, s\theta) = k$$

- Making the following substitutions yields an expression in terms of a single veritable u , Using this substitutions, transcendental equation are converted into polynomial equation

$$\begin{cases} u = \tan\dfrac{\theta}{2} \\ \cos\theta = \dfrac{1-u^2}{1+u^2} \\ \sin\theta = \dfrac{2u}{1+u^2} \end{cases}$$

Algebraic Solution by Reduction to Polynomial - Example

- Transcendental equation

$$\underbrace{ac\theta + bs\theta = c}$$

- Substitute $c\theta, s\theta$ with the following equations

$$\left\{\begin{array}{l} \cos\theta = \dfrac{1-u^2}{1+u^2} \\ \sin\theta = \dfrac{2u}{1+u^2} \end{array}\right.$$

- yields

$$a(1-u^2) + 2bu = c(1+u^2)$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

Algebraic Solution by Reduction to Polynomial - Example

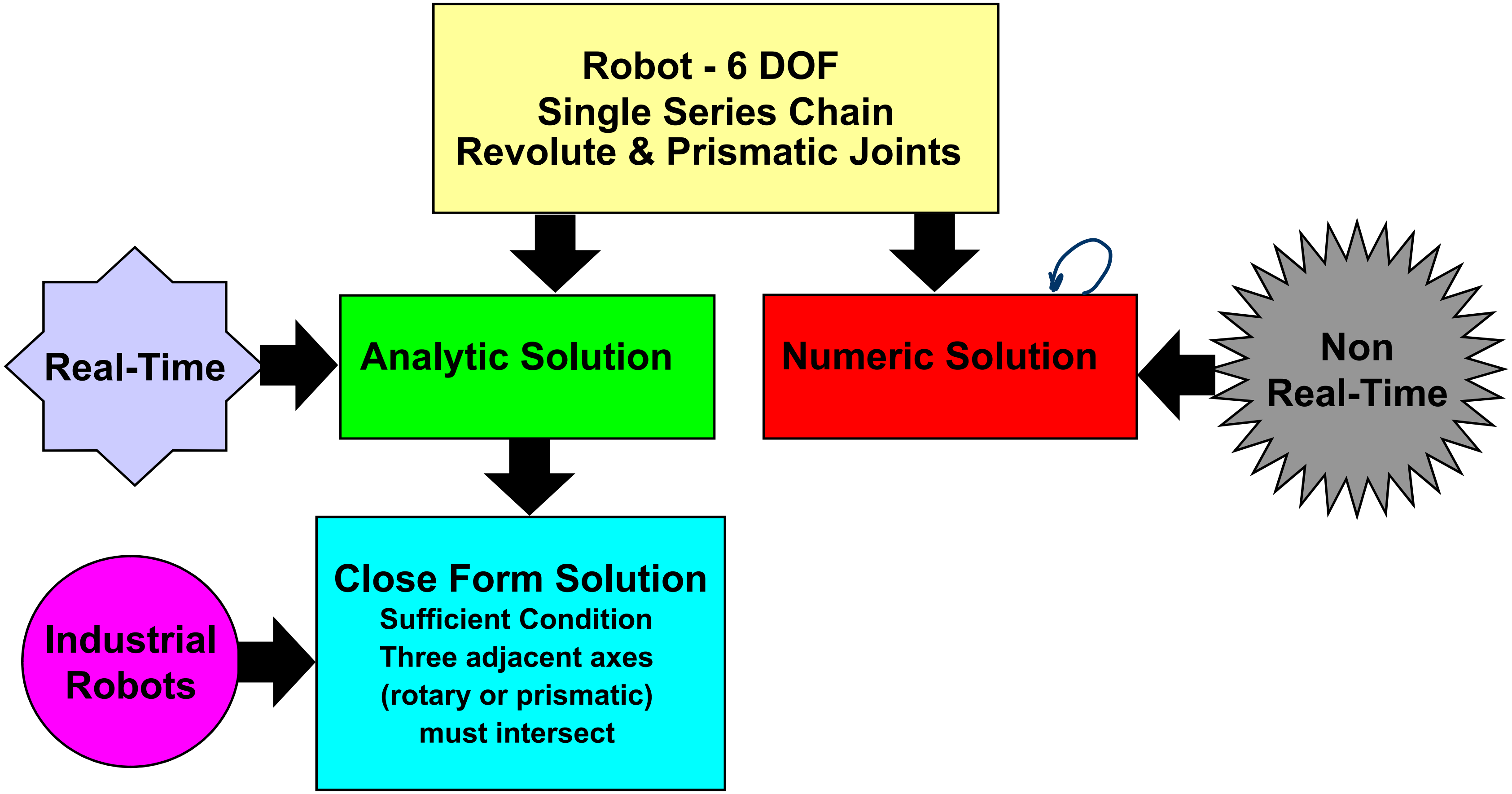
- Which is solved by the quadratic formula to be

$$u = \frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a + c}$$

$$\rightarrow \qquad \theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 - a^2 - c^2}}{a + c}\right)$$

- Note
  - If u is complex there is no real solution to the original transcendental equation
  - If $a + c = 0$ then $\theta = 180^o$

Solvability



Robot - 6 DOF
Single Series Chain
Revolute & Prismatic Joints

Real-Time

Analytic Solution

Numeric Solution

Non
Real-Time

Industrial
Robots

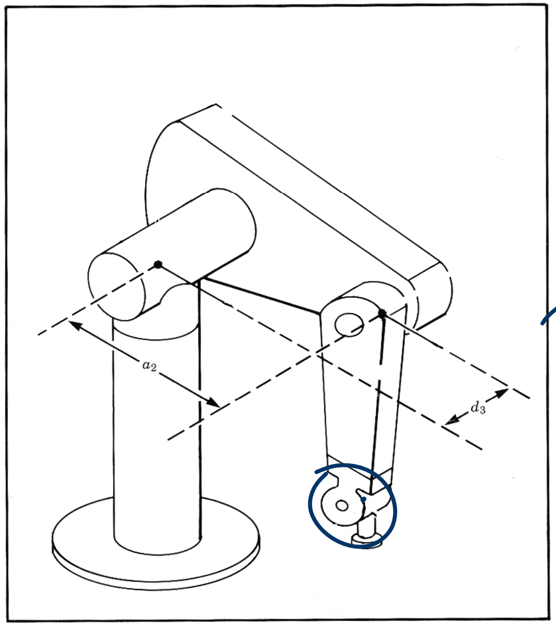
Close Form Solution
Sufficient Condition
Three adjacent axes
(rotary or prismatic)
must intersect

