ROBOT ANALYSIS

The Mechanics of Serial and Parallel Manipulators

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CHAPTER 9

DYNAMICS OF SERIAL MANIPULATORS

9.1 INTRODUCTION

In this chapter we extend our study from kinematics and statics to the dynamics of serial manipulators. For some applications, such as arc welding (Fig. 9.1), it is necessary to move the end effector of a manipulator from point to point rapidly. The dynamics of the manipulator plays an important role in achieving such high-speed performance. The purpose of this chapter is to develop a set of equations that describe the dynamical behavior of a manipulator. The development of a dynamical model is important in several ways. First, a dynamical model can be used for computer simulation of a robotic system. By examining the behavior of the model under various operating conditions, it is possible to predict how a robotic system will behave when it is built. Various manufacturing automation tasks can be examined without the need of a real system. Second, it can be used for the development of suitable control strategies. A sophisticated controller requires the use of a realistic dynamical model to achieve optimal performance under high-speed operations. Some control schemes rely directly on a dynamic model to compute actuator torques required to follow a desired trajectory. Third, the dynamics analysis of a manipulator reveals all the joint reaction forces (and moments) needed for the design and sizing of links, bearings, and actuators.

There are two types of dynamical problems: direct dynamics and inverse dynamics. The *direct dynamics problem* is to find the response of a robot arm corresponding to some applied torques and/or forces. That is, given a vector of joint torques or forces, we wish to compute the resulting motion of the

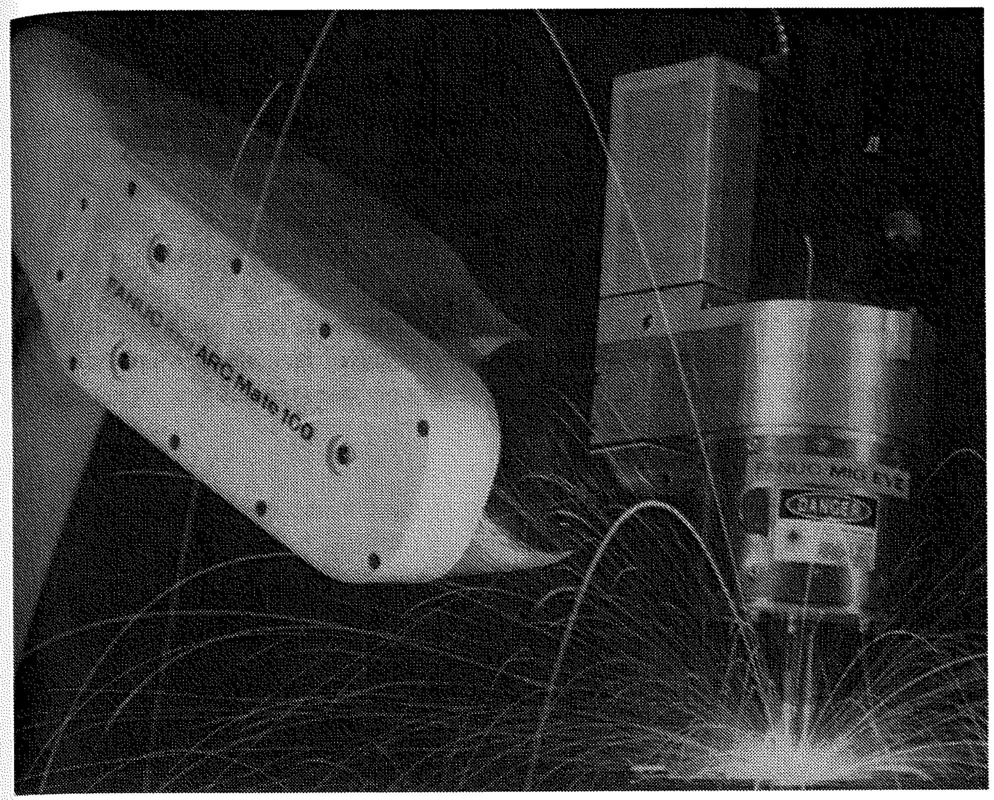

FIGURE 9.1. Arc welding robot. (Courtesy of Fanuc Robotics North America, Inc., Rochester Hills, Michigan.)

manipulator as a function of time. The *inverse dynamics problem* is to find the actuator torques and/or forces required to generate a desired trajectory of the manipulator. The problem can be formulated in the joint space, $\mathbf{q}(t)$, or the end effector space, $\mathbf{x}(t)$. The two formulations are related by the Jacobian matrix and its time derivative. In general, the efficiency of computation for direct dynamics is not as critical since it is used primarily for computer simulations of a manipulator. On the other hand, an efficient inverse dynamical model becomes extremely important for real-time feedforward control of a manipulator.

The dynamical equations of motion can be formulated by several methods. One approach is application of the Newton and Euler laws. Writing Newton's and Euler's equations once for each body of a mechanical system results in a system of equations that contains both the applied forces and the forces of constraint. The latter can be eliminated by considering the geometric and kinematic equations describing the nature of constraints. Another approach is application of the principle of *d'Alembert* or *Hamilton*. Alternatively, one can apply *Lagrange's equations of motion* (Goldstein, 1980; Paul, 1981) or *Kane's method* (Kane and Levinson, 1980, 1985). The advantage of employing the Lagrangian approach is that it eliminates the forces of constraint at the outset. However, these forces of constraint must be restored at a later time if they are to be used for the purpose of design. On the other hand, the Newton–Euler approach produces a larger system of equations, and these equations can be solved simultaneously for all the forces, including the forces of constraint.

Recently, there has been an increasing interest in the development of general-purpose computer programs for dynamical analysis of mechanical systems. For example, the following programs were developed using the Lagrangian formulation.

- *ADAMS*: developed by Chace et al. at the University of Michigan and marketed by Mechanical Dynamics, Inc. (1981).
- *DADS*: developed by Haug et al. (Haug, 1989) at the University of Iowa and marketed by Computer Aided Design Software, Inc. (1995).
- *DYMAC*: developed by Paul et al. (Paul, 1979) at the University of Pennsylvania.
- *IMP*: developed by Uicker et al. (Uicker, 1965; Sheth and Uicker, 1972) at the University of Wisconsin.

Other computer programs, such as *NBOD2* developed at NASA Goddard Space Flight Center (Frisch, 1974) and *SD-EXACT* developed by Rosenthal and Sherman (1983), are based on Eulerian approach and Kane's method. General-purpose computer programs are great for computer simulations. However, they are not necessarily suitable for real-time control of a robot manipulator. Thus, more efficient methodologies specifically tailored for robotic systems have been proposed. These include the recursive Lagrangian equations (Hollerbach, 1980), the recursive Newton–Euler equations (Armstrong, 1979; Luh et al., 1980; Orin et al., 1979), and the generalized d'Alembert equations (Fu et al., 1987, Lee et al., 1983).

Dynamics is a huge subject by itself. Obviously, we will not be able to cover the subject in great detail. In what follows, we review some fundamental laws associated with the dynamics of a rigid body and present the recursive Newton–Euler and Lagrangian methods of analysis. In addition, the effects of rotor inertias are discussed. Several examples are used to demonstrate the principles.

9.2 MASS PROPERTIES

In this section, the center of mass, inertia matrix, parallel axis theorem, and principal moments of inertia of a rigid body are defined.

9.2.1 Center of Mass

Mass is a quantity of matter that forms a body of a certain shape and size. Referring to Fig. 9.2, $A(x, y, z)$ is a Cartesian reference frame, $\mathbf{u}$ and $\mathbf{w}$ are two unit vectors, dV represents a differential volume of the material body B, ρ is the material density, and $\mathbf{p}$ is the position vector of the differential mass $\rho\, dV$ with respect to the reference frame A. The *center of mass* of such a material body is defined as the point C, whose position vector $\mathbf{p}_c$ satisfies the following condition:

$$\mathbf{p}_c = \frac{1}{m}\int_V \mathbf{p}\rho\, dV, \tag{9.1}$$

where $m = \int_V \rho\, dV$ is the total mass of the material body B.

9.2.2 Inertia Matrix

The *second moment*, $\mathbf{I}_u^O$, of a rigid body B relative to a line L_u that passes through a reference point O and is parallel to a unit vector $\mathbf{u}$ is defined as

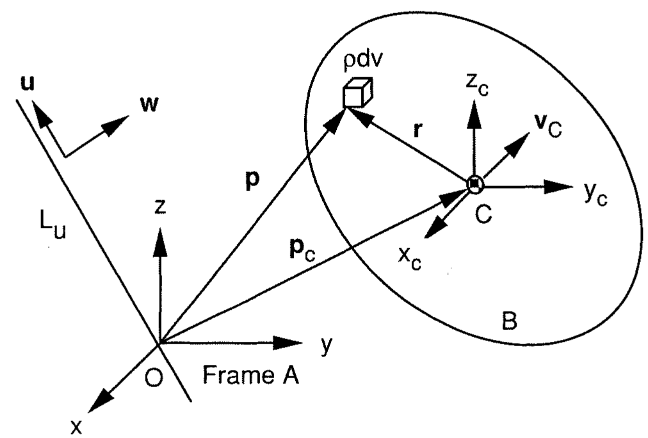

FIGURE 9.2. Moments of mass about a reference point.

(Roberson and Schwertassek, 1988)

$$\mathbf{I}_u^O = \int_V \mathbf{p} \times (\mathbf{u} \times \mathbf{p}) \rho \, dV, \tag{9.2}$$

where the trailing superscript O denotes the reference point and the trailing subscript u denotes the direction of the reference line. Expanding the triple product in Eq. (9.2) yields

$$\mathbf{I}_u^O = \int_V [\mathbf{p}^2\mathbf{u} - (\mathbf{p}^{\mathrm{T}}\mathbf{u})\mathbf{p}] \rho \, dV. \tag{9.3}$$

The scalar product of $\mathbf{I}_u^O$ with a unit vector $\mathbf{w}$ is called the *product of inertia* of B relative to O for $\mathbf{u}$ and $\mathbf{w}$,

$$I_{uw} = \mathbf{I}_u^O \cdot \mathbf{w} = \int_V [(\mathbf{u}^{\mathrm{T}}\mathbf{w})\mathbf{p}^2 - (\mathbf{p}^{\mathrm{T}}\mathbf{u})(\mathbf{p}^{\mathrm{T}}\mathbf{w})] \rho \, dV. \tag{9.4}$$

It follows from the definition above that $I_{uw} = I_{wu}$. In particular, when $\mathbf{u}$ and $\mathbf{w}$ represent the same vector, the corresponding product of inertia, I_{uu}, is called the *moment of inertia* of B about L_u:

$$I_{uu} = \int_V [\mathbf{p}^2 - (\mathbf{p}^{\mathrm{T}}\mathbf{u})^2] \rho \, dV = m r_a^2, \tag{9.5}$$

where $r_a = \mathbf{p}^2 - (\mathbf{p}^{\mathrm{T}}\mathbf{u})^2 = (\mathbf{u} \times \mathbf{p})^2$ is a nonnegative real quantity called the *radius of gyration* of B with respect to L_u.

Equation (9.3) can be written in matrix form as

$$\mathbf{I}_u^O = I_B^O \, \mathbf{u}, \tag{9.6}$$

where

$$I_B^O \equiv \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{9.7}$$

is called the *inertia matrix* or *inertia tensor* of B about O, and

$$I_{xx} = \int_V (y^2 + z^2) \rho \, dV,$$

$$I_{yy} = \int_V (z^2 + x^2) \rho \, dV,$$

$$I_{zz} = \int_V (x^2 + y^2) \rho \, dV,$$

$$I_{xy} = I_{yx} = -\int_V xy \, \rho \, dV,$$

$$I_{yz} = I_{zy} = -\int_V yz \, \rho \, dV,$$

$$I_{xz} = I_{zx} = -\int_V xz \, \rho \, dV, \tag{9.8}$$

where x, y, and z are the coordinates of a differential volume of mass $\rho \, dV$ with respect to a reference frame A whose origin is located at O. Note that each element of I_B^O represents either a moment of inertia or a product of inertia of B about the coordinate axes of the reference frame A.

The inertia matrix is symmetric. Its elements depend on the choice of a reference point and a reference frame. For brevity, we often omit the trailing superscript whenever the reference point is clearly understood or is the center of mass of a rigid body. For a rigid body of simple geometry, the inertia matrix can be computed by using the volumetric integration given by Eq. (9.8). For objects of irregular shape, the inertia matrix is often determined experimentally.

9.2.3 Parallel Axis Theorem

Let $C(x_c, y_c, z_c)$ be a Cartesian coordinate frame attached to the center of mass C of a rigid body B with its coordinate axes parallel to those of A, as shown in Fig. 9.2. Then it can be shown that

$$\begin{aligned}
I_{xx}^O &= I_{xx}^C + m(y_c^2 + z_c^2), \\
I_{yy}^O &= I_{yy}^C + m(z_c^2 + x_c^2), \\
I_{zz}^O &= I_{zz}^C + m(x_c^2 + y_c^2), \\
I_{xy}^O &= I_{xy}^C + m x_c y_c, \\
I_{yz}^O &= I_{yz}^C + m y_c z_c, \\
I_{zx}^O &= I_{xz}^C + m z_c x_c,
\end{aligned} \tag{9.9}$$

where x_c, y_c, and z_c are the coordinates of the center of mass in frame A. Equation (9.9) is called the *parallel axes theorem*.

9.2.4 Principal Moments of Inertia

We have shown that the inertia matrix depends on the choice of a reference point and the orientation of a reference frame. It turns out that for a certain orientation of a reference frame, the products of inertia will vanish. These special coordinate axes are called the *principal axes*, and the corresponding moments of inertia are called the *principal moments of inertia*.

Let I_B^O be the inertia matrix of a rigid body B about a point O expressed in a reference frame A. Also, let L_u be a principal axis that passes through the origin O and points in the direction of $\mathbf{u}$. By definition, $\mathbf{u}$ is parallel to the vector of the second moment of B about L_u. That is,

$$I_B^O \mathbf{u} = \lambda \mathbf{u}. \tag{9.10}$$

Equation (9.10) contains three linear homogeneous equations in three unknowns: u_x, u_y, and u_z. The condition for existence of nontrivial solutions is

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0. \tag{9.11}$$

Hence the eigenvalues and eigenvectors of the inertia matrix I_B^O correspond to the principal moments of inertia and the directions of principal axes, respectively. In general, corresponding to each reference point there exists at least one set of three mutually perpendicular principal axes of inertia (Kane and Levinson, 1985).

Example 9.2.1 *Inertia Matrix of a Rectangular Bar* Consider a rectangular bar of cross section $a \times b$ and length c as shown in Fig. 9.3. Assuming that the material of the bar is homogeneous, the mass m of the bar is equal to ρabc. It can be shown that the axes of the center-of-mass coordinate system shown in Fig. 9.3 are already aligned with the principal axes of the bar. Hence the products of inertia are all zero, and the resulting inertia matrix is

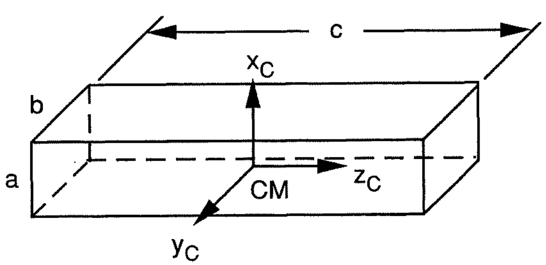

FIGURE 9.3. Rectangular bar.

$$I_B^C = \frac{m}{12}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}. \tag{9.12}$$

For a slender rod, a and b are much smaller than c. The inertia matrix can be approximated by

$$I_B^C = \frac{mc^2}{12}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.13}$$

9.3 MOMENTUM

In this section the linear and angular momentum of a rigid body are defined.

