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- **6.1** Review of Power Series
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Chapter 6 in Review

Up to this point in our study of differential equations we have primarily solved linear equations of order two (or higher) that have constant coefficients. The only exception was the Cauchy-Euler equation in Section 4.7. In applications, higher-order linear equations with variable coefficients are just as important as, if not mor than, differential equations with constant coefficients. As pointed out in Section 4.7, even a simple linear second-order equation with variable coefficients such a y'' + xy = 0 does not possess solutions that are elementary functions. But this is not to say that we *can't* find two linearly independent solutions of y'' + xy = 0; we can. In Sections 6.2 and 6.4 we shall see that the functions that are solutions of this equation are defined by infinite series.

In this chapter we shall study two infinite-series methods for finding solutio of homogeneous linear second-order DEs $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, where the variable coefficients $a_2(x)$, $a_1(x)$, and $a_0(x)$ are, for the most part, simple polynomial functions.



6.1 REVIEW OF POWER SERIES

REVIEW MATERIAL

- Infinite series of constants, *p*-series, harmonic series, alternating harmonic series, geometric series, tests for convergence especially the ratio test
- Power series, Taylor series, Maclaurin series (See any calculus text)

INTRODUCTION In Section 4.3 we saw that solving a homogeneous linear DE with constant coefficients was essentially a problem in algebra. By finding the roots of the auxiliary equation, we could write a general solution of the DE as a linear combination of the elementary functions $e^{\alpha x}$, $x^k e^{\alpha x} \cos \beta x$, and $x^k e^{\alpha x} \sin \beta x$. But as was pointed out in the introduction to Section 4.7, most linear higher-order DEs with variable coefficients cannot be solved in terms of elementary functions. A usual course of action for equations of this sort is to assume a solution in the form of an infinite series and proceed in a manner similar to the method of undetermined coefficients that possess solutions in the form of a power series, and so it is appropriate that we begin this chapter with a review of that topic.

Power Series Recall from calculus that **power series** in x - a is an infinit series of the form

The index of summation need not start at n = 0.

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Such a series also said to be a **power series centered at** *a*. For example, the power series $\sum_{n=0}^{\infty} (x + 1)^n$ is centered at a = -1. In the next section we will be concerned principally with power series in *x*, in other words, power series that are centered at a = 0. For example,

$$\sum_{n=0}^{\infty} 2^n x^n = 1 + 2x + 4x^2 + \cdots$$

is a power series in *x*.

Important Facts The following bulleted list summarizes some important facts about power series $\sum_{n=0}^{\infty} c_n (x - a)^n$.

- **Convergence** A power series is **convergent** at a specified value of x if its sequence of partial sums $\{S_N(x)\}$ converges, that is, $\lim_{N \to \infty} S_N(x) = \lim_{N \to \infty} \sum_{n=0}^{N} c_n (x a)^n$ exists. If the limit does not exist at x, then the series is said to be **divergent**.
- **Interval of Convergence** Every power series has an **interval of convergence**. The interval of convergence is the set of *all* real numbers *x* for which the series converges. The center of the interval of convergence is the center *a* of the series.
- Radius of Convergence The radius *R* of the interval of convergence of a power series is called its radius of convergence. If *R* > 0, then a power series converges for |*x* − *a*| < *R* and diverges for |*x* − *a*| > *R*. If the series converges only at its center *a*, then *R* = 0. If the series converges for all *x*, then we write *R* = ∞. Recall, the absolute-value inequality |*x* − *a*| < *R* is equivalent to the simultaneous inequality *a* − *R* < *x* < *a* + *R*. A power series may or may not converge at the endpoints *a* − *R* and *a* + *R* of this interval.
- Absolute Convergence Within its interval of convergence a power series converges absolutely. In other words, if x is in the interval of convergence and is not an endpoint of the interval, then the series of absolute values $\sum_{n=0}^{\infty} |c_n(x-a)^n|$ converges. See Figure 6.1.1.



FIGURE 6.1.1 Absolute convergence within the interval of convergence and divergence outside of this interval

• **Ratio Test** Convergence of power series can often be determined by the **ratio test.** Suppose $c_n \neq 0$ for all n in $\sum_{n=0}^{\infty} c_n (x - a)^n$, and that

$$\lim_{n \to \infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = L.$$

If L < 1, the series converges absolutely; if L > 1 the series diverges; and if L = 1 the test is inconclusive. The ratio test is always inconclusive at an endpoint $a \pm R$.

EXAMPLE 1 Interval of Convergence

Find the interval and radius of convergence for $\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n n}$.

SOLUTION The ratio test gives

$$\lim_{n \to \infty} \left| \frac{\frac{(x-3)^{n+1}}{2^{n+1}(n+1)}}{\frac{(x-3)^n}{2^n n}} \right| = |x-3| \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2} |x-3|$$

The series converges absolutely for $\frac{1}{2}|x-3| < 1$ or |x-3| < 2 or 1 < x < 5. This last inequality defines the *open* interval of convergence. The series diverges for |x-3| > 2, that is, for x > 5 or x < 1. At the left endpoint x = 1 of the open interval of convergence, the series of constants $\sum_{n=1}^{\infty} ((-1)^n/n)$ is convergent by the alternating series test. At the right endpoint x = 5, the series $\sum_{n=1}^{\infty} (1/n)$ is the divergent harmonic series. The interval of convergence of the series is [1, 5), and the radius of convergence is R = 2.

A Power Series Defines a Functio A power series defines a function that is, f(x) = ∑_{n=0}[∞] c_n(x - a)ⁿ whose domain is the interval of convergence of the series. If the radius of convergence is R > 0 or R = ∞, then f is continuous, differentiable, and integrable on the intervals (a - R, a + R) or (-∞, ∞), respectively. Moreover, f'(x) and ∫f(x) dx can be found by term-by-term differentiation and integration. Convergence at an endpoint may be either lost by differentiation or gained through integration. If

$$y = \sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

is a power series in x, then the first two derivatives are $y' = \sum_{n=0}^{\infty} nx^{n-1}$ and $y'' = \sum_{n=0}^{\infty} n(n-1)x^{n-2}$. Notice that the first term in the first derivative and the first two terms in the second derivative are zero. We omit these zero terms and write

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \cdots$$
(1)

Be sure you understand the two results given in (1); especially note where the index of summation starts in each series. These results are important and will be used in all examples in the next section.

- Identity Property If $\sum_{n=0}^{\infty} c_n (x-a)^n = 0$, R > 0, for all numbers x in some open interval, then $c_n = 0$ for all n.
- Analytic at a Point A function f is said to be analytic at a point a if it can be represented by a power series in x a with either a positive or an infinite radius of conve gence. In calculus it is seen that infinitel

differentiable functions such as e^x , $\sin x$, $\cos x$, $e^x \ln(1 + x)$, and so on, can be represented by Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{1!} (x-a)^2 + \cdots$$

or by a Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{1!} x^2 + \cdots$$

You might remember some of the following Maclaurin series representations.

Maclaurin Series	Interval of Convergence
$e^{x} = 1 + \frac{x}{1!} + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n}$	$(-\infty,\infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty,\infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty,\infty)$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	[-1, 1] (2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty,\infty)$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty,\infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	(-1, 1]

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$
 (-1, 1)

These results can be used to obtain power series representations of other functions. For example, if we wish to find the Maclaurin series representatio of, say, e^{x^2} we need only replace x in the Maclaurin series for e^x :

$$e^{x^2} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$$

Similarly, to obtain a Taylor series representation of $\ln x$ centered at a = 1 we replace x by x - 1 in the Maclaurin series for $\ln(1 + x)$:

$$\ln x = \ln(1 + (x - 1)) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \frac{(x - 1)^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x - 1)^n.$$

The interval of convergence for the power series representation of e^{x^2} is the same as that of e^x , that is, $(-\infty, \infty)$. But the interval of convergence of the Taylor series of $\ln x$ is now (0, 2]; this interval is (-1, 1] shifted 1 unit to the right.

• Arithmetic of Power Series Power series can be combined through the operations of addition, multiplication, and division. The procedures for powers series are similar to the way in which two polynomials are added, multiplied, and divided —that is, we add coefficients of like powers of *x*, use the distributive law and collect like terms, and perform long division.

You can also verify that the interval of convergence is (0, 2] by using the ratio test.

EXAMPLE 2 Multiplication of Power Series

Find a power series representation of $e^x \sin x$.

SOLUTION We use the power series for e^x and $\sin x$:

$$e^{x}\sin x = \left(1 + x + \frac{x^{2}}{2} + \frac{x^{3}}{6} + \frac{x^{4}}{24} + \cdots\right)\left(x - \frac{x^{3}}{6} + \frac{x^{5}}{120} - \frac{x^{7}}{5040} + \cdots\right)$$
$$= (1)x + (1)x^{2} + \left(-\frac{1}{6} + \frac{1}{2}\right)x^{3} + \left(-\frac{1}{6} + \frac{1}{6}\right)x^{4} + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^{5} + \cdots$$
$$= x + x^{2} + \frac{x^{3}}{3} - \frac{x^{5}}{30} + \cdots$$

Since the power series of e^x and $\sin x$ both converge on $(-\infty, \infty)$, the product series converges on the same interval. Problems involving multiplication or division of power series can be done with minimal fuss using a computer algebra system.

Shifting the Summation Index For the three remaining sections of this chapter, it is crucial that you become adept at simplifying the sum of two or more power series, each series expressed in summation notation, to an expression with a single Σ . As the next example illustrates, combining two or more summations as a single summation often requires a reindexing, that is, a shift in the index of summation.

EXAMPLE 3 Addition of Power Series

Write

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

as one power series.

SOLUTION In order to add the two series given in summation notation, it is necessary that both indices of summation start with the same number and that the powers of x in each series be "in phase," in other words, if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power. Note that in the given problem, the first series starts with x^0 whereas the second series starts with x^1 . By writing the first term of the first series outside of the summation notation,

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2 \cdot 1c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$
(3)

we see that both series on the right side start with the same power of x, namely, x^1 . Now to get the same summation index we are inspired by the exponents of x; we let k = n - 2 in the first series and at the same time let k = n + 1 in the second series. For n = 3 in k = n - 2 we get k = 1, and for n = 0 in k = n + 1 we get k = 1, and so the right-hand side of (3) becomes

$$2c_{2} + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^{k} + \sum_{k=1}^{\infty} c_{k-1}x^{k}.$$
(4)

Remember the summation index is a "dummy" variable; the fact that k = n - 2 in one case and k = n + 1 in the other should cause no confusion if you keep in mind that it is the *value* of the summation index that is important. In both cases k takes on the same successive values k = 1, 2, 3, ... when n takes on the values n = 2, 3, 4, ... for k = n - 1 and n = 0, 1, 2, ... for k = n + 1. We are now in a position to add the series in (4) term-by-term:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2c_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)c_{k+2} + c_{k-1}]x^k.$$
(5)

If you are not totally convinced of the result in (5), then write out a few terms on both sides of the equality.

A Preview The point of this section is to remind you of the salient facts about power series so that you are comfortable using power series in the next section to fin solutions of linear second-order DEs. In the last example in this section we tie up many of the concepts just discussed; it also gives a preview of the method that will used in Section 6.2. We purposely keep the example simple by solving a linear first order equation. Also suspend, for the sake of illustration, the fact that you already know how to solve the given equation by the integrating-factor method in Section 2.3.

EXAMPLE 4 A Power Series Solution

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the differential equation y' + y = 0.

SOLUTION We break down the solution into a sequence of steps.

(*i*) First calculate the derivative of the assumed solution:

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \leftarrow$$
 see the first line in (1)

(*ii*) Then substitute y and y' into the given DE:

$$y' + y = \sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n.$$

(*iii*) Now shift the indices of summation. When the indices of summation have the same starting point and the powers of *x* agree, combine the summations:

$$y' + y = \sum_{\substack{n=1\\k=n-1}}^{\infty} c_n n x^{n-1} + \sum_{\substack{n=0\\k=n}}^{\infty} c_n x^n$$
$$= \sum_{\substack{k=0\\k=0}}^{\infty} c_{k+1}(k+1) x^k + \sum_{\substack{k=0\\k=0}}^{\infty} c_k x^k$$
$$= \sum_{\substack{k=0\\k=0}}^{\infty} [c_{k+1}(k+1) + c_k] x^k.$$

(*iv*) Because we want y' + y = 0 for all x in some interval,

$$\sum_{k=0}^{\infty} [c_{k+1}(k+1) + c_k] x^k = 0$$

is an identity and so we must have $c_{k+1}(k+1) + c_k = 0$, or

$$c_{k+1} = -\frac{1}{k+1}c_k, \quad k = 0, 1, 2, \ldots$$

(v) By letting k take on successive integer values starting with k = 0, we fin

$$c_{1} = -\frac{1}{1}c_{0} = -c_{0}$$

$$c_{2} = -\frac{1}{2}c_{1} = -\frac{1}{2}(-c_{0}) = \frac{1}{2}c_{0}$$

$$c_{3} = -\frac{1}{3}c_{2} = -\frac{1}{3}\left(\frac{1}{2}c_{0}\right) = -\frac{1}{3\cdot 2}c_{0}$$

$$c_{4} = -\frac{1}{4}c_{2} = -\frac{1}{4}\left(-\frac{1}{3\cdot 2}c_{0}\right) = \frac{1}{4\cdot 3\cdot 2}c_{0}$$

and so on, where c_0 is arbitrary.

(vi) Using the original assumed solution and the results in part (v) we obtain a formal power series solution

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

= $c_0 - c_0 x + \frac{1}{2} c_0 x^2 - c_0 \frac{1}{3 \cdot 2} x^3 + c_0 \frac{1}{4 \cdot 3 \cdot 2} x^4 - \cdots$
= $c_0 \bigg[1 - x + \frac{1}{2} x^2 - \frac{1}{3 \cdot 2} x^3 + \frac{1}{4 \cdot 3 \cdot 2} x^4 - \cdots \bigg].$

It should be fairly obvious that the pattern of the coefficients in part (v) is $c_k = c_0(-1)^k/k!, k = 0, 1, 2, \dots$ so that in summation notation we can write

$$y = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k.$$
 (8)

From the first power series representation in (2) the solution in (8) is recognized as $y = c_0 e^{-x}$. Had you used the method of Section 2.3, you would have found that $y = ce^{-x}$ is a solution of y' + y = 0 on the interval $(-\infty, \infty)$. This interval is also the interval of convergence of the power series in (8).

EXERCISES 6.1

In Problems 1–10 find the interval and radius of convergence for the given power series.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ **2.** $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ **3.** $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$ **4.** $\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n$

5.
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{10^k} (x-5)^k$$
 6. $\sum_{k=0}^{\infty} k! (x-1)^k$

7.
$$\sum_{k=1}^{\infty} \frac{1}{k^2 + k} (3x - 1)^k$$

8. $\sum_{k=0}^{\infty} 3^{-k} (4x - 5)^k$
9. $\sum_{k=1}^{\infty} \frac{2^{5k}}{5^{2k}} (\frac{x}{3})^k$
10. $\sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} x^{2n+1}$

In Problems 11–16 use an appropriate series in (2) to find the Maclaurin series of the given function. Write your answer in summation notation.