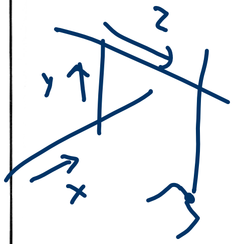
Pieper’s Solution - Three consecutive Axes Intersect

- **Pieper’s Solution** - Closed form solution for a serial 6 DOF in which **three consecutive axes intersect at a point** (including robots with three consecutive parallel axes, since they meet at a point at infinity)

- Pieper’s method applies to the majority of commercially available industrial robots

- Example: (Puma 560)
  - All 6 joints are revolute joints
  - The last 3 joints are intersecting

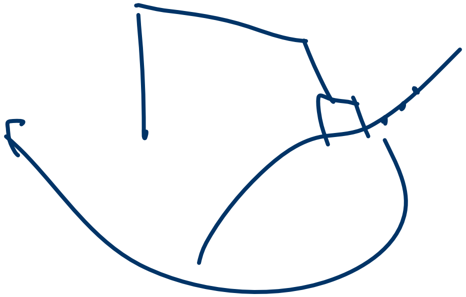




Pieper's Solution - Three consecutive Axes Intersect

- ***Given:***
  - ***Manipulator Geometry:*** 6 DOF & DH parameters
    - All 6 joints are revolute joints
    - The last 3 joints are intersecting
  - ***Goal Point Definition***: The position and orientation of the wrist in space

$$ {}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{(t)} $$



- ***Problem:***

What are the joint angles ($\underline{\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6}$) as a function of the goal (wrist position and orientation)

Pieper's Solution - Three consecutive Axes Intersect

- When the last three axes of a 6 DOF robot intersect, the origins of link frame {4}, {5}, and {6} are all located at the point of intersection. This point is given in the base coordinate system as

$$\nearrow\underbrace{{}^{0}P_{4org}} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,\boxed{{}^{3}P_{4org}}\;\cancel{\underbrace{{}^{3}_{4}T\;{}^{4}_{5}T\;{}^{5}_{6}T}}$$

- From the general forward kinematics method for determining homogeneous transforms using DH parameters, we know:

$$\longrightarrow\;{}^{\overset{3}{i-1}}_{\underset{4}{i}}T = \begin{array}{c}\begin{array}{cccc} & {}^{i-1}_{i}R & & {}^{i-1}P_{iorg}\end{array}\\ \left[\begin{array}{ccc|c} c\theta_i & -s\theta_i & 0 & a_{i-1}\\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i\\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i\\ 0 & 0 & 0 & 1\end{array}\right]\end{array}$$

Pieper's Solution - Three consecutive Axes Intersect

- For i=4

$$
\begin{array}{cc}
 & \begin{array}{cc} \qquad\qquad {}^3_4R \qquad\qquad & \boxed{{}^3P_{4org}}\ \checkmark \end{array} \\
{}^3_4T = & \left[\begin{array}{ccc|c}
c\theta_4 & -s\theta_4 & 0 & a_3 \\
s\theta_4\, c\alpha_3 & c\theta_4\, c\alpha_3 & -s\alpha_3 & -s\alpha_3 d_4 \\
s\theta_4\, s\alpha_3 & c\theta_4\, s\alpha_3 & c\alpha_3 & c\alpha_3 d_4 \\
0 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

- Using the fourth column and substituting for ${}^3P_{4org}$, we find

$$
\underbrace{{}^0P_{4org}}_{} = \underbrace{{}^0_{\cancel{1}}T\,{}^1_{\cancel{2}}T\,{}^2_{\cancel{3}}T}_{}\,\underbrace{{}^3P_{4org}}_{} = {}^0_1T\,{}^1_2T\,{}^2_3T\underbrace{\begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}}_{{}^3P_{4ORG}}
$$

$$
{}^0P_{4org} = {}^0_1T\,{}^1_2T\,\underbrace{{}^2_3T\,{}^3P_{4org}}_{} = {}^0_1T\,{}^1_2T \overset{\downarrow}{\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}}
$$

Pieper's Solution - Three consecutive Axes Intersect

- where

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^2_3T \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 c\alpha_2 & c\theta_3 c\alpha_2 & -s\alpha_2 & -s\alpha_2 d_3 \\ s\theta_3 s\alpha_2 & c\theta_3 s\alpha_2 & c\alpha_2 & c\alpha_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

$$\left\{ \begin{aligned} f_1(\theta_3) &= a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ f_2(\theta_3) &= a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ f_3(\underbrace{\theta_3}_{\uparrow}) &= a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \end{aligned} \right.$$

Pieper’s Solution - Three consecutive Axes Intersect

- Repeating the same process again

$$ {}^0P_{4org} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3P_{4org} = {}^0_1T\,\underbrace{{}^1_2T}_{} \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

$$ \downarrow $$

$$ \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = {}^1_2T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_1 \\ s\theta_2\, c\alpha_1 & c\theta_2\, c\alpha_2 & -s\alpha_1 & -s\alpha_1 d_2 \\ s\theta_2\, s\alpha_1 & c\theta_2\, s\alpha_1 & c\alpha_1 & c\alpha_1 d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

Pieper's Solution - Three consecutive Axes Intersect

$$(*)\left\{\begin{aligned} g_1(\theta_2) &= c_2 f_1 + s_2 f_2 + a_1 \\ g_2(\theta_2) &= s_2 s\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 \\ g_3(\theta_2) &= \underbrace{s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1}_{} \end{aligned}\right.$$

