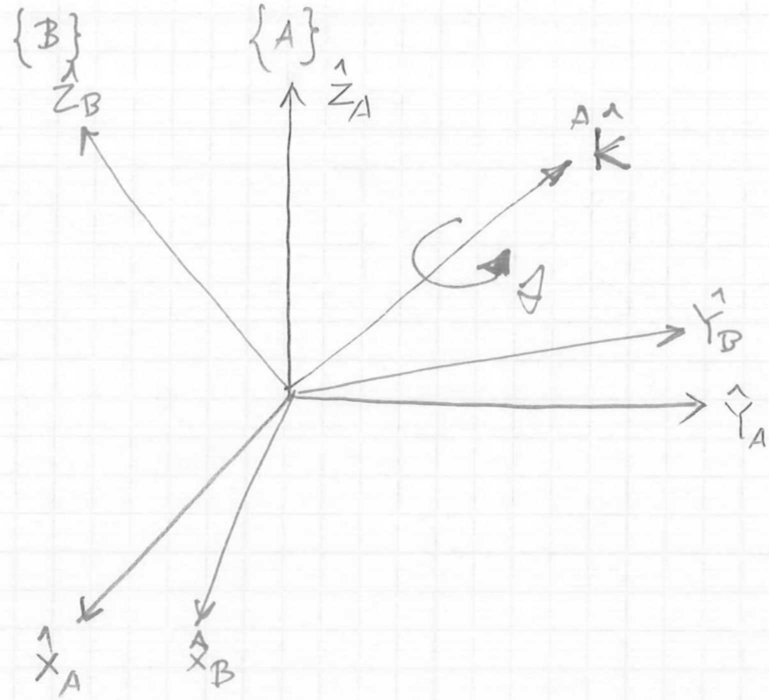
EQUIVALENT ANGLE-AXIS REPRESENTATION



1. START WITH THE FRAME COINCIDENT WITH A KNOWN FRAME {A};

2. ROTATE {B} ABOUT THE VECTOR $^A\hat{K}$ BY AN ANGLE θ ACCORDING TO THE RIGHTHAND RULE

$$^A\hat{K} = \underbrace{\begin{bmatrix} k_x & k_y & k_z \end{bmatrix}}_{\text{UNIT VECTORS}}^T$$

$$\left[\sqrt{k_x^2 + k_y^2 + k_z^2} = 1\right]$$

$$R_k(\theta) = \begin{bmatrix} k_x k_x v\theta + c\theta & k_x k_y v\theta - k_z s\theta & k_x k_z v\theta + k_y s\theta \\ k_x k_y v\theta + k_z s\theta & k_y k_y v\theta + c\theta & k_y k_z v\theta - k_x s\theta \\ k_x k_z v\theta - k_y s\theta & k_y k_z v\theta + k_x s\theta & k_z k_z v\theta + c\theta \end{bmatrix}$$

$$c\theta = \cos\theta$$

$$s\theta = \sin\theta$$

$$v\theta = 1 - \cos\theta$$

INVERS PROBLEM

COMPUTE $\hat{k}$ AND θ

GIVEN ROTATION MATRIX

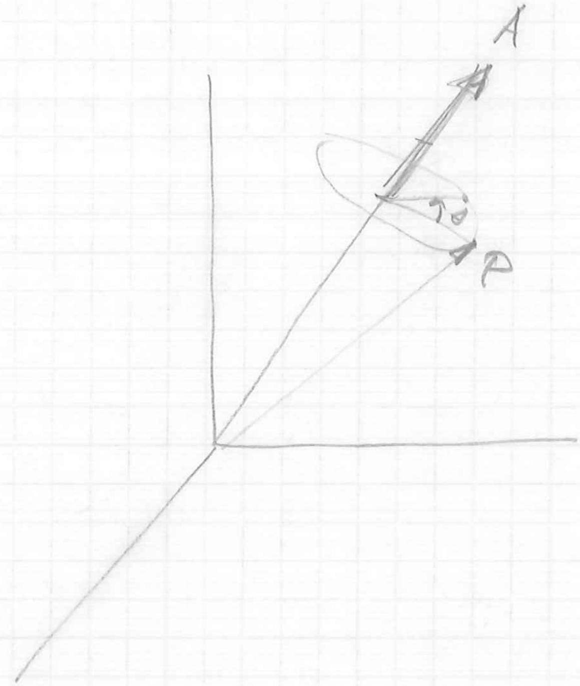
$$^A_B R_k(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\theta = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