9.3.1 Linear Momentum

The *linear momentum* of a mass element $\rho\, dV$ about a point O, expressed in a reference frame A, as shown in Fig. 9.2, is defined as

$$d\mathbf{l}^O = \frac{d\mathbf{p}}{dt}\rho\, dV. \tag{9.14}$$

Hence the total linear momentum of the material body B about O is given by

$$\mathbf{l}^O = \int_V \frac{d\mathbf{p}}{dt}\rho\, dV. \tag{9.15}$$

With reference to the center of mass, the position vector $\mathbf{p}$ of the mass element $\rho\, dV$ can be written as

$$\mathbf{p} = \mathbf{p}_c + \mathbf{r}, \tag{9.16}$$

where $\mathbf{r} = {}^A R_C\, {}^C\mathbf{r}$ denotes the position of the mass element with respect to the center of mass C and expressed in the reference frame A. Here the rotation matrix ${}^A R_C$ is used to transform the vector ${}^C\mathbf{r}$ from frame C to A.

Substituting Eq. (9.16) into (9.15), we obtain

$$\mathbf{l}^O = \int_V \frac{d\mathbf{p}_c}{dt}\rho\, dV + \int_V \frac{d\mathbf{r}}{dt}\rho\, dV. \tag{9.17}$$

For a rigid body of constant mass, it can be shown that the integral and the time derivative in the second term on the right-hand side of Eq. (9.17) can be interchanged. Hence following the definition of a center of mass, the second term vanishes and Eq. (9.17) reduces to

$$\mathbf{l}^O = \frac{d\mathbf{p}_c}{dt}\int_V \rho\, dV = m\mathbf{v}_c, \tag{9.18}$$

where $\mathbf{v}_c \equiv d\mathbf{p}_c/dt$ denotes the linear velocity of the center of mass with respect to the reference frame A. Equation (9.18) implies that the total linear momentum of a rigid body is equal to the linear momentum of a point mass with mass m located at the center of mass.

9.3.2 Angular Momentum

Referring to Fig. 9.2, the *angular momentum* $d\mathbf{h}^O$ of a mass element $\rho\, dV$ about a reference point O and expressed in a reference frame A is defined as the moment of its linear momentum about O:

$$d\mathbf{h}^O = \left(\mathbf{p} \times \frac{d\mathbf{p}}{dt}\right)\rho\, dV. \tag{9.19}$$

Therefore, the total angular momentum of B about O is

$$\mathbf{h}^O = \int_V \left(\mathbf{p} \times \frac{d\mathbf{p}}{dt}\right)\rho\, dV. \tag{9.20}$$

Substituting Eq. (9.16) into (9.20), we obtain

$$\begin{aligned}\mathbf{h}^O = {} & \left(\mathbf{p}_c \times \frac{d\mathbf{p}_c}{dt}\right)\int_V \rho\, dV + \int_V \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right)\rho\, dV \\ & + \mathbf{p}_c \times \left(\frac{d}{dt}\int_V \mathbf{r}\rho\, dV\right) + \int_V \mathbf{r}\rho\, dV \times \left(\frac{d\mathbf{p}_c}{dt}\right).\end{aligned} \tag{9.21}$$

The last two terms in the expression above vanish since both contain the factor

$$\int_V \mathbf{r}\rho\, dV = 0.$$

Hence the total angular momentum about O is given by

$$\mathbf{h}^O = m(\mathbf{p}_c \times \mathbf{v}_c) + \mathbf{h}^C, \tag{9.22}$$

where

$$\mathbf{h}^C = \int_V \left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right)\rho\, dV = \int_V \mathbf{r} \times (\boldsymbol{\omega}_B \times \mathbf{r})\rho\, dV \tag{9.23}$$

denotes the angular momentum of motion about the center of mass, $\boldsymbol{\omega}_B$ is the angular velocity of B, and $\mathbf{v}_c$ denotes the linear velocity of the center of mass with respect to the reference frame A.

Equation (9.22) states that the total angular momentum of B about the origin O is equal to the angular momentum of a point mass with mass m concentrated at the center of mass, plus the angular momentum of rotation about the center of mass. Since Eq. (9.23) takes the same form as Eq. (9.2), it follows from Eq. (9.6) that $\mathbf{h}^C$ can be written as

$$\mathbf{h}^C = I_B^C\, \boldsymbol{\omega}_B, \tag{9.24}$$

where I_B^C denotes the inertia matrix of B about the center of mass C and expressed in the reference frame A. In this book, unless otherwise specified, the center of mass is taken as the reference point and the trailing superscript is omitted.

9.4 TRANSFORMATION OF INERTIA MATRIX

The angular momentum given in Eq. (9.24) can be expressed in any reference frame. Expressing Eq. (9.24) in the fixed reference frame A and dropping the trailing superscript C, we obtain

$${}^A\mathbf{h} = {}^A I_B\, {}^A\boldsymbol{\omega}_B, \tag{9.25}$$

where a leading superscript denotes the frame in which a vector or an inertia matrix is expressed. Expressing Eq. (9.24) in a body-fixed, center-of-mass, coordinate frame C, we have

$${}^C\mathbf{h} = {}^C I_B\, {}^C\boldsymbol{\omega}_B, \tag{9.26}$$

where ${}^C\boldsymbol{\omega}_B$ denotes the angular velocity of body B relative to the fixed frame A and expressed in the body frame C. That is, ${}^C\boldsymbol{\omega}_B = {}^A R_C^{\mathrm{T}}\, {}^A\boldsymbol{\omega}_B$, where ${}^A R_C$ is a rotation matrix describing the orientation of C relative to A.

Since $\mathbf{h}$ is a vector, its transformation follows that of a vector. Specifically,

$${}^A\mathbf{h} = {}^A R_C\, {}^C\mathbf{h}. \tag{9.27}$$

Substituting Eqs. (9.25) and (9.26) into (9.27), we obtain

$${}^A I_B\, {}^A\boldsymbol{\omega}_B = {}^A R_C\, {}^C I_B\, {}^C\boldsymbol{\omega}_B. \tag{9.28}$$

Substituting ${}^C\omega_B = {}^AR_C^{\mathrm{T}}\,{}^A\omega_B$ into Eq. (9.28) results in

$$ {}^AI_B = {}^AR_C\,{}^CI_B\,{}^AR_C^{\mathrm{T}}. \tag{9.29} $$

Equation (9.29) transforms an inertia matrix expressed in one reference frame into another. Both inertia matrices are taken about the center of mass C. The elements of ${}^CI_B$ are constant since they are expressed in the body-fixed coordinate frame C. However, the elements of ${}^AI_B$ are not because they are expressed in the fixed frame A and the orientation of B relative to A may be time dependent. The dependence of ${}^AI_B$ arises from the rotation matrix ${}^AR_C$.

9.5 KINETIC ENERGY

Referring to Fig. 9.2, the *kinetic energy* dK of a mass element $\rho\, dV$ with respect to a reference frame A is defined as

$$ dK = \tfrac{1}{2}\mathbf{v}^{\mathrm{T}}\mathbf{v}\rho\, dV, \tag{9.30} $$

where $\mathbf{v}$ denotes the velocity of a mass element $\rho\, dV$. Therefore, the total kinetic energy of B is given by

$$ K = \frac{1}{2}\int_V \mathbf{v}^{\mathrm{T}}\mathbf{v}\rho\, dV. \tag{9.31} $$

We can express $\mathbf{v}$ in terms of the linear velocity of the center of mass and the angular velocity of the moving body as

$$ \mathbf{v} = \mathbf{v}_c + \boldsymbol{\omega}_B \times \mathbf{r}. \tag{9.32} $$

Substituting Eq. (9.32) into (9.31), we obtain

$$ \begin{aligned} K = {} & \frac{1}{2}\mathbf{v}_c^{\mathrm{T}}\mathbf{v}_c \int_V \rho\, dV + (\mathbf{v}_c \times \boldsymbol{\omega}_B)^{\mathrm{T}} \int_V \mathbf{r}\rho\, dV \\ & + \frac{1}{2}\boldsymbol{\omega}_B^{\mathrm{T}}\left(\int_V \mathbf{r}\times(\boldsymbol{\omega}_B \times \mathbf{r})\rho\, dV\right). \end{aligned} \tag{9.33} $$

The second term in Eq. (9.33) vanishes because it follows the definition of the center of mass. The integrand in the third term represents the angular momentum of B about the center of mass. Hence Eq. (9.33) reduces to

$$ K = \tfrac{1}{2}\mathbf{v}_c^{\mathrm{T}} m \mathbf{v}_c + \tfrac{1}{2}\boldsymbol{\omega}_B^{\mathrm{T}} I_B \boldsymbol{\omega}_B. \tag{9.34} $$

In words, the kinetic energy of a rigid body B is equal to the kinetic energy of a point mass of mass m located at the center of mass, plus the kinetic energy of rotation about the center of mass.

9.6 NEWTON–EULER LAWS

In this section the Newton–Euler laws of motion are reviewed. Based on these laws, a recursive method of analysis is developed in the next section.

9.6.1 General Reference Point

We assume that there exists an *inertia frame* with respect to which the Newton–Euler laws can be applied. As shown in Fig. 9.4, let $A(x, y, z)$ be an inertia frame, $\mathbf{l}^O$ the linear momentum of a rigid body B about the origin O and expressed in A, and $\mathbf{h}^O$ the corresponding angular momentum. Also let $\mathbf{f}^O$ and $\mathbf{n}^O$ be the resultants of forces and moments exerted on the rigid body B about the origin O. Then the Newton–Euler laws can be stated as

$$ \mathbf{f}^O = \frac{d\mathbf{l}^O}{dt}, \tag{9.35} $$

$$ \mathbf{n}^O = \frac{d\mathbf{h}^O}{dt}. \tag{9.36} $$

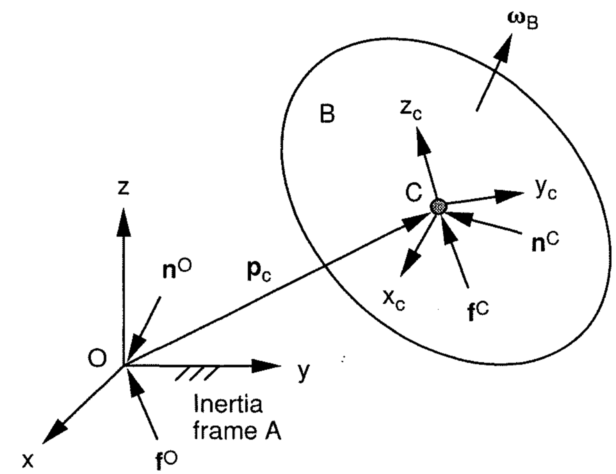

FIGURE 9.4. Resultant force and moment acting on a rigid body.

Equations (9.35) into (9.36) are two fundamental dynamical equations of motion. The main difference between dynamical and kinematical equations is that dynamical equations apply only in an inertia frame, whereas kinematical equations are valid in any frame of reference. For engineering applications, any frame that is fixed to the ground can be considered as an inertia frame. In this book we use the terms *inertia frame*, *base frame*, and *fixed frame* interchangeably. Any vector that is described in a fixed frame is called *absolute* as opposed to *relative*.

9.6.2 Center of Mass as the Reference Point

When an arbitrary point is taken as the reference point, it may be inconvenient to apply the basic laws of motion. In what follows, we show that when the center of mass is used as the reference point, the motion of a rigid body can naturally be split into two parts: a linear motion of its center of mass plus a rotational motion of the rigid body about the center of mass.

Referring to Fig. 9.4, let C be the center of mass of a rigid body B. First, we apply Newton's law. Substituting Eq. (9.18) into (9.35) gives

$$\mathbf{f}^O = \frac{d(m\mathbf{v}_c)}{dt}. \tag{9.37}$$

For a body of constant mass, Eq. (9.37) reduces to

$$\mathbf{f}^O = m\frac{d\mathbf{v}_c}{dt}. \tag{9.38}$$

Equation (9.38) is called *Newton's equation of motion* for the center of mass.

Next, we consider the rotational motion of the rigid body B. Differentiating Eq. (9.22) with respect to time yields

$$\frac{d\mathbf{h}^O}{dt} = \frac{d\mathbf{h}^C}{dt} + m\left(\mathbf{p}_c \times \frac{d\mathbf{v}_c}{dt}\right). \tag{9.39}$$

Let $\mathbf{f}^C$ and $\mathbf{n}^C$ be the resultants of forces and moments exerted at the center of mass C as shown in Fig. 9.4. Then it can be shown that

$$\mathbf{f}^O = \mathbf{f}^C, \tag{9.40}$$

$$\mathbf{n}^O = \mathbf{n}^C + \mathbf{p}_c \times \mathbf{f}^C. \tag{9.41}$$

Substituting Eqs. (9.39) and (9.41) into (9.36), we obtain

$$\mathbf{n}^C + \mathbf{p}_c \times \mathbf{f}^C = \frac{d\mathbf{h}^C}{dt} + m\left(\mathbf{p}_c \times \frac{d\mathbf{v}_c}{dt}\right). \tag{9.42}$$

Because of Eqs. (9.38) and (9.40), Eq. (9.42) reduces further to

$$\mathbf{n}^C = \frac{d\mathbf{h}^C}{dt}. \tag{9.43}$$

In words, the rate of change of angular momentum of B about its center of mass C is equal to the resulting moment exerted at the same point.

The derivative of $\mathbf{h}^C$ can be developed most conveniently in the body-fixed, center-of-mass, coordinate frame C because the inertia components of B are constant in C. Substituting Eq. (9.26) into (9.43), dropping the trailing superscript C, and expressing the resulting equation in frame C yields

$${}^C\mathbf{n} = \frac{d({}^CI_B\ {}^C\omega_B)}{dt}. \tag{9.44}$$

Note that the differentiation of $\mathbf{h}^C$ in Eq. (9.44) is taken with respect to the inertia frame A. Applying Eq. (4.19) to (9.44) yields

$${}^C\mathbf{n} = {}^CI_B\ {}^C\dot{\omega}_B + {}^C\omega_B \times ({}^CI_B\ {}^C\omega_B). \tag{9.45}$$

Equation (9.45) is called *Euler's equation of motion* for the center-of-mass coordinate frame.

Euler's equation of motion can also be written in the fixed frame A. Multiplying both sides of Eq. (9.45) by ${}^AR_C$ and making use of the relationships ${}^C\dot{\omega}_B = {}^AR_C^{\mathrm{T}}\ {}^A\dot{\omega}_B$ and ${}^C\omega_B = {}^AR_C^{\mathrm{T}}\ {}^A\omega_B$, we obtain

$${}^A\mathbf{n} = ({}^AR_C\ {}^CI_B\ {}^AR_C^{\mathrm{T}})\ {}^A\dot{\omega}_B + {}^A\omega_B \times [({}^AR_C\ {}^CI_B\ {}^AR_C^{\mathrm{T}})\ {}^A\omega_B], \tag{9.46}$$

or simply

$${}^A\mathbf{n} = {}^AI_B\ {}^A\dot{\omega}_B + {}^A\omega_B \times ({}^AI_B\ {}^A\omega_B). \tag{9.47}$$

Equation (9.47) is called *Euler's equation of motion* for a nonbody fixed frame, a coordinate frame that is located instantaneously at the center of mass with its coordinate axes parallel to those of the inertia frame A. Although Eqs. (9.45) and (9.47) have similar form, they are fundamentally different. The inertia elements in Eq. (9.45) are constant, whereas those in Eq. (9.47) are time dependent. Hence we often use Eq. (9.45) instead of (9.47) to avoid any possible confusion.

For the direct dynamics problem, the resulting forces are given and the motion of a rigid body is obtained by integrating the differential equations (9.38) and (9.45). For the inverse dynamics problem, the motion of a rigid body is prescribed as a function of time, and the forces required to produce that motion are obtained by substituting the position, velocity, and acceleration of the rigid body directly into Eqs. (9.38) and (9.45) or (9.47).

Special Case When the axes of the center-of-mass coordinate frame coincide with the principal axes of B, Eq. (9.45) reduces further to

$$\begin{aligned} n_x &= I_{xx}\dot{\omega}_x - \omega_y\omega_z(I_{yy} - I_{zz}), \\ n_y &= I_{yy}\dot{\omega}_y - \omega_z\omega_x(I_{zz} - I_{xx}), \\ n_z &= I_{zz}\dot{\omega}_z - \omega_x\omega_y(I_{xx} - I_{yy}), \end{aligned} \tag{9.48}$$

where I_{xx}, I_{yy}, and I_{zz} are the principal moments of inertia about the center-of-mass coordinate frame.