- Repeating the same process for the last time

$$ {}^{0}P_{4org} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}P_{4org} = {}^{0}_{1}T \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_2(\theta_2) \\ 1 \end{bmatrix} $$

$$ {}^{0}P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} $$

Pieper's Solution - Three consecutive Axes Intersect

- ***Frame {0}*** - The frame attached to the base of the robot or link 0 called frame {0} This frame does not move and for the problem of arm kinematics can be considered as the ***reference frame.***

- Assign {0} to match {1} when the first joint veritable is zero

$$\theta_1 \neq 0 \qquad \alpha_0 = d_1 = a_0 = 0$$

$${}^0_1T$$

$${}^0P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$

$$\rightarrow \quad {}^0P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ \boxed{g_3} \\ 1 \end{bmatrix} \begin{matrix} \text{—}\left(\qquad\right)^2 \\ \text{—}\left(\qquad\right)^2 \\ \text{—}\left(\qquad\right)^2 \\ \end{matrix}$$

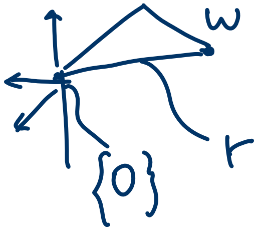
Pieper's Solution - Three consecutive Axes Intersect

- Through algebraic manipulation of these equations, we can solve for the desired joint angles ($\theta_1, \theta_2, \theta_3$).

- The first step is to square the magnitude of the distance from the frame {0} origin to frame {4} origin.

$$\rightarrow \quad r^2 = \left({}^0P_{4orgx}\right)^2 + \left({}^0P_{4orgy}\right)^2 + \left({}^0P_{4orgz}\right)^2 = \underbrace{g_1^2 + g_2^2 + g_3^2}$$

$$x^2 \quad + \quad y^2 \quad + \quad z^2$$



- Using the previously define function for g_i we have

$$r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + a_1\left(c_2 f_1 - s_2 f_2\right)$$

Pieper's Solution - Three consecutive Axes Intersect

$$(*)\quad \begin{cases} \rightarrow\; r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + a_1\left(c_2 f_1 - s_2 f_2\right) \\ \rightarrow\; Z = \underbrace{{}^0P_{4orgz}} = \underbrace{g_3} = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 \end{cases}$$

- Applying a substitution of temporary variables, we can write the magnitude squared term along with the z-component of the {0} frame origin to the {4} frame origin distance.

$$\begin{cases} r^2 = \overbrace{\left(k_1 c_2 + k_2 s_2\right)}^{/\,o} 2a_1 + k_3 \\ Z = \underbrace{\left(k_1 s_2 - k_2 c_2\right) s\underset{\uparrow}{\alpha_1}}_{'\,o} + k_4 \end{cases}$$

$$\begin{cases} k_1 = f_1 \\ k_2 = -f_2 \\ k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 \\ k_4 = f_3 c\alpha_1 + d_2 c\alpha_1 \end{cases}$$

- These equations are useful because dependence on θ_1 has been eliminated, and dependence on θ_2 takes a simple form

Pieper's Solution - Three consecutive Axes Intersect

- Consider 3 cases while solving for θ_3 :

$\underbrace{x, y, z}$

θ_3

- $\underline{\textbf{Case 1}}$ - $\underbrace{a_1 = 0}$

√

$$\rightarrow r^2 = k_3$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$$

$$\left\{ \begin{aligned} f_1(\theta_3) &= a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ f_2(\theta_3) &= a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ f_3(\underbrace{\theta_3}) &= a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \end{aligned} \right.$$

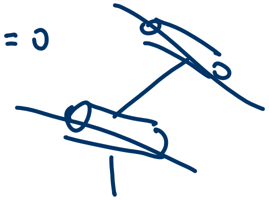
- Solution Methodology - Reduction to Ploynomial => Quadratic Equation

$$\underbrace{u = \tan\frac{\theta}{2}} \qquad \underbrace{\cos\theta = \frac{1-u^2}{1+u^2}} \qquad \underbrace{\sin\theta = \frac{2u}{1+u^2}}$$

Pieper's Solution - Three consecutive Axes Intersect

- **Case 2 -** $\underbrace{s\alpha_1 = 0}$

$\alpha = 0$



$$Z = k_4$$

$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1$$

$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

- Solution Methodology - Reduction to Ploynomial => Quadratic Equation

$$u = \tan\frac{\theta}{2} \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad \sin\theta = \frac{2u}{1+u^2}$$

Pieper’s Solution - Three consecutive Axes Intersect

- **Case 3 (General case)** : We can find θ_3 through the following algebraic manipulation:

$$+\underline{\left\{\begin{array}{l}\left(\dfrac{r^2-k_3}{2a_1}=(k_1c_2+k_2s_2)\right)^2 \\ \left(\dfrac{Z-k_4}{s\alpha_1}=(k_1s_2-k_2c_2)\right)^2\end{array}\right.} \qquad \theta_2$$

- squaring both sides, we find

$$\left(\frac{r^2-k_3}{2a_1}\right)^2=(k_1c_2+k_2s_2)^2=k_1^2c_2^2+k_2^2s_2^2+2k_1k_2c_2s_2$$

$$\left(\frac{Z-k_4}{s\alpha_1}\right)^2=(k_1s_2-k_2c_2)^2=k_1^2s_2^2+k_2^2c_2^2-2k_1k_2c_2s_2$$