11.
$$e^{-x/2}$$
 12. xe^{3x}

 13. $\frac{1}{2+x}$
 14. $\frac{x}{1+x^2}$

15. $\ln(1-x)$ **16.** $\sin x^2$

Answers to selected odd-numbered problems begin on page ANS-9.

In Problems 17 and 18 use an appropriate series in (2) to fin the Taylor series of the given function centered at the indicated value of a. Write your answer in summation notation.

17. sinx, a = 2π [*Hint*: Use periodicity.]
18. ln x; a = 2 [*Hint*: x = 2[1 + (x - 2)/2]]

In Problems 19 and 20 the given function is analytic at a = 0. Use appropriate series in (2) and multiplication to find the first four nonzero terms of the Maclaurin series of the given function.

19.
$$\sin x \cos x$$
 20. $e^{-x} \cos x$

In Problems 21 and 22 the given function is analytic at a = 0. Use appropriate series in (2) and long division to fin the first four nonzero terms of the Maclaurin series of the given function.

21. $\sec x$ **22.** $\tan x$

If desired we could switch back to *n* as the index of summation.

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In Problems 23 and 24 use a substitution to shift the summation index so that the general term of given power series involves x^k .

23.
$$\sum_{n=1}^{\infty} nc_n x^{n+2}$$

24. $\sum_{n=3}^{\infty} (2n-1)c_n x^{n-3}$

In Problems 25–30 proceed as in Example 3 to rewrite the given expression using a single power series whose general term involves x^k .

25.
$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

26.
$$\sum_{n=1}^{\infty} nc_n x^{n-1} + 3 \sum_{n=0}^{\infty} c_n x^{n+2}$$

- **27.** $\sum_{n=1}^{n} 2nc_n x^{n-1} + \sum_{n=0}^{n} 6c_n x^{n+1}$
- **28.** $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2}$

29.
$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

30.
$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + 2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3\sum_{n=1}^{\infty} nc_n x^n$$

In Problems 31–34 verify by direct substitution that the given power series is a solution of the indicated differential equation. [*Hint*: For a power x^{2n+1} let k = n + 1.]

31.
$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}, \quad y' + 2xy = 0$$

32. $y = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad (1 + x^2)y' + 2xy = 0$
33. $y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \quad (x + 1)y'' + y' = 0$
34. $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}, \quad xy'' + y' + xy = 0$

In Problems 35–38 proceed as in Example 4 and find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the given linear first order differential equation.

35.
$$y' - 5y = 0$$
36. $4y' + y = 0$ **37.** $y' = xy$ **38.** $(1 + x)y' + y = 0$

Discussion Problems

- **39.** In Problem 19, find an easier way than multiplying two power series to obtain the Maclaurin series representation of sin *x* cos *x*.
- **40.** In Problem 21, what do you think is the interval of convergence for the Maclaurin series of sec *x*?

6.2 SOLUTIONS ABOUT ORDINARY POINTS

REVIEW MATERIAL

• Power series, analytic at a point, shifting the index of summation in Section 6.1

INTRODUCTION At the end of the last section we illustrated how to obtain a power series solution of a linear first-order differential equation. In this section we turn to the more important problem of finding power series solutions of linear second-order equations. More to the point, we are going to find solutions of linear second-order equations in the form of power series whose center is a number x_0 that is an **ordinary point** of the DE. We begin with the definition of an ordinary point.

 \equiv A Definition If we divide the homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
⁽¹⁾

by the lead coefficient $a_2(x)$ we obtain the standard form

$$y'' + P(x)y' + Q(x)y = 0.$$
 (2)

We have the following definition

DEFINITION 6.2.1 Ordinary and Singular Points

A point $x = x_0$ is said to be an **ordinary point** of the differential of the differential equation (1) if both coefficients P(x) and Q(x) in the standard form (2) are analytic at x_0 . A point that is *not* an ordinary point of (1) is said to be a **singular point** of the DE.

EXAMPLE 1 Ordinary Points

(a) A homogeneous linear second-order differential equation with constant coefficients such as

$$y'' + y = 0$$
 and $y'' + 3y' + 2y = 0$,

can have no singular points. In other words, every finite value^{*} of x is an ordinary point of such equations.

(b) Every finite value of x is an ordinary point of the differential equation

$$y'' + e^{x}y' + (\sin x)y = 0.$$

Specifically x = 0 is an ordinary point of the DE, because we have already seen in (2) of Section 6.1 that both e^x and sin x are analytic at this point.

The negation of the second sentence in Definition 6.2.1 stipulates that if *at least* one of the coefficient functions P(x) and Q(x) in (2) fails to be analytic at x_0 , then x_0 is a singular point.

EXAMPLE 2 Singular Points

(a) The differential equation

$$y'' + xy' + (\ln x)y = 0$$

is already in standard form. The coefficient functions ar

$$P(x) = x$$
 and $Q(x) = \ln x$.

Now P(x) = x is analytic at every real number, and $Q(x) = \ln x$ is analytic at every *positive* real number. However, since $Q(x) = \ln x$ is discontinuous at x = 0 it cannot be represented by a power series in x, that is, a power series centered at 0. We conclude that x = 0 is a singular point of the DE.

(b) By putting xy'' + y' + xy = 0 in the standard form

$$y'' + \frac{1}{x}y' + y = 0,$$

we see that P(x) = 1/x fails to be analytic at x = 0. Hence x = 0 is a singular point of the equation.

Polynomial Coefficients We will primarily be interested in the case when the coefficients $a_2(x)$, $a_1(x)$, and $a_0(x)$ in (1) are polynomial functions with no common factors. A polynomial function is analytic at any value of x, and a rational function is analytic except at points where its denominator is zero. Thus, in (2) both coefficients

$$P(x) = \frac{a_1(x)}{a_2(x)}$$
 and $Q(x) = \frac{a_0(x)}{a_2(x)}$

^{*}For our purposes, ordinary points and singular points will always be finite points. It is possible for a ODE to have, say, a singular point at infinit.

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are analytic except at those numbers for which $a_2(x) = 0$. It follows, then, that

A number $x = x_0$ is an ordinary point of (1) if $a_2(x_0) \neq 0$, whereas $x = x_0$ is a singular point of (1) if $a_2(x_0) = 0$.

EXAMPLE 3 Ordinary and Singular Points

(a) The only singular points of the differential equation

$$(x^2 - 1)y'' + 2xy' + 6y = 0$$

are the solutions of $x^2 - 1 = 0$ or $x = \pm 1$. All other values of x are ordinary points.

(b) Inspection of the Cauchy-Euler

$$\int a_2(x) = x^2 = 0 \text{ at } x = 0$$
$$x^2 y'' + y = 0$$

shows that it has a singular point at x = 0. All other values of x are ordinary points.

(c) Singular points need not be real numbers. The equation

$$(x^2 + 1)y'' + xy' - y = 0$$

has singular points at the solutions of $x^2 + 1 = 0$ —namely, $x = \pm i$. All other values of *x*, real or complex, are ordinary points.

We state the following theorem about the existence of power series solutions without proof.

THEOREM 6.2.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (1), we can always find two linearly independent solutions in the form of a power series centered at x_0 , that is,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

A power series solution converges at least on some interval defined by $|x - x_0| < R$, where *R* is the distance from x_0 to the closest singular point.

A solution of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is said to be a **solution about the** ordinary point x_0 . The distance *R* in Theorem 6.2.1 is the *minimum* value or *lower bound* for the radius of convergence.

EXAMPLE 4 Minimum Radius of Convergence

Find the minimum radius of convergence of a power series solution of the secondorder differential equation

$$(x^2 - 2x + 5)y'' + xy' - y = 0$$

(a) about the ordinary point x = 0, (b) about the ordinary point x = -1.

SOLUTION By the quadratic formula we see from $x^2 - 2x + 5 = 0$ that the singular points of the given differential equation are the complex numbers $1 \pm 2i$.



FIGURE 6.2.1 Distance from singular points to the ordinary point 0 in Example 4

Before working through this example, we recommend that you reread Example 4 of Section 6.1.

(a) Because x = 0 is an ordinary point of the DE, Theorem 6.2.1 guarantees that we can find two power series solutions centered at 0. That is, solutions that look like $y = \sum_{n=0}^{\infty} c_n x^n$ and, moreover, we know without actually finding these solutions that each series must converge *at least* for $|x| < \sqrt{5}$, where $R = \sqrt{5}$ is the distance in the complex plane from either of the numbers 1 + 2i (the point (1, 2)) or 1 - 2i (the point (1, -2)) to the ordinary point 0 (the point (0, 0)). See Figure 6.2.1.

(b) Because x = -1 is an ordinary point of the DE, Theorem 6.2.1 guarantees that we can find two power series solutions that look like $y = \sum_{n=0}^{\infty} c_n (x + 1)^n$. Each of power series converges at least for $|x + 1| < 2\sqrt{2}$ since the distance from each of the singular points to -1 (the point (-1, 0)) is $R = \sqrt{8} = 2\sqrt{2}$.

In part (a) of Example 4, *one* of the two power series solutions centered at 0 of the differential equation is valid on an interval much larger than $(-\sqrt{5}, \sqrt{5})$; in actual fact this solution is valid on the interval $(-\infty, \infty)$ because it can be shown that one of the two solutions about 0 reduces to a polynomial.

Note In the examples that follow as well as in the problems of Exercises 6.2 we will, for the sake of simplicity, find only power series solutions about the ordinary point x = 0. If it is necessary to find a power series solutions of an ODE about an ordinary point $x_0 \neq 0$, we can simply make the change of variable $t = x - x_0$ in the equation (this translates $x = x_0$ to t = 0), find solutions of the new equation of the form $y = \sum_{n=0}^{\infty} c_n t^n$, and then resubstitute $t = x - x_0$.

Finding a Power Series Solution Finding a power series solution of a homogeneous linear second-order ODE has been accurately described as "the method of undetermined *series* coefficients" since the procedure is quite analogous to what we did in Section 4.4. In case you did not work through Example 4 of Section 6.1 here, in brief, is the idea. Substitute $y = \sum_{n=0}^{\infty} c_n x^n$ into the differential equation, combine series as we did in Example 3 of Section 6.1, and then equate the all coefficients to the right-hand side of the equation to determine the coefficients c_n . But because the right-hand side is zero, the last step requires, by the *identity property* in the bulleted list in Section 6.1, that all coefficients of *x* must be equated to zero. No, this does *not* mean that all coefficients *are* zero; this would not make sense, after all Theorem 6.2.1 guarantees that we can find two solutions. We will see in Example 5 how the single assumption that $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ leads to two sets of coefficients so that we have two distinct power series $y_1(x)$ and $y_2(x)$, both expanded about the ordinary point x = 0. The general solution of the differential equation is $y = C_1 y_1(x) + C_2 y_2(x)$; indeed, it can be shown that $C_1 = c_0$ and $C_2 = c_1$.

EXAMPLE 5 Power Series Solutions

Solve y'' + xy = 0.

SOLUTION Since there are no singular points, Theorem 6.2.1 guarantees two power series solutions centered at 0 that converge for $|x| < \infty$. Substituting $y = \sum_{n=0}^{\infty} c_n x^n$ and the second derivative $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$ (see (1) in Section 6.1) into the differential equation give

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}.$$
 (3)

We have already added the last two series on the right-hand side of the equality in (3) by shifting the summation index. From the result given in (5) of Section 6.1

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty} [(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0.$$
 (4)

At this point we invoke the identity property. Since (4) is identically zero, it is necessary that the coefficient of each power of x be set equal to zero—that is, $2c_2 = 0$ (it is the coefficient of x^0), and

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0, \qquad k = 1, 2, 3, \dots$$
 (5)

Now $2c_2 = 0$ obviously dictates that $c_2 = 0$. But the expression in (5), called a **recurrence relation**, determines the c_k in such a manner that we can choose a certain subset of the set of coefficients to be *nonzero*. Since $(k + 1)(k + 2) \neq 0$ for all values of k, we can solve (5) for c_{k+2} in terms of c_{k-1} :

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \dots$$
 (6)

This relation generates consecutive coefficients of the assumed solution one at a time as we let k take on the successive integers indicated in (6):

$$k = 1, \qquad c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \qquad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \qquad c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{7 \cdot 8} = 0 \qquad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10 \cdot 11} = 0 \qquad \leftarrow c_8 \text{ is zero}$$

and so on. Now substituting the coefficients just obtained into the original assumption

 $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + \cdots,$ we get

$$y = c_0 + c_1 x + 0 - \frac{c_0}{2 \cdot 3} x^3 - \frac{c_1}{3 \cdot 4} x^4 + 0 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + 0 - \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 - \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + 0 + \cdots$$

After grouping the terms containing c_0 and the terms containing c_1 , we obtain $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_{1}(x) = 1 - \frac{1}{2 \cdot 3}x^{3} + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}x^{6} - \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}x^{9} + \dots = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{2 \cdot 3 \cdot \dots (3k-1)(3k)}x^{3k}$$
$$y_{2}(x) = x - \frac{1}{3 \cdot 4}x^{4} + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}x^{7} - \frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}x^{10} + \dots = x + \sum_{k=1}^{\infty} \frac{(-1)^{k}}{3 \cdot 4 \cdot \dots (3k)(3k+1)}x^{3k+1}$$

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Because the recursive use of (6) leaves c_0 and c_1 completely undetermined, they can be chosen arbitrarily. As was mentioned prior to this example, the linear combination $y = c_0y_1(x) + c_1y_2(x)$ actually represents the general solution of the differential equation. Although we know from Theorem 6.2.1 that each series solution converges for $|x| < \infty$, that is, on the interval $(-\infty, \infty)$. This fact can also be verified by the ratio test

The differential equation in Example 5 is called **Airy's equation** and is named after the English mathematician and astronomer **George Biddel Airy** (1801–1892). Airy's differential equation is encountered in the study of diffraction of light, diffraction of radio waves around the surface of the Earth, aerodynamics, and the deflectio of a uniform thin vertical column that bends under its own weight. Other common forms of Airy's equation are y'' - xy = 0 and $y'' + \alpha^2 xy = 0$. See Problem 41 in Exercises 6.4 for an application of the last equation.

EXAMPLE 6 Power Series Solution

Solve $(x^2 + 1)y'' + xy' - y = 0$.