$$\hat{k} = \frac{1}{2\sin\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

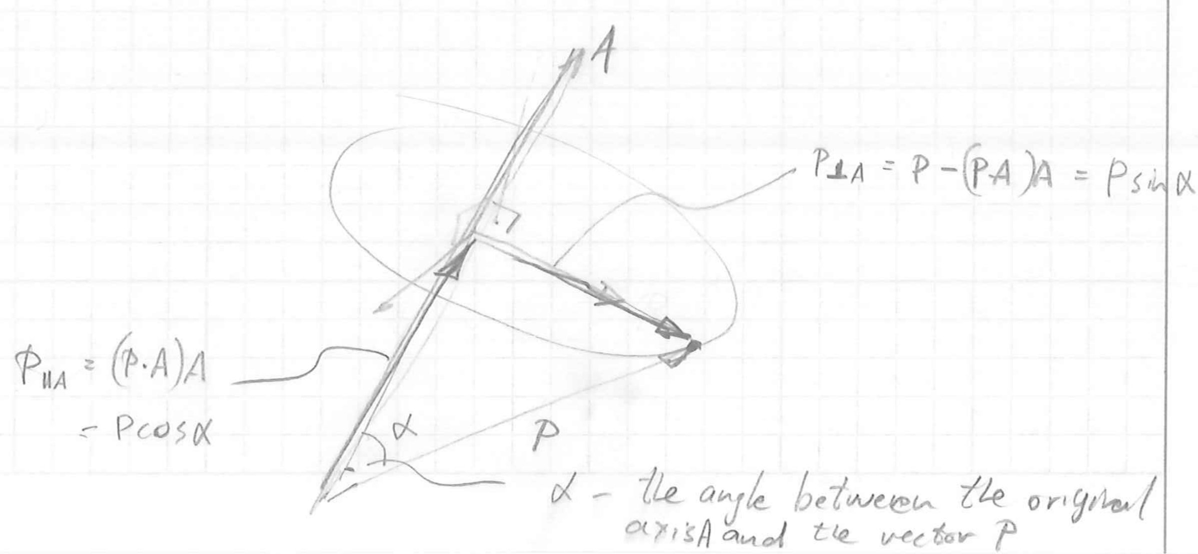
THE SOLUTION FAILS IF $\theta = 0^\circ$, $\theta = 180$

ROTATION ABOUT AN ARBITRARY AXIS



+ Rotat a vector P through an angle θ about an arbitrary axis whose direction is represented by a unit vector A

+ Decompose P into componnets that are
  - Parraller to A
  - Perpendiculer to A



scallar multyply by A → vector along A with a magnitude of of the projection of p on A

$$\vec{P}_{\parallel A} = \underbrace{(\vec{P} \bullet \vec{A})}\vec{A}$$

dot product $\boxed{PA\cos\alpha}$ projection of P on A (scalar)