Pieper’s Solution - Three consecutive Axes Intersect

- Adding these two equations together and simplifying using the trigonometry identity (Reduction to Ploynomial), we find a fourth order equation for $\underbrace{\theta_3}$

$$\left(\frac{r^2-k_3}{2a_1}\right)^2+\left(\frac{Z-k_4}{s\alpha_1}\right)^2=k_1^2+k_2^2$$

$$\left\{\begin{aligned} k_1 &= f_1 \\ k_2 &= -f_2 \\ k_3 &= f_1^2+f_2^2+f_3^2+a_1^2+d_2^2+2d_2f_3 \\ k_4 &= f_3c\alpha_1+d_2c\alpha_1 \end{aligned}\right.$$

$$\left\{\begin{aligned} f_1(\theta_3) &= a_3c_3+d_4s\alpha_3s_3+a_2 \\ f_2(\theta_3) &= a_3c\alpha_2s_3-d_4s\alpha_3c\alpha_2c_3-d_4s\alpha_2c\alpha_3-d_3s\alpha_2 \\ f_3(\underset{\uparrow}{\theta_3}) &= a_3s\alpha_2s_3-d_4s\alpha_3s\alpha_2c_3+d_4c\alpha_2c\alpha_3+d_3c\alpha_2 \end{aligned}\right.$$

Pieper's Solution - Three consecutive Axes Intersect

- With θ_3 solved, substitute into r^2, Z to find $\underline{\theta_2}$ ✓

$$\begin{cases} r^2 = (k_1 c_2 + k_2 s_2)2a_1 + k_3 \\ Z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 \end{cases}$$

$$\begin{cases} k_1 = f_1 \\ k_2 = -f_2 \\ k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 \\ k_4 = f_3 c\alpha_1 + d_2 c\alpha_1 \end{cases}$$

$$\begin{cases} f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \end{cases}$$

Pieper’s Solution - Three consecutive Axes Intersect

- With $\underbrace{\theta_2,\theta_3}$ solved, substitute into ${}^{0}P_{4org}$ to find

$${}^{0}P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

$$x = {}^{0}P_{4orgx} = c_1 g_1 - s_1 g_2$$

$$y = {}^{0}P_{4orgy} = s_1 g_1 + c_1 g_2$$

$$\underbrace{\theta_1}$$

$$\left\{\begin{aligned} g_1(\theta_2) &= c_2 f_1 + s_2 f_2 + a_1 \\ g_2(\theta_2) &= s_2 s\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 \\ g_3(\theta_2) &= s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 \end{aligned}\right.$$

$$\left\{\begin{aligned} f_1(\theta_3) &= a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 \\ f_2(\theta_3) &= a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 \\ f_3(\theta_3) &= a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 \end{aligned}\right.$$

- Solve for θ_1 using the reduction to polynomial method

Pieper's Solution - Three consecutive Axes Intersect

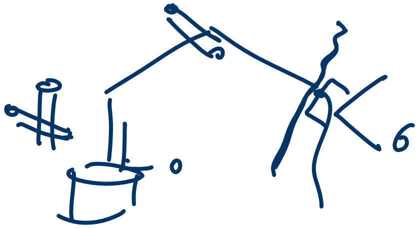
- To complete our solution we need to solve for $\underbrace{\theta_4, \theta_5, \theta_6}$
- Since the last three axes intersect these joint angle affect the orientation of only the last link. We can compute them based only upon the rotation portion of the specified goal ${}^0_6R$

$${}^0_4R^{-1} \qquad {}^0_4R^{-1} \qquad I$$

$$\underbrace{{}^0_6R} = \underbrace{{}^0_4R\Big|_{\theta_4=0}}\ \underbrace{{}^4_6R}$$

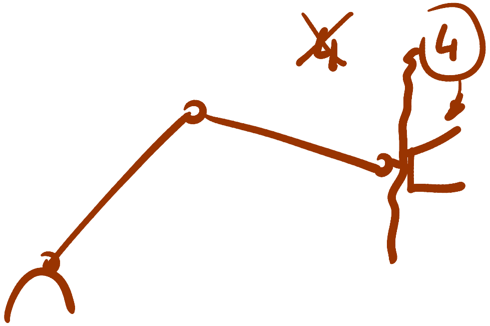
$$\underbrace{{}^4_6R\Big|_{\theta_4=0}} = \underbrace{{}^0_4R^{-1}\Big|_{\theta_4=0}}_{\checkmark}\ \underbrace{{}^0_6R}$$

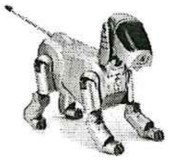
$\theta_1, \theta_2, \theta_3$ ✓



- $\underbrace{{}^0_4R\Big|_{\theta_4=0}}$ - The orientation of link frame {4} relative to the base frame {0} when $\theta_4 = 0$
- $\underbrace{\theta_4, \theta_5, \theta_6}$ are the Euler angles applied to $\underbrace{{}^4_6R\Big|_{\theta_4=0}}$ ↑

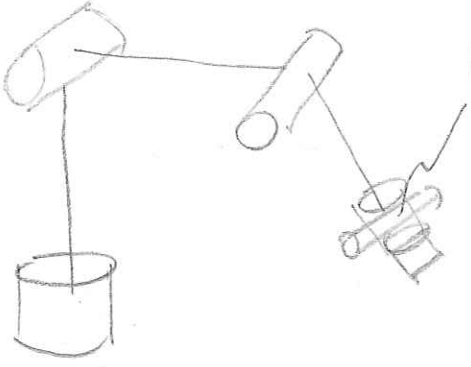
Inverse Manipulator Kinematics (4/4)





INVERSE KINEMATICS - HYBRID APPROUCH

$$
{}^0_6T = \underset{\theta_1}{\underset{|}{{}^0_1T}}\ \underset{\theta_2}{\underset{\downarrow}{{}^1_2T}}\ \underset{\theta_3}{\underset{\downarrow}{{}^2_3T}}\ \underset{\theta_4}{\underset{\downarrow}{{}^3_4T}}\ \underset{\theta_5}{\underset{\downarrow}{{}^4_5T}}\ \underset{\theta_6}{\underset{\downarrow}{{}^5_6T}} = \underbrace{\left[\begin{array}{c|c} {}^0_6R & {}^0P_6 \\ \hline & \end{array}\right]}_{\text{GIVEN}}
$$