SOLUTION As we have already seen on page 240, the given differential equation has singular points at $x = \pm i$, and so a power series solution centered at 0 will converge at least for |x| < 1, where 1 is the distance in the complex plane from 0 to either *i* or -i. The assumption $y = \sum_{n=0}^{\infty} c_n x^n$ and its first two derivatives lead t

$$(x^{2}+1)\sum_{n=2}^{\infty}n(n-1)c_{n}x^{n-2} + x\sum_{n=1}^{\infty}nc_{n}x^{n-1} - \sum_{n=0}^{\infty}c_{n}x^{n}$$

$$= \sum_{n=2}^{\infty}n(n-1)c_{n}x^{n} + \sum_{n=2}^{\infty}n(n-1)c_{n}x^{n-2} + \sum_{n=1}^{\infty}nc_{n}x^{n} - \sum_{n=0}^{\infty}c_{n}x^{n}$$

$$= 2c_{2}x^{0} - c_{0}x^{0} + 6c_{3}x + c_{1}x - c_{1}x + \sum_{n=2}^{\infty}n(n-1)c_{n}x^{n}$$

$$+ \sum_{n=4}^{\infty}n(n-1)c_{n}x^{n-2} + \sum_{n=2}^{\infty}nc_{n}x^{n} - \sum_{n=2}^{\infty}c_{n}x^{n}$$

$$= 2c_{2} - c_{0} + 6c_{3}x + \sum_{k=2}^{\infty}[k(k-1)c_{k} + (k+2)(k+1)c_{k+2} + kc_{k} - c_{k}]x^{k}$$

$$= 2c_{2} - c_{0} + 6c_{3}x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_{k} + (k+2)(k+1)c_{k+2}]x^{k} = 0.$$

From this identity we conclude that $2c_2 - c_0 = 0$, $6c_3 = 0$, and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0.$$

Thus

$$c_{2} = \frac{1}{2} c_{0}$$

$$c_{3} = 0$$

$$c_{k+2} = \frac{1-k}{k+2} c_{k}, \qquad k = 2, 3, 4, \dots$$

Substituting k = 2, 3, 4, ... into the last formula gives

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2 \cdot 4}c_0 = -\frac{1}{2^2 2!}c_0$$

$$c_{5} = -\frac{2}{5}c_{3} = 0 \qquad \leftarrow c_{3} \text{ is zero}$$

$$c_{6} = -\frac{3}{6}c_{4} = \frac{3}{2 \cdot 4 \cdot 6}c_{0} = \frac{1 \cdot 3}{2^{3}3!}c_{0}$$

$$c_{7} = -\frac{4}{7}c_{5} = 0 \qquad \leftarrow c_{5} \text{ is zero}$$

$$c_{8} = -\frac{5}{8}c_{6} = -\frac{3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}c_{0} = -\frac{1 \cdot 3 \cdot 5}{2^{4}4!}c_{0}$$

$$c_{9} = -\frac{6}{9}c_{7} = 0, \qquad \leftarrow c_{7} \text{ is zero}$$

$$c_{10} = -\frac{7}{10}c_{8} = \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}c_{0} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^{5}5!}c_{0}$$

and so on. Therefore

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$$

= $c_0 \bigg[1 + \frac{1}{2} x^2 - \frac{1}{2^2 2!} x^4 + \frac{1 \cdot 3}{2^3 3!} x^6 - \frac{1 \cdot 3 \cdot 5}{2^4 4!} x^8 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 5!} x^{10} - \cdots \bigg] + c_1 x$
= $c_0 y_1(x) + c_1 y_2(x)$.

The solutions are the polynomial $y_2(x) = x$ and the power series

$$y_1(x) = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{2n}, \qquad |x| < 1.$$

EXAMPLE 7 T

7 Three-Term Recurrence Relation

If we seek a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ for the differential equation y'' - (1 + x)y = 0,

we obtain $c_2 = \frac{1}{2}c_0$ and the three-term recurrence relation

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \qquad k = 1, 2, 3, \dots$$

It follows from these two results that all coefficients c_n , for $n \ge 3$, are expressed in terms of *both* c_0 and c_1 . To simplify life, we can first choose $c_0 \ne 0$, $c_1 = 0$; this yields coefficients for one solution expressed entirely in terms of c_0 . Next, if we choose $c_0 = 0$, $c_1 \ne 0$, then coefficients for the other solution are expressed in terms of c_1 . Using $c_2 = \frac{1}{2}c_0$ in both cases, the recurrence relation for $k = 1, 2, 3, \ldots$ gives

$$c_{0} \neq 0, c_{1} = 0$$

$$c_{2} = \frac{1}{2}c_{0}$$

$$c_{3} = \frac{c_{1} + c_{0}}{2 \cdot 3} = \frac{c_{0}}{2 \cdot 3} = \frac{c_{0}}{6}$$

$$c_{4} = \frac{c_{2} + c_{1}}{3 \cdot 4} = \frac{c_{0}}{2 \cdot 3 \cdot 4} = \frac{c_{0}}{24}$$

$$c_{5} = \frac{c_{3} + c_{2}}{4 \cdot 5} = \frac{c_{0}}{4 \cdot 5} \left[\frac{1}{6} + \frac{1}{2}\right] = \frac{c_{0}}{30}$$

$$c_{0} = 0, c_{1} \neq 0$$

$$c_{2} = \frac{1}{2}c_{0} = 0$$

$$c_{3} = \frac{c_{1} + c_{0}}{2 \cdot 3} = \frac{c_{1}}{2 \cdot 3} = \frac{c_{1}}{6}$$

$$c_{4} = \frac{c_{2} + c_{1}}{3 \cdot 4} = \frac{c_{1}}{3 \cdot 4} = \frac{c_{1}}{12}$$

$$c_{5} = \frac{c_{3} + c_{2}}{4 \cdot 5} = \frac{c_{0}}{4 \cdot 5 \cdot 6} = \frac{c_{1}}{120}$$

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and so on. Finally, we see that the general solution of the equation is $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$
$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots$$

Each series converges for all finite values of x.

Nonpolynomial Coefficients The next example illustrates how to find a power series solution about the ordinary point $x_0 = 0$ of a differential equation when its coefficients are not polynomials. In this example we see an application of the multiplication of two power series.

EXAMPLE 8 DE with Nonpolynomial Coefficient

Solve $y'' + (\cos x)y = 0$.

and

SOLUTION We see that x = 0 is an ordinary point of the equation because, as we have already seen, $\cos x$ is analytic at that point. Using the Maclaurin series for $\cos x$ given in (2) of Section 6.1, along with the usual assumption $y = \sum_{n=0}^{\infty} c_n x^n$ and the results in (1) of Section 6.1 we fin

$$y'' + (\cos x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \sum_{n=0}^{\infty} c_n x^n$$

= $2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots)$
= $2c_2 + c_0 + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{1}{2}c_0\right)x^2 + \left(20c_5 + c_3 - \frac{1}{2}c_1\right)x^3 + \cdots = 0.$

It follows that

$$2c_2 + c_0 = 0$$
, $6c_3 + c_1 = 0$, $12c_4 + c_2 - \frac{1}{2}c_0 = 0$, $20c_5 + c_3 - \frac{1}{2}c_1 = 0$,

and so on. This gives $c_2 = -\frac{1}{2}c_0$, $c_3 = -\frac{1}{6}c_1$, $c_4 = \frac{1}{12}c_0$, $c_5 = \frac{1}{30}c_1$, By grouping terms, we arrive at the general solution $y = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{12}x^4 - \cdots$$
 and $y_2(x) = x - \frac{1}{6}x^3 + \frac{1}{30}x^5 - \cdots$

Because the differential equation has no finite singular points, both power series converge for $|x| < \infty$.

Solution Curves The approximate graph of a power series solution $y(x) = \sum_{n=0}^{\infty} c_n x^n$ can be obtained in several ways. We can always resort to graphing the terms in the sequence of partial sums of the series — in other words, the graphs of the polynomials $S_N(x) = \sum_{n=0}^{N} c_n x^n$. For large values of N, $S_N(x)$ should give us an indication of the behavior of y(x) near the ordinary point x = 0. We can also obtain an approximate or numerical solution curve by using a solver as we did in Section 4.10. For example, if you carefully scrutinize the series solutions of Airy's equation in



FIGURE 6.2.2 Numerical solution curves for Airy's DE

Example 5, you should see that $y_1(x)$ and $y_2(x)$ are, in turn, the solutions of the initial-value problems

$$y'' + xy = 0, \quad y(0) = 1, \quad y'(0) = 0,$$

$$y'' + xy = 0, \quad y(0) = 0, \quad y'(0) = 1.$$
(11)

The specified initial conditions "pick out" the solutions $y_1(x)$ and $y_2(x)$ from $y = c_0y_1(x) + c_1y_2(x)$, since it should be apparent from our basic series assumption $y = \sum_{n=0}^{\infty} c_n x^n$ that $y(0) = c_0$ and $y'(0) = c_1$. Now if your numerical solver requires a system of equations, the substitution y' = u in y'' + xy = 0 gives y'' = u' = -xy, and so a system of two first-order equations equivalent to Airy's equation is

$$y' = u$$

$$u' = -xy.$$
(12)

Initial conditions for the system in (12) are the two sets of initial conditions in (11) rewritten as y(0) = 1, u(0) = 0, and y(0) = 0, u(0) = 1. The graphs of $y_1(x)$ and $y_2(x)$ shown in Figure 6.2.2 were obtained with the aid of a numerical solver. The fact that the numerical solution curves appear to be oscillatory is consistent with the fact that Airy's equation appeared in Section 5.1 (page 197) in the form mx'' + ktx = 0 as a model of a spring whose "spring constant" K(t) = kt increases with time.

REMARKS

(*i*) In the problems that follow, do not expect to be able to write a solution in terms of summation notation in each case. Even though we can generate as many terms as desired in a series solution $y = \sum_{n=0}^{\infty} c_n x^n$ either through the use of a recurrence relation or, as in Example 8, by multiplication, it might not be possible to deduce any general term for the coefficients c_n . We might have to settle, as we did in Examples 7 and 8, for just writing out the first few terms of the series.

(*ii*) A point x_0 is an ordinary point of a *nonhomogeneous* linear second-order DE y'' + P(x)y' + Q(x)y = f(x) if P(x), Q(x), and f(x) are analytic at x_0 . Moreover, Theorem 6.2.1 extends to such DEs; in other words, we can fin power series solutions $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ of nonhomogeneous linear DEs in the same manner as in Examples 5–8. See Problem 26 in Exercises 6.2.

EXERCISES 6.2

In Problems 1 and 2 without actually solving the given differential equation, find the minimum radius of convergence of power series solutions about the ordinary point x = 0. About the ordinary point x = 1.

1.
$$(x^2 - 25)y'' + 2xy' + y = 0$$

2.
$$(x^2 - 2x + 10)y'' + xy' - 4y = 0$$

In Problems 3–6 find two power series solutions of the given differential equation about the ordinary point x = 0. Compare the series solutions with the solutions of the differential equations obtained using the method of Section 4.3. Try to explain any differences between the two forms of the solutions.

3.
$$y'' + y = 0$$

4. $y'' - y = 0$
5. $y'' - y' = 0$
6. $y'' + 2y' = 0$

Answers to selected odd-numbered problems begin on page ANS-9.

In Problems 7–18 find two power series solutions of the given differential equation about the ordinary point x = 0.

7. y'' - xy = 08. $y'' + x^2y = 0$ 9. y'' - 2xy' + y = 010. y'' - xy' + 2y = 011. $y'' + x^2y' + xy = 0$ 12. y'' + 2xy' + 2y = 013. (x - 1)y'' + y' = 014. (x + 2)y'' + xy' - y = 015. y'' - (x + 1)y' - y = 016. $(x^2 + 1)y'' - 6y = 0$ 17. $(x^2 + 2)y'' + 3xy' - y = 0$ 18. $(x^2 - 1)y'' + xy' - y = 0$

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In Problems 19–22 use the power series method to solve the given initial-value problem.

19.
$$(x - 1)y'' - xy' + y = 0, y(0) = -2, y'(0) = 6$$

20. $(x + 1)y'' - (2 - x)y' + y = 0, y(0) = 2, y'(0) = -1$
21. $y'' - 2xy' + 8y = 0, y(0) = 3, y'(0) = 0$
22. $(x^2 + 1)y'' + 2xy' = 0, y(0) = 0, y'(0) = 1$

In Problems 23 and 24 use the procedure in Example 8 to find two power series solutions of the given differential equation about the ordinary point x = 0.

23.
$$y'' + (\sin x)y = 0$$

24. $y'' + e^{x}y' - y = 0$

Discussion Problems

- **25.** Without actually solving the differential equation $(\cos x)y'' + y' + 5y = 0$, find the minimum radius of convergence of power series solutions about the ordinary point x = 0. About the ordinary point x = 1.
- **26.** How can the power series method be used to solve the *nonhomogeneous* equation y'' xy = 1 about the ordinary point x = 0? Of $y'' 4xy' 4y = e^{x}$? Carry out your ideas by solving both DEs.
- 27. Is x = 0 an ordinary or a singular point of the differential equation $xy'' + (\sin x)y = 0$? Defend your answer with sound mathematics. [*Hint*: Use the Maclaurin series of sin x and then examine $(\sin x)/x$.

28. Is x = 0 an ordinary point of the differential equation $y'' + 5xy' + \sqrt{xy} = 0$?

Computer Lab Assignments

- **29.** (a) Find two power series solutions for y'' + xy' + y = 0 and express the solutions $y_1(x)$ and $y_2(x)$ in terms of summation notation.
 - (b) Use a CAS to graph the partial sums $S_N(x)$ for $y_1(x)$. Use N = 2, 3, 5, 6, 8, 10. Repeat using the partial sums $S_N(x)$ for $y_2(x)$.
 - (c) Compare the graphs obtained in part (b) with the curve obtained by using a numerical solver. Use the initial-conditions $y_1(0) = 1$, $y'_1(0) = 0$, and $y_2(0) = 0$, $y'_2(0) = 1$.
 - (d) Reexamine the solution $y_1(x)$ in part (a). Express this series as an elementary function. Then use (5) of Section 4.2 to find a second solution of the equation. Verify that this second solution is the same as the power series solution $y_2(x)$.
- 30. (a) Find one more nonzero term for each of the solutions y₁(x) and y₂(x) in Example 8.
 - (b) Find a series solution y(x) of the initial-value problem $y'' + (\cos x)y = 0$, y(0) = 1, y'(0) = 1.
 - (c) Use a CAS to graph the partial sums $S_N(x)$ for the solution y(x) in part (b). Use N = 2, 3, 4, 5, 6, 7.
 - (d) Compare the graphs obtained in part (c) with the curve obtained using a numerical solver for the initial-value problem in part (b).

6.3 SOLUTIONS ABOUT SINGULAR POINTS

REVIEW MATERIAL

- Section 4.2 (especially (5) of that section)
- The definition of a singular point in Definition 6.2.1

INTRODUCTION The two differential equations

y'' + xy = 0 and xy'' + y = 0

are similar only in that they are both examples of simple linear second-order DEs with variable coefficients. That is all they have in common. Since x = 0 is an *ordinary point* of y'' + xy = 0, we saw in Section 6.2 that there was no problem in finding two distinct power series solutions centered at that point. In contrast, because x = 0 is a *singular point* of xy'' + y = 0, finding two infinit series—notice that we did not say *power series*—solutions of the equation about that point becomes a more difficult task

The solution method that is discussed in this section does not always yield two infinite series solutions. When only one solution is found, we can use the formula given in (5) of Section 4.2 to find a second solution

EXAMPLE A Singular point x_0 of a linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
⁽¹⁾

is further classified as either regular or irregular. The classification again depends on the functions P and Q in the standard form

$$y'' + P(x)y' + Q(x)y = 0.$$
 (2)

DEFINITION 6.3.1 Regular and Irregular Singular Points

A singular point $x = x_0$ is said to be a **regular singular point** of the differential equation (1) if the functions $p(x) = (x - x_0) P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0 . A singular point that is not regular is said to be an **irregular singular point** of the equation.