9.7 RECURSIVE NEWTON–EULER FORMULATION

In this section we present a recursive Newton–Euler formulation for the dynamical analysis of serial manipulators. The Newton–Euler formulation incorporates all the forces acting on the individual links of a robot arm. Hence the resulting dynamical equations include all the forces of constraint between two adjacent links. These forces of constraint are useful for sizing the links and bearings during the design stage. The method consists of a *forward computation* of the velocities and accelerations of each link, followed by a *backward computation* of the forces and moments in each joints.

The forces and moments acting on a typical link i of a serial manipulator are shown in Fig. 9.5. For the purpose of analysis, the following notations are employed:

$\mathbf{f}_{i,i-1}$: resulting force exerted on link i by link $i-1$ at point O_{i-1}.
$\mathbf{f}_i^*$: inertia force exerted at the center of mass of link i.
${}^iI_i$: inertia matrix of link i about its center of mass and expressed in the ith link frame.
$\mathbf{n}_{i,i-1}$: resulting moment exerted on link i by link $i-1$ at point O_{i-1}.
$\mathbf{n}_i^*$: inertia moment exerted at the center of mass of link i.
$\mathbf{p}_i$: position vector of the origin of the ith link frame with respect to the base link frame, $\mathbf{p}_i = \overline{O_0 O_i}$.
$\mathbf{p}_{ci}$: position vector of the center of mass of the ith link with respect to the base link frame, $\mathbf{p}_{ci} = \overline{O_0 C_i}$.
$\mathbf{r}_i$: position vector of the origin of the ith link frame with respect to the $(i-1)$th link frame, $\mathbf{r}_i = \overline{O_{i-1} O_i}$.
$\mathbf{r}_{ci}$: position vector of the center of mass of link i with respect to the ith link frame, $\mathbf{r}_{ci} = \overline{O_i C_i}$.
$\mathbf{v}_i$: absolute linear velocity of the origin O_i.

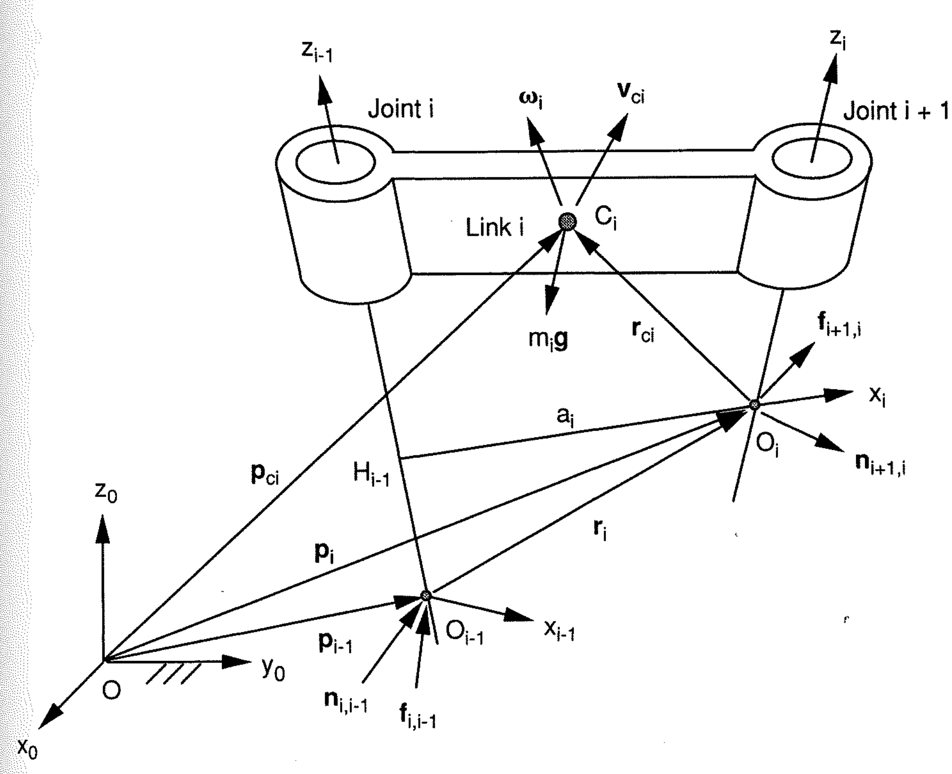

FIGURE 9.5. Forces and moments exerted on link i.

$\mathbf{v}_{ci}$: absolute linear velocity of the center of mass of link i.
$\dot{\mathbf{v}}_i$: absolute linear acceleration of the origin O_i.
$\dot{\mathbf{v}}_{ci}$: absolute linear acceleration of the center of mass of link i.
$\mathbf{z}_i$: unit vector pointing along the z_i-axis.
$\boldsymbol{\omega}_i$: absolute angular velocity of link i.
$\dot{\boldsymbol{\omega}}_i$: absolute angular acceleration of link i.

9.7.1 Forward Computation

We first compute the angular velocity, angular acceleration, linear velocity, and linear acceleration of each link in terms of its preceding link. These velocities can be computed in a recursive manner, starting at the first moving link and ending at the end-effector link. The initial conditions for the base link are $\mathbf{v}_0 = \dot{\mathbf{v}}_0 = \boldsymbol{\omega}_0 = \dot{\boldsymbol{\omega}}_0 = 0$.

(a) Angular Velocity Propagation. Due to the serial construction of a manipulator, the angular velocity of link i relative to link $i-1$ is equal to

$\mathbf{z}_{i-1}\dot{\theta}_i$ for a revolute joint and 0 for a prismatic joint, where $\mathbf{z}_{i-1}$ denotes a unit vector pointing along the ith joint axis. Hence the angular velocity of link i can be written as

$$\boldsymbol{\omega}_i = \begin{cases} \boldsymbol{\omega}_{i-1} + \mathbf{z}_{i-1}\dot{\theta}_i & \text{for a revolute joint,} \\ \boldsymbol{\omega}_{i-1} & \text{for a prismatic joint.} \end{cases} \tag{9.49}$$

Expressing Eq. (9.49) in the ith link frame, we obtain

$$^{i}\boldsymbol{\omega}_i = \begin{cases} ^{i}R_{i-1}(^{i-1}\boldsymbol{\omega}_{i-1} + {}^{i-1}\mathbf{z}_{i-1}\dot{\theta}_i) & \text{for a revolute joint,} \\ ^{i}R_{i-1}\,{}^{i-1}\boldsymbol{\omega}_{i-1} & \text{for a prismatic joint,} \end{cases} \tag{9.50}$$

where

$$^{i}R_{i-1} = \begin{bmatrix} c\theta_i & s\theta_i & 0 \\ -c\alpha_i s\theta_i & c\alpha_i c\theta_i & s\alpha_i \\ s\alpha_i s\theta_i & -s\alpha_i c\theta_i & c\alpha_i \end{bmatrix}, \tag{9.51}$$

and $^{i-1}\mathbf{z}_{i-1} = [0, 0, 1]^{\mathrm{T}}$ is a unit vector pointing along the ith joint axis and expressed in the $(i-1)$th link coordinate system.

(b) Angular Acceleration Propagation. The angular acceleration of link i is obtained by differentiating Eq. (9.49) with respect to time:

$$\dot{\boldsymbol{\omega}}_i = \begin{cases} \dot{\boldsymbol{\omega}}_{i-1} + \mathbf{z}_{i-1}\ddot{\theta}_i + \boldsymbol{\omega}_{i-1} \times \mathbf{z}_{i-1}\dot{\theta}_i & \text{for a revolute joint,} \\ \dot{\boldsymbol{\omega}}_{i-1} & \text{for a prismatic joint.} \end{cases} \tag{9.52}$$

Expressing Eq. (9.52) in the ith link frame, we obtain

$$^{i}\dot{\boldsymbol{\omega}}_i = \begin{cases} ^{i}R_{i-1}(^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + {}^{i-1}\mathbf{z}_{i-1}\ddot{\theta}_i \\ \quad + {}^{i-1}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\mathbf{z}_{i-1}\dot{\theta}_i) & \text{for a revolute joint,} \\ ^{i}R_{i-1}\,{}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} & \text{for a prismatic joint.} \end{cases} \tag{9.53}$$

Equation (9.53) provides a recursive formula for computing the angular acceleration of link i in terms of link $i-1$.

(c) Linear Velocity Propagation. Referring to Fig. 9.5, we observe that (1) if the ith joint is a revolute joint, link i does not translate along the ith joint axis, and (2) if the ith joint is a prismatic joint, there is a translational velocity of $\dot{d}_i$ along the ith joint axis. Hence the velocity of O_i can be written in terms of O_{i-1} as follows:

$$\mathbf{v}_i = \begin{cases} \mathbf{v}_{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i & \text{for a revolute joint,} \\ \mathbf{v}_{i-1} + \boldsymbol{\omega}_i \times \mathbf{r}_i + \mathbf{z}_{i-1}\dot{d}_i & \text{for a prismatic joint.} \end{cases} \tag{9.54}$$

We may express Eq. (9.54) in the ith link frame as

$$^{i}\mathbf{v}_i = \begin{cases} ^{i}R_{i-1}\,{}^{i-1}\mathbf{v}_{i-1} + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{r}_i & \text{for a revolute joint,} \\ ^{i}R_{i-1}(^{i-1}\mathbf{v}_{i-1} + {}^{i-1}\mathbf{z}_{i-1}\dot{d}_i) + {}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{r}_i & \text{for a prismatic joint,} \end{cases} \tag{9.55}$$

where

$$^{i}\mathbf{r}_i = \begin{bmatrix} a_i \\ d_i s\alpha_i \\ d_i c\alpha_i \end{bmatrix}. \tag{9.56}$$

The vector $^{i}\mathbf{r}_i$ is a constant vector for a revolute joint and a variable for a prismatic joint. Equation (9.55) is a recursive formula for computing the linear velocity of the origin of link i in terms of link $i-1$.

(d) Linear Acceleration Propagation. Linear acceleration of the origin O_i of frame i can be obtained by differentiating Eq. (9.54) with respect to time:

$$\dot{\mathbf{v}}_i = \begin{cases} \dot{\mathbf{v}}_{i-1} + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_i) & \text{for a revolute joint,} \\ \dot{\mathbf{v}}_{i-1} + \mathbf{z}_{i-1}\ddot{d}_i + \dot{\boldsymbol{\omega}}_i \times \mathbf{r}_i \\ \quad + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \mathbf{r}_i) + 2\boldsymbol{\omega}_i \times (\mathbf{z}_{i-1}\dot{d}_i) & \text{for a prismatic joint.} \end{cases} \tag{9.57}$$

Expressing Eq. (9.57) in the ith link frame, we obtain

$$^{i}\dot{\mathbf{v}}_i = \begin{cases} ^{i}R_{i-1}\,{}^{i-1}\dot{\mathbf{v}}_{i-1} + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\mathbf{r}_i \\ \quad + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{r}_i) & \text{for a revolute joint,} \\ ^{i}R_{i-1}(^{i-1}\dot{\mathbf{v}}_{i-1} + {}^{i-1}\mathbf{z}_{i-1}\ddot{d}_i) + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\mathbf{r}_i \\ \quad + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{r}_i) \\ \quad + 2\,{}^{i}\boldsymbol{\omega}_i \times ({}^{i}R_{i-1}\,{}^{i-1}\mathbf{z}_{i-1}\dot{d}_i) & \text{for a prismatic joint.} \end{cases} \tag{9.58}$$

Equation (9.58) is a recursive formula for computing the linear acceleration of link i in terms of link $i-1$.

(e) Linear Acceleration of the Center of Mass. The linear acceleration of the center of mass is computed by:

$$^{i}\dot{\mathbf{v}}_{ci} = {}^{i}\dot{\mathbf{v}}_i + {}^{i}\dot{\boldsymbol{\omega}}_i \times {}^{i}\mathbf{r}_{ci} + {}^{i}\boldsymbol{\omega}_i \times ({}^{i}\boldsymbol{\omega}_i \times {}^{i}\mathbf{r}_{ci}). \tag{9.59}$$

(f) Acceleration of Gravity. Finally, we transform the acceleration of gravity from the $(i-1)$ link frame to the ith link frame as follows:

$$ {}^{i}\mathbf{g} = {}^{i}R_{i-1}\,{}^{i-1}\mathbf{g}. \tag{9.60}$$

9.7.2 Backward Computation

Once the velocities and accelerations of the links are found, the joint forces can be computed one link at a time starting from the end-effector link and ending at the base link. We first apply Eqs. (9.38) and (9.45) to compute the inertia force and inertia moment exerted at the center of mass of link i:

$$ {}^{i}\mathbf{f}_i^* \equiv -m_i\,{}^{i}\dot{\mathbf{v}}_{ci}, \tag{9.61}$$

$$ {}^{i}\mathbf{n}_i^* \equiv -\,{}^{i}I_i\,{}^{i}\dot{\omega}_i - {}^{i}\omega_i \times ({}^{i}I_i\,{}^{i}\omega_i). \tag{9.62}$$

Next, we write the force and moment balance equations about the center of mass of link i. Referring to Fig. 9.5, we have

$$ {}^{i}\mathbf{f}_i^* + {}^{i}\mathbf{f}_{i,i-1} - {}^{i}\mathbf{f}_{i+1,i} + m_i\,{}^{i}\mathbf{g} = 0, \tag{9.63}$$

$$ {}^{i}\mathbf{n}_i^* + {}^{i}\mathbf{n}_{i,i-1} - {}^{i}\mathbf{n}_{i+1,i} - ({}^{i}\mathbf{r}_i + {}^{i}\mathbf{r}_{ci}) \times {}^{i}\mathbf{f}_{i,i-1} + {}^{i}\mathbf{r}_{ci} \times {}^{i}\mathbf{f}_{i+1,i} = 0. \tag{9.64}$$

Writing the Eqs. (9.63) and (9.64) in recursive forms, we obtain

$$ {}^{i}\mathbf{f}_{i,i-1} = {}^{i}\mathbf{f}_{i+1,i} - m_i\,{}^{i}\mathbf{g} - {}^{i}\mathbf{f}_i^*, \tag{9.65}$$

$$ {}^{i}\mathbf{n}_{i,i-1} = {}^{i}\mathbf{n}_{i+1,i} + ({}^{i}\mathbf{r}_i + {}^{i}\mathbf{r}_{ci}) \times {}^{i}\mathbf{f}_{i,i-1} - {}^{i}\mathbf{r}_{ci} \times {}^{i}\mathbf{f}_{i+1,i} - {}^{i}\mathbf{n}_i^*. \tag{9.66}$$

Once the reaction force and moment are computed in the ith link frame, they are converted into the $(i-1)$th link frame by the following transformations:

$$ {}^{i-1}\mathbf{f}_{i,i-1} = {}^{i-1}R_i\,{}^{i}\mathbf{f}_{i,i-1}, \tag{9.67}$$

$$ {}^{i-1}\mathbf{n}_{i,i-1} = {}^{i-1}R_i\,{}^{i}\mathbf{n}_{i,i-1}. \tag{9.68}$$

Equations (9.65) through (9.68) can be used to solve for ${}^{i}\mathbf{f}_{i,i-1}$ and ${}^{i}\mathbf{n}_{i,i-1}$ recursively, starting from the end-effector link. For the end-effector link, ${}^{n}\mathbf{f}_{n+1,n}$ and ${}^{n}\mathbf{n}_{n+1,n}$ represent the end-effector output force and moment and are considered as known.

9.7.3 Joint Torque Equations

Actuator torques or forces, τ_i, are obtained by projecting the forces of constraint onto their corresponding joint axes; that is,

$$ \tau_i = \begin{cases} {}^{i-1}\mathbf{n}_{i,i-1}^{T}\,{}^{i-1}\mathbf{z}_{i-1} & \text{for a revolute joint,} \\ {}^{i-1}\mathbf{f}_{i,i-1}^{T}\,{}^{i-1}\mathbf{z}_{i-1} & \text{for a prismatic joint.} \end{cases} \tag{9.69}$$

If there are viscous forces in the joints, the actuator torques or forces are computed as follows:

$$ \tau_i = \begin{cases} {}^{i-1}\mathbf{n}_{i,i-1}^{T}\,{}^{i-1}\mathbf{z}_{i-1} + b_i\dot{\theta}_i & \text{for a revolute joint,} \\ {}^{i-1}\mathbf{f}_{i,i-1}^{T}\,{}^{i-1}\mathbf{z}_{i-1} + b_i\dot{d}_i & \text{for a prismatic joint.} \end{cases} \tag{9.70}$$

where b_i is the damping coefficient for joint i.