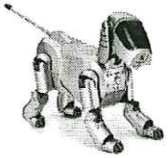


WRIST CENTER

3 INTERSECTING AXIS

Given 3 intersecting axis 4,5,6 (origines of 4,56 are at the same point)

$$
{}^0P_6 = {}^6P_4
$$



$$ {}^0_6T = \underbrace{{}^0_1T\;{}^1_2T\;{}^2_3T \;\Big|\; {}^3_4T}_{} \;\Big|\; {}^4_5T\;{}^5_6T $$

Problem 1

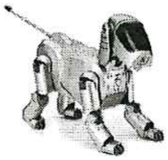
position problem

Problem 2

orientation problem

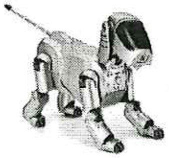
$$ {}^3_4T $$

$$ \{\text{Frame } 3\} \xrightarrow[R_{x_3}(\alpha_3)]{\alpha_3} \{\text{Frame}(R)\} \xrightarrow[D_{x_3}(a_3)]{a_3} \{\text{Frame}(Q)\} \xrightarrow[(R_{z_4}\theta_4)]{\theta_4} \{\text{Frame}(P)\} \xrightarrow[D_{z_4}(d_4)]{d_4} \{\text{Frame}(4)\} $$



$$ {}^6_6T = \underbrace{{}^0_1T\,{}^1_2T\,{}^2_3T\; R_{x3}(\alpha_\alpha)\, D_{x3}(a_3)}_{\substack{\text{Problem 1} \\ \text{position problem}}} \Big| \underbrace{R_{z4}(\theta_4)\, D_{z4}(d_4)\; {}^4_5T\; {}^5_6T}_{\substack{\text{Problem 2} \\ \text{Oriantation problem}}} $$

$$ {}^0_6T = {}^0_1T\; {}^1_2T\; {}^2_3T\; \underset{\underset{\text{set } \theta_4=0}{\uparrow}}{{}^3_4T}\Big|_{\theta_4=0} \;\Big|\; \underset{\underset{\text{set } \alpha_3=0}{\uparrow}}{{}^3_4T}\Big|_{\alpha_3=0}\; {}^4_5T\; {}^5_6T $$

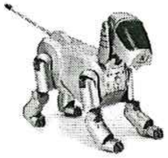


$${}^0_6R = {}^0_3R \; {}^3_4R \; {}^4_6R$$

$$\downarrow$$

$${}^0_6R = {}^0_3R \left(R(\alpha_3) \underset{\underset{D_{x_3}(a_3)}{\uparrow}}{I} \;\Big|\; R(\theta_4) \underset{\underset{D_{z_4}(d_4)}{\uparrow}}{I} \right) {}^4_6R$$

$${}^0_6R = \left[{}^0_3R \; R_{x_3}(\alpha_3) \right] \left[R_{z_4}(\theta_4) \; {}^4_6R \right]$$



Solving for $\theta_4, \theta_5, \theta_6$

$$\underbrace{R_{z4}(\theta_4)\,{}^4_6R}_{} = \underbrace{\left[{}^0_3R\,R_{x3}(\alpha_3)\right]^{-1}}_{} \underbrace{{}^0_6R}_{}$$

Solved in Problem 1

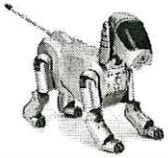
Known $\theta_1, \theta_2, \theta_3$

Desired oriantation given for every point on the trajectory

$$\downarrow$$

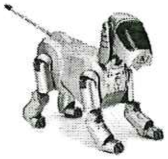
$$R_{z4}(\theta_4)\,{}^4_5R(\theta_5)\,{}^5_6R(\theta_6) = \bar{R}_D \longrightarrow$$

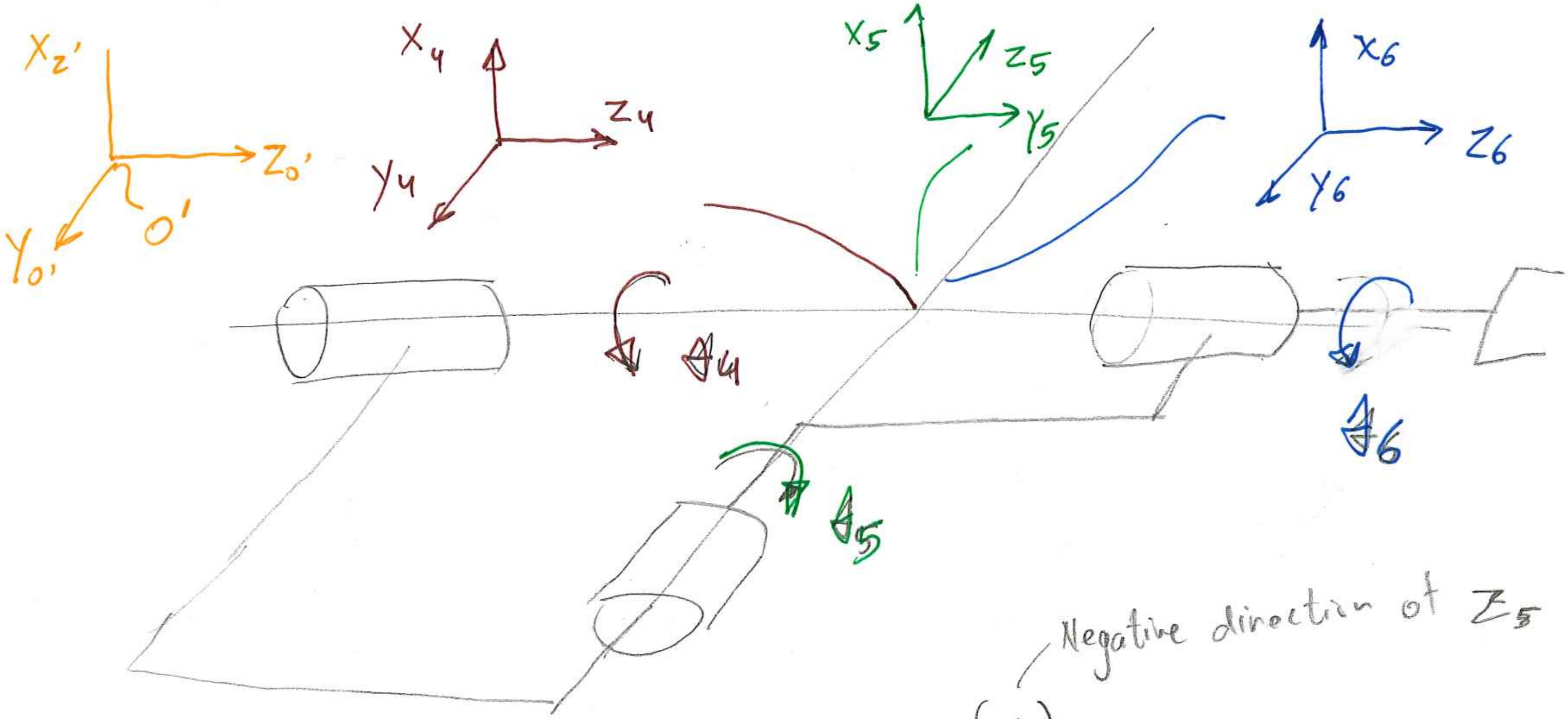
[Desired orintation of the wrist taking into account the contribution of the first 3 angles to the orintation]