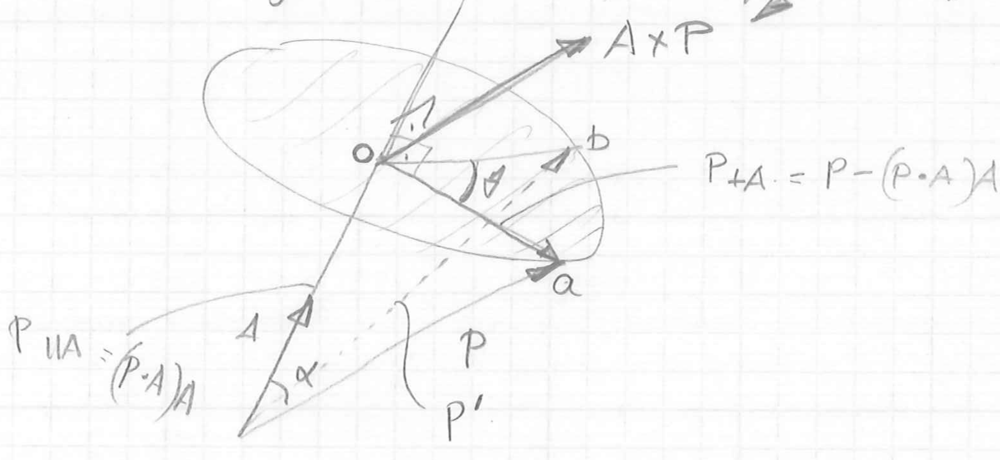
$$\vec{P} = \vec{P}_{\perp A} + \vec{P}_{\parallel A}$$

$$\vec{Oa} = \vec{P}_{\perp A} = \vec{P} - \vec{P}_{\parallel A} = \vec{P} - (\vec{P} \cdot \vec{A})\vec{A} = \|P\| \sin\alpha$$

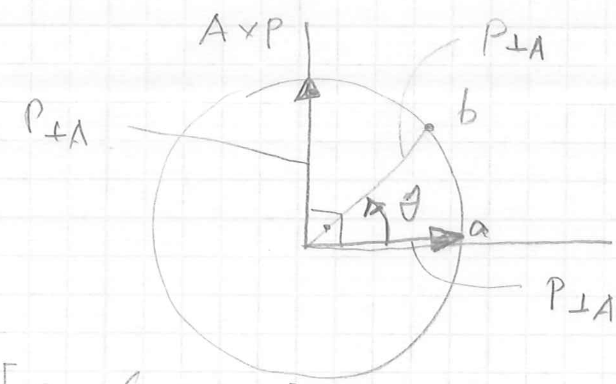
SIDE OPPOSITE THE ANGLE α

A VECTOR that is perpendicular to the plane defined by A, and p, is $A \times P$

Note the the magnitude of $|P_{\perp A}|$ is equal to magnitude of $\|A \times P|$



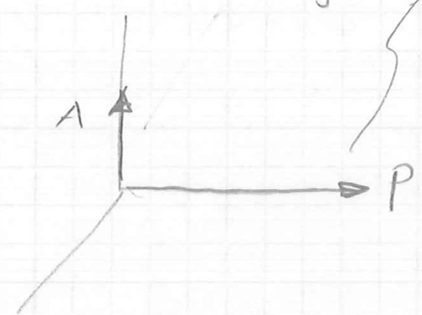
Express the rotation of $P_{\perp A}$ through an angle θ as



$$ob = \left[P - (P \cdot A)A\right]\cos\theta + (A \times P)\sin\theta$$

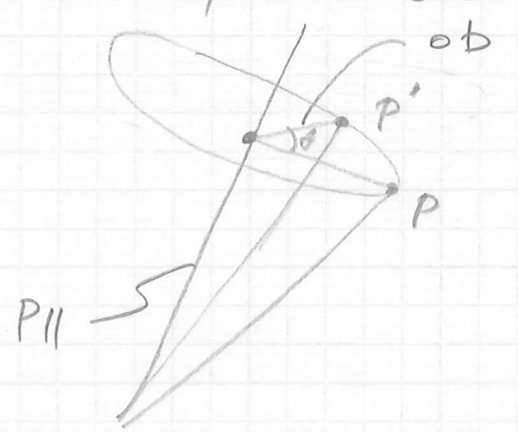
Example: multiplying a vector P by a unite vector A creats a vector that is perpendicular the the plane of A and P and has the same magnitude as P

$$\begin{cases} A = 0i + 1j + 0k \\ P = 5i + 0j + 0k \end{cases}$$



$$\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{vmatrix} = 0i + 0j - 5k$$

The new vector p' which results from rotating vector P by θ is expressed as



$$p' = ob + P_{||} = \overbrace{\left[P - (P \cdot A)A\right]\cos\theta + (A \times P)\sin\theta}^{ob} + (P \cdot A)A$$

$$= P\cos\theta + (A \times P)\sin\theta + (P \cdot A)A\,(1 - \cos\theta)$$

$$P' = P\cos\theta + (A \times P)\sin\theta + (P \cdot A)A(1-\cos\theta)$$

Replacin $A \times P$ with its matrix equivalent

The cross product $A \times P$

$$A \times P = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

The projection of P onto Q

$$\text{proj}\ P = \frac{1}{\|Q\|^2} \begin{bmatrix} Q_x^2 & Q_x Q_y & Q_x Q_z \\ Q_x Q_y & Q_y^2 & Q_y Q_z \\ Q_x Q_z & Q_y Q_z & Q_z^2 \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P\cos\theta + \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} P\sin\theta$$

$$+ \begin{bmatrix} A_x^2 & A_x A_y & A_x A_z \\ A_x A_y & A_y^2 & A_y A_z \\ A_x A_z & A_y A_z & A_z^2 \end{bmatrix} P(1-\cos\theta)$$

combining the terms and setting $c = \cos\theta$, $s = \sin\theta$ gives us the following formula for matrix $R_A(\theta)$ that rotates a vector through an angle θ about the axis A

$$P' = R_A(\theta)\, P$$

$$(*) \quad R_A(\theta) = R(A,\theta) =$$

$$= \begin{bmatrix} c+(1-c)A_x^2 & (1-c)A_xA_y - sA_z & (1-c)A_xA_z + sA_y \\ (1-c)A_xA_y + sA_z & c+(1-c)A_y^2 & (1-c)A_yA_z - sA_x \\ (1-c)A_xA_z - sA_y & (1-c)A_yA_z + sA_x & c+(1-c)A_z^2 \end{bmatrix}$$

From the general rotation transformation we can obtain each of the elementary rotation transformation. For example

$$R(x,\theta) = Rot(k,\theta) \qquad k_x = 1;\ k_y = 0;\ k_z = 0$$

$$R(y,\theta) = Rot(k,\theta) \qquad k_x = 0 \quad k_y = 1 \quad k_z = 0$$

$$R(z,\theta) = Rot(k,\theta) \qquad k_x = 0 \quad k_y = 0 \quad k_z = 1$$

substituting these values of k into eq. * we obtain

$$\begin{bmatrix} \overbrace{c+(1-c)1}^{1} & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}$$

EQUIVALENT ANGLE & AXIS OF ROTATION

Given any arbitrary rotation transformation we can use eq(*) to obtain an axis K about which an equivalent rotation θ is made as follows

Given a rotation matrix R

$$R = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

We may equate R to $Rot(k, \theta)$

$$\begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix} = \begin{bmatrix} k_x^2 \,\mathrm{VERS}\theta + C & k_x k_y \,\mathrm{VERS}\theta - k_z S & k_x k_z \,\mathrm{VERS}\theta + k_y S \\ k_x k_y \,\mathrm{VERS}\theta + k_z S & k_y^2 \,\mathrm{VERS}\theta + C & k_y k_z \,\mathrm{VERS}\theta - k_x S \\ k_x k_z \,\mathrm{VERS}\theta - k_y S & k_y k_z \,\mathrm{VERS}\theta + k_x S & k_z^2 \,\mathrm{VERS}\theta + C \end{bmatrix} \tag{+}$$

$$\mathrm{VERS}\,\theta = (1 - c)$$

Summing the diagonal terms of ~~the~~ (+) we obtain

$$n_x + o_y + a_z = \left(k_x^2 \,\mathrm{VERS}\theta + C\right) + \left(k_y^2 \,\mathrm{VERS}\theta + C\right) + \left(k_z^2 \,\mathrm{VERS}\theta + C\right)$$

$$= \underbrace{\left(k_x^2 + k_y^2 + k_z^2\right)}_{1} \underbrace{\mathrm{VERS}\,\theta}_{(1-c)} + 3C = 1 + 2C$$

and the cosine of the angle of rotation is

$$C = \cos\theta = \frac{1}{2}\left(n_x + o_y + a_z - 1\right) \qquad (1.76)$$

Differencing pairs of off-diagonal terms in eq (+) we obtain

(X)

$$(3,2) - (2,3) \rightarrow \quad o_z - a_y = 2k_x S$$

$$(1,3) - (3,1) \rightarrow \quad a_x - n_z = 2k_y S$$

$$(2,1) - (1,2) \rightarrow \quad n_y - o_x = 2k_z S$$

Squaring and adding Eq (+) we obtain an expression for $\sin\theta$

$$(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2 = 4\underbrace{(k_x^2 + k_y^2 + k_z^2)}_{1} s^2$$

$$\rightarrow S = \sin\theta = \pm\frac{1}{2}\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2} \qquad 1.81$$

We may define the rotation to be positive about the vector k such that $0 < \theta < 180$ In this case the + sign is appropriate in Eq( ) and thus the ange of the rotation θ is uniquely define as

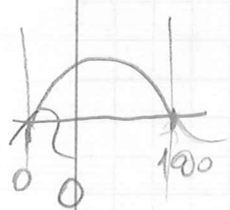
$$\tan\theta = \frac{\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}}{(n_x + o_y + a_z - 1)}$$

The components of k may be obtained from Eq.(X)

$$(\dagger) \begin{cases} k_x = \dfrac{o_z - a_y}{2s} \\[2ex] k_y = \dfrac{a_x - n_z}{2s} \\[2ex] k_z = \dfrac{n_y - o_x}{2s} \end{cases}$$

Notes

1 When the angle of rotation (θ) is very small the axis of rotation is physically not well defined due to the small magnitude of both the numerator and denominator in Eq.($\dagger$)



If the resulting angle is small the vector **k** should be renormalized to ensure the $|k| = 1$

2 When the angle of rotation approaches 180° the vector k is once again poorly defined by eq($\dagger$) as the magnitude of the sine is again decreasing

3. The axis of rotation is, however physically well defined in this case.

~~when~~

4. When $\theta > 150°$, the denominator of Eqs (X) is less tan 1. As the angle increases to 180 the rapidly decreasing magnitude of both numerator and denominator leads to cosiderable inaccuracies in the determination of $\mathbf{K}$,

(5) At $\theta = 180$, Eq (X) are of the form 0/0, yielding no information at all about a physically defined vector $\mathbf{K}$

(6) If the angle of rotation is greater than 90°, then we must follow a different approuch in determining K Equating the diagonal elements of Eq. (+) we obtain

$$k_x^2(1-c) + c = n_x$$
$$k_y^2(1-c) + c = n_y$$
$$k_z^2(1-c) + c = n_z$$

$$\odot \quad \begin{cases} k_x = \pm\sqrt{\dfrac{n_x - \cos\theta}{1-\cos\theta}} \\ k_y = \pm\sqrt{\dfrac{o_y - \cos\theta}{1-\cos\theta}} \\ k_z = \pm\sqrt{\dfrac{a_z - \cos\theta}{1-\cos\theta}} \end{cases}$$

The largest component of k defined by Eq. ⊙ corresponding to the most positive component of n_x, o_y and a_z. For this largest element, the sign of the radical can be obtained from Eq (+). As the sine of the angle of rotation θ must be positive, then the sign of the component of k defined by Eq.(+) must be the same as the sign of the left hand side of these equations. Thus we may combine Eq ⊙ with the information contained in Eq (+) as follows

$$\boxed{\cdot} \quad \begin{cases} k_x = \operatorname{sgn}(o_z - a_y)\sqrt{\dfrac{n_x - \cos\theta}{1-\cos\theta}} \qquad 1.92 \\ k_y = \operatorname{sgn}(a_x - n_z)\sqrt{\dfrac{o_y - \cos\theta}{1-\cos\theta}} \\ k_z = \operatorname{sgn}(n_y - o_x)\sqrt{\dfrac{a_z - \cos\theta}{1-\cos\theta}} \end{cases}$$

where $sgn(e) = +1$ if $e \geq 0$
and $sgn(e) = -1$ if $e \leq 0$

only the largest element of k is determined from eq. $\boxed{\cdot}$ corresponding to the most positive element of n_x, o_y and a_z. The remaining element are more accurately determined by the following equations formed by suming pairs of off-diagonal ellements of Eq (+)

$$n_y + o_x = 2k_x k_y \, VERS\,\theta \qquad (1) \quad 1.95$$

$$o_z + a_y = 2k_y k_z \, VERS\,\theta \qquad (2) \quad 1.96$$

$$n_z + a_x = 2k_z k_x \, VERS\,\theta \qquad (3) \quad 1.97$$

If k_x is largest then

$$k_y = \frac{n_y + o_x}{2k_x \, VERS\,\theta} \qquad \text{from Eq (1)} \quad 1.98$$

$$k_z = \frac{a_x + n_z}{2k_x \, VERS} \qquad \text{from Eq (3)} \quad 1.99$$

If k_y is largest then

$$k_x = \frac{n_y + o_x}{2k_y \, VERS\,\theta} \qquad \text{from Eq (1)} \quad 1.100$$

$$k_z = \frac{a_x + n_z}{2k_x \, VERS\,\theta} \qquad \text{from Eq (3)} \quad 1.101$$

If k_z is the largest then

$$k_x = \frac{a_x + n_z}{2k_z \, VERS\,\theta} \qquad \text{from (3)} \quad 1.102$$

$$k_y = \frac{o_z + a_y}{2k_z \, VERS\,\theta} \qquad \text{from Eq. (2)} \quad 1.103$$

Example

Determine the equivalent axis and angle of rotation matrix given in Eq. (4)

$$Rot(y,90)\ Rot(z,90) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

We first determin $\cos\theta$ from Eq. 1.76

$$\cos(\theta) = \frac{1}{2}\left(0+0+0-1\right) = \frac{1}{2}$$

and the $\sin\theta$ from Eq. 1.81

$$\sin\theta = \pm\frac{1}{2}\sqrt{(1-0)^2+(1-0)^2+(1-0)^2} = \frac{\sqrt{3}}{2}$$

Thus $\quad \theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = 120°$

As $\theta > 90$, we determine the largest component of k corresponding to the largest element on the diagonal. As all diagonal elements are equal in this example we may pick any one. We will pick k_x given in eq

$$k_x = +\sqrt{\frac{0+0.5}{1+0.5}} = \frac{1}{\sqrt{3}}$$

As we have determined k_x we may determine k_y and k_z from Eq. 1.98 and 1.99 respectively

$$k_y = \frac{1+0}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

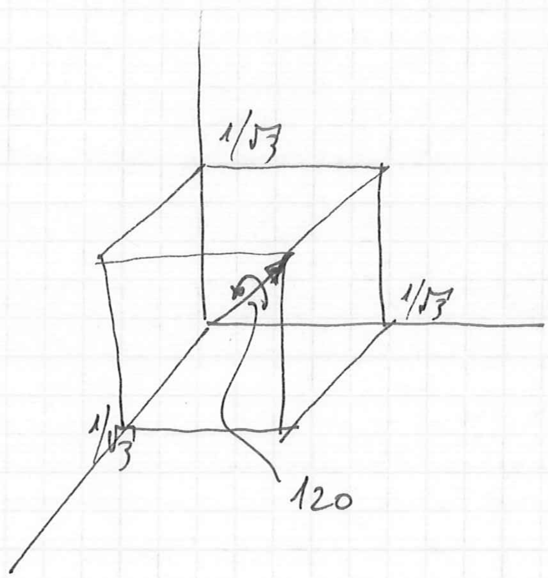
$$k_z = \frac{1+0}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

In summary then

$$Rot(y, 90)\, Rot(z, 90) = Rot(k, 120)$$

where

$$k = \frac{1}{\sqrt{3}} i + \frac{1}{\sqrt{3}} j + \frac{1}{\sqrt{3}} k$$



Any combination of rotation is allways equivalent to a single rotation about some axis k by an angle θ, an important result that we will make use of later