The second sentence in Definition 6.3.1 indicates that if one or both of the functions $p(x) = (x - x_0) P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ fail to be analytic at x_0 , then x_0 is an irregular singular point.

Polynomial Coefficients As in Section 6.2, we are mainly interested in linear equations (1) where the coefficients $a_2(x)$, $a_1(x)$, and $a_0(x)$ are polynomials with no common factors. We have already seen that if $a_2(x_0) = 0$, then $x = x_0$ is a singular point of (1), since at least one of the rational functions $P(x) = a_1(x)/a_2(x)$ and $Q(x) = a_0(x)/a_2(x)$ in the standard form (2) fails to be analytic at that point. But since $a_2(x)$ is a polynomial and x_0 is one of its zeros, it follows from the Factor Theorem of algebra that $x - x_0$ is a factor of $a_2(x)$. This means that after $a_1(x)/a_2(x)$ and $a_0(x)/a_2(x)$ are reduced to lowest terms, the factor $x - x_0$ must remain, to some positive integer power, in one or both denominators. Now suppose that $x = x_0$ is a singular point of (1) but both the functions defined by the products $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2Q(x)$ are analytic at x_0 . We are led to the conclusion that multiplying P(x) by $x - x_0$ and Q(x) by $(x - x_0)^2$ has the effect (through cancellation) that $x - x_0$ is regular by a quick visual check of denominators:

If $x - x_0$ appears at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x), then $x = x_0$ is a regular singular point.

Moreover, observe that if $x = x_0$ is a regular singular point and we multiply (2) by $(x - x_0)^2$, then the original DE can be put into the form

$$(x - x_0)^2 y'' + (x - x_0)p(x)y' + q(x)y = 0,$$
(3)

where *p* and *q* are analytic at $x = x_0$.

EXAMPLE 1 Classification of Singula Points

It should be clear that x = 2 and x = -2 are singular points of

$$(x^{2} - 4)^{2}y'' + 3(x - 2)y' + 5y = 0.$$

After dividing the equation by $(x^2 - 4)^2 = (x - 2)^2(x + 2)^2$ and reducing the coefficients to lowest terms, we find th

$$P(x) = \frac{3}{(x-2)(x+2)^2}$$
 and $Q(x) = \frac{5}{(x-2)^2(x+2)^2}$

We now test P(x) and Q(x) at each singular point.

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For x = 2 to be a regular singular point, the factor x - 2 can appear at most to the first power in the denominator of P(x) and at most to the second power in the denominator of Q(x). A check of the denominators of P(x) and Q(x) shows that both these conditions are satisfied, so x = 2 is a regular singular point. Alternatively, we are led to the same conclusion by noting that both rational functions

$$p(x) = (x - 2)P(x) = \frac{3}{(x + 2)^2}$$
 and $q(x) = (x - 2)^2 Q(x) = \frac{5}{(x + 2)^2}$

are analytic at x = 2.

Now since the factor x - (-2) = x + 2 appears to the second power in the denominator of P(x), we can conclude immediately that x = -2 is an irregular singular point of the equation. This also follows from the fact that

$$p(x) = (x + 2)P(x) = \frac{3}{(x - 2)(x + 2)}$$

is not analytic at x = -2

In Example 1, notice that since x = 2 is a regular singular point, the original equation can be written as

$$(x-2)^2 y'' + (x-2) \frac{y(x)}{(x+2)^2} y' + \frac{y(x)}{(x+2)^2} y' + \frac{y(x)}{(x+2)^2} y = 0.$$

As another example, we can see that x = 0 is an irregular singular point of $x^3y'' - 2xy' + 8y = 0$ by inspection of the denominators of $P(x) = -2/x^2$ and $Q(x) = 8/x^3$. On the other hand, x = 0 is a regular singular point of xy'' - 2xy' + 8y = 0, since x - 0 and $(x - 0)^2$ do not even appear in the respective denominators of P(x) = -2 and Q(x) = 8/x. For a singular point $x = x_0$ any nonnegative power of $x - x_0$ less than one (namely, zero) and any nonnegative power less than two (namely, zero and one) in the denominators of P(x) and Q(x), respectively, imply that x_0 is a regular singular point. A singular point can be a complex number. You should verify that x = 3i and x = -3i are two regular singular points of $(x^2 + 9)y'' - 3xy' + (1 - x)y = 0$.

Note Any second-order Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$, where *a*, *b*, and *c* are real constants, has a regular singular point at x = 0. You should verify that two solutions of the Cauchy-Euler equation $x^2y'' - 3xy' + 4y = 0$ on the interval $(0, \infty)$ are $y_1 = x^2$ and $y_2 = x^2 \ln x$. If we attempted to find a power series solution about the regular singular point x = 0 (namely, $y = \sum_{n=0}^{\infty} c_n x^n$), we would succeed in obtaining only the polynomial solution $y_1 = x^2$. The fact that we would not obtain the second solution is not surprising because $\ln x$ (and consequently $y_2 = x^2 \ln x$) is not analytic at x = 0—that is, y_2 does not possess a Taylor series expansion centered at x = 0.

Method of Frobenius To solve a differential equation (1) about a regular singular point, we employ the following theorem due to the eminent German mathematician **Ferdinand Georg Frobenius** (1849–1917).

THEOREM 6.3.1 Frobenius' Theorem

If $x = x_0$ is a regular singular point of the differential equation (1), then there exists at least one solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r},$$
 (4)

where the number *r* is a constant to be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Notice the words *at least* in the first sentence of Theorem 6.3.1. This means that in contrast to Theorem 6.2.1, Theorem 6.3.1 gives us no assurance that *two* series solutions of the type indicated in (4) can be found. The **method of Frobenius**, findin series solutions about a regular singular point x_0 , is similar to the power-series method in the preceding section in that we substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ into the given differential equation and determine the unknown coefficients c_n by a recurrence relation. However, we have an additional task in this procedure: Before determining the coefficients, we must find the unknown exponent *r*. If *r* is found to be a number that is not a nonnegative integer, then the corresponding solution $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ is not a power series.

As we did in the discussion of solutions about ordinary points, we shall always assume, for the sake of simplicity in solving differential equations, that the regular singular point is x = 0.

EXAMPLE 2 Two Series Solutions

Because x = 0 is a regular singular point of the differential equation

$$3xy'' + y' - y = 0, (5)$$

we try to find a solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Now

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$
 and $y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$,

so

$$3xy'' + y' - y = 3\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$
$$= \sum_{n=0}^{\infty} (n+r)(3n+3r-2)c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r}$$
$$= x^r \bigg[r(3r-2)c_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)(3n+3r-2)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \bigg]$$
$$\underbrace{k=n-1} \qquad k=n$$

$$= x^{r} \left[r(3r-2)c_{0}x^{-1} + \sum_{k=0}^{\infty} \left[(k+r+1)(3k+3r+1)c_{k+1} - c_{k} \right] x^{k} \right] = 0,$$

which implies that $r(3r-2)c_0 = 0$ and $(k+r+1)(3k+3r+1)c_{k+1} - c_k = 0$, k = 0, 1, 2, ...Because nothing is gained by taking $c_0 = 0$, we must then have

 $r(3r - 2) = 0 \tag{6}$

$$c_k = 0.1.2$$

and

$$c_{k+1} = \frac{c_k}{(k+r+1)(3k+3r+1)}, \qquad k = 0, 1, 2, \dots$$
(7)
en substituted in (7) the two values of *r* that satisfy the quadratic equation

When substituted in (7), the two values of *r* that satisfy the quadratic equation (6), $r_1 = \frac{2}{3}$ and $r_2 = 0$, give two different recurrence relations:

$$r_1 = \frac{2}{3}, \qquad c_{k+1} = \frac{c_k}{(3k+5)(k+1)}, \qquad k = 0, 1, 2, \dots$$
 (8)

$$r_2 = 0, \qquad c_{k+1} = \frac{c_k}{(k+1)(3k+1)}, \qquad k = 0, 1, 2, \dots$$
 (9)

From (8) we fin

$$c_{1} = \frac{c_{0}}{5 \cdot 1}$$

$$c_{2} = \frac{c_{1}}{8 \cdot 2} = \frac{c_{0}}{2!5 \cdot 8}$$

$$c_{3} = \frac{c_{2}}{11 \cdot 3} = \frac{c_{0}}{3!5 \cdot 8 \cdot 11}$$

$$c_{4} = \frac{c_{3}}{14 \cdot 4} = \frac{c_{0}}{4!5 \cdot 8 \cdot 11 \cdot 14}$$

$$\vdots$$

$$c_{n} = \frac{c_{0}}{n!5 \cdot 8 \cdot 11 \cdot \cdots (3n + 2)}$$
From (9) we fin

$$c_{1} = \frac{c_{0}}{1 \cdot 1}$$

$$c_{2} = \frac{c_{0}}{1 \cdot 1}$$

$$c_{2} = \frac{c_{0}}{2!1 \cdot 4}$$

$$c_{2} = \frac{c_{1}}{2 \cdot 4} = \frac{c_{0}}{2!1 \cdot 4}$$

$$c_{3} = \frac{c_{2}}{3 \cdot 7} = \frac{c_{0}}{3!1 \cdot 4 \cdot 7}$$

$$c_{4} = \frac{c_{3}}{4 \cdot 10} = \frac{c_{0}}{4!1 \cdot 4 \cdot 7 \cdot 10}$$

$$\vdots$$

$$c_{n} = \frac{c_{0}}{n!1 \cdot 4 \cdot 7 \cdot \cdots (3n - 2)}$$

Here we encounter something that did not happen when we obtained solutions about an ordinary point; we have what looks to be two different sets of coefficients, but each set contains the *same* multiple c_0 . If we omit this term, the series solutions are

$$y_1(x) = x^{2/3} \left[1 + \sum_{n=1}^{\infty} \frac{1}{n! 5 \cdot 8 \cdot 11 \cdots (3n+2)} x^n \right]$$
(10)

$$y_2(x) = x^0 \left[1 + \sum_{n=1}^{\infty} \frac{1}{n! \cdot 4 \cdot 7 \cdot \cdot \cdot (3n-2)} x^n \right].$$
(11)

By the ratio test it can be demonstrated that both (10) and (11) converge for all values of *x*—that is, $|x| < \infty$. Also, it should be apparent from the form of these solutions that neither series is a constant multiple of the other, and therefore $y_1(x)$ and $y_2(x)$ are linearly independent on the entire *x*-axis. Hence by the superposition principle, $y = C_1y_1(x) + C_2y_2(x)$ is another solution of (5). On any interval that does not contain the origin, such as $(0, \infty)$, this linear combination represents the general solution of the differential equation.

Indicial Equation Equation (6) is called the **indicial equation** of the problem, and the values $r_1 = \frac{2}{3}$ and $r_2 = 0$ are called the **indicial roots**, or **exponents**, of the singularity x = 0. In general, after substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the given differential equation and simplifying, the indicial equation is a quadratic equation in *r* that results from equating the *total coefficient of the lowest power of x to zero*. We solve for the two values of *r* and substitute these values into a recurrence relation such as (7). Theorem 6.3.1 guarantees that at least one solution of the assumed series form can be found.

It is possible to obtain the indicial equation in advance of substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the differential equation. If x = 0 is a regular singular point of (1), then by Definition 6.3.1 both functions p(x) = xP(x) and $q(x) = x^2Q(x)$, where *P* and *Q* are defined by the standard form (2), are analytic at x = 0; that is, the power series expansions

$$p(x) = xP(x) = a_0 + a_1x + a_2x^2 + \cdots$$
 and $q(x) = x^2Q(x) = b_0 + b_1x + b_2x^2 + \cdots$ (12)

are valid on intervals that have a positive radius of convergence. By multiplying (2) by x^2 , we get the form given in (3):

$$x^{2}y'' + x[xP(x)]y' + [x^{2}Q(x)]y = 0.$$
(13)

After substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ and the two series in (12) into (13) and carrying out the multiplication of series, we find the general indicial equation to b

$$r(r-1) + a_0 r + b_0 = 0, (14)$$

where a_0 and b_0 are as defined in (12). See Problems 13 and 14 in Exercises 6.3

EXAMPLE 3 Two Series Solutions

Solve 2xy'' + (1 + x)y' + y = 0.

SOLUTION Substituting $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ gives

$$2xy'' + (1+x)y' + y = 2\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} (n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r+1)c_n x^{n+r} = x^r \left[r(2r-1)c_0 x^{-1} + \sum_{n=1}^{\infty} (n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} (n+r+1)c_n x^n \right] k=n = x^r \left[r(2r-1)c_0 x^{-1} + \sum_{k=0}^{\infty} [(k+r+1)(2k+2r+1)c_{k+1} + (k+r+1)c_k]x^k \right],$$

which implies that r(2r-1) =

(16)

and $(k + r + 1)(2k + 2r + 1)c_{k+1} + (k + r + 1)c_k = 0$,

 $k = 0, 1, 2, \dots$ From (15) we see that the indicial roots are $r_1 = \frac{1}{2}$ and $r_2 = 0$. For $r_1 = \frac{1}{2}$ we can divide by $k + \frac{3}{2}$ in (16) to obtain

$$c_{k+1} = \frac{-c_k}{2(k+1)}, \qquad k = 0, 1, 2, \dots,$$
 (17)

whereas for $r_2 = 0$, (16) becomes

$$c_{k+1} = \frac{-c_k}{2k+1}, \qquad k = 0, 1, 2, \dots$$
 (18)

From (17) we fin From (18) we fin

$$c_{1} = \frac{-c_{0}}{2 \cdot 1} \qquad c_{1} = \frac{-c_{0}}{1}$$

$$c_{2} = \frac{-c_{1}}{2 \cdot 2} = \frac{c_{0}}{2^{2} \cdot 2!} \qquad c_{2} = \frac{-c_{1}}{3} = \frac{c_{0}}{1 \cdot 3}$$

$$c_{3} = \frac{-c_{2}}{2 \cdot 3} = \frac{-c_{0}}{2^{3} \cdot 3!} \qquad c_{3} = \frac{-c_{2}}{5} = \frac{-c_{0}}{1 \cdot 3 \cdot 5}$$

$$c_{4} = \frac{-c_{3}}{2 \cdot 4} = \frac{c_{0}}{2^{4} \cdot 4!} \qquad c_{4} = \frac{-c_{3}}{7} = \frac{c_{0}}{1 \cdot 3 \cdot 5 \cdot 7}$$

$$\vdots \qquad \vdots \qquad c_{n} = \frac{(-1)^{n}c_{0}}{2^{n}n!} \qquad c_{n} = \frac{(-1)^{n}c_{0}}{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots (2n-1)}$$

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Thus for the indicial root $r_1 = \frac{1}{2}$ we obtain the solution

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} x^n \right] = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{n+1/2},$$

where we have again omitted c_0 . The series converges for $x \ge 0$; as given, the series is not defined for negative values of x because of the presence of $x^{1/2}$. For $r_2 = 0$ a second solution is

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)} x^n, \qquad |x| < \infty.$$

On the interval $(0, \infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$.