$$R_{z4}(\theta_4)\,{}^{4}_{5}R(\theta_5)\,{}^{5}_{6}R(\theta_6) = \overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}^{\bar{R}_D}$$

solve for $\theta_4, \theta_5, \theta_6$ using the Z-Y-Z problem

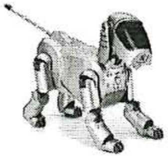




Negative direction of Z_5

$(-)$

$$R_{Euler} = R_z(\theta_4)\ R_{Y4}(\theta_5)\ R_{Z4}(\theta_6)$$



$$R_{ZYZ}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \alpha & \beta & \gamma \\ \downarrow & \downarrow & \downarrow \\ \theta_4 & -\theta_5 & \theta_6 \end{matrix}$$

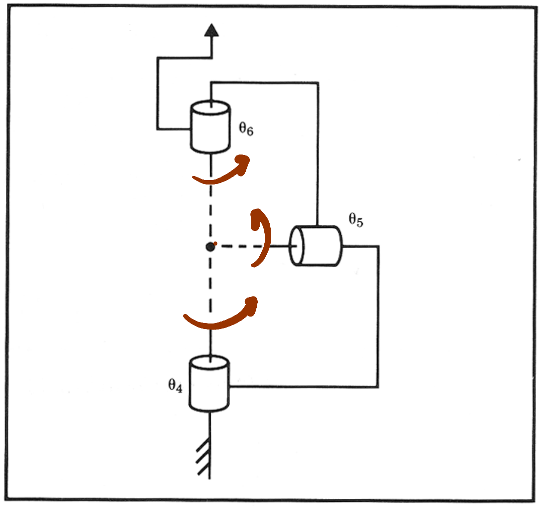
Note: !

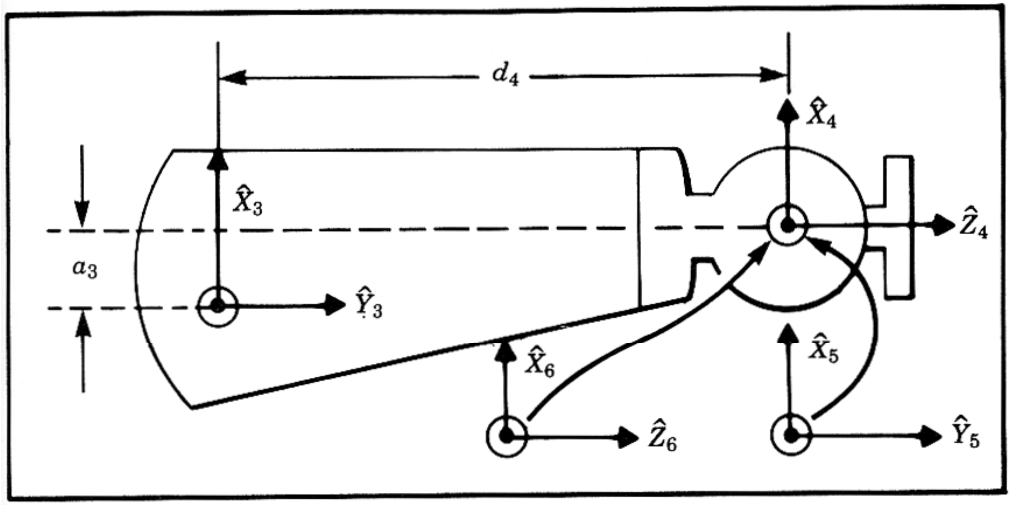
$$= \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

Central Topic - Inverse Manipulator Kinematics - Examples

- ***Algebraic Solution*** *(closed form) - Piepers Method (Continue) - Last three consecutive axes intersect at one point*

- Consider a 3 DOF (wrist) non-planar robot whose axes all intersect at a point.