Given the desired joint velocities and joint accelerations, we compute the velocities and accelerations of the links followed by the forces of constraint, recursively. The instantaneous velocities and accelerations of link i are computed from those of link $i-1$ by Eqs. (9.50), (9.53), (9.58), and (9.59). The process starts with the first moving link and ends at the end-effector link. Once the velocities and accelerations of the links are found, the reaction forces between two adjacent links are solved by a backward procedure. Namely, the forces of constraint at joint i are calculated from those of joint $i+1$ by Eqs. (9.65) and (9.66), and the process begins with the end-effector link and ends at the first moving link.

Example 9.7.1 *Newton–Euler Dynamics of a Planar 2-DOF Manipulator* Let us consider the planar 2-dof manipulator shown in Fig. 6.4 as an example. The D-H transformation matrices are

$$ {}^{0}A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_1c\theta_1 \\ s\theta_1 & c\theta_1 & 0 & a_1s\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{1}A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{9.71}$$

Assuming that the links are homogeneous, the vectors ${}^{i}\mathbf{r}_i$ and ${}^{i}\mathbf{r}_{ci}$ are given by

$$ {}^{i}\mathbf{r}_i = \begin{bmatrix} a_i \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad {}^{i}\mathbf{r}_{ci} = \begin{bmatrix} -a_i/2 \\ 0 \\ 0 \end{bmatrix} \qquad \text{for } i = 1, 2. \tag{9.72}$$

Let the two links be square beams of relatively small cross-sectional area. Then the inertia matrix of link i about its center of mass coordinate frame is given by

$$
{}^iI_i = \frac{m_i a_i^2}{12}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } i = 1, 2.
$$

Assuming that the acceleration of gravity points in the $-y_0$-direction, ${}^0\mathbf{g} = [0, -g_c, 0]^T$. We now apply the Newton–Euler method to calculate the link velocities and accelerations, and then forces and moments, recursively.

(a) Forward computation ($i = 1, 2$). First, we compute the velocities and accelerations of link 1. Substituting ${}^0\omega_0 = {}^0\mathbf{v}_0 = {}^0\dot{\omega}_0 = {}^0\dot{\mathbf{v}}_0 = \mathbf{0}$ into Eqs. (9.50), (9.53), (9.58), and (9.59), we obtain

$$
{}^1\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix},
$$

$$
{}^1\dot{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix},
$$

$$
{}^1\dot{\mathbf{v}}_1 = a_1 \begin{bmatrix} -\dot{\theta}_1^2 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix},
$$

$$
{}^1\dot{\mathbf{v}}_{c1} = \frac{a_1}{2} \begin{bmatrix} -\dot{\theta}_1^2 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix}.
$$

The acceleration of gravity expressed in the first link frame is

$$
{}^1\mathbf{g} = {}^1R_0\,{}^0\mathbf{g} = [-g_c s\theta_1 \quad -g_c c\theta_1 \quad 0]^T.
$$

Next, we compute the velocities and accelerations of link 2. Substituting the velocities and accelerations of link 1 into Eqs. (9.50), (9.53), (9.58), and (9.59), we obtain

$$
{}^2\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},
$$

$$
{}^2\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix},
$$

$$
{}^2\dot{\mathbf{v}}_2 = \begin{bmatrix} a_1(\ddot{\theta}_1 s\theta_2 - \dot{\theta}_1^2 c\theta_2) - a_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ a_1(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1^2 s\theta_2) + a_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix},
$$

$$
{}^2\dot{\mathbf{v}}_2^c = \begin{bmatrix} a_1(\ddot{\theta}_1 s\theta_2 - \dot{\theta}_1^2 c\theta_2) - \frac{1}{2}a_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ a_1(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1^2 s\theta_2) + \frac{1}{2}a_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}.
$$

The acceleration of gravity expressed in the second link frame is

$$
{}^2\mathbf{g} = {}^2R_1\,{}^1\mathbf{g} = [-g_c s\theta_{12} \quad -g_c c\theta_{12} \quad 0]^T.
$$

(b) Backward computation ($i = 2, 1$). For the backward computation, we first compute the forces exerted on link 2 and then link 1. Assuming that there are no externally applied forces, ${}^2\mathbf{f}_{3,2} = {}^2\mathbf{n}_{3,2} = \mathbf{0}$. Substituting ${}^2\mathbf{f}_{3,2} = {}^2\mathbf{n}_{3,2} = \mathbf{0}$ along with the velocities and accelerations of link 2 obtained from the forward computation into Eqs. (9.61), (9.62), (9.65), and (9.66) for $i = 2$, we obtain

$$
{}^2\mathbf{f}_2^* = -m_2 \begin{bmatrix} a_1(\ddot{\theta}_1 s\theta_2 - \dot{\theta}_1^2 c\theta_2) - \frac{1}{2}a_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ a_1(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1^2 s\theta_2) + \frac{1}{2}a_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix},
$$

$$
{}^2\mathbf{n}_2^* = -\frac{m_2 a_2^2}{12} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix},
$$

$$
{}^2\mathbf{f}_{2,1} = m_2 \begin{bmatrix} a_1(\ddot{\theta}_1 s\theta_2 - \dot{\theta}_1^2 c\theta_2) - \frac{1}{2}a_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + g_c s\theta_{12} \\ a_1(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1^2 s\theta_2) + \frac{1}{2}a_2(\ddot{\theta}_1 + \ddot{\theta}_2) + g_c c\theta_{12} \\ 0 \end{bmatrix},
$$

$$
{}^2\mathbf{n}_{2,1} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3}m_2 a_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + \frac{1}{2}m_2 a_1 a_2(\ddot{\theta}_1 c\theta_2 + \dot{\theta}_1^2 s\theta_2) + \frac{1}{2}m_2 g_c a_2 c\theta_{12} \end{bmatrix}.
$$

Substituting the foregoing forces and moments obtained for link 2 along with the velocities and accelerations of link 1 into Eqs. (9.61), (9.62), (9.65), and (9.66) for $i = 1$, and making use of Eqs. (9.67) and (9.68), we obtain

$$
{}^1\mathbf{f}_1^* = -\frac{m a_1}{2} \begin{bmatrix} -\dot{\theta}_1^2 \\ \ddot{\theta}_1 \\ 0 \end{bmatrix},
$$

$$
{}^1\mathbf{n}_1^* = -\frac{m_1 a_1^2}{12}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix},
$$

$$
{}^1\mathbf{f}_{1,0} = \begin{bmatrix} m_2[-a_1\dot{\theta}_1^2 - \frac{1}{2}a_2(\dot{\theta}_1+\dot{\theta}_2)^2 c\theta_2 - \frac{1}{2}a_2(\ddot{\theta}_1+\ddot{\theta}_2)s\theta_2 + g_c s\theta_1] \\ +m_1(-\frac{1}{2}a_1\dot{\theta}_1^2 + g_c s\theta_1) \\ \\ m_2[a_1\ddot{\theta}_1 - \frac{1}{2}a_2(\dot{\theta}_1+\dot{\theta}_2)^2 s\theta_2 + \frac{1}{2}a_2(\ddot{\theta}_1+\ddot{\theta}_2)c\theta_2 + g_c c\theta_1] \\ +m_1(\frac{1}{2}a_1\ddot{\theta}_1 + g_c c\theta_1) \\ \\ 0 \end{bmatrix},
$$

$$
{}^1\mathbf{n}_{1,0} = \begin{bmatrix} 0 \\ \\ 0 \\ \\ (\frac{1}{3}m_1a_1^2 + \frac{1}{3}m_2a_2^2 + m_2a_1^2 + m_2a_1a_2c\theta_2)\ddot{\theta}_1 \\ +(\frac{1}{3}m_2a_2^2 + \frac{1}{2}m_2a_1a_2c\theta_2)\ddot{\theta}_2 - m_2a_1a_2s\theta_2(\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2) \\ +\frac{1}{2}m_1g_ca_1c\theta_1 + m_2g_ca_1c\theta_1 + \frac{1}{2}m_2g_ca_2c\theta_{12} \end{bmatrix}.
$$

(c) Joint torques computation. Finally, we apply Eq. (9.69) to compute the required joint torques. This results in two dynamical equations:

$$
\begin{aligned}
\tau_1 = {} & \left[\left(\tfrac{1}{3}m_1 + m_2\right)a_1^2 + m_2a_1a_2c\theta_2 + \tfrac{1}{3}m_2a_2^2\right]\ddot{\theta}_1 \\
& + \left(\tfrac{1}{2}m_2a_1a_2c\theta_2 + \tfrac{1}{3}m_2a_2^2\right)\ddot{\theta}_2 - m_2a_1a_2s\theta_2\left(\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}\dot{\theta}_2^2\right) \\
& + g_c\left[\left(\tfrac{1}{2}m_1 + m_2\right)a_1c\theta_1 + \tfrac{1}{2}m_2a_2c\theta_{12}\right], \qquad (9.73)
\end{aligned}
$$

$$
\begin{aligned}
\tau_2 = {} & \left(\tfrac{1}{2}m_2a_1a_2c\theta_2 + \tfrac{1}{3}m_2a_2^2\right)\ddot{\theta}_1 + \tfrac{1}{3}m_2a_2^2\ddot{\theta}_2 + \tfrac{1}{2}m_2a_1a_2s\theta_2\dot{\theta}_1^2 \\
& + \tfrac{1}{2}m_2g_ca_2c\theta_{12}. \qquad (9.74)
\end{aligned}
$$

As expected, the dynamical equation for the second joint is simpler than that for first joint. The various dynamical effects, including the Coriolis and centrifugal velocity coupling, and gravitational effects are demonstrated clearly in this example. It can be seen from Eqs. (9.73) and (9.74) that the dynamical model is fairly complex even for such a simple 2-dof manipulator. Using the Newton–Euler recursive method, all the joint reaction forces are also found.

9.8 LAGRANGIAN FORMULATION

The Newton–Euler equations of motion contain all the forces of constraint between adjacent links as variables. Therefore, additional operations are needed to eliminate these forces of constraint in order to obtain closed-form equations. The Lagrangian method, on the other hand, formulates the equations of motion by using a set of *generalized coordinates*. It eliminates all or some of the forces of constraint at the outset. The following notations are used throughout this section:

- $\mathbf{G}$: vector of gravitational forces.
- I_i: inertia matrix of link i about its center of mass and expressed in the base link frame.
- J_i: link i Jacobian matrix.
- J_{vi}: Jacobian submatrix associated with the linear velocity of the center of mass of link i.
- $J_{\omega i}$: Jacobian submatrix associated with the angular velocity vector of link i.
- K: kinetic energy of a mechanical system.
- L: Lagrange function, $L = K - U$.
- M: manipulator inertia matrix.
- M_{ij}: (i, j) element of M.
- n: number of generalized coordinates.
- ${}^k\mathbf{p}_{ci}^*$: position vector of the center of mass of the ith link with respect to the kth link frame and expressed in the fixed base frame.
- Q_i: generalized active force corresponding to the ith generalized coordinate.
- $\mathbf{Q}$: vector of generalized forces, $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_n]^{\mathrm{T}}$.
- q_i: ith generalized coordinate.
- $\mathbf{q}$: vector of generalized coordinates, $\mathbf{q} = [q_1, q_2, \ldots, q_n]^{\mathrm{T}}$.
- U: potential energy of a mechanical system.
- $\mathbf{V}$: velocity coupling vector.
- δW: virtual work.

The *Lagrangian function* is defined as the difference between the kinetic and potential energy of a mechanical system:

$$
L = K - U. \qquad (9.75)
$$

Note that the kinetic energy depends on both location and velocity of the links of a manipulator system, whereas the potential energy depends only on the location of the links. Lagrange's equations of motion are formulated in terms of the Lagrangian function as (Goldstein, 1980)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \quad i = 1, 2, \ldots, n. \tag{9.76}$$

In what follows, we first define the generalized coordinates. Then we formulate expressions for the kinetic energy, potential energy, and generalized force of a robot manipulator.

9.8.1 Generalized Coordinates

Various constraints exist in a mechanical system. A kinematic constraint imposes some conditions on the relative motion between a pair of bodies. Perhaps the most frequently encountered *constraints* are those provided by the *joints* that physically connect several links to form a mechanical system.

Constraints can be classified into holonomic and nonholonomic constraints. A kinematic constraint is said to be *holonomic* if the conditions of constraint can be expressed as algebraic equations of their coordinates, and possibly the time, of the form

$$f(\mathbf{x}_1, \mathbf{x}_2, \ldots, t) = 0, \tag{9.77}$$

where $\mathbf{x}_i$ denotes the coordinates of a particle or a rigid body. A constraint that cannot be expressed in the foregoing form is said to be *nonholonomic*. The equations of constraint can be derived from the geometry of a joint. For example, the constraints imposed by a spherical joint can be stated as the position vector of the center of the sphere of one body always being equal to that of the enclosing socket of the other. Hence the constraints imposed by a spherical joint are holonomic. Similarly, the constraints provided by revolute, prismatic, and cylindrical joints are also holonomic.

The *configuration* of a mechanical system is known completely if the position and orientation of all the bodies in the system with respect to a reference frame are known. Since a rigid body has 6 degrees of freedom, a mechanical system with m moving bodies requires $6m$ coordinates to specify its configuration completely in a three-dimensional space. In a mechanical system such as a robot arm, however, these bodies are often subject to mechanical constraints imposed by the joints. As a result, these $6m$ coordinates are no longer independent. Fortunately, most of the constraints encountered in a robotic system are holonomic. If there exist c holonomic constraints, we

may use these constraints to eliminate c of the $6m$ coordinates. Hence we are left with $n = 6m - c$ independent coordinates and the system is said to have n degrees of freedom. The elimination of dependent coordinates can also be accomplished by the introduction of n new independent variables, say $q_1, q_2, \ldots, q_n$, such that the $6m$ old coordinates can be expressed in terms of the n new independent variables. We call these n new independent variables a set of *independent generalized coordinates*. Thus the number of independent generalized coordinates is equal to the number of degrees of freedom of a multibody mechanical system. We observe that given a mechanical system, the generalized coordinates can be defined in several different ways.

Lagrangian equations of motion formulated in terms of a set of independent generalized coordinates and generalized forces are called *Lagrangian equations of the second type*. Using the second type of formulation, all the forces of constraint in the joints do not appear in the equations, and the number of equations is exactly equal to the number of degrees of freedom. In formulating the equations of motion, however, it is sometimes more convenient to employ more coordinates than the number of degrees of freedom. Under such a situation, the coordinates are no longer independent and the resulting equations of motion must be solved along with an appropriate set of constraint equations using, for example, the method of Lagrangian multipliers. We call such a set of nonindependent coordinates *redundant Lagrangian coordinates*. Equations of motion formulated in terms of a set of redundant Lagrangian coordinates are called *Lagrangian equations of the first type*. The first type of formulation will contain some unknown forces of constraint as the Lagrange multipliers.

Lagrange's equations of the first type are applicable to mechanical systems with either holonomic or nonholonomic constraints. The equations of constraint and their first and second derivatives must be adjoined to the equations of motion to produce a number of equations that is equal to the number of unknowns. In this regard, Lagrange's equations of the first type are more suitable for modeling the dynamics of parallel manipulators, where there are numerous kinematical constraints due to the presence of several closed loops.

Although Lagrange's equations of the second type are only applicable to mechanical systems with holonomic constraints, they are particularly suitable for modeling the dynamics of serial manipulators. For a serial manipulator, it turns out that the number of joints is equal to the number of degrees of freedom. Therefore, the joint variables, $\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$, constitute a set of independent generalized coordinates. Each component of $\mathbf{q}$ represents either the joint angle of a revolute joint or the translational distance of a prismatic joint. Consequently, a generalized coordinate q_i does not necessarily have the dimension of length, and the corresponding generalized force, Q_i, does not necessarily have the dimension of force. However, the product $Q_i^T q_i$ always has the dimension of work.

9.8.2 Kinetic Energy

Let us examine the kinetic energy of a typical link as shown in Fig. 9.5. Applying Eq. (9.34), the kinetic energy of link i can be written as

$$K_i = \tfrac{1}{2}\mathbf{v}_{ci}^T m_i \mathbf{v}_{ci} + \tfrac{1}{2}\boldsymbol{\omega}_i^T I_i \boldsymbol{\omega}_i. \tag{9.78}$$

The velocity vectors and the inertia matrix in Eq. (9.78) can be expressed in any reference frame. Let I_i be the inertia matrix of link i about its center of mass and expressed in the base frame, and ${}^iI_i$ be the inertia matrix of link i about its center of mass and expressed in the link frame i. Then following Eq. (9.29), we have

$$I_i = {}^0R_i \, {}^iI_i ({}^0R_i)^T. \tag{9.79}$$

We note that ${}^iI_i$ is time invariant. However, I_i depends on the robot arm posture, because it is expressed in the base frame and the orientation of link i with respect to the base is a function of joint variables.

The velocity of the center of mass and the angular velocity of link i can be found by using the recursive method developed in the preceding section. Alternatively, they can also be found by applying the theory of instantaneous screw motion. Either way, we can express them in matrix form as

$$\dot{\mathbf{x}}_{ci} = J_i \dot{\mathbf{q}}, \tag{9.80}$$

where

$$\dot{\mathbf{x}}_{ci} = \begin{bmatrix} \mathbf{v}_{ci} \\ \boldsymbol{\omega}_i \end{bmatrix} \quad \text{and} \quad J_i = \begin{bmatrix} J_{vi} \\ J_{\omega i} \end{bmatrix}.$$

Here J_i is a $6 \times n$ matrix that maps the instantaneous joint rates into the instantaneous velocity of the center of mass and the angular velocity of link i, and J_{vi} and $J_{\omega i}$ are two $3 \times n$ submatrices of J_i. We call J_i the *link Jacobian matrix* and J_{vi} and $J_{\omega i}$ the *link Jacobian submatrices*.

Let J_{vi}^j and $J_{\omega i}^j$ be the jth column vectors of J_{vi} and $J_{\omega i}$, respectively. Then applying the theory of instantaneous screw motion for $j \le i$, we obtain

$$J_{vi}^j = \begin{cases} \mathbf{z}_{j-1} \times {}^{j-1}\mathbf{p}_{ci}^* & \text{for a revolute joint,} \\ \mathbf{z}_{j-1} & \text{for a prismatic joint,} \end{cases} \tag{9.81}$$

$$J_{\omega i}^j = \begin{cases} \mathbf{z}_{j-1} & \text{for a revolute joint,} \\ \mathbf{0} & \text{for a prismatic joint.} \end{cases} \tag{9.82}$$

where ${}^{j-1}\mathbf{p}_{ci}^*$ is a position vector defined from the origin of the $j-1$ link frame to the center of mass of link i and expressed in the base frame. We note that J_{vi}^j and $J_{\omega i}^j$ denote the partial rate of change of the velocity of the center of mass and the angular velocity of link i with respect to the jth joint motion. Since the motion of link i depends only on joints 1 through i, the two column vectors above are set to zero for $j > i$. Furthermore, since both $\mathbf{z}_{j-1}$ and ${}^{j-1}\mathbf{p}_{ci}^*$ depend on $\mathbf{q}$, the submatrices J_{vi} and $J_{\omega i}$ are configuration dependent. Using the notations above, J_{vi} and $J_{\omega i}$ can be written as

$$J_{vi} = \left[J_{vi}^1, J_{vi}^2, \ldots, J_{vi}^i, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\right], \tag{9.83}$$

$$J_{\omega i} = \left[J_{\omega i}^1, J_{\omega i}^2, \ldots, J_{\omega i}^i, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\right]. \tag{9.84}$$

Substituting Eq. (9.80) into (9.78) and then summing over all links, we obtain an expression for the kinetic energy of the system as

$$\begin{aligned} K &= \frac{1}{2}\sum_{i=1}^{n}(\mathbf{v}_{ci}^T m_i \mathbf{v}_{ci} + \boldsymbol{\omega}_i^T I_i \boldsymbol{\omega}_i) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left[(J_{vi}\dot{\mathbf{q}})^T m_i (J_{vi}\dot{\mathbf{q}}) + (J_{\omega i}\dot{\mathbf{q}})^T I_i (J_{\omega i}\dot{\mathbf{q}})\right] \\ &= \frac{1}{2}\dot{\mathbf{q}}^T \left[\sum_{i=1}^{n}(J_{vi}^T m_i J_{vi} + J_{\omega i}^T I_i J_{\omega i})\right]\dot{\mathbf{q}}. \end{aligned} \tag{9.85}$$

For convenience, we define an $n \times n$ *manipulator inertia matrix* as

$$M = \sum_{i=1}^{n}(J_{vi}^T m_i J_{vi} + J_{\omega i}^T I_i J_{\omega i}). \tag{9.86}$$

In this way, the total kinetic energy of a robot arm can be expressed in terms of the manipulator inertia matrix and the vector of joint rates:

$$K = \tfrac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}}. \tag{9.87}$$

We note that the manipulator inertia matrix M defined in Eq. (9.86) involves the link Jacobian submatrices J_{vi} and $J_{\omega i}$. Therefore, the manipulator inertia matrix is configuration dependent. Similar to the inertia matrix of a rigid body, the manipulator inertia matrix is a symmetric, positive-definite matrix. The quadratic form of the equation indicates that the kinetic energy is always positive unless the system is at rest.

9.8.3 Potential Energy

The potential energy stored in link i of a robot arm is defined as the amount of work required to raise the center of mass of link i from a horizontal reference plane to its present position under the influence of gravity. With reference to the inertia frame, the work required to displace link i to position $\mathbf{p}_{ci}$ is given by $-m_i\mathbf{g}^{\mathrm{T}}\mathbf{p}_{ci}$. Hence the total potential energy stored in a robot arm is

$$U = -\sum_{i=1}^{n} m_i \mathbf{g}^{\mathrm{T}}\mathbf{p}_{ci}. \tag{9.88}$$

9.8.4 Generalized Forces

In this section we investigate various contributions to the vector of generalized forces. Except for gravitational and inertia forces, the generalized forces account for all the other forces acting on a robot arm that are consistent with the mechanical constraints. The vector of generalized forces, $\mathbf{Q} = [Q_1, Q_2, \ldots, Q_n]^{\mathrm{T}}$, is defined by the principle of virtual work as

$$\delta W = \mathbf{Q}^{\mathrm{T}}\delta\mathbf{q}. \tag{9.89}$$

We first consider the case in which actuators exert forces or torques at the joints and external force and moment are applied at the end effector. Let $\boldsymbol{\tau} = [\tau_1, \ldots, \tau_n]^{\mathrm{T}}$ be an n-dimensional vector of joint torques generated by the actuators and $\mathbf{F}_e = [\mathbf{f}_e^{\mathrm{T}}, \mathbf{n}_e^{\mathrm{T}}]^{\mathrm{T}}$ be a six-dimensional vector of resultant force and moment exerted at the end effector. Then the virtual work produced by these forces and moments is

$$\delta W = \boldsymbol{\tau}^{\mathrm{T}}\,\delta\mathbf{q} + \mathbf{F}_e^{\mathrm{T}}\,\delta\mathbf{x}, \tag{9.90}$$

where $\delta\mathbf{x}$ denotes a six-dimensional virtual displacement vector of the end effector. Substituting the relation $\delta\mathbf{x} = J\,\delta\mathbf{q}$ into Eq. (9.90) and then equating the resulting virtual work to that of Eq. (9.89) yields the vector of generalized forces as

$$\mathbf{Q} = \boldsymbol{\tau} + J^{\mathrm{T}}\mathbf{F}_e. \tag{9.91}$$

The contribution of friction to the generalized force vector can also be formulated. Frictional force is a highly nonlinear phenomenon that is difficult to model accurately (Armstrong-Helouvry, 1991), yet it can have significant effects on system dynamics. In a grease- or oil-lubricated bearing, there are four regimes of lubrication: static friction, boundary lubrication, partial fluid lubrication, and full fluid lubrication. In the fourth regime, frictional force is proportional to the relative velocity between the contacting bodies. It can

be modeled by a simple expression: $-b_i\dot{q}_i$. Therefore the virtual work contributed by this type of frictional forces is given by

$$\delta W = -\mathbf{f}_r^{\mathrm{T}}\,\delta\mathbf{q}, \tag{9.92}$$

where $\mathbf{f}_r = [b_1\dot{q}_1, b_2\dot{q}_2, \ldots, b_n\dot{q}_n]^{\mathrm{T}}$ denotes the frictional torques or forces in the joints and the minus sign indicates that the direction of frictional torque or force is always opposite to the joint velocity. Adding this contribution to the vector of generalized forces in Eq. (9.91), we obtain

$$\mathbf{Q} = \boldsymbol{\tau} + J^{\mathrm{T}}\mathbf{F}_e - \mathbf{f}_r. \tag{9.93}$$

We notice that in the absence of friction and externally applied force, the vector of generalized forces and the vector of joint torques are equivalent (i.e., $\mathbf{Q} = \boldsymbol{\tau}$). In this case, the components of the generalized force vector are the actuator forces for prismatic joints and torques for revolute joints.

9.8.5 General Form of Dynamical Equations

Now we are in a position to formulate the dynamical equations of a serial manipulator. First, we substitute Eqs. (9.87) and (9.88) into (9.75) to obtain a compact expression for the Lagrangian function:

$$L = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}M\dot{\mathbf{q}} + \sum_{i=1}^{n} m_i\mathbf{g}^{\mathrm{T}}\mathbf{p}_{ci}. \tag{9.94}$$

Next, we differentiate the Lagrangian function with respect to q_i, $\dot{q}_i$, and t to formulate the dynamical equations of motion. To facilitate the derivation, we expand the term for the kinetic energy into a sum of scalars. Let M_{ij} be the (i, j) element of the manipulator inertia matrix M; then Eq. (9.94) can be written as

$$L = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}\dot{q}_i\dot{q}_j + \sum_{i=1}^{n} m_i\mathbf{g}^{\mathrm{T}}\mathbf{p}_{ci}. \tag{9.95}$$

Since the potential energy does not depend on $\dot{q}_i$, taking the partial derivative of Eq. (9.95) with respect to $\dot{q}_i$, we obtain

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} M_{ij}\dot{q}_j. \tag{9.96}$$

$\frac{\partial L}{\partial \dot{q}_i} = \sum_{j=1}^{n} M_{ij} \dot{q}_j$

Taking the total derivative of Eq. (9.96) with respect to time yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\left(\frac{dM_{ij}}{dt}\right)\dot{q}_j = \sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j. \tag{9.97}$$

Taking the partial derivative of Eq. (9.95) with respect to q_i yields

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\frac{\partial}{\partial q_i}\left(\sum_{j=1}^{n}\sum_{k=1}^{n} M_{jk}\dot{q}_j\dot{q}_k\right) + \sum_{j=1}^{n} m_j \mathbf{g}^{\mathrm{T}}\left(\frac{\partial \mathbf{p}_{cj}}{\partial q_i}\right). \tag{9.98}$$

Note that the partial derivative of $\mathbf{p}_{cj}$ with respect to q_i is equal to the ith column vector of the link Jacobian submatrix J_{vj}. Hence Eq. (9.98) can be written as

$$\frac{\partial L}{\partial q_i} = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n} m_j \mathbf{g}^{\mathrm{T}} J_{vj}^{i}. \tag{9.99}$$

Finally, we substitute Eqs. (9.97) and (9.99) into (9.76) to obtain the dynamical equations of motion:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$

$$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + V_i + G_i = Q_i \qquad \text{for } i = 1, 2, \ldots, n. \tag{9.100}$$

where

$\sum_{j=1}^{n} M_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j - \frac{1}{2}\frac{\partial}{\partial q_i}\sum\sum M_{jk}\dot{q}_j\dot{q}_k$

$$V_i = \sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k, \tag{9.101}$$

$$G_i = -\sum_{j=1}^{n} m_j \mathbf{g}^{\mathrm{T}} J_{vj}^{i}. \tag{9.102}$$

The first term in Eq. (9.100) accounts for the inertia forces, the second term represents the *Coriolis* and *centrifugal* forces, and the third term gives the gravitational effects. The n scalar equations given by Eq. (9.100) can be written in matrix form as

$$M\ddot{\mathbf{q}} + \mathbf{V} + \mathbf{G} = \mathbf{Q}, \tag{9.103}$$

where $\mathbf{V} = [V_1, \ldots, V_n]^{\mathrm{T}}$, $\mathbf{G} = [G_1, \ldots, G_n]^{\mathrm{T}}$, and $\mathbf{Q} = [Q_1, \ldots, Q_n]^{\mathrm{T}}$.

Equation (9.103) is called the *general form of dynamical equations*. The vector $\mathbf{V}$ is called the *velocity coupling vector*. The vector $\mathbf{G}$ is called the vector of gravitational forces. There are two distinct types of velocity coupling between joints. The velocity-squared terms correspond to the centrifugal forces, and the velocity product terms correspond to the Coriolis forces. The manipulator inertia matrix M is symmetric and positive definite and therefore is always invertible. The off-diagonal terms of M represent the acceleration coupling effect between joints.

Example 9.8.1 *Lagrangian Dynamics of a Planar 2-DOF Manipulator* In this example we formulate Lagrange's equations of motion for the planar 2-dof manipulator shown in Fig. 6.4. We note that the link coordinate axes are aligned with the principal axes of each link. The two D-H transformation matrices are given by Eq. (9.71). The center of mass of link i, expressed in link frame i, is given by Eq. (9.72). Let θ_1 and θ_2 be a set of two independent generalized coordinates. We compute the link inertia matrices, link Jacobian matrices, gravitational effects, and Lagrange's equations of motion as follows.

(a) Link inertia matrices. Assuming that the moving links are homogeneous with a relatively small cross section, the inertia matrix of link i about its center of mass and expressed in the ith link frame is

$${}^{i}I_i = \frac{1}{12}m_i a_i^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{for } i = 1, 2. \tag{9.104}$$

The inertia matrices of links 1 and 2 about their respective centers of mass and expressed in the base frame are obtained by substituting Eq. (9.104) for $i = 1$ and 2 into (9.79). As a result, we obtain

$$I_1 = \frac{1}{12}m_1 a_1^2\begin{bmatrix} s^2\theta_1 & -s\theta_1 c\theta_1 & 0 \\ -s\theta_1 c\theta_1 & c^2\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.105}$$

$$I_2 = \frac{1}{12}m_2 a_2^2\begin{bmatrix} s^2\theta_{12} & -s\theta_{12} c\theta_{12} & 0 \\ -s\theta_{12} c\theta_{12} & c^2\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.106}$$

(b) Link Jacobian matrices. The position vectors of the centers of mass of links 1 and 2 with respect to the various link frames and expressed in the base frame are given by

$${}^{0}\mathbf{p}_{c1}^{*} = \begin{bmatrix} \frac{1}{2}a_1 c\theta_1 \\ \frac{1}{2}a_1 s\theta_1 \\ 0 \end{bmatrix}, \tag{9.107}$$

$$
{}^{1}\mathbf{p}^{*}_{c2} = \begin{bmatrix} \frac{1}{2}a_2 c\overset{\theta_2}{\cancel{\theta_{12}}} \\ \frac{1}{2}a_2 s\cancel{\theta_{12}}\ \theta_2 \\ 0 \end{bmatrix} \tag{9.108}
$$

$$
{}^{0}\mathbf{p}^{*}_{c2} = \begin{bmatrix} a_1 c\theta_1 + \frac{1}{2}a_2 c\theta_{12} \\ a_1 s\theta_1 + \frac{1}{2}a_2 s\theta_{12} \\ 0 \end{bmatrix}. \tag{9.109}
$$

The link Jacobian submatrices, J_{vi} and $J_{\omega i}$, are obtained by substituting Eqs. (9.107) through (9.109) into (9.83) and (9.84):

$$
J_{v1} = \begin{bmatrix} -\frac{1}{2}a_1 s\theta_1 & 0 \\ \frac{1}{2}a_1 c\theta_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{9.110}
$$

$$
J_{\omega 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{9.111}
$$

$$
J_{v2} = \begin{bmatrix} -a_1 s\theta_1 - \frac{1}{2}a_2 s\theta_{12} & -\frac{1}{2}a_2 \overset{s\theta_2}{\cancel{s\theta_{12}}} \\ a_1 c\theta_1 + \frac{1}{2}a_2 c\theta_{12} & \frac{1}{2}a_2 \cancel{c\theta_{12}}\ c\theta_2 \\ 0 & 0 \end{bmatrix}, \tag{9.112}
$$

$$
J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{9.113}
$$

(c) Manipulator inertia matrix. The manipulator inertia matrix is obtained by substituting Eqs. (9.105), (9.106), and (9.110) through (9.113) into (9.86):

$$
\begin{aligned}
M &= J_{v1}^{\mathrm{T}} m_1 J_{v1} + J_{\omega 1}^{\mathrm{T}} I_1 J_{\omega 1} + J_{v2}^{\mathrm{T}} m_2 J_{v2} + J_{\omega 2}^{\mathrm{T}} I_2 J_{\omega 2} \\
&= \begin{bmatrix} \frac{1}{3}m_1 a_1^2 + m_2(a_1^2 + a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2) & m_2(\frac{1}{2}a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2) \\ m_2(\frac{1}{2}a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2) & \frac{1}{3}m_2 a_2^2 \end{bmatrix}.
\end{aligned} \tag{9.114}
$$

?

(d) Velocity coupling vector. Taking the partial derivatives of the manipulator inertia matrix, Eq. (9.114), with respect to θ in accordance with Eq. (9.101) yields

$$
\begin{aligned}
V_1 &= \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial M_{1j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_1}\right)\dot{\theta}_j\dot{\theta}_k \\
&= -m_2 a_1 a_2 s\theta_2\left(\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2\right),
\end{aligned} \tag{9.115}
$$

$$
V_2 = \sum_{j=1}^{2}\sum_{k=1}^{2}\left(\frac{\partial M_{2j}}{\partial \theta_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial \theta_2}\right)\dot{\theta}_j\dot{\theta}_k = \frac{1}{2}m_2 a_1 a_2 s\theta_2\dot{\theta}_1^2. \tag{9.116}
$$

(e) Gravitational vector. The gravitational terms are obtained by substituting Eqs. (9.110) and (9.112) into (9.102),

$$
G_1 = \tfrac{1}{2}m_1 g_c a_1 c\theta_1 + m_2 g_c a_1 c\theta_1 + \tfrac{1}{2}m_2 g_c a_2 c\theta_{12}, \tag{9.117}
$$

$$
G_2 = \tfrac{1}{2}m_2 g_c a_2 c\theta_{12}. \tag{9.118}
$$

(f) Lagrange's equations of motion. Assuming that there are no external forces exerted at the end effector and the joint friction is negligible, the vector of joint torques and the vector of generalized forces are equivalent. Lagrange's equations of motion are obtained by substituting Eqs. (9.114) through (9.118) into (9.100). This results in the following two dynamical equations of motion:

$$
\begin{aligned}
\tau_1 &= \left[\left(\tfrac{1}{3}m_1 + m_2\right)a_1^2 + m_2 a_1 a_2 c\theta_2 + \tfrac{1}{3}m_2 a_2^2\right]\ddot{\theta}_1 \\
&\quad + \left(\tfrac{1}{2}m_2 a_1 a_2 c\theta_2 + \tfrac{1}{3}m_2 a_2^2\right)\ddot{\theta}_2 - m_2 a_1 a_2 s\theta_2\left(\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}\dot{\theta}_2^2\right) \\
&\quad + g_c\left[\left(\tfrac{1}{2}m_1 + m_2\right)a_1 c\theta_1 + \tfrac{1}{2}m_2 a_2 c\theta_{12}\right],
\end{aligned} \tag{9.119}
$$

$$
\begin{aligned}
\tau_2 &= \left(\tfrac{1}{2}m_2 a_1 a_2 c\theta_2 + \tfrac{1}{3}m_2 a_2^2\right)\ddot{\theta}_1 + \tfrac{1}{3}m_2 a_2^2\ddot{\theta}_2 + \tfrac{1}{2}m_2 a_1 a_2 s\theta_2\dot{\theta}_1^2 \\
&\quad + \tfrac{1}{2}m_2 g_c a_2 c\theta_{12}.
\end{aligned} \tag{9.120}
$$

Hence we have arrived at the same equations obtained by using the recursive Newton–Euler formulation. Using the Lagrangian formulation, forces of constraint do not appear in the equations of motion.

Example 9.8.2 *Langrangian Dynamics of a SCARA Arm* Let us study the dynamics of a SCARA arm as a second example. The SCARA arm is constructed with four parallel joint axes. The first two and the fourth are revolute joints, and the third is a prismatic joint. To simplify the problem, we consider the motion of the first three moving links and combine the mass of the fourth link and the load, if any, with the third link. In this way, we will be dealing with a pure position problem. Figure 9.6 shows a schematic diagram of the

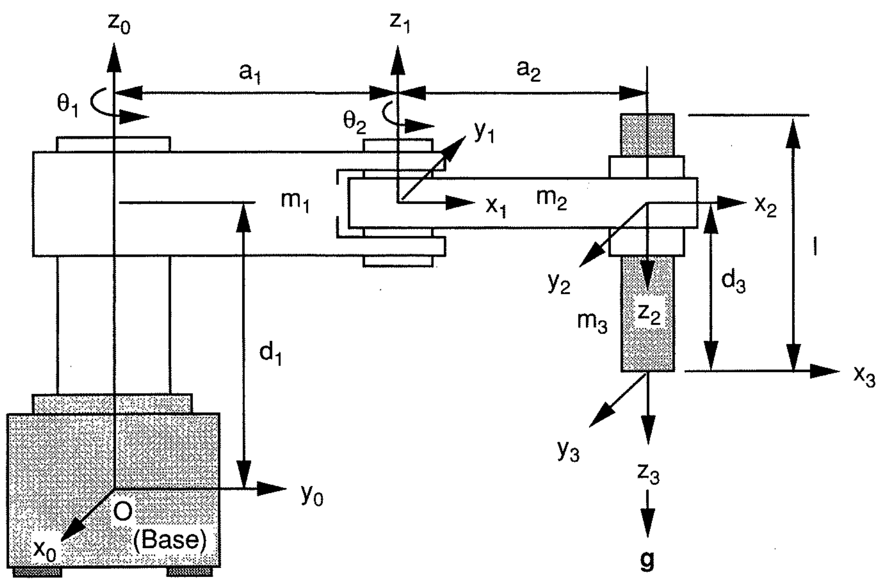

FIGURE 9.6. Schematic diagram of a SCARA arm.

first three moving links in which the coordinate axes of each link frame are aligned with the principal axes of the link. The Denavit–Hartenberg parameters are listed in Table 2.2, and the D-H transformation matrices are given by Eqs. (2.9) to (2.11).

Assuming that all links are homogeneous with relatively small cross section, the position vectors of the centers of mass are given by

$$
\begin{aligned}
{}^1\mathbf{p}_{c1} &= [-a_1/2, 0, 0]^{\mathrm{T}}, \\
{}^2\mathbf{p}_{c2} &= [-a_2/2, 0, 0]^{\mathrm{T}}, \\
{}^3\mathbf{p}_{c3} &= [0, 0, -\ell/2]^{\mathrm{T}},
\end{aligned}
$$

where ℓ is the third link length. Let θ_1, θ_2, and d_3 be a set of independent generalized coordinates. We compute the link inertia matrices, link Jacobian matrices, and the gravitational effects for the links, and substitute them into Eq. (9.100) to obtain Lagrange's equations of motion as follows.

(a) Link inertia matrices. The link inertia matrices about their centers of mass and expressed in their respective link frames are

$$
{}^1I_1 = \frac{1}{12} m_1 a_1^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.121}
$$

$$
{}^2I_2 = \frac{1}{12} m_2 a_2^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.122}
$$

$$
{}^3I_3 = \frac{1}{12} m_3 \ell^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.123}
$$

The link inertia matrices about their centers of mass and expressed in the base frame are obtained by substituting Eqs. (9.121) through (9.123) along with their rotation matrices into (9.79):

$$
I_1 = \frac{1}{12} m_1 a_1^2 \begin{bmatrix} \mathrm{s}^2\theta_1 & -\mathrm{s}\theta_1 \mathrm{c}\theta_1 & 0 \\ -\mathrm{s}\theta_1 \mathrm{c}\theta_1 & \mathrm{c}^2\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.124}
$$

$$
I_2 = \frac{1}{12} m_2 a_2^2 \begin{bmatrix} \mathrm{s}^2\theta_{12} & -\mathrm{s}\theta_{12} \mathrm{c}\theta_{12} & 0 \\ -\mathrm{s}\theta_{12} \mathrm{c}\theta_{12} & \mathrm{c}^2\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{9.125}
$$

$$
I_3 = \frac{1}{12} m_3 \ell^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9.126}
$$

(b) Link Jacobian matrices. The position vectors of the centers of mass of links 1, 2, and 3 with respect to the various link frames and expressed in the base frame are

$$
{}^0\mathbf{p}_{c1}^* = \begin{bmatrix} \frac{1}{2} a_1 \mathrm{c}\theta_1 \\ \frac{1}{2} a_1 \mathrm{s}\theta_1 \\ d_1 \end{bmatrix}, \tag{9.127}
$$

$$
{}^1\mathbf{p}_{c2}^* = \begin{bmatrix} \frac{1}{2} a_2 \mathrm{c}\theta_{12} \\ \frac{1}{2} a_2 \mathrm{s}\theta_{12} \\ 0 \end{bmatrix}, \tag{9.128}
$$

$$
{}^0\mathbf{p}_{c2}^* = \begin{bmatrix} a_1 \mathrm{c}\theta_1 + \frac{1}{2} a_2 \mathrm{c}\theta_{12} \\ a_1 \mathrm{s}\theta_1 + \frac{1}{2} a_2 \mathrm{s}\theta_{12} \\ d_1 \end{bmatrix}, \tag{9.129}
$$

$$ {}^2\mathbf{p}^*_{c3} = \begin{bmatrix} 0 \\ 0 \\ d_3 - \frac{1}{2}\ell \end{bmatrix}, \tag{9.130} $$

$$ {}^1\mathbf{p}^*_{c3} = \begin{bmatrix} a_2 c\theta_{12} \\ a_2 s\theta_{12} \\ -d_3 + \frac{1}{2}\ell \end{bmatrix}, \tag{9.131} $$

$$ {}^0\mathbf{p}^*_{c3} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} \\ a_1 s\theta_1 + a_2 s\theta_{12} \\ d_1 - d_3 + \frac{1}{2}\ell \end{bmatrix}. \tag{9.132} $$

The link Jacobian submatrices, J_{vi} and $J_{\omega i}$, are obtained by substituting the equations above into Eqs. (9.83) and (9.84):

$$ J_{v1} = \begin{bmatrix} -\frac{1}{2}a_1 s\theta_1 & 0 & 0 \\ \frac{1}{2}a_1 c\theta_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{9.133} $$

$$ J_{\omega 1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{9.134} $$

$$ J_{v2} = \begin{bmatrix} -a_1 s\theta_1 - \frac{1}{2}a_2 s\theta_{12} & -\frac{1}{2}a_2 s\theta_{12} & 0 \\ a_1 c\theta_1 + \frac{1}{2}a_2 c\theta_{12} & \frac{1}{2}a_2 c\theta_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{9.135} $$

$$ J_{\omega 2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \tag{9.136} $$

$$ J_{v3} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} & -a_2 s\theta_{12} & 0 \\ a_1 c\theta_1 + a_2 c\theta_{12} & a_2 c\theta_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{9.137} $$

$$ J_{\omega 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}. \tag{9.138} $$

(c) Manipulator inertia matrix. The manipulator inertia matrix is obtained by substituting Eqs. (9.124) through (9.126) and (9.133) through (9.138) into (9.86):

$$ \begin{aligned} M &= J_{v1}^{\mathrm{T}} m_1 J_{v1} + J_{\omega 1}^{\mathrm{T}} I_1 J_{\omega 1} + J_{v2}^{\mathrm{T}} m_2 J_{v2} + J_{\omega 2}^{\mathrm{T}} I_2 J_{\omega 2} + J_{v3}^{\mathrm{T}} m_3 J_{v3} + J_{\omega 3}^{\mathrm{T}} I_3 J_{\omega 3} \\ &= m_1 \begin{bmatrix} \frac{1}{3}a_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad + m_2 \begin{bmatrix} a_1^2 + a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2 & \frac{1}{2}a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2 & 0 \\ \frac{1}{2}a_1 a_2 c\theta_2 + \frac{1}{3}a_2^2 & \frac{1}{3}a_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad + m_3 \begin{bmatrix} a_1^2 + 2a_1 a_2 c\theta_2 + a_2^2 & a_1 a_2 c\theta_2 + a_2^2 & 0 \\ a_1 a_2 c\theta_2 + a_2^2 & a_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{9.139} $$

(d) Velocity coupling vector. Taking the partial derivatives of the manipulator inertia matrix with respect to q_i in accordance with Eq. (9.101) yields

$$ \begin{aligned} V_1 &= \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial M_{1j}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_1}\right)\dot q_j \dot q_k \\ &= -(m_2 + 2m_3)a_1 a_2 s\theta_2 \left(\dot\theta_1\dot\theta_2 + \frac{1}{2}\dot\theta_2^2\right), \end{aligned} \tag{9.140} $$

$$ \begin{aligned} V_2 &= \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial M_{2j}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_2}\right)\dot q_j \dot q_k \\ &= \left(\frac{1}{2}m_2 + m_3\right) a_1 a_2 s\theta_2 \dot\theta_1^2, \end{aligned} \tag{9.141} $$

$$ V_3 = \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial M_{3j}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{jk}}{\partial q_3}\right)\dot q_j \dot q_k = 0. \tag{9.142} $$

(e) Gravitational vector. Assuming that the acceleration of gravity points in the negative z_0-direction (i.e., $\mathbf{g} = [0, 0, -g_c]^{\mathrm{T}}$), the gravitational terms are obtained by substituting Eqs. (9.133), (9.135), and (9.137) into (9.102):

$$ G_1 = -\sum_{j=1}^{3} m_j \mathbf{g}^{\mathrm{T}} J_{vj}^1 = 0, \tag{9.143} $$

$$ G_2 = -\sum_{j=1}^{3} m_j \mathbf{g}^{\mathrm{T}} J_{vj}^2 = 0, \tag{9.144} $$

$$G_3 = -\sum_{j=1}^{3} m_j \mathbf{g}^{\mathrm{T}} J_{vj}^{3} = -m_3 g_c. \tag{9.145}$$

(f) Lagrange's equations of motion. Assuming that there are no external forces and moments exerted at the end effector and that the joint friction is negligible, the vector of joint torques and the vector of generalized forces are equivalent. Substituting Eqs. (9.139) through (9.145) into (9.100) yields

$$\begin{aligned}\tau_1 &= \left[\left(\tfrac{1}{3}m_1 + m_2 + m_3\right) a_1^2 + (m_2 + 2m_3)a_1 a_2 \mathrm{c}\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right) a_2^2\right] \ddot{\theta}_1 \\ &\quad + \left[\left(\tfrac{1}{2}m_2 + m_3\right) a_1 a_2 \mathrm{c}\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right) a_2^2\right] \ddot{\theta}_2 \\ &\quad - (m_2 + 2m_3) a_1 a_2 \mathrm{s}\theta_2 \left(\dot{\theta}_1 \dot{\theta}_2 + \tfrac{1}{2}\dot{\theta}_2^2\right),\end{aligned} \tag{9.146}$$

$$\begin{aligned}\tau_2 &= \left[\left(\tfrac{1}{2}m_2 + m_3\right) a_1 a_2 \mathrm{c}\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right) a_2^2\right] \ddot{\theta}_1 + \left(\tfrac{1}{3}m_2 + m_3\right) a_2^2 \ddot{\theta}_2 \\ &\quad + \left(\tfrac{1}{2}m_2 + m_3\right) a_1 a_2 \mathrm{s}\theta_2 \dot{\theta}_1^2,\end{aligned} \tag{9.147}$$

$$\tau_3 = m_3 \ddot{d}_3 - m_3 g_c. \tag{9.148}$$

Equations (9.146) through (9.148) are the dynamical equations of motion for the 3-dof SCARA arm. The model is slightly more complex than the planar 2-dof manipulator. This is because the construction of first two joint axes of the SCARA arm is essentially the same as that of the planar 2-dof manipulator. In a SCARA manipulator, the first two joint axes control the horizontal position of the end effector, while the third prismatic joint controls the vertical position. The gravitational effects do not appear in the first two equations, because the acceleration of gravity is parallel to the first two joint axes. We observe that the motion of the third joint is completely independent of the first two, and the third link merely acts as a load to the motion of the first two joints.

9.9 INERTIA EFFECTS OF THE ROTORS

In previous sections we have assumed that each joint in a serial manipulator is driven directly by a motor and that the inertia effects of gears and rotors are negligible. Strictly speaking, the resulting equations are valid only for direct-drive manipulators. In a non-direct-drive manipulator, typically each joint is driven by a motor through a gear reduction unit. Although the inertias of gears and rotors are relatively small, their effects on the dynamics of a manipulator can be significant. This is because their inertia effects, when reflected in the joint space, are functions of the squares or products of the gear ratios (Tsai and Chang, 1994). Therefore, a more accurate dynamical model should take

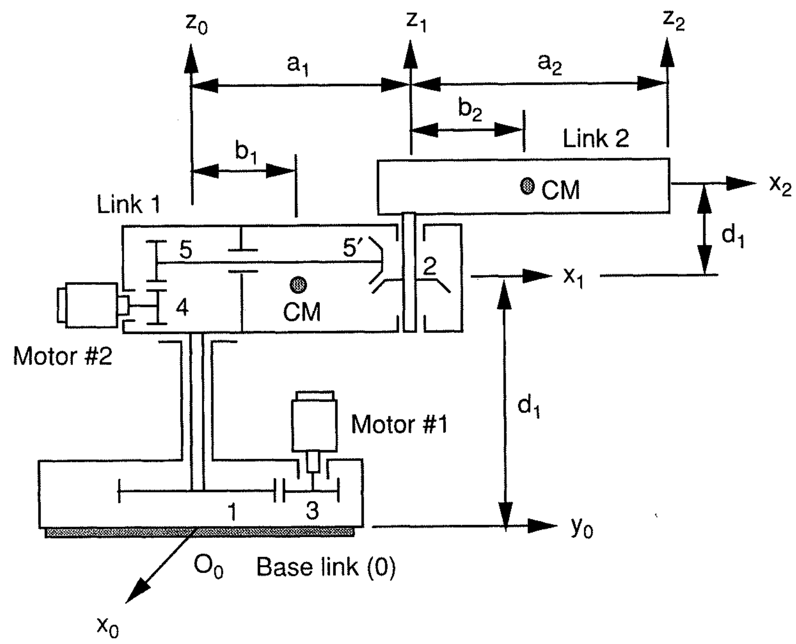

FIGURE 9.7. Schematic diagram of a geared 2-dof robot.

this into consideration. In this section we use a planar 2-dof manipulator as an example to illustrate the principle.

Figure 9.7 shows a geared planar 2-dof manipulator, where link 1 rotates with respect to the fixed base about the z_0-axis, and link 2 rotates with respect to link 1 about the z_1-axis. For brevity, only a few gears are shown. In practice, there may be multiple stages of gear reduction in each transmission line. As shown in the figure, motor 1 drives the first moving link through a gear pair attached to the shafts of motor 1 and link 1, and motor 2 drives link 2 through a spur gear pair attached to motor 2 and an intermediate shaft 5 followed by a bevel gear pair attached to the other end of the intermediate shaft 5 and link 2. Motor 1 is mounted on the fixed base, while motor 2 is mounted on the rear end of link 1. Assuming that the acceleration of gravity points in the negative z_0-direction, we wish to develop a dynamical model for the manipulator.

9.9.1 Kinematic Analysis

In Chapter 7, we have shown that the kinematic analysis of geared robotic mechanisms can be achieved in two basic steps. The first step involves the identification of an *equivalent open-loop chain* and the derivation of a kinematic relationship between the location of the end effector and the joint angles of the equivalent open-loop chain. The second step involves the derivation of a kinematic relationship between the joint angles and the input actuator displacements. The kinematic relation between the end-effector location and the joint angles has been described in Chapter 2. In this section we apply the theory of fundamental circuits to derive the kinematic relation between the joint angles and the input actuator displacements. As shown in Fig. 9.7, the equivalent open-loop chain consists of three *primary links*: base link 0, link 1, and link 2. All the other links are called *secondary links*. Link 0 carries gear 3; link 1 carries gears 4 and 5; link 2 does not carry any secondary link.

We now apply the theory of fundamental circuits to derive the kinematic relationship between the joint angles and the input actuator displacements. There are three gear pairs. Link 0 serves as the carrier for the 3–1 gear pair; link 1 serves as the carrier for the 4–5 and 5′–2 gear pairs. Let N_{ij} be the gear ratio between gears i and j. The fundamental circuit equations can be written as

$$f(3,1,0): \quad \theta_{3,0} = -N_{13}\theta_{1,0}, \tag{9.149}$$

$$f(4,5,1): \quad \theta_{4,1} = -N_{54}\theta_{5,1}, \tag{9.150}$$

$$f(5',2,1): \quad \theta_{5,1} = -N_{25}\theta_{2,1}, \tag{9.151}$$

where $\theta_{i,j}$ denotes the relative rotation of link i with respect to link j.

Since $\theta_{1,0}$ and $\theta_{2,1}$ are the joint angles of the equivalent open-loop chain, we should express all the other angular displacements in terms of these two joint angles. Substituting Eq. (9.151) into (9.150) gives

$$\theta_{4,1} = N_{54}N_{25}\theta_{2,1}. \tag{9.152}$$

Combining Eqs. (9.149), (9.151), and (9.152), we obtain

$$\begin{bmatrix} \theta_{3,0} \\ \theta_{4,1} \\ \theta_{5,1} \end{bmatrix} = \begin{bmatrix} -N_{13} & 0 \\ 0 & N_{54}N_{25} \\ 0 & -N_{25} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}. \tag{9.153}$$

For brevity, we have used θ_i to replace $\theta_{i,i-1}$ for $i = 1$ and 2 in Eq. (9.153). Hence, given the joint angles, one can compute the corresponding rotations of the rotors and the intermediate shaft.

9.9.2 Kinetic Energy of a Revolving Rotor

Before we formulate the dynamical equations of motion, we study the kinetic energy of a rotor j that is carried by a primary link i, as shown in Fig. 9.8. We assume that the mass of the rotor is symmetrically distributed about its axis of rotation. This is a reasonable assumption, since rotors, gears, and shafts are all symmetric about their axes of rotation.

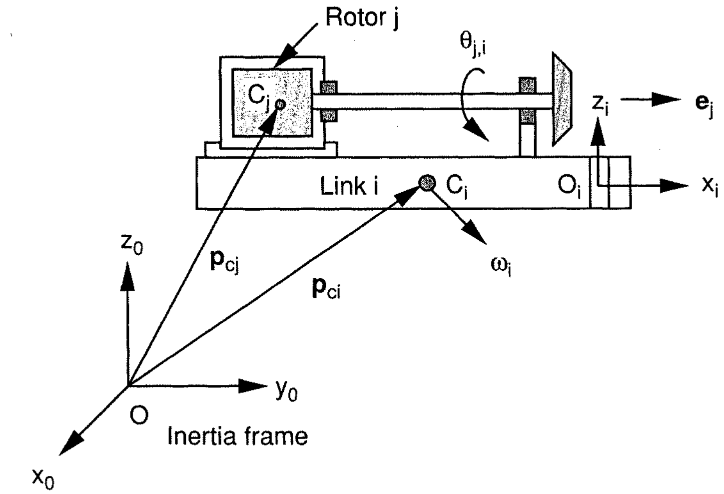

FIGURE 9.8. Typical rotor j that is carried by link i.

Let $\mathbf{v}_{cj}$ be the absolute velocity of the center of mass of the rotor, $\boldsymbol{\omega}_j$ the absolute angular velocity of the rotor, and I_j the inertia matrix of the rotor about its center of mass and expressed in the inertia frame. Then the kinetic energy of link j can be written as

$$K_j = \tfrac{1}{2}\left[\mathbf{v}_{cj}^{\mathrm{T}} m_j \mathbf{v}_{cj} + \boldsymbol{\omega}_j^{\mathrm{T}} I_j \boldsymbol{\omega}_j\right]. \tag{9.154}$$

The absolute angular velocity of link j can be expressed in terms of the absolute angular velocity of the carrier, $\boldsymbol{\omega}_i$, and the angular velocity of the rotor relative to the carrier; that is,

$$\boldsymbol{\omega}_j = \boldsymbol{\omega}_i + \dot{\theta}_{j,i}\mathbf{e}_j, \tag{9.155}$$

where $\mathbf{e}_j$ denotes the direction of the rotor axis and $\theta_{j,i}$ denotes the relative rotation of link j with respect to link i.

Substituting Eq. (9.155) into (9.154) and making use of the symmetric property of the inertia matrix, we obtain

$$K_j = \tfrac{1}{2}\left[\mathbf{v}_{cj}^{\mathrm{T}} m_j \mathbf{v}_{cj} + \boldsymbol{\omega}_i^{\mathrm{T}} I_j \boldsymbol{\omega}_i + 2\boldsymbol{\omega}_i^{\mathrm{T}} I_j \mathbf{e}_j \dot{\theta}_{j,i} + \mathbf{e}_j^{\mathrm{T}} I_j \mathbf{e}_j \dot{\theta}_{j,i}^2\right]. \tag{9.156}$$

Here $\mathbf{e}_j^{\mathrm{T}} I_j \mathbf{e}_j = I_{j,z}$ is called the *axial moment of inertia* of the rotor j. We observe that only the last two terms on the right-hand side of Eq. (9.156) depend exclusively on the relative rotation of the rotor. For an axisymmetric rotor, the center of mass always lies on its axis of rotation and therefore can be considered as a point fixed on the carrier. Further, due to symmetry, the rotor inertia matrix, I_j, is invariant in the link frame i. Hence the contributions of the first two terms can conveniently be combined with link i to form an equivalent link. In this way, the inertia effects of a rotor due to its relative rotation with respect to the carrier can be written as

$$\hat{K}_j = \tfrac{1}{2} I_{j,z}\dot{\theta}_{j,i}^2 + I_{j,z}(\boldsymbol{\omega}_i^{\mathrm{T}}\mathbf{e}_j)\dot{\theta}_{j,i}. \tag{9.157}$$

In general, $\dot{\theta}_{j,i}$ is a linear function of the joint rates of the equivalent open-loop chain. For a manipulator with each joint individually driven by an actuator, $\dot{\theta}_{j,i}$ is related to the joint rate, $\dot{q}_{i+1,i}$, by a simple gear ratio, N. Substituting $\dot{\theta}_{j,i} = N\dot{q}_{i+1,i}$ into Eq. (9.157) gives

$$\hat{K}_j = \tfrac{1}{2} N^2 I_{j,z}\dot{q}_{i+1,i}^2 + N I_{j,z}(\boldsymbol{\omega}_i^{\mathrm{T}}\mathbf{e}_j)\dot{q}_{i+1,i}. \tag{9.158}$$

It is worth noting that the common wisdom of simply adding $N^2 I_{j,z}$ to the inertia of a robotic system is valid only if the carrier is stationary or its angular velocity is perpendicular to the rotor axis of rotation, $\boldsymbol{\omega}_i^{\mathrm{T}}\mathbf{e}_j = 0$.

9.9.3 Dynamic Analysis

In this section we perform the dynamic analysis of the example manipulator. First, we apply the recursive method to compute the angular velocity, the velocity of the center of mass, and the potential energy of each link. Then we substitute these quantities directly into Lagrange's equations of motion. We provide just sufficient information leading to the solution without detailed derivations.

The first equivalent link consists of link 1, rotor 4, intermediate shaft 5, and the gears attached to them. The second equivalent link consists of link 2 and the bevel gear attached to it. Let m_i be the mass of the equivalent link i. To simplify the analysis, we assume that the inertia matrix of an equivalent link i about its combined center of mass and expressed in the link i frame takes the following form:

$${}^{i}I_i = \begin{bmatrix} I_{i,x} & 0 & 0 \\ 0 & I_{i,y} & 0 \\ 0 & 0 & I_{i,z} \end{bmatrix}.$$

(a) Kinetic and Potential Energies of Link 1. The initial conditions of the base link are ${}^{0}\boldsymbol{\omega}_0 = {}^{0}\mathbf{v}_0 = \mathbf{0}$. The angular velocity and the velocity of the center of mass of link 1 expressed in link 1 frame are

$${}^{1}\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix},$$

$${}^{1}\mathbf{v}_{c1} = \begin{bmatrix} 0 \\ b_1\dot{\theta}_1 \\ 0 \end{bmatrix},$$

where b_1 locates the combined center of mass of link 1 along the x_1-axis as shown in Fig. 9.7. Note that the acceleration information is not needed for the Lagrangian formulation. Therefore, the kinetic energy of link 1 is given by

$$K_1 = \tfrac{1}{2}\,{}^{1}\mathbf{v}_{c1}^{\mathrm{T}}\, m_1\, {}^{1}\mathbf{v}_{c1} + \tfrac{1}{2}\,{}^{1}\boldsymbol{\omega}_1^{\mathrm{T}}\, {}^{1}I_1\, {}^{1}\boldsymbol{\omega}_1 = \tfrac{1}{2}(m_1 b_1^2 + I_{1,z})\dot{\theta}_1^2, \tag{9.159}$$

and the potential energy of link 1 is given by

$$U_1 = m_1 g_c d_1. \tag{9.160}$$

(b) Kinetic and Potential Energies of Link 2. The angular velocity and the linear velocity of the combined center of mass of link 2 are computed and expressed in frame 2 as

$${}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},$$

$${}^{2}\mathbf{v}_{c2} = \begin{bmatrix} a_1 s\theta_2\dot{\theta}_1 \\ a_1 c\theta_2\dot{\theta}_1 + b_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix},$$

where b_2 locates the combined center of mass of link 2 along the x_2-axis as shown in Fig. 9.7. Therefore, the kinetic energy of link 2 is given by

$$\begin{aligned} K_2 &= \tfrac{1}{2}\,{}^{2}\mathbf{v}_{c2}^{\mathrm{T}}\, m_2\, {}^{2}\mathbf{v}_{c2} + \tfrac{1}{2}\,{}^{2}\boldsymbol{\omega}_2^{\mathrm{T}}\, {}^{2}I_2\, {}^{2}\boldsymbol{\omega}_2 \\ &= \tfrac{1}{2} m_2[a_1^2\dot{\theta}_1^2 + b_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2a_1 b_2 c\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)] + \tfrac{1}{2} I_{2,z}(\dot{\theta}_1 + \dot{\theta}_2)^2, \end{aligned} \tag{9.161}$$

and the potential energy of link 2 is given by

$$U_2 = m_2 g_c(d_1 + d_2). \tag{9.162}$$

(c) Kinetic Energies of the Rotors and Gears. We note that $\boldsymbol{\omega}_i^{\mathrm{T}}\mathbf{e}_j = 0$ for both $i = 0$ and 1. Substituting Eq. (9.153) into (9.157), we obtain the additional kinetic energies contributed by the rotors and gears:

$$K_r = \tfrac{1}{2}[N_{13}^2 I_{3,z}\dot{\theta}_1^2 + (N_{54}^2 I_{4,z} + I_{5,z})N_{25}^2\dot{\theta}_2^2]. \tag{9.163}$$

(d) Lagrangian Function and Its Derivatives. Substituting Eqs. (9.159) through (9.163) into (9.75), we obtain the Lagrangian function

$$\begin{aligned} L = {} & \tfrac{1}{2}[(\kappa_1 + 2m_2a_1b_2c\theta_2)\dot{\theta}_1^2 + \kappa_2\dot{\theta}_2^2 + 2(\kappa_4 + m_2a_1b_2c\theta_2)\dot{\theta}_1\dot{\theta}_2] \\ & - m_1g_cd_1 - m_2g_c(d_1 + d_2), \end{aligned} \tag{9.164}$$

where

$$\begin{aligned} \kappa_1 &= \kappa_3 + \kappa_4 + N_{13}^2 I_{3,z} + m_2a_1^2, \\ \kappa_2 &= \kappa_4 + N_{54}^2 N_{25}^2 I_{4,z} + N_{25}^2 I_{5,z}, \\ \kappa_3 &= I_{1,z} + m_1b_1^2, \\ \kappa_4 &= I_{2,z} + m_2b_2^2. \end{aligned}$$

We note that κ_3 represents the mass moment of inertia of link 1 about the z_0-axis, and κ_4 represents the mass moment of inertia of link 2 about the z_1-axis. The effects of rotor and gear inertias are clearly shown as functions of the squares of their respective gear ratios.

Taking the partial derivatives of L with respect to θ_1, θ_2, $\dot{\theta}_1$ and $\dot{\theta}_2$ yields

$$\frac{\partial L}{\partial \theta_1} = 0, \tag{9.165}$$

$$\frac{\partial L}{\partial \theta_2} = -m_2a_1b_2s\theta_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2), \tag{9.166}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (\kappa_1 + 2m_2a_1b_2c\theta_2)\dot{\theta}_1 + (\kappa_4 + m_2a_1b_2c\theta_2)\dot{\theta}_2, \tag{9.167}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \kappa_2\dot{\theta}_2 + (\kappa_4 + m_2a_1b_2c\theta_2)\dot{\theta}_1. \tag{9.168}$$

Taking the total derivatives of Eqs. (9.167) and (9.168) with respect to time yields

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = {} & (\kappa_1 + 2m_2a_1b_2c\theta_2)\ddot{\theta}_1 + (\kappa_4 + m_2a_1b_2c\theta_2)\ddot{\theta}_2 \\ & - m_2a_1b_2s\theta_2\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2), \end{aligned} \tag{9.169}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = (\kappa_4 + m_2a_1b_2c\theta_2)\ddot{\theta}_1 + \kappa_2\ddot{\theta}_2 - m_2a_1b_2s\theta_2\dot{\theta}_1\dot{\theta}_2. \tag{9.170}$$

(e) Lagrange's Equations of Motion. Substituting Eqs. (9.165), (9.166), (9.169), and (9.170) into (9.76) produces Lagrange's equations of motion as

$$M(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) = \boldsymbol{\tau}, \tag{9.171}$$

where

$$\begin{aligned} \boldsymbol{\theta} &= \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \\ M(\boldsymbol{\theta}) &= \begin{bmatrix} \kappa_1 + 2m_2a_1b_2c\theta_2 & \kappa_4 + m_2a_1b_2c\theta_2 \\ \kappa_4 + m_2a_1b_2c\theta_2 & \kappa_2 \end{bmatrix}, \\ \mathbf{V}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) &= \begin{bmatrix} -m_2a_1b_2s\theta_2\dot{\theta}_2(2\dot{\theta}_1 + \dot{\theta}_2) \\ m_2a_1b_2s\theta_2\dot{\theta}_1^2 \end{bmatrix}, \\ \mathbf{G}(\boldsymbol{\theta}) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Equation (9.171) is the dynamical model for the two-link manipulator. Although the rotor inertias may be small, their effects can be significant since they are multiplied by the squares or products of the gear ratios. A gear reduction on the order of 50 to 100 is commonly used in an industrial robot. For example, for a gear ratio of 60, the inertia of a rotor will be multiplied by 3600.

9.10 END-EFFECTOR SPACE DYNAMICAL EQUATIONS

In previous sections we have derived the dynamical equations of motion in terms of the joint angles, $\mathbf{q}$, or in the *joint space*. We assume that a desired trajectory of the end effector can be expressed in terms of the joint angles, velocities, and accelerations. Based on the joint space formulation, various control schemes have been developed. However, in practice, we often wish to program the end-effector trajectories in the Cartesian space, $\mathbf{x}$, and for the joint-based control schemes to work these Cartesian space trajectories should be converted into joint space trajectories. Theoretically, the conversion can be accomplished by applying

$$\begin{aligned} \mathbf{q}_d &= \text{inverse kinematics of } \mathbf{x}_d, \\ \dot{\mathbf{q}}_d &= J^{-1}\dot{\mathbf{x}}_d, \\ \ddot{\mathbf{q}}_d &= \dot{J}^{-1}\dot{\mathbf{x}}_d + J^{-1}\ddot{\mathbf{x}}_d, \end{aligned} \tag{9.172}$$

where J is the Jacobian matrix and the subscript d is used to indicate a desired quantity.

In reality, it is quite difficult to compute the equations above efficiently for real-time control purpose. Therefore, usually only the desired joint angles, $\mathbf{q}_d$, are computed from the inverse kinematics, and the joint velocities and accelerations are computed numerically by the first and second differences. However, for certain control techniques (Khatib, 1983), it may be desirable to express the dynamical equations in the *end-effector space*. This can be accomplished by the following procedure.

The end-effector velocity is related to the joint velocity by

$$\dot{\mathbf{x}} = J\dot{\mathbf{q}}. \tag{9.173}$$

Assuming that J is a nonsingular square matrix, we substitute the inverse transformation of Eq. (9.173) into (9.87) to obtain an expression for the kinetic energy in terms of the end-effector velocity:

$$K = \tfrac{1}{2}\dot{\mathbf{x}}^{\mathrm{T}}\tilde{M}\dot{\mathbf{x}}, \tag{9.174}$$

where

$$\tilde{M} = (J^{-1})^{\mathrm{T}} M J^{-1}, \tag{9.175}$$

is the manipulator inertia matrix expressed in the end-effector space. We refer to $\tilde{M}$ as the *end-effector space inertia matrix* or *Cartesian inertia matrix.*

Taking the derivative of Eq. (9.173) with respect to time yields

$$\ddot{\mathbf{x}} = J\ddot{\mathbf{q}} + \dot{J}\dot{\mathbf{q}}. \tag{9.176}$$

Multiplying both sides of Eq. (9.176) by J^{-1} and rearranging yields

$$\ddot{\mathbf{q}} = J^{-1}(\ddot{\mathbf{x}} - \dot{J}\dot{\mathbf{q}}). \tag{9.177}$$

Multiplying Eq. (9.103) by $J^{-\mathrm{T}}$ and then substituting Eqs. (9.175) and (9.177) into the resulting equation, we obtain

$$\tilde{M}\ddot{\mathbf{x}} + \tilde{\mathbf{V}} + \tilde{\mathbf{G}} = \tilde{\mathbf{Q}}, \tag{9.178}$$

where

$$\tilde{\mathbf{V}} = J^{-\mathrm{T}}(\mathbf{V} - MJ^{-1}\dot{J}\dot{\mathbf{q}}),$$
$$\tilde{\mathbf{G}} = J^{-\mathrm{T}}\mathbf{G},$$
$$\tilde{\mathbf{Q}} = J^{-\mathrm{T}}\mathbf{Q}.$$

Hence once the dynamical equations are derived in the joint space, they can be converted into the end-effector space. Although the equations of motion above are expressed in the end-effector space, some of the terms, such as $\tilde{\mathbf{V}}$, $\tilde{\mathbf{G}}$, and $\tilde{\mathbf{Q}}$, are still written as functions of the joint variables, $\mathbf{q}$. Due to the nonlinearity of the inverse kinematics, it is practically impossible to express everything in terms of the end-effector variables, $\mathbf{x}$. We note that as the robot arm approaches a singular configuration, the Jacobian matrix is not invertible and certain quantities in the end-effector space become very large.

9.11 SUMMARY

In this chapter we first reviewed the inertia properties, the momentum, and the kinetic energy of a rigid body. It was shown that the angular momentum and the kinetic energy of a rigid body can be divided into two parts: one associated with the motion of the center of mass and the other with the motion of the rigid body about its center of mass. Next, we reviewed the Newton–Euler laws. Both the Newton and Euler equations of motion were derived. Then we presented two methods for the dynamical analysis of serial manipulators. The recursive Newton–Euler formulation consists of a forward computation followed by a backward computation. In the forward computation, link velocities and accelerations are calculated, one link at a time, from link 1 to link n, using the kinematical equations derived in Chapter 4. In the backward computation, joint reaction forces are calculated one link at a time from link n back to link 1 using the Newton–Euler equations of motion. Although the recursive method is more tedious, it renders all the joint reaction forces that may be useful for sizing the links and bearings during the design phase. Lagrange's method formulates the problem with all the forces of constraint eliminated at the outset. The link Jacobian submatrices, the manipulator inertia matrix, and the derivation of the generalized forces have been described and a general matrix form of the dynamical equations of motion was presented. The effects of rotor inertia were also discussed. It has been shown that rotor inertias, which have been ignored in most textbooks, may have significant effects on the dynamics of a manipulator. Finally, the transformation of dynamical equations into the end-effector space was described briefly.

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EXERCISES

1. Derive the parallel axis theorem given by Eq. (9.9).

2. The inertia matrix of a rectangular bar about a center-of-mass coordinate frame, (x_c, y_c, z_c), is given by Eq. (9.12). What is the inertia matrix about O expressed in the (x, y, z) coordinate frame, as shown in Fig. 9.9?

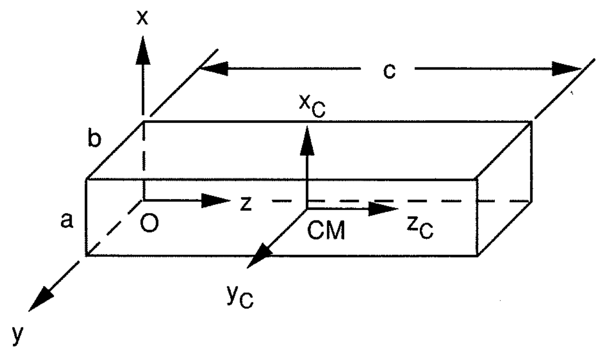

FIGURE 9.9. Inertia matrix of a rectangular bar.

3. The inertia matrix of a rectangular bar about a center-of-mass coordinate frame, (x_c, y_c, z_c), is given by Eq. (9.12). What is the inertia matrix about the center of mass and expressed in an (x, y, z) coordinate frame that is rotated with respect to the (x_c, y_c, z_c) frame by an angle ϕ about the z_c-axis, as shown in Fig. 9.10?

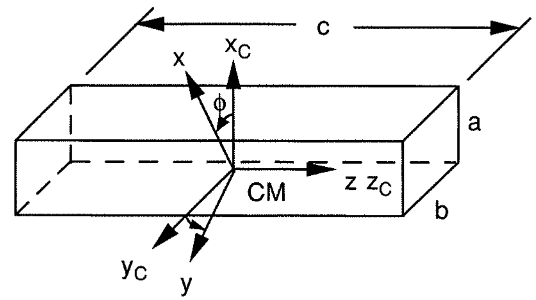

FIGURE 9.10. Inertia matrix of a rectangular bar.

4. Figure 9.11 shows a 2-dof $2R$ pointer, in which the first joint axis points up vertically along the positive z_0-axis and the second joint axis intersects the first perpendicularly. Assuming that the second moving link is a slender homogeneous rod of mass m, what is the inertia matrix of this link about O expressed in the (x_0, y_0, z_0) coordinate frame?

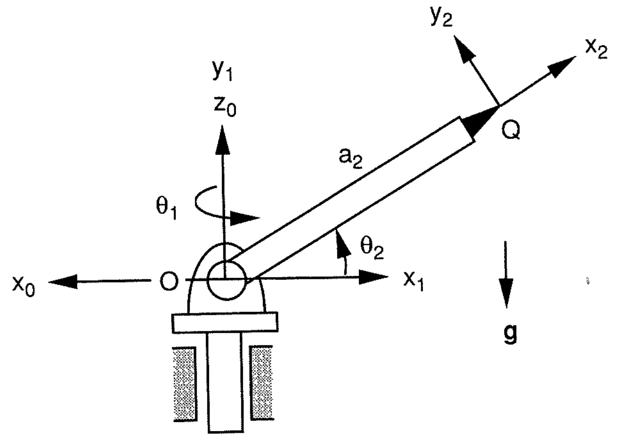

FIGURE 9.11. A 2-dof pointer.

5. Consider the 2-dof $2R$ pointer shown in Fig. 9.11. Assuming that the inertia of the first moving link is negligible and that the second moving link is a slender homogeneous rod of mass m, calculate the angular momentum of the system about the origin O expressed in the (x_0, y_0, z_0) coordinate frame.

6. Show that when the axes of a center-of-mass coordinate system coincide with the principal axes of a rigid body, Euler's equations of motion reduce to Eq. (9.48).

7. For the 2-dof $2R$ pointer shown in Fig. 9.11, assume that the inertia of the first moving link is negligible and that the second moving link is a slender homogeneous rod of mass m. Develop the dynamical equations of motion by the recursive Newton–Euler method. Identify the contributions due to Coriolis, centrifugal, and gravitational effects.

8. For the planar 3-dof manipulator shown in Fig. 2.3, assume that the acceleration of gravity points in the negative z_0-direction and that the three moving links are slender homogeneous rods of masses m_1, m_2, and m_3, respectively. Derive the dynamical equations of motion by the recursive Newton–Euler method. Express the resulting equations in matrix form.

9. Figure 9.12 shows a spatial 3-dof, $3R$ manipulator in which the second joint axis intersects the first perpendicularly and the third joint axis is parallel to the second. Assuming that the link inertias are negligible and that there is a point mass m attached to the end effector at point Q, derive the dynamical equations of motion by the recursive Newton–Euler method.

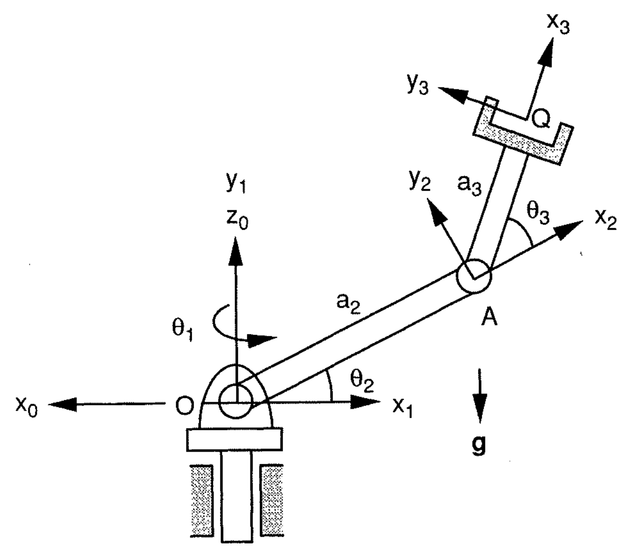

FIGURE 9.12. Spatial 3-dof, 3R manipulator.

10. Consider the 2-dof pointer shown in Fig. 9.11. Assuming that the inertia of the first moving link is negligible and that the second moving link is a slender homogeneous rod of mass m, derive the dynamical equations of motion by the Lagrangian method using θ_1 and θ_2 as the generalized coordinates.

11. Describe two possible sets of generalized coordinates for the spatial $3R$ manipulator shown in Fig. 9.12.

12. Derive the dynamical equations of motion for the spatial 3-dof manipulator shown in Fig. 9.12 by the Lagrangian method, assuming that the link inertias are negligible and that there is a point mass m attached to the end effector at point Q.