EXAMPLE 4 Only One Series Solution

Solve xy'' + y = 0.

SOLUTION From xP(x) = 0, $x^2Q(x) = x$ and the fact that 0 and x are their own power series centered at 0 we conclude that $a_0 = 0$ and $b_0 = 0$, so from (14) the indicial equation is r(r - 1) = 0. You should verify that the two recurrence relations corresponding to the indicial roots $r_1 = 1$ and $r_2 = 0$ yield exactly the same set of coefficients. In other words, in this case the method of Frobenius produces only a single series solution

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^{n+1} = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$$

Three Cases For the sake of discussion let us again suppose that x = 0 is a regular singular point of equation (1) and that the indicial roots r_1 and r_2 of the singularity are real. When using the method of Frobenius, we distinguish three cases corresponding to the nature of the indicial roots r_1 and r_2 . In the first two cases the symbol r_1 denotes the largest of two distinct roots, that is, $r_1 > r_2$. In the last case $r_1 = r_2$.

Case I: If r_1 and r_2 are distinct and the difference $r_1 - r_2$ is not a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0.$$

This is the case illustrated in Examples 2 and 3.

Next we assume that the difference of the roots is N, where N is a positive integer. In this case the second solution *may* contain a logarithm.

Case II: If r_1 and r_2 are distinct and the difference $r_1 - r_2$ is a positive integer, then there exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0,$$
 (19)

$$y_2(x) = Cy_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \qquad b_0 \neq 0,$$
 (20)

where C is a constant that could be zero.

Finally, in the last case, the case when $r_1 = r_2$, a second solution will *always* contain a logarithm. The situation is analogous to the solution of a Cauchy-Euler equation when the roots of the auxiliary equation are equal.

Case III: If r_1 and r_2 are equal, then there always exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0,$$
 (21)

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}.$$
 (22)

Finding a Second Solution When the difference $r_1 - r_2$ is a positive integer (Case II), we *may* or *may not* be able to find two solutions having the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. This is something that we do not know in advance but is determined after we have found the indicial roots and have carefully examined the recurrence relation that defines the coefficients c_n . We just may be lucky enough to find two solutions that involve only powers of x, that is, $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ (equation (19)) and $y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$ (equation (20) with C = 0). See Problem 31 in Exercises 6.3. On the other hand, in Example 4 we see that the difference of the indicial roots is a positive integer $(r_1 - r_2 = 1)$ and the method of Frobenius failed to give a second series solution. In this situation equation (20), with $C \neq 0$, indicates what the second solution looks like. Finally, when the difference $r_1 - r_2$ is a zero (Case III), the method of Frobenius fails to give a second series solution; the second solution (22) always contains a logarithm and can be shown to be equivalent to (20) with C = 1. One way to obtain the second solution with the logarithmic term is to use the fact that

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$
(23)

is also a solution of y'' + P(x)y' + Q(x)y = 0 whenever $y_1(x)$ is a known solution. We illustrate how to use (23) in the next example.

EXAMPLE 5 Example 4 Revisited Using a CAS

Find the general solution of xy'' + y = 0.

SOLUTION From the known solution given in Example 4,

$$y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots,$$

we can construct a second solution $y_2(x)$ using formula (23). Those with the time, energy, and patience can carry out the drudgery of squaring a series, long division, and integration of the quotient by hand. But all these operations can be done with relative ease with the help of a CAS. We give the results:

$$y_{2}(x) = y_{1}(x) \int \frac{e^{-j0dx}}{[y_{1}(x)]^{2}} dx = y_{1}(x) \int \frac{dx}{\left[x - \frac{1}{2}x^{2} + \frac{1}{12}x^{3} - \frac{1}{144}x^{4} + \cdots\right]^{2}}$$

$$= y_{1}(x) \int \frac{dx}{\left[x^{2} - x^{3} + \frac{5}{12}x^{4} - \frac{7}{72}x^{5} + \cdots\right]} \quad \leftarrow \text{ after squaring}$$

$$= y_{1}(x) \int \left[\frac{1}{x^{2}} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots\right] dx \quad \leftarrow \text{ after long division}$$

$$= y_{1}(x) \left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^{2} + \cdots\right] \quad \leftarrow \text{ after integrating}$$

$$= y_{1}(x) \ln x + y_{1}(x) \left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^{2} + \cdots\right],$$

 \equiv

or
$$y_2(x) = y_1(x) \ln x + \left[-1 - \frac{1}{2}x + \frac{1}{2}x^2 + \cdots \right].$$
 \leftarrow after multiplying out

On the interval $(0, \infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$.

Note that the final form of y_2 in Example 5 matches (20) with C = 1; the series in the brackets corresponds to the summation in (20) with $r_2 = 0$.

REMARKS

(*i*) The three different forms of a linear second-order differential equation in (1), (2), and (3) were used to discuss various theoretical concepts. But on a practical level, when it comes to actually solving a differential equation using the method of Frobenius, it is advisable to work with the form of the DE given in (1).

(*ii*) When the difference of indicial roots $r_1 - r_2$ is a positive integer $(r_1 > r_2)$, it sometimes pays to iterate the recurrence relation using the smaller root r_2 first. See Problems 31 and 32 in Exercises 6.3

(iii) Because an indicial root r is a solution of a quadratic equation, it could be complex. We shall not, however, investigate this case.

(*iv*) If x = 0 is an irregular singular point, then we might not be able to fin *any* solution of the DE of form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

EXERCISES 6.3

In Problems 1–10 determine the singular points of the given differential equation. Classify each singular point as regular or irregular.

1.
$$x^3y'' + 4x^2y' + 3y = 0$$

$$2. \ x(x+3)^2 y^2 - y = 0$$

3.
$$(x^2 - 9)^2 y'' + (x + 3)y' + 2y = 0$$

4.
$$y'' - \frac{1}{x}y' + \frac{1}{(x-1)^3}y = 0$$

5
$$(r^3 + 4r)v'' - 2rv' + 6v = 0$$

6.
$$x^2(x-5)^2y'' + 4xy' + (x^2-25)y = 0$$

7.
$$(x^2 + x - 6)y'' + (x + 3)y' + (x - 2)y = 0$$

8.
$$x(x^2 + 1)^2 y'' + y = 0$$

9.
$$x^{3}(x^{2} - 25)(x - 2)^{2}y'' + 3x(x - 2)y' + 7(x + 5)y = 0$$

10.
$$(x^3 - 2x^2 + 3x)^2 y'' + x(x - 3)^2 y' - (x + 1)y = 0$$

In Problems 11 and 12 put the given differential equation into form (3) for each regular singular point of the equation. Identify the functions p(x) and q(x).

11.
$$(x^2 - 1)y'' + 5(x + 1)y' + (x^2 - x)y = 0$$

12. $xy'' + (x + 3)y' + 7x^2y = 0$

Answers to selected odd-numbered problems begin on page ANS-10.

In Problems 13 and 14, x = 0 is a regular singular point of the given differential equation. Use the general form of the indicial equation in (14) to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius

13.
$$x^2y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$$

14. $xy'' + y' + 10y = 0$

In Problems 15–24, x = 0 is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent series solutions about x = 0. Form the general solution on $(0, \infty)$.

15. 2xy'' - y' + 2y = 016. 2xy'' + 5y' + xy = 017. $4xy'' + \frac{1}{2}y' + y = 0$ 18. $2x^2y'' - xy' + (x^2 + 1)y = 0$ 19. 3xy'' + (2 - x)y' - y = 020. $x^2y'' - (x - \frac{2}{9})y = 0$ 21. 2xy'' - (3 + 2x)y' + y = 0

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22.
$$x^2y'' + xy' + (x^2 - \frac{4}{9})y = 0$$

23. $9x^2y'' + 9x^2y' + 2y = 0$
24. $2x^2y'' + 3xy' + (2x - 1)y = 0$

In Problems 25–30, x = 0 is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the method of Frobenius to obtain at least one series solution about x = 0. Use (23) where necessary and a CAS, if instructed, to fin a second solution. Form the general solution on $(0, \infty)$.

25.
$$xy'' + 2y' - xy = 0$$

26. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$
27. $xy'' - xy' + y = 0$
28. $y'' + \frac{3}{x}y' - 2y = 0$
29. $xy'' + (1 - x)y' - y = 0$
30. $xy'' + y' + y = 0$

In Problems 31 and 32, x = 0 is a regular singular point of the given differential equation. Show that the indicial roots of the singularity differ by an integer. Use the recurrence relation found by the method of Frobenius first with the larger root r_1 . How many solutions did you find? Next use the recurrence relation with the smaller root r_2 . How many solutions did you find

31.
$$xy'' + (x - 6)y' - 3y = 0$$

32.
$$x(x-1)y'' + 3y' - 2y = 0$$

33. (a) The differential equation $x^4y'' + \lambda y = 0$ has an irregular singular point at x = 0. Show that the substitution t = 1/x yields the DE

$$\frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \lambda y = 0,$$

which now has a regular singular point at t = 0.

- (b) Use the method of this section to find two series solutions of the second equation in part (a) about the regular singular point t = 0.
- (c) Express each series solution of the original equation in terms of elementary functions.

Mathematical Model

34. Buckling of a Tapered Column In Example 4 of Section 5.2 we saw that when a constant vertical compressive force or load P was applied to a thin column of uniform cross section, the deflection y(x) was a solution of the boundary-value problem

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$
(24)

The assumption here is that the column is hinged at both ends. The column will buckle or deflect only when the compressive force is a critical load P_n .

(a) In this problem let us assume that the column is of length *L*, is hinged at both ends, has circular cross sections, and is tapered as shown in Figure 6.3.1(a). If the column, a truncated cone, has a linear taper y = cx as shown in cross section in Figure 6.3.1(b), the moment of inertia of a cross section with respect to an axis perpendicular to the *xy*-plane is $I = \frac{1}{4}\pi r^4$, where r = y and y = cx. Hence we can write $I(x) = I_0(x/b)^4$, where $I_0 = I(b) = \frac{1}{4}\pi (cb)^4$. Substituting I(x) into the differential equation in (24), we see that the deflection in this case is determined from the BVP

$$x^4 \frac{d^2 y}{dx^2} + \lambda y = 0, \quad y(a) = 0, \quad y(b) = 0,$$

where $\lambda = Pb^4/EI_0$. Use the results of Problem 33 to find the critical loads P_n for the tapered column. Use an appropriate identity to express the buckling modes $y_n(x)$ as a single function.

(b) Use a CAS to plot the graph of the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 when b = 11 and a = 1.



FIGURE 6.3.1 Tapered column in Problem 34

Discussion Problems

35. Discuss how you would define a regular singular point for the linear third-order differential equation

$$a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0.$$

36. Each of the differential equations

$$x^{3}y'' + y = 0$$
 and $x^{2}y'' + (3x - 1)y' + y = 0$

has an irregular singular point at x = 0. Determine whether the method of Frobenius yields a series solution of each differential equation about x = 0. Discuss and explain your findings

37. We have seen that x = 0 is a regular singular point of any Cauchy-Euler equation $ax^2y'' + bxy' + cy = 0$. Are the indicial equation (14) for a Cauchy-Euler equation and its auxiliary equation related? Discuss.

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6.4 SPECIAL FUNCTIONS

REVIEW MATERIAL

• Sections 6.2 and 6.3

INTRODUCTION In the *Remarks* at the end of Section 2.3 we mentioned the branch of mathematics called **special functions**. Perhaps a better title for this field of applied mathematics might be *named functions* because many of the functions studied bear proper names: Bessel functions, Legendre functions, Airy functions, Chebyshev polynomials, Hermite polynomials, Jacobi polynomials, Laguerre polynomials, Gauss' hypergeometric function, Mathieu functions, and so on. Historically, special functions were often the by-product of necessity: Someone needed a solution of a very specialized differential equation that arose from an attempt to solve a physical problem. In effect, a special function was determined or defined by the differential equation and many properties of the function could be discerned from the series form of the solution.

In this section we use the methods of Sections 6.2 and 6.3 to find solutions of two differential equations

$$x^{2}y'' + xy' + (x^{2} - \nu^{2})y = 0$$
⁽¹⁾

$$(1 - x2)y'' - 2xy' + n(n + 1)y = 0$$
(2)

that arise in advanced studies of applied mathematics, physics, and engineering. They are called, respectively, **Bessel's equation of order** ν , named after the German mathematician and astronomer **Friedrich Wilhelm Bessel** (1784–1846), and **Legendre's equation of order** n, named after the French mathematician **Adrien-Marie Legendre** (1752–1833). When we solve (1) we shall assume that $\nu \ge 0$, whereas in (2) we shall consider only the case when n in a nonnegative integer.