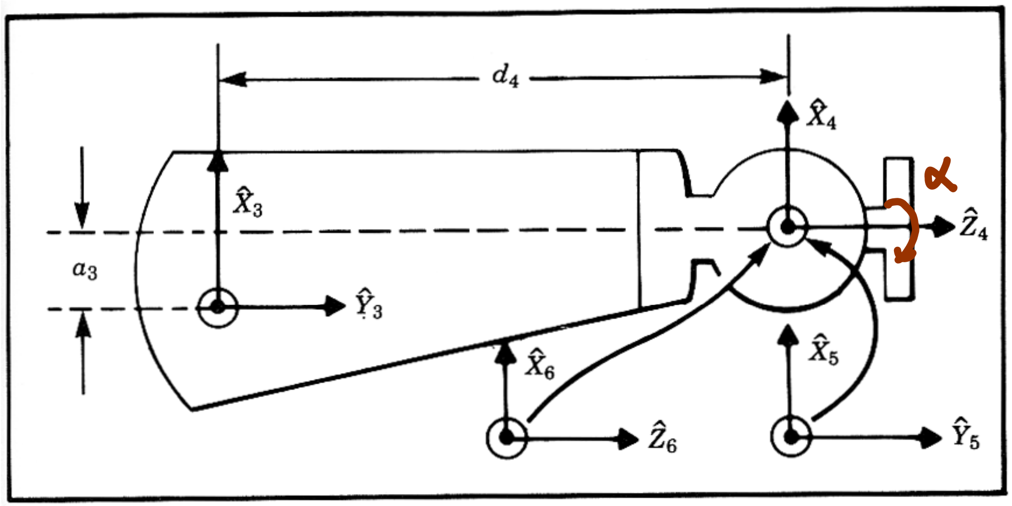
Mapping - Rotated Frames - Z-Y-Z Euler Angles

Start with frame {4}.

- Rotate frame {4} about $\hat{Z}_4$ by an angle α
- Rotate frame {4} about $\hat{Y}_4$ by an angle β
- Rotate frame {4} about $\hat{Z}_4$ by an angle γ

Euler Angles

Note - *Each rotation is preformed about an axis of the moving reference frame {4}, rather then a fixed reference.*



Mapping - Rotated Frames - X-Y-Z Euler Angles

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_Z(\gamma) = \overbrace{\begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{} \overbrace{\begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix}}^{} \overbrace{\begin{bmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{}$$

$$\rightarrow \quad R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$

$${}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = {}^{4}_{6}R_{\underbrace{\theta_4=0}_{}}$$

Three consecutive Axes Intersect - wrist

✓

$$\underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\textbf{Goal}} = \underbrace{\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ \underbrace{-s\beta c\gamma}_{r_{31}} & \underbrace{s\beta s\gamma}_{r_{32}} & c\beta \end{bmatrix}}_{\textbf{Direct Kinematics}}$$

α, β, γ

Three consecutive Axes Intersect - wrist

- Solve for $\boxed{\beta}$ using element r_{31}, r_{32}, r_{33}

$$r_{31}^2 + r_{32}^2 = \overset{\downarrow}{\underline{s\beta^2}}\left(\underbrace{c\alpha^2 + s\alpha^2}_{1}\right)$$

$$r_{33} = \overset{\downarrow}{\underline{c\beta}}$$

$$s\beta = \pm\sqrt{r_{31}^2 + r_{32}^2}$$

- Using the Atan2 function, we find

$$\underline{\beta} = \mathrm{Atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right)$$

Three consecutive Axes Intersect - wrist

- Solve for α using elements $\underbrace{r_{23}, r_{13}}$

$$\downarrow$$

$$r_{13} = \underbrace{c\alpha} s\beta$$

$$r_{23} = \underbrace{s\alpha} s\beta$$

$$\uparrow$$

$$\alpha = \text{Atan2}\left(r_{23} / \underbrace{s\beta}_{\uparrow}, r_{13} / \underbrace{s\beta}_{\uparrow}\right)$$

Three consecutive Axes Intersect - wrist

- Solve for γ using elements r_{32}, r_{31}

$$r_{32} = s\beta \overset{\downarrow}{s\gamma}$$

$$r_{31} = -s\beta \overset{\downarrow}{c\gamma}$$

$$\gamma = \text{Atan2}\left(r_{32} / \underset{\uparrow}{s\beta}, -r_{31} / \underset{\uparrow}{s\beta}\right)$$

Three consecutive Axes Intersect - wrist

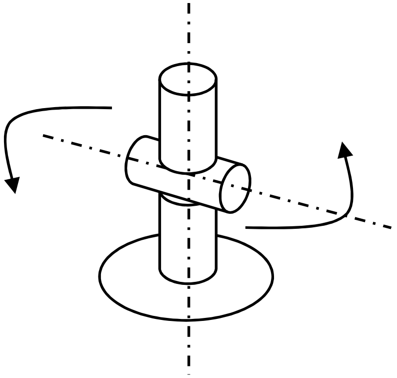
- Note: Two answers exist for angle β which will result in two answers each for angles α and γ .

$$\left\{\begin{array}{l} \beta = \text{Atan2}\left(\pm\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right) \\ \\ \alpha = \text{Atan2}\left(r_{23} / s\beta, r_{13} / s\beta\right) \\ \\ \gamma = \text{Atan2}\left(r_{32} / \underset{\uparrow}{s\beta}, -r_{31} / s\beta\right) \end{array}\right.$$

- If $\underbrace{\beta = 0^o}_{}, \underbrace{\beta = 180^o}_{} \Rightarrow s\beta = 0$ the solution degenerates

Three consecutive Axes Intersect - wrist

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}$$



$$\underbrace{\beta = 0^o}$$

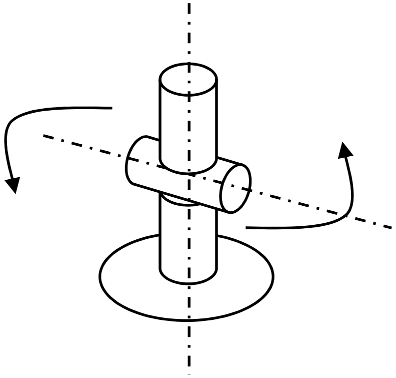
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\gamma - s\alpha s\gamma & -c\alpha s\gamma - s\alpha c\gamma & 0 \\ s\alpha c\gamma + c\alpha s\gamma & -s\alpha s\gamma + c\alpha c\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c(\overset{\swarrow}{\alpha} + \overset{\swarrow}{\gamma}) & -s(\alpha + \gamma) & 0 \\ s(\alpha + \gamma) & c(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We are left with $(\gamma + \alpha)$ for every case. This means we can't solve for either, just their difference.