Solution of Bessel's Equation Because x = 0 is a regular singular point of Bessel's equation, we know that there exists at least one solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting the last expression into (1) gives

$$x^{2}y'' + xy' + (x^{2} - \nu^{2})y = \sum_{n=0}^{\infty} c_{n}(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_{n}(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_{n}x^{n+r+2} - \nu^{2}\sum_{n=0}^{\infty} c_{n}x^{n+r}$$
$$= c_{0}(r^{2} - r + r - \nu^{2})x^{r} + x^{r}\sum_{n=1}^{\infty} c_{n}[(n+r)(n+r-1) + (n+r) - \nu^{2}]x^{n} + x^{r}\sum_{n=0}^{\infty} c_{n}x^{n+2}$$
$$= c_{0}(r^{2} - \nu^{2})x^{r} + x^{r}\sum_{n=1}^{\infty} c_{n}[(n+r)^{2} - \nu^{2}]x^{n} + x^{r}\sum_{n=0}^{\infty} c_{n}x^{n+2}.$$
(3)

From (3) we see that the indicial equation is $r^2 - \nu^2 = 0$, so the indicial roots are $r_1 = \nu$ and $r_2 = -\nu$. When $r_1 = \nu$, (3) becomes

$$x^{\nu} \sum_{n=1}^{\infty} c_n n(n+2\nu) x^n + x^{\nu} \sum_{n=0}^{\infty} c_n x^{n+2}$$

$$= x^{\nu} \left[(1+2\nu) c_1 x + \sum_{n=2}^{\infty} c_n n(n+2\nu) x^n + \sum_{n=0}^{\infty} c_n x^{n+2} \right]$$

$$= x^{\nu} \left[(1+2\nu) c_1 x + \sum_{k=0}^{\infty} [(k+2)(k+2+2\nu) c_{k+2} + c_k] x^{k+2} \right] = 0.$$

Therefore by the usual argument we can write $(1 + 2\nu)c_1 = 0$ and

$$(k+2)(k+2+2\nu)c_{k+2} + c_k = 0$$

$$c_{k+2} = \frac{-c_k}{(k+2)(k+2+2\nu)}, \qquad k = 0, 1, 2, \dots$$
(4)

or

The choice $c_1 = 0$ in (4) implies that $c_3 = c_5 = c_7 = \cdots = 0$, so for $k = 0, 2, 4, \ldots$ we find, after letting k + 2 = 2n, $n = 1, 2, 3, \ldots$, that

$$c_{2n} = -\frac{c_{2n-2}}{2^2 n(n+\nu)}.$$
 (5)

Thus $c_2 = -\frac{c_0}{2^2 \cdot 1 \cdot (1+u)}$

$$c_{4} = -\frac{c_{2}}{2^{2} \cdot 2(2+\nu)} = \frac{c_{0}}{2^{4} \cdot 1 \cdot 2(1+\nu)(2+\nu)}$$

$$c_{6} = -\frac{c_{4}}{2^{2} \cdot 3(3+\nu)} = -\frac{c_{0}}{2^{6} \cdot 1 \cdot 2 \cdot 3(1+\nu)(2+\nu)(3+\nu)}$$

$$\vdots$$

$$c_{2n} = \frac{(-1)^{n}c_{0}}{2^{2n}n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \quad n = 1, 2, 3, \dots$$
(6)

It is standard practice to choose c_0 to be a specific value, namel,

$$c_0 = \frac{1}{2^{\nu} \Gamma(1 + \nu)}$$

where $\Gamma(1 + \nu)$ is the gamma function. See Appendix I. Since this latter function possesses the convenient property $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$, we can reduce the indicated product in the denominator of (6) to one term. For example,

$$\Gamma(1 + \nu + 1) = (1 + \nu)\Gamma(1 + \nu)$$

$$\Gamma(1 + \nu + 2) = (2 + \nu)\Gamma(2 + \nu) = (2 + \nu)(1 + \nu)\Gamma(1 + \nu)$$

Hence we can write (6) as

$$c_{2n} = \frac{(-1)^n}{2^{2n+\nu}n!(1+\nu)(2+\nu)\cdots(n+\nu)\Gamma(1+\nu)} = \frac{(-1)^n}{2^{2n+\nu}n!\Gamma(1+\nu+n)}$$

for $n = 0, 1, 2, \dots$

Bessel Functions of the First Kind Using the coefficients c_{2n} just obtained and $r = \nu$, a series solution of (1) is $y = \sum_{n=0}^{\infty} c_{2n} x^{2n+\nu}$. This solution is usually denoted by $J_{\nu}(x)$:

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}.$$
 (7)

If $\nu \ge 0$, the series converges at least on the interval $[0, \infty)$. Also, for the second exponent $r_2 = -\nu$ we obtain, in exactly the same manner,

$$J_{-\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}.$$
(8)

The functions $J_{\nu}(x)$ and $J_{-\nu}(x)$ are called **Bessel functions of the first kind** of order ν and $-\nu$, respectively. Depending on the value of ν , (8) may contain negative powers of *x* and hence converges on $(0, \infty)$.^{*}

^{*}When we replace x by |x|, the series given in (7) and (8) converge for $0 < |x| < \infty$.

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FIGURE 6.4.1 Bessel functions of the first kind for n = 0, 1, 2, 3, 4

Now some care must be taken in writing the general solution of (1). When $\nu = 0$, it is apparent that (7) and (8) are the same. If $\nu > 0$ and $r_1 - r_2 = \nu - (-\nu) = 2\nu$ is not a positive integer, it follows from Case I of Section 6.3 that $J_{\nu}(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of (1) on $(0, \infty)$, and so the general solution on the interval is $y = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x)$. But we also know from Case II of Section 6.3 that when $r_1 - r_2 = 2\nu$ is a positive integer, a second series solution of (1) may exist. In this second case we distinguish two possibilities. When $\nu = m$ = positive integer, $J_{-m}(x)$ defined by (8) and $J_m(x)$ are not linearly independent solutions. It can be shown that J_{-m} is a constant multiple of J_m (see Property (*i*) on page 262). In addition, $r_1 - r_2 = 2\nu$ can be a positive integer when ν is half an odd positive integer. It can be shown in this latter event that $J_{\nu}(x)$ and $J_{-\nu}(x)$ are linearly independent. In other words, the general solution of (1) on $(0, \infty)$ is

$$y = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x), \qquad \nu \neq \text{ integer.}$$
(9)

The graphs of $y = J_0(x)$ and $y = J_1(x)$ are given in Figure 6.4.1.

EXAMPLE 1 Bessel's Equation of Order $\frac{1}{2}$

By identifying $\nu^2 = \frac{1}{4}$ and $\nu = \frac{1}{2}$, we can see from (9) that the general solution of the equation $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$ on $(0, \infty)$ is $y = c_1J_{1/2}(x) + c_2J_{-1/2}(x)$.

Bessel Functions of the Second Kind If $\nu \neq$ integer, the function define by the linear combination

$$Y_{\nu}(x) = \frac{\cos \nu \pi J_{\nu}(x) - J_{-\nu}(x)}{\sin \nu \pi}$$
(10)

and the function $J_{\nu}(x)$ are linearly independent solutions of (1). Thus another form of the general solution of (1) is $y = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x)$, provided that $\nu \neq$ integer. As $\nu \rightarrow m, m$ an integer, (10) has the indeterminate form 0/0. However, it can be shown by L'Hôpital's Rule that $\lim_{\nu \to m} Y_{\nu}(x)$ exists. Moreover, the function

$$Y_m(x) = \lim_{n \to \infty} Y_\nu(x)$$

and $J_m(x)$ are linearly independent solutions of $x^2y'' + xy' + (x^2 - m^2)y = 0$. Hence for *any* value of ν the general solution of (1) on $(0, \infty)$ can be written as

$$y = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x). \tag{11}$$

 $Y_{\nu}(x)$ is called the **Bessel function of the second kind** of order ν . Figure 6.4.2 shows the graphs of $Y_0(x)$ and $Y_1(x)$.

EXAMPLE 2 Bessel's Equation of Order 3

By identifying $\nu^2 = 9$ and $\nu = 3$, we see from (11) that the general solution of the equation $x^2y'' + xy' + (x^2 - 9)y = 0$ on $(0, \infty)$ is $y = c_1J_3(x) + c_2Y_3(x)$.

DES Solvable in Terms of Bessel Functions Sometimes it is possible to transform a differential equation into equation (1) by means of a change of variable. We can then express the solution of the original equation in terms of Bessel functions. For example, if we let $t = \alpha x$, $\alpha > 0$, in

$$x^{2}y'' + xy' + (\alpha^{2}x^{2} - \nu^{2})y = 0, \qquad (12)$$

then by the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \alpha \frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \alpha^2 \frac{d^2y}{dt^2}$$



FIGURE 6.4.2 Bessel functions of the second kind for n = 0, 1, 2, 3, 4

Accordingly, (12) becomes

$$\left(\frac{t}{\alpha}\right)^{2} \alpha^{2} \frac{d^{2}y}{dt^{2}} + \left(\frac{t}{\alpha}\right) \alpha \frac{dy}{dt} + (t^{2} - \nu^{2})y = 0 \quad \text{or} \quad t^{2} \frac{d^{2}y}{dt^{2}} + t\frac{dy}{dt} + (t^{2} - \nu^{2})y = 0.$$

The last equation is Bessel's equation of order ν with solution $y = c_1 J_{\nu}(t) + c_2 Y_{\nu}(t)$. By resubstituting $t = \alpha x$ in the last expression, we find that the general solution of (12) is

$$y = c_1 J_{\nu}(\alpha x) + c_2 Y_{\nu}(\alpha x).$$
 (13)

Equation (12), called the **parametric Bessel equation of order** ν , and its general solution (13) are very important in the study of certain boundary-value problems involving partial differential equations that are expressed in cylindrical coordinates.

Modified Bessel Functions Another equation that bears a resemblance to (1) is the **modified Bessel equation of orde** ν ,

$$x^{2}y'' + xy' - (x^{2} + \nu^{2})y = 0.$$
(14)

This DE can be solved in the manner just illustrated for (12). This time if we let t = ix, where $i^2 = -1$, then (14) becomes

$$t^{2}\frac{d^{2}y}{dt^{2}} + t\frac{dy}{dt} + (t^{2} - \nu^{2})y = 0.$$

Because solutions of the last DE are $J_{\nu}(t)$ and $Y_{\nu}(t)$, *complex-valued* solutions of (14) are $J_{\nu}(ix)$ and $Y_{\nu}(ix)$. A real-valued solution, called the **modified Bessel func-**tion of the first kin of order ν , is defined in terms of $J_{\nu}(ix)$:

$$I_{\nu}(x) = i^{-\nu} J_{\nu}(ix). \tag{15}$$

See Problem 21 in Exercises 6.4.

Analogous to (10), the **modified Bessel function of the second kind** of order $\nu \neq$ integer is defined to b

$$K_{\nu}(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_{\nu}(x)}{\sin \nu \pi},$$
(16)

and for integer $\nu = n$,

$$K_n(x) = \lim_{\nu \to n} K_{\nu}(x).$$

Because I_{ν} and K_{ν} are linearly independent on the interval $(0, \infty)$ for any value of ν , the general solution of (14) on that interval is

$$y = c_1 I_{\nu}(x) + c_2 K_{\nu}(x). \tag{17}$$

The graphs of $y = I_0(x)$, $y = I_1(x)$, and $y = I_2(x)$ are given in Figure 6.4.3 and the graphs of $y = K_0(x)$, $y = K_1(x)$, and $y = K_2(x)$ are given in Figure 6.4.4. Unlike the Bessel functions of the first and second kinds, the modified Bessel functions of the first and second kind are not oscillatory. Figures 6.4.3 and 6.4.4 also illustrate the fact that the modified Bessel functions $I_n(x)$ and $K_n(x)$, n = 0, 1, 2, ... have no real zeros in the interval $(0, \infty)$. Also notice that the modified Bessel functions of the second kind $K_n(x)$ like the Bessel functions of the second kind $Y_n(x)$ become unbounded as $x \rightarrow 0^+$.

A change of variable in (14) gives us the **parametric form** of the modifie Bessel equation of order ν :

$$x^2y'' + xy' - (\alpha^2 x^2 + \nu^2)y = 0.$$

The general solution of the last equation on the interval $(0, \infty)$ is

$$y = c_1 I_{\nu}(\alpha x) + c_2 K_{\nu}(\alpha x).$$



FIGURE 6.4.3 Modified Besse functions of the first kind for n = 0, 1, 2

y

3

2

1 0.5

2.5

1.5



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Yet another equation, important because many DEs fit into its form by appropriate choices of the parameters, is

$$y'' + \frac{1 - 2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)y = 0, \qquad p \ge 0.$$
(18)

Although we shall not supply the details, the general solution of (18),

$$y = x^{a} \left[c_{1} J_{p}(bx^{c}) + c_{2} Y_{p}(bx^{c}) \right],$$
 (19)

can be found by means of a change in both the independent and the dependent variables: $z = bx^c$, $y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$. If *p* is not an integer, then Y_p in (19) can be replaced by J_{-p} .

EXAMPLE 3 Using (18)

Find the general solution of xy'' + 3y' + 9y = 0 on $(0, \infty)$.

SOLUTION By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0,$$

we can make the following identifications with (18)

1 - 2a = 3, $b^2c^2 = 9$, 2c - 2 = -1, and $a^2 - p^2c^2 = 0$.

The first and third equations imply that a = -1 and $c = \frac{1}{2}$. With these values the second and fourth equations are satisfied by taking b = 6 and p = 2. From (19) we find that the general solution of the given DE on the interval $(0, \infty)$ is $y = x^{-1}[c_1J_2(6x^{1/2}) + c_2Y_2(6x^{1/2})]$.

EXAMPLE 4 The Aging Spring Revisited

Recall that in Section 5.1 we saw that one mathematical model for the free undamped motion of a mass on an aging spring is given by $mx'' + ke^{-\alpha t}x = 0$, $\alpha > 0$. We are now in a position to find the general solution of the equation. It is left as a problem

to show that the change of variables $s = \frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}$ transforms the differential equation of the aging spring into

$$s^2 \frac{d^2x}{ds^2} + s \frac{dx}{ds} + s^2 x = 0.$$

The last equation is recognized as (1) with $\nu = 0$ and where the symbols x and s play the roles of y and x, respectively. The general solution of the new equation is $x = c_1 J_0(s) + c_2 Y_0(s)$. If we resubstitute s, then the general solution of $mx'' + ke^{-\alpha t}x = 0$ is seen to be

$$x(t) = c_1 J_0 \left(\frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}\right) + c_2 Y_0 \left(\frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}\right).$$

=

See Problems 33 and 39 in Exercises 6.4.

The other model that was discussed in Section 5.1 of a spring whose characteristics change with time was mx'' + ktx = 0. By dividing through by *m*, we see that the equation $x'' + \frac{k}{m}tx = 0$ is Airy's equation $y'' + \alpha^2 xy = 0$. See Example 5 in Section 6.2. The general solution of Airy's differential equation can also be written in terms of Bessel functions. See Problems 34, 35, and 40 in Exercises 6.4. **Properties** We list below a few of the more useful properties of Bessel functions of order m, m = 0, 1, 2, ...

(i)
$$J_{-m}(x) = (-1)^m J_m(x),$$
 (ii) $J_m(-x) = (-1)^m J_m(x),$
(iii) $J_m(0) = \begin{cases} 0, & m > 0\\ 1, & m = 0, \end{cases}$ (iv) $\lim_{x \to 0^+} Y_m(x) = -\infty.$

Note that Property (*ii*) indicates that $J_m(x)$ is an even function if *m* is an even integer and an odd function if *m* is an odd integer. The graphs of $Y_0(x)$ and $Y_1(x)$ in Figure 6.4.2 illustrate Property (*iv*), namely, $Y_m(x)$ is unbounded at the origin. This last fact is not obvious from (10). The solutions of the Bessel equation of order 0 can be obtained by using the solutions $y_1(x)$ in (21) and $y_2(x)$ in (22) of Section 6.3. It can be shown that (21) of Section 6.3 is $y_1(x) = J_0(x)$, whereas (22) of that section is

$$y_2(x) = J_0(x) \ln x - \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) \left(\frac{x}{2}\right)^{2k}$$

The Bessel function of the second kind of order 0, $Y_0(x)$, is then defined to be the linear combination $Y_0(x) = \frac{2}{\pi}(\gamma - \ln 2)y_1(x) + \frac{2}{\pi}y_2(x)$ for x > 0. That is,

$$Y_0(x) = \frac{2}{\pi} J_0(x) \left[\gamma + \ln \frac{x}{2} \right] - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{k} \right) \left(\frac{x}{2} \right)^{2k},$$

where $\gamma = 0.57721566...$ is **Euler's constant.** Because of the presence of the logarithmic term, it is apparent that $Y_0(x)$ is discontinuous at x = 0.

EXAMPLE 1 Numerical Values The first five nonnegative zeros of $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ are given in Table 6.4.1. Some additional function values of these four functions are given in Table 6.4.2.

$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
2.4048	0.0000	0.8936	2.1971
5.5201	3.8317	3.9577	5.4297
8.6537	7.0156	7.0861	8.5960
11.7915	10.1735	10.2223	11.7492
14.9309	13.3237	13.3611	14.8974

TABLE 6.4.1 Zeros of J_0 , J_1 , Y_0 , and Y_1

TABLE 6.4.2	Numerical	Values	of J_0, J_1	$, Y_0,$	and 1	Y_1
-------------	-----------	--------	---------------	----------	-------	-------

x	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0	1.0000	0.0000	_	—
1	0.7652	0.4401	0.0883	-0.7812
2	0.2239	0.5767	0.5104	-0.1070
3	-0.2601	0.3391	0.3769	0.3247
4	-0.3971	-0.0660	-0.0169	0.3979
5	-0.1776	-0.3276	-0.3085	0.1479
6	0.1506	-0.2767	-0.2882	-0.1750
7	0.3001	-0.0047	-0.0259	-0.3027
8	0.1717	0.2346	0.2235	-0.1581
9	-0.0903	0.2453	0.2499	0.1043
10	-0.2459	0.0435	0.0557	0.2490
11	-0.1712	-0.1768	-0.1688	0.1637
12	0.0477	-0.2234	-0.2252	-0.0571
13	0.2069	-0.0703	-0.0782	-0.2101
14	0.1711	0.1334	0.1272	-0.1666
15	-0.0142	0.2051	0.2055	0.0211

Differential Recurrence Relation Recurrence formulas that relate Bessel functions of different orders are important in theory and in applications. In the next example we derive a **differential recurrence relation**.

EXAMPLE 5 Derivation Using the Series Definitio

Derive the formula $xJ'_{\nu}(x) = \nu J_{\nu}(x) - xJ_{\nu+1}(x)$.

SOLUTION It follows from (7) that

$$\begin{aligned} xJ_{\nu}'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^{n}(2n+\nu)}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\ &= \nu \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} + 2\sum_{n=0}^{\infty} \frac{(-1)^{n}n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \\ &= \nu J_{\nu}(x) + x \sum_{n=1}^{\infty} \frac{(-1)^{n}}{(n-1)!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu-1} \\ &= \nu J_{\nu}(x) - x \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!\Gamma(2+\nu+k)} \left(\frac{x}{2}\right)^{2k+\nu+1} = \nu J_{\nu}(x) - x J_{\nu+1}(x). \end{aligned}$$

The result in Example 5 can be written in an alternative form. Dividing $xJ'_{\nu}(x) - \nu J_{\nu}(x) = -xJ_{\nu+1}(x)$ by x gives

$$J'_{\nu}(x) - \frac{\nu}{x} J_{\nu}(x) = -J_{\nu+1}(x).$$

This last expression is recognized as a linear first-order differential equation in $J_{\nu}(x)$. Multiplying both sides of the equality by the integrating factor $x^{-\nu}$ then yields

$$\frac{d}{dx}[x^{-\nu}J_{\nu}(x)] = -x^{-\nu}J_{\nu+1}(x).$$
(20)

It can be shown in a similar manner that

$$\frac{d}{dx}[x^{\nu}J_{\nu}(x)] = x^{\nu}J_{\nu-1}(x).$$
(21)

See Problem 27 in Exercises 6.4. The differential recurrence relations (20) and (21) are also valid for the Bessel function of the second kind $Y_{\nu}(x)$. Observe that when $\nu = 0$, it follows from (20) that

$$J'_0(x) = -J_1(x)$$
 and $Y'_0(x) = -Y_1(x)$. (22)

An application of these results is given in Problem 39 of Exercises 6.4.

Bessel Functions of Half-Integral Order When the order is half an odd integer, that is, $\pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots$, Bessel functions of the first and second kinds can be expressed in terms of the elementary functions $\sin x$, $\cos x$, and powers of x. Let's consider the case when $\nu = \frac{1}{2}$. From (7)

$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(1 + \frac{1}{2} + n)} \left(\frac{x}{2}\right)^{2n+1/2}$$

In view of the property $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$ and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ the values of $\Gamma(1 + \frac{1}{2} + n)$ for n = 0, n = 1, n = 2, and n = 3 are, respectively,

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1 + \frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1 + \frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5 \cdot 3}{2^3}\sqrt{\pi} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^3 4 \cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(1 + \frac{7}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7 \cdot 5}{2^6 \cdot 2!}\sqrt{\pi} = \frac{7 \cdot 6 \cdot 5!}{2^6 \cdot 6 \cdot 2!}\sqrt{\pi} = \frac{7!}{2^7 3!}\sqrt{\pi}.$$
general,
$$\Gamma\left(1 + \frac{1}{2} + n\right) = \frac{(2n+1)!}{2^{2n+1}n!}\sqrt{\pi}.$$

In general,

Hence
$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \frac{(2n+1)!}{2^{2n+1}n!} \sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

From (2) of Section 6.1 you should recognize that the infinite series in the last line is the Maclaurin series for $\sin x$, and so we have shown that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$
 (23)

We leave it as an exercise to show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$
 (24)

See Figure 6.4.5 and Problems 31, 32, and 38 in Exercises 6.4.

If *n* is an integer, then $\nu = n + \frac{1}{2}$ is half an odd integer. Because $\cos(n + \frac{1}{2}) \pi = 0$ and $\sin(n + \frac{1}{2})\pi = \cos n\pi = (-1)^n$, we see from (10) that $Y_{n+1/2}(x) =$ $(-1)^{n+1}J_{-(n+1/2)}(x)$. For n = 0 and n = -1 we have, in turn, $Y_{1/2}(x) = -J_{-1/2}(x)$ and $Y_{-1/2}(x) = J_{1/2}(x)$. In view of (23) and (24) these results are the same as

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \cos x$$
 (25)

(26)

and

and

Spherical Bessel Functions Bessel functions of half-integral order are used to define two more important functions:

 $Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$
 and $y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x).$ (27)

The function $j_n(x)$ is called the **spherical Bessel function of the first kind** and $y_n(x)$ is the spherical Bessel function of the second kind. For example, for n = 0 the expressions in (27) become

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \sin x = \frac{\sin x}{x}$$
$$y_0(x) = \sqrt{\frac{\pi}{2x}} Y_{1/2}(x) = -\sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \cos x = -\frac{\cos x}{x}.$$



FIGURE 6.4.5 Bessel functions of order $\frac{1}{2}$ (blue) and order $-\frac{1}{2}$ (red)

Copyright 2012 Cengage Learning. All Rights Reserved. May not be copied, scanned, or duplicated, in whole or in part. Due to electronic rights, some third party content may be suppressed from the eBook and/or eChapter(s). Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it It is apparent from (27) and Figure 6.4.2 for $n \ge 0$ the spherical Bessel of the second kind $y_n(x)$ becomes unbounded as $x \to 0^+$.

Spherical Bessel functions arise in the solution of a special partial differential equation expressed in spherical coordinates. See Problem 54 in Exercises 6.4 and Problem 13 in Exercises 13.3.

Example 1 Solution of Legendre's Equation Since x = 0 is an ordinary point of Legendre's equation (2), we substitute the series $y = \sum_{k=0}^{\infty} c_k x^k$, shift summation indices, and combine series to get

$$(1 - x^{2})y'' - 2xy' + n(n+1)y = [n(n+1)c_{0} + 2c_{2}] + [(n-1)(n+2)c_{1} + 6c_{3}]x$$
$$+ \sum_{j=2}^{\infty} [(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_{j}]x^{j} = 0$$
which implies that
$$n(n+1)c_{0} + 2c_{2} = 0$$

which implies that

$$(n-1)(n+2)c_1 + 6c_3 = 0$$

(j+2)(j+1)c_{j+2} + (n-j)(n+j+1)c_j = 0

or

$$c_{2} = -\frac{n(n+1)}{2!}c_{0}$$

$$c_{3} = -\frac{(n-1)(n+2)}{3!}c_{1}$$

$$c_{j+2} = -\frac{(n-j)(n+j+1)}{(j+2)(j+1)}c_{j}, \qquad j = 2, 3, 4, \dots$$
(28)

If we let *j* take on the values $2, 3, 4, \ldots$, the recurrence relation (28) yields

$$c_{4} = -\frac{(n-2)(n+3)}{4 \cdot 3}c_{2} = \frac{(n-2)n(n+1)(n+3)}{4!}c_{0}$$

$$c_{5} = -\frac{(n-3)(n+4)}{5 \cdot 4}c_{3} = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}c_{1}$$

$$c_{6} = -\frac{(n-4)(n+5)}{6 \cdot 5}c_{4} = -\frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}c_{0}$$

$$c_{7} = -\frac{(n-5)(n+6)}{7 \cdot 6}c_{5} = -\frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}c_{1}$$

and so on. Thus for at least |x| < 1 we obtain two linearly independent power series solutions:

$$y_{1}(x) = c_{0} \left[1 - \frac{n(n+1)}{2!} x^{2} + \frac{(n-2)n(n+1)(n+3)}{4!} x^{4} - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!} x^{6} + \cdots \right]$$
(29)
$$y_{2}(x) = c_{1} \left[x - \frac{(n-1)(n+2)}{3!} x^{3} + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^{5} - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!} x^{7} + \cdots \right].$$

Notice that if n is an even integer, the first series terminates, whereas $y_2(x)$ is an infinite series. For example, if n = 4, then

$$y_1(x) = c_0 \left[1 - \frac{4 \cdot 5}{2!} x^2 + \frac{2 \cdot 4 \cdot 5 \cdot 7}{4!} x^4 \right] = c_0 \left[1 - 10x^2 + \frac{35}{3} x^4 \right].$$

Similarly, when n is an odd integer, the series for $y_2(x)$ terminates with x^n ; that is, when n is a nonnegative integer, we obtain an nth-degree polynomial solution of Legendre's equation.

Because we know that a constant multiple of a solution of Legendre's equation is also a solution, it is traditional to choose specific values for c_0 or c_1 , depending on whether *n* is an even or odd positive integer, respectively. For n = 0 we choose $c_0 = 1$, and for n = 2, 4, 6, ...

$$c_0 = (-1)^{n/2} \frac{1 \cdot 3 \cdots (n-1)}{2 \cdot 4 \cdots n},$$

whereas for n = 1 we choose $c_1 = 1$, and for $n = 3, 5, 7, \ldots$

$$c_1 = (-1)^{(n-1)/2} \frac{1 \cdot 3 \cdots n}{2 \cdot 4 \cdots (n-1)}.$$

For example, when n = 4, we have

$$y_1(x) = (-1)^{4/2} \frac{1 \cdot 3}{2 \cdot 4} \left[1 - 10x^2 + \frac{35}{3}x^4 \right] = \frac{1}{8} (35x^4 - 30x^2 + 3).$$

Elegendre Polynomials These specific *n*th-degree polynomial solutions are called **Legendre polynomials** and are denoted by $P_n(x)$. From the series for $y_1(x)$ and $y_2(x)$ and from the above choices of c_0 and c_1 we find that the first several Legendre polynomials are

$$P_{0}(x) = 1, P_{1}(x) = x, P_{1}(x) = x, P_{2}(x) = \frac{1}{2}(3x^{2} - 1), P_{3}(x) = \frac{1}{2}(5x^{3} - 3x), (30) P_{4}(x) = \frac{1}{8}(35x^{4} - 30x^{2} + 3), P_{5}(x) = \frac{1}{8}(63x^{5} - 70x^{3} + 15x).$$

Remember, $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, ... are, in turn, particular solutions of the differential equations

The graphs, on the interval [-1, 1], of the six Legendre polynomials in (30) are given in Figure 6.4.6.

Properties You are encouraged to verify the following properties using the Legendre polynomials in (30).

(i)
$$P_n(-x) = (-1)^n P_n(x)$$

(ii) $P_n(1) = 1$ (iii) $P_n(-1) = (-1)^n$
(iv) $P_n(0) = 0$, $n \text{ odd}$ (v) $P'_n(0) = 0$, $n \text{ even}$

Property (*i*) indicates, as is apparent in Figure 6.4.6, that $P_n(x)$ is an even or odd function according to whether *n* is even or odd.

Recurrence Relation Recurrence relations that relate Legendre polynomials of different degrees are also important in some aspects of their applications. We state, without proof, the three-term recurrence relation

$$(k+1)P_{k+1}(x) - (2k+1)xP_k(x) + kP_{k-1}(x) = 0,$$
(32)

which is valid for k = 1, 2, 3, ... In (30) we listed the first six Legendre polynomials. If, say, we wish to find $P_6(x)$, we can use (32) with k = 5. This relation expresses $P_6(x)$ in terms of the known $P_4(x)$ and $P_5(x)$. See Problem 45 in Exercises 6.4.



FIGURE 6.4.6 Legendre polynomials for n = 0, 1, 2, 3, 4, 5

Another formula, although not a recurrence relation, can generate the Legendre polynomials by differentiation. **Rodrigues' formula** for these polynomials is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \qquad n = 0, 1, 2, \dots.$$
(33)

See Problem 48 in Exercises 6.4.

REMARKS

Although we have assumed that the parameter *n* in Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$, represented a nonnegative integer, in a more general setting *n* can represent any real number. Any solution of Legendre's equation is called a **Legendre function**. If *n* is *not* a nonnegative integer, then both Legendre functions $y_1(x)$ and $y_2(x)$ given in (29) are infinit series convergent on the open interval (-1, 1) and divergent (unbounded) at $x = \pm 1$. If *n* is a nonnegative integer, then as we have just seen one of the Legendre functions in (29) is a polynomial and the other is an infinite series convergent for -1 < x < 1. You should be aware of the fact that Legendre's equation possesses solutions that are bounded on the *closed* interval [-1, 1] only in the case when $n = 0, 1, 2, \ldots$. More to the point, the only Legendre functions that are bounded on the closed interval [-1, 1] are the Legendre polynomials $P_n(x)$ or constant multiples of these polynomials. See Problem 47 in Exercises 6.4 and Problem 24 in Chapter 6 in Review.

EXERCISES 6.4

Bessel's Equation

In Problems 1–6 use (1) to find the general solution of the given differential equation on $(0, \infty)$.

- 1. $x^2y'' + xy' + (x^2 \frac{1}{9})y = 0$
- **2.** $x^2y'' + xy' + (x^2 1)y = 0$
- 3. $4x^2y'' + 4xy' + (4x^2 25)y = 0$
- 4. $16x^2y'' + 16xy' + (16x^2 1)y = 0$
- 5. xy'' + y' + xy = 0

$$6. \ \frac{d}{dx}[xy'] + \left(x - \frac{4}{x}\right)y = 0$$

In Problems 7–10 use (12) to find the general solution of the given differential equation on $(0, \infty)$.

7.
$$x^2y'' + xy' + (9x^2 - 4)y = 0$$

8. $x^2y'' + xy' + (36x^2 - \frac{1}{4})y = 0$

Answers to selected odd-numbered problems begin on page ANS-11.

9.
$$x^2y'' + xy' + (25x^2 - \frac{4}{9})y = 0$$

10. $x^2y'' + xy' + (2x^2 - 64)y = 0$

In Problems 11 and 12 use the indicated change of variable to find the general solution of the given differential equation on $(0, \infty)$.

11.
$$x^2y'' + 2xy' + \alpha^2 x^2 y = 0; \quad y = x^{-1/2}v(x)$$

12. $x^2y'' + (\alpha^2 x^2 - \nu^2 + \frac{1}{4})y = 0; \quad y = \sqrt{x}v(x)$

In Problems 13–20 use (18) to find the general solution of the given differential equation on $(0, \infty)$.

13.
$$xy'' + 2y' + 4y = 0$$

14. $xy'' + 3y' + xy = 0$
15. $xy'' - y' + xy = 0$
16. $xy'' - 5y' + xy = 0$
17. $x^2y'' + (x^2 - 2)y = 0$
18. $4x^2y'' + (16x^2 + 1)y = 0$

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19. $xy'' + 3y' + x^3y = 0$

20. $9x^2y'' + 9xy' + (x^6 - 36)y = 0$

- **21.** Use the series in (7) to verify that $I_{\nu}(x) = i^{-\nu} J_{\nu}(ix)$ is a real function.
- **22.** Assume that *b* in equation (18) can be pure imaginary, that is, $b = \beta i, \beta > 0, i^2 = -1$. Use this assumption to express the general solution of the given differential equation in terms the modified Bessel functions I_n and K_n .

(a)
$$y'' - x^2 y = 0$$

(b) $xy'' + y' - 7x^3 y = 0$

In Problems 23-26 first use (18) to express the general solution of the given differential equation in terms of Bessel functions. Then use (23) and (24) to express the general solution in terms of elementary functions.

- **23.** y'' + y = 0
- **24.** $x^2y'' + 4xy' + (x^2 + 2)y = 0$
- **25.** $16x^2y'' + 32xy' + (x^4 12)y = 0$
- **26.** $4x^2y'' 4xy' + (16x^2 + 3)y = 0$
- 27. (a) Proceed as in Example 5 to show that

$$xJ'_{\nu}(x) = -\nu J_{\nu}(x) + xJ_{\nu-1}(x).$$

[*Hint*: Write $2n + \nu = 2(n + \nu) - \nu$.]

- (b) Use the result in part (a) to derive (21).
- **28.** Use the formula obtained in Example 5 along with part (a) of Problem 27 to derive the recurrence relation

$$2\nu J_{\nu}(x) = x J_{\nu+1}(x) + x J_{\nu-1}(x)$$

In Problems 29 and 30 use (20) or (21) to obtain the given result.

29.
$$\int_0^x r J_0(r) dr = x J_1(x)$$
 30. $J_0'(x) = J_{-1}(x) = -J_1(x)$

- **31.** Proceed as on page 264 to derive the elementary form of $J_{-1/2}(x)$ given in (24).
- **32.** Use the recurrence relation in Problem 28 along with (23) and (24) to express $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$ and $J_{-5/2}(x)$ in terms of sin *x*, cos *x*, and powers of *x*.

33. Use the change of variables
$$s = \frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}$$
 to show

that the differential equation of the aging spring $mx'' + ke^{-\alpha t}x = 0, \alpha > 0$, becomes

$$s^2 \frac{d^2x}{ds^2} + s \frac{dx}{ds} + s^2 x = 0.$$

- **34.** Show that $y = x^{1/2}w(\frac{2}{3}\alpha x^{3/2})$ is a solution of Airy's differential equation $y'' + \alpha^2 xy = 0$, x > 0, whenever w is a solution of Bessel's equation of order $\frac{1}{3}$, that is, $t^2w'' + tw' + (t^2 \frac{1}{9})w = 0$, t > 0. [*Hint*: After differentiating, substituting, and simplifying, then let $t = \frac{2}{3}\alpha x^{3/2}$.]
- **35.** (a) Use the result of Problem 34 to express the general solution of Airy's differential equation for x > 0 in terms of Bessel functions.
 - (b) Verify the results in part (a) using (18).
- **36.** Use the Table 6.4.1 to find the first three positive eigenvalues and corresponding eigenfunctions of the boundary-value problem

$$xy'' + y' + \lambda xy = 0,$$

y(x), y'(x) bounded as $x \rightarrow 0^+, y(2) = 0.$

[*Hint*: By identifying $\lambda = \alpha^2$, the DE is the parametric Bessel equation of order zero.]

37. (a) Use (18) to show that the general solution of the differential equation xy" + λy = 0 on the interval (0,∞) is

$$y = c_1 \sqrt{x} J_1 \left(2\sqrt{\lambda x} \right) + c_2 \sqrt{x} Y_1 \left(2\sqrt{\lambda x} \right).$$

(b) Verify by direct substitution that $y = \sqrt{x}J_1(2\sqrt{x})$ is a particular solution of the DE in the case $\lambda = 1$.

Computer Lab Assignments

- **38.** Use a CAS to graph $J_{3/2}(x)$, $J_{-3/2}(x)$, $J_{5/2}(x)$, and $J_{-5/2}(x)$.
- **39.** (a) Use the general solution given in Example 4 to solve the IVP

$$4x'' + e^{-0.1t}x = 0$$
, $x(0) = 1$, $x'(0) = -\frac{1}{2}$.

Also use $J'_0(x) = -J_1(x)$ and $Y'_0(x) = -Y_1(x)$ along with Table 6.4.1 or a CAS to evaluate coefficients

- (b) Use a CAS to graph the solution obtained in part (a) for $0 \le t \le \infty$.
- **40.** (a) Use the general solution obtained in Problem 35 to solve the IVP

$$4x'' + tx = 0$$
, $x(0.1) = 1$, $x'(0.1) = -\frac{1}{2}$.

Use a CAS to evaluate coefficients

- (b) Use a CAS to graph the solution obtained in part (a) for $0 \le t \le 200$.
- **41.** Column Bending Under Its Own Weight A uniform thin column of length *L*, positioned vertically with one

end embedded in the ground, will deflect, or bend away, from the vertical under the influence of its own weight when its length or height exceeds a certain critical value. It can be shown that the angular deflection $\theta(x)$ of the column from the vertical at a point P(x) is a solution of the boundary-value problem:

$$EI\frac{d^2\theta}{dx^2} + \delta g(L-x)\theta = 0, \quad \theta(0) = 0, \quad \theta'(L) = 0,$$

where *E* is Young's modulus, *I* is the cross-sectional moment of inertia, δ is the constant linear density, and *x* is the distance along the column measured from its base. See Figure 6.4.7. The column will bend only for those values of *L* for which the boundary-value problem has a nontrivial solution.

- (a) Restate the boundary-value problem by making the change of variables t = L x. Then use the results of a problem earlier in this exercise set to express the general solution of the differential equation in terms of Bessel functions.
- (b) Use the general solution found in part (a) to find a solution of the BVP and an equation which define the critical length L, that is, the smallest value of L for which the column will start to bend.
- (c) With the aid of a CAS, find the critical length L of a solid steel rod of radius r = 0.05 in., $\delta g = 0.28 A$ lb/in., $E = 2.6 \times 10^7$ lb/in.², $A = \pi r^2$, and $I = \frac{1}{4}\pi r^4$.



FIGURE 6.4.7 Beam in Problem 41

42. Buckling of a Thin Vertical Column In Example 4 of Section 5.2 we saw that when a constant vertical compressive force, or load, P was applied to a thin column of uniform cross section and hinged at both ends, the deflection y(x) is a solution of the BVP:

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

(a) If the bending stiffness factor *EI* is proportional to *x*, then EI(x) = kx, where *k* is a constant of proportionality. If EI(L) = kL = M is the maximum stiffness factor, then k = M/L and so EI(x) = Mx/L.

Use the information in Problem 37 to find a solution of

$$M\frac{x}{L}\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

if it is known that $\sqrt{x} Y_1(2\sqrt{\lambda x})$ is *not* zero at x = 0.

- (b) Use Table 6.4.1 to find the Euler load P_1 for the column.
- (c) Use a CAS to graph the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 . For simplicity assume that $c_1 = 1$ and L = 1.
- **43.** Pendulum of Varying Length For the simple pendulum described on page 220 of Section 5.3, suppose that the rod holding the mass m at one end is replaced by a flexible wire or string and that the wire is strung over a pulley at the point of support O in Figure 5.3.3. In this manner, while it is in motion in a vertical plane, the mass m can be raised or lowered. In other words, the length l(t) of the pendulum varies with time. Under the same assumptions leading to equation (6) in Section 5.3, it can be shown* that the differential equation for the displacement angle θ is now

$$l\theta'' + 2l'\theta' + g\sin\theta = 0.$$

(a) If *l* increases at constant rate *v* and if $l(0) = l_0$, show that a linearization of the foregoing DE is

$$(l_0 + vt)\theta'' + 2v\theta' + g\theta = 0.$$
(34)

(b) Make the change of variables $x = (l_0 + vt)/v$ and show that (34) becomes

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{g}{vx}\theta = 0.$$

- (c) Use part (b) and (18) to express the general solution of equation (34) in terms of Bessel functions.
- (d) Use the general solution obtained in part (c) to solve the initial-value problem consisting of equation (34) and the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$. [*Hints*: To simplify calculations, use a further change of variable $u = \frac{2}{2}\sqrt{g(l_0 + vt)} = 2\sqrt{\frac{g}{2}}x^{1/2}$.

hange of variable
$$u = \frac{2}{v}\sqrt{g(l_0 + vt)} = 2\sqrt{\frac{g}{v}}x^{1/2}$$
.

Also, recall that (20) holds for both $J_1(u)$ and $Y_1(u)$. Finally, the identity

$$J_1(u)Y_2(u) - J_2(u)Y_1(u) = -\frac{2}{\pi u}$$

will be helpful.]

(problem continues on page 270)

^{*}See *Mathematical Methods in Physical Sciences*, Mary Boas, John Wiley & Sons, Inc., 1966. Also see the article by Borelli, Coleman, and Hobson in *Mathematics Magazine*, vol. 58, no. 2, March 1985.

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- (e) Use a CAS to graph the solution $\theta(t)$ of the IVP in part (d) when $l_0 = 1$ ft, $\theta_0 = \frac{1}{10}$ radian, and $v = \frac{1}{60}$ ft/s. Experiment with the graph using different time intervals such as [0, 10], [0, 30], and so on.
- (f) What do the graphs indicate about the displacement angle $\theta(t)$ as the length *l* of the wire increases with time?

Legendre's Equation

- 44. (a) Use the explicit solutions $y_1(x)$ and $y_2(x)$ of Legendre's equation given in (29) and the appropriate choice of c_0 and c_1 to find the Legendre polynomials $P_6(x)$ and $P_7(x)$.
 - (b) Write the differential equations for which $P_6(x)$ and $P_7(x)$ are particular solutions.
- **45.** Use the recurrence relation (32) and $P_0(x) = 1$, $P_1(x) = x$, to generate the next six Legendre polynomials.
- 46. Show that the differential equation

$$\sin\theta \frac{d^2y}{d\theta^2} + \cos\theta \frac{dy}{d\theta} + n(n+1)(\sin\theta)y = 0$$

can be transformed into Legendre's equation by means of the substitution $x = \cos \theta$.

47. Find the first three positive values of λ for which the problem

$$(1 - x^2)y'' - 2xy' + \lambda y = 0,$$

y(0) = 0, y(x), y'(x) bounded on [-1,1]

has nontrivial solutions.

Computer Lab Assignments

- **48.** For purposes of this problem ignore the list of Legendre polynomials given on page 266 and the graphs given in Figure 6.4.3. Use Rodrigues' formula (33) to generate the Legendre polynomials $P_1(x), P_2(x), \ldots, P_7(x)$. Use a CAS to carry out the differentiations and simplifications
- **49.** Use a CAS to graph $P_1(x)$, $P_2(x)$, ..., $P_7(x)$ on the interval [-1, 1].
- **50.** Use a root-findin application to fin the zeros of $P_1(x)$, $P_2(x)$, ..., $P_7(x)$. If the Legendre polynomials are built-in functions of your CAS, fin zeros of Legendre polynomials of higher degree. Form a conjecture about the location of the zeros of any Legendre polynomial $P_n(x)$, and then investigate to see whether it is true.

Miscellaneous Differential Equations

51. The differential equation

$$y'' - 2xy' + 2\alpha y = 0$$

is known as **Hermite's equation of order** α after the French mathematician **Charles Hermite** (1822–1901). Show that the general solution of the equation is $y(x) = c_0y_1(x) + c_1y_2(x)$, where

$$y_1(x) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^k \alpha (\alpha - 2) \dots (\alpha - 2k + 2)}{(2k)!} x^{2k}$$
$$y_2(x) = x + \sum_{k=1}^{\infty} (-1)^k \frac{2^k (\alpha - 1)(\alpha - 3) \dots (\alpha - 2k + 1)}{(2k + 1)!} x^{2k+1}$$

are power series solutions centered at the ordinary point 0.

- **52.** (a) When $\alpha = n$ is a nonnegative integer, Hermite's differential equation always possesses a polynomial solution of degree *n*. Use $y_1(x)$, given in Problem 51, to find polynomial solutions for n = 0, n = 2, and n = 4. Then use $y_2(x)$ to find polynomial solutions for n = 1, n = 3, and n = 5.
 - (b) A Hermite polynomial $H_n(x)$ is defined to be the *n*th degree polynomial solution of Hermite's equation multiplied by an appropriate constant so that the coefficient of x^n in $H_n(x)$ is 2^n . Use the polynomial solutions in part (a) to show that the first six Hermite polynomials are

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = 32x^5 - 160x^3 + 120x.$$

53. The differential equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a parameter, is known as **Chebyshev's** equation after the Russian mathematician **Pafnuty Chebyshev** (1821–1894). When $\alpha = n$ is a nonnegative integer, Chebyshev's differential equation always possesses a polynomial solution of degree *n*. Find a fifth degree polynomial solution of this differential equation.

54. If *n* is an integer, use the substitution $R(x) = (\alpha x)^{-1/2}Z(x)$ to show that the general solution of the differential equation

$$x^{2}R'' + 2xR' + [\alpha^{2}x^{2} - n(n+1)]R = 0$$

on the interval $(0, \infty)$ is $R(x) = c_1 j_n(\alpha x) + c_2 y_n(\alpha x)$, where $j_n(\alpha x)$ and $y_n(\alpha x)$ are the spherical Bessel functions of the first and second kind defined in (27)

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