Three consecutive Axes Intersect - wrist

- One possible convention is to choose $\alpha = 0^o$

- The solution can be calculated to be



$$\underline{\beta = 0}$$

$$\rightarrow \alpha = 0$$

$$\gamma = \text{Atan2}(-r_{12}, r_{11}) = \text{Atan2}(\underline{s\gamma, c\gamma})$$

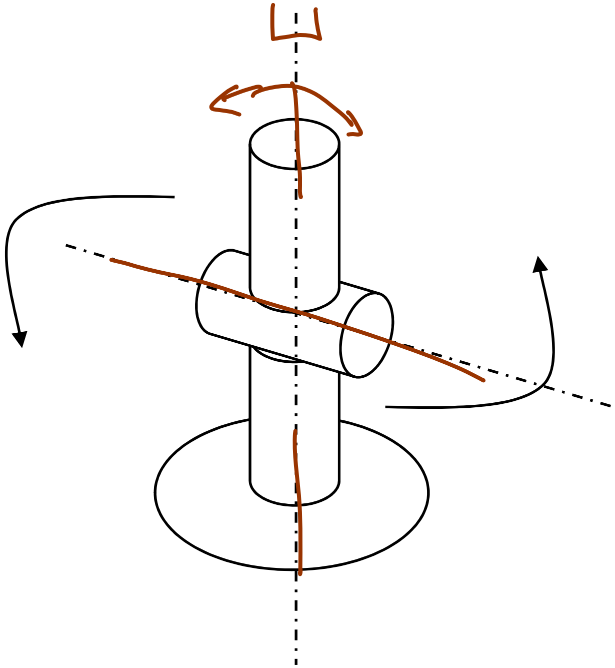
$$\beta = 180$$

$$\rightarrow \alpha = 0$$

$$\gamma = \text{Atan2}(r_{12}, -r_{11}) = \text{Atan2}(\underline{s\gamma, c\gamma})$$

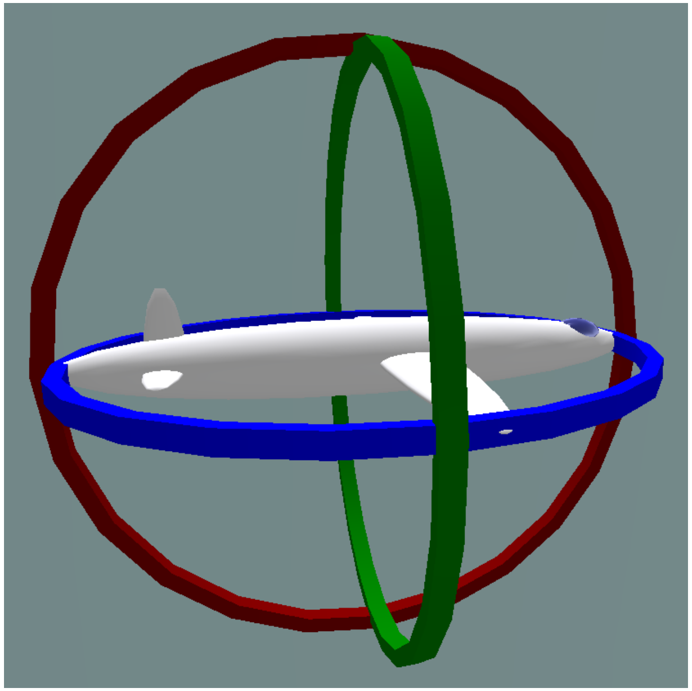
Three consecutive Axes Intersect - wrist

- For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector's orientation never changes.

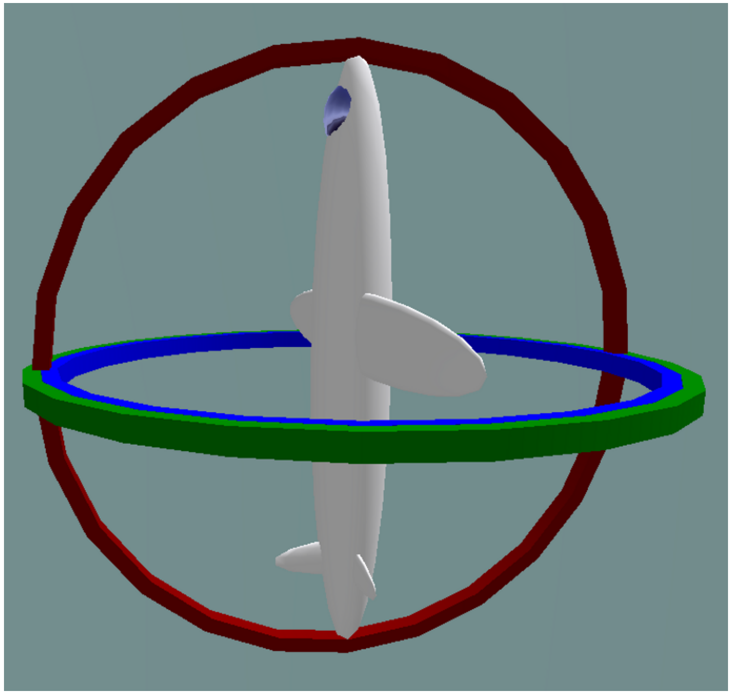


Gimbal Lock

http://youtu.be/zc8b2Jo7mno



Normal situation
The three gimbals are independent



Gimbal lock:
Two out of the three gimbals are in the same plane, one degree of freedom is lost

Gimbal Lock – Robotics

- In robotics, gimbal lock is commonly referred to as "wrist flip", due to the use of a "triple-roll wrist" in robotic arms, where three axes of the wrist, controlling yaw, pitch, and roll, all pass through a common point.

- An example of a wrist flip, also called a wrist singularity, is when the path through which the robot is traveling causes the first and third axes of the robot's wrist to line up. The second wrist axis then attempts to spin 180° in zero time to maintain the orientation of the end effector. The result of a singularity can be quite dramatic and can have adverse effects on the robot arm, the end effector, and the process.

- The importance of non-singularities in robotics has led the American National Standard for Industrial Robots and Robot Systems — Safety Requirements to define it as "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities".