C H A P T E R 9

Series Solutions of Differential Equations

Various analytical methods have been presented in previous chapters for solving ordinary differential equations to obtain exact solutions. However, in applied mathematics, science, and engineering applications, there are a large number of differential equations, especially those with variable coefficients, that cannot be solved exactly in terms of elementary functions, such as exponential, logarithmic, and trigonometric functions. For many of these differential equations, it is possible to find solutions in terms of series.

For example, Bessel's differential equation of the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)\,y = 0,$$

where ν is an arbitrary real or complex number, finds many applications in engineering disciplines. Some examples include heat conduction in a cylindrical object, vibration of a thin circular or annular membrane, and electromagnetic waves in a cylindrical waveguide. Bessel's equation cannot be solved exactly in terms of elementary functions; it can be solved using series, which were first defined by Daniel Bernoulli and then generalized by Friedrich Bessel and are known as Bessel functions.

The objective of this chapter is to present the essential techniques for solving such ordinary differential equations, in particular second-order linear ordinary differential equations with variable coefficients.

Before explaining how series can be used to solve ordinary differential equations, some relevant results on power series are briefly reviewed in the following section.

9.1 Review of Power Series

Definition — Power Series

A *power series* is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots, \qquad (1)$$

where $a_0, a_1, a_2, \ldots$ are constants, and x_0 is a fixed number.

This series usually arises as the Taylor series of some function $f(x)$. If $x_0 = 0$, the power series becomes

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Convergence of a Power Series

Power series (1) is *convergent* at x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n$$

exists and is finite. Otherwise, the power series is *divergent*. A power series will converge for some values of x and may diverge for other values. Series (1) is always convergent at $x = x_0$.

If power series (1) is convergent for all x in the interval $|x-x_0| < r$ and is divergent whenever $|x-x_0| > r$, where $0 \leqslant r \leqslant \infty$, then r is called the *radius of convergence* of the power series.

The radius of convergence r is given by

$$r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

if this limit exists.

Four very important power series are

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad -\infty < x < \infty,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad -\infty < x < \infty,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad -\infty < x < \infty.$$

Operations of Power Series

Suppose functions $f(x)$ and $g(x)$ can be expanded into power series as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad \text{for } |x-x_0| < r_1,$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n, \qquad \text{for } |x-x_0| < r_2.$$

Then, for $|x-x_0| < r$, $r = \min(r_1, r_2)$,

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n,$$

i.e., the power series of the sum or difference of the functions can be obtained by termwise addition and substraction. For multiplication,

$$f(x)\,g(x) = \left[\sum_{m=0}^{\infty} a_m (x-x_0)^m\right]\left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m b_n (x-x_0)^{m+n}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_m b_{n-m}\right)(x-x_0)^n,$$

and for division,

$$\frac{f(x)}{g(x)} = \frac{\sum\limits_{n=0}^{\infty} a_n (x-x_0)^n}{\sum\limits_{n=0}^{\infty} b_n (x-x_0)^n} = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$\Longrightarrow \quad \sum_{n=0}^{\infty} a_n (x-x_0)^n = \left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right]\left[\sum_{n=0}^{\infty} c_n (x-x_0)^n\right],$$

in which c_n can be obtained by expanding the right-hand side and comparing coefficients of $(x-x_0)^n$, $n=0,1,2,\ldots$.

If the power series of $f(x)$ is convergent in the interval $|x-x_0| < r_1$, then $f(x)$ is continuous and has continuous derivatives of all orders in this interval. The derivatives can be obtained by differentiating the power series termwise

$$f'(x) = \sum_{n=1}^{\infty} a_n n (x-x_0)^{n-1}, \qquad \text{for } |x-x_0| < r_1.$$

The integral of $f(x)$ can be obtained by integrating the power series termwise

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n (x-x_0)^{n+1}}{n+1} + C, \qquad \text{for } |x-x_0| < r_1,$$

$$\xrightarrow{n+1=m} \sum_{m=1}^{\infty} \frac{a_{m-1}(x-x_0)^m}{m} + C. \qquad \text{Change the index of summation.}$$

Definition — Analytic Function

A function $f(x)$ defined in the interval I containing x_0 is said to be *analytic* at x_0 if $f(x)$ can be expressed as a power (Taylor) series $f(x) = \sum\limits_{n=0}^{\infty} a_n (x-x_0)^n$, which has a positive radius of convergence.

Example 9.1

Determine the radius of convergence for

(1) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n}(x-1)^n$ (2) $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}x^n$

(1) $a_n = \dfrac{1}{2^n n}$, $\quad a_{n+1} = \dfrac{1}{2^{n+1}(n+1)}$,

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\dfrac{1}{2^n n}}{\dfrac{1}{2^{n+1}(n+1)}} = \lim_{n\to\infty}\frac{2^{n+1}(n+1)}{2^n n}$$

$$= \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right) = 2.$$

(2) $a_n = \dfrac{(n!)^3}{(3n)!}$, $\quad a_{n+1} = \dfrac{[(n+1)!]^3}{[3(n+1)]!}$,

$$r = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\dfrac{(n!)^3}{(3n)!}}{\dfrac{[(n+1)\cdot n!]^3}{(3n+3)!}}$$

$$= \lim_{n\to\infty}\left[\frac{(n!)^3}{(3n)!}\,\frac{(3n+3)(3n+2)(3n+1)(3n)!}{(n+1)^3(n!)^3}\right]$$

$$= \lim_{n\to\infty}\frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = \lim_{n\to\infty}\frac{27(n+1)\left(n+\frac{2}{3}\right)\left(n+\frac{1}{3}\right)}{(n+1)^3} = 27.$$

Example 9.2

Expand $\dfrac{1}{x(x+1)}$ as a power series in $x-1$.

Letting $t = x-1$ yields

$$\frac{1}{x(x+1)} = \frac{1}{(t+1)(t+2)} = \frac{1}{1+t} - \frac{1}{2+t} = \frac{1}{1+t} - \frac{1}{2}\,\frac{1}{1+\frac{t}{2}}$$

$$= \sum_{n=0}^{\infty}(-t)^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{t}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right)t^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \Big(1 - \frac{1}{2^{n+1}}\Big)(x-1)^n.$$

Since the interval of convergence of the power series of $\dfrac{1}{1+t}$ is $-1<t<1$ and that of $\dfrac{1}{1+t/2}$ is $-2<t<2$, hence the region of convergence of the power series of $\dfrac{1}{x(x+1)}$ is $-1<t<1$ or $0<(x=t+1)<2$.

Example 9.3

Expand $\dfrac{1}{(1-x)^3}$ as a power series in x.

$$\frac{1}{(1-x)^3} = \frac{1}{2}\Big(\frac{1}{1-x}\Big)'' = \frac{1}{2}\Big(\sum_{n=0}^{\infty} x^n\Big)'', \quad -1<x<1,$$

$$= \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad -1<x<1,$$

$$\xrightarrow{n-2=m} \frac{1}{2}\sum_{m=0}^{\infty} (m+2)(m+1)x^m.$$ Change the index of summation.

Example 9.4

Expand $\ln(1+x)$ as a power series in x.

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \Big[\sum_{n=0}^{\infty} (-1)^n x^n\Big]\,dx, \quad -1<x<1,$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \xrightarrow{n+1=m} \sum_{m=1}^{\infty} (-1)^{m-1}\frac{x^m}{m}, \quad -1<x<1.$$

9.2 Series Solution about an Ordinary Point

Two simple ordinary differential equations with closed-form solutions are considered first as motivating examples.

Motivating Example 1

Consider the first-order ordinary differential equation

$$y' - y = 0.$$

Let the solution of the equation be in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x|<r,$$

for some $r>0$, where a_n are constants to be determined. Differentiating $y(x)$ with respect to x yields

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1} \xrightarrow{n-1=m} \sum_{m=0}^{\infty} a_{m+1}(m+1)x^m.$$ Change the index of summation.

Substituting into the differential equation leads to

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \implies \sum_{n=0}^{\infty} \big[(n+1)a_{n+1} - a_n\big]x^n = 0.$$

For this equation to be true, the coefficient of x^n, $n=0,1,\ldots$, must be zero:

$$x^0: \quad a_1 - a_0 = 0 \implies a_1 = a_0,$$

$$x^1: \quad 2a_2 - a_1 = 0 \implies a_2 = \frac{1}{2}a_1 = \frac{1}{2!}a_0,$$

$$x^2: \quad 3a_3 - a_2 = 0 \implies a_3 = \frac{1}{3}a_2 = \frac{1}{3}\cdot\frac{1}{2!}a_0 = \frac{1}{3!}a_0,$$

$$\vdots$$

$$x^n: \quad (n+1)a_{n+1} - a_n = 0 \implies a_{n+1} = \frac{1}{n+1}a_n = \frac{1}{n+1}\cdot\frac{1}{n!}a_0 = \frac{1}{(n+1)!}a_0.$$

Hence, the solution is

$$y(x) = a_0 + a_0 x + \frac{1}{2!}a_0 x^2 + \frac{1}{3!}a_0 x^3 + \cdots + \frac{1}{n!}a_0 x^2 + \cdots$$

$$= a_0\Big(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots\Big)$$

$$= a_0 e^x, \quad a_0 \text{ is an arbitrary constant,}$$

which recovers the general solution of $y'-y=0$.

Motivating Example 2

Consider the second-order ordinary differential equation

$$y'' + y = 0.$$

Suppose that the solution is in the form of a power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x|<r$, for some $r>0$, where a_n are constants to be determined. Differentiating $y(x)$ with respect to x twice yields

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1},$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} \xrightarrow{n-2=m} \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m.$$

Substituting into the differential equation leads to

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \implies \sum_{n=0}^{\infty} \big[(n+2)(n+1)a_{n+1} + a_n\big]x^n = 0.$$

For this equation to be true, the coefficient of x^n, $n=0,1,\ldots$, must be zero:

$$\begin{aligned}
x^0:&\quad 2\cdot 1\, a_2 + a_0 = 0 \implies a_2 = -\frac{1}{2!}a_0,\\
x^1:&\quad 3\cdot 2\, a_3 + a_1 = 0 \implies a_3 = -\frac{1}{3!}a_1,\\
x^2:&\quad 4\cdot 3\, a_4 + a_2 = 0 \implies a_4 = -\frac{1}{4\cdot 3}a_2 = \frac{1}{4!}a_0,\\
x^3:&\quad 5\cdot 4\, a_5 + a_3 = 0 \implies a_5 = -\frac{1}{5\cdot 4}a_3 = \frac{1}{5!}a_1,\\
x^4:&\quad 6\cdot 5\, a_6 + a_4 = 0 \implies a_6 = -\frac{1}{6\cdot 5}a_4 = -\frac{1}{6!}a_0,\\
x^5:&\quad 7\cdot 6\, a_7 + a_5 = 0 \implies a_7 = -\frac{1}{7\cdot 6}a_5 = -\frac{1}{7!}a_1,\\
&\quad \vdots
\end{aligned}$$

In general, for $k=1,2,3,\ldots$,

$$a_{2k} = (-1)^k \frac{1}{(2k)!}a_0, \qquad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}a_1.$$

Hence, the solution is

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\\
&= a_0 \cos x + a_1 \sin x, \qquad a_0 \text{ and } a_1 \text{ are arbitrary constants,}
\end{aligned}$$

which recovers the general solution of $y''+y=0$.

Remarks: These two examples show that it is possible to solve an ordinary differential equation using power series.

Definition — Ordinary Point

Consider the nth-order linear ordinary differential equation

$$y^{(n)}(x) + p_{n-1}(x)\,y^{(n-1)}(x) + p_{n-2}(x)\,y^{(n-2)}(x) + \cdots + p_0(x)\,y(x) = f(x).$$

A point x_0 is called an *ordinary point* of the given differential equation if each of the coefficients $p_0(x)$, $p_1(x)$, …, $p_{n-1}(x)$ and $f(x)$ is analytic at $x=x_0$, i.e., $p_i(x)$, for $i=0,1,\ldots,n-1$, and $f(x)$ can be expressed as power series about x_0 that are convergent for $|x-x_0|<r$, $r>0$,

$$p_i(x) = \sum_{n=0}^{\infty} p_{i,n}(x-x_0)^n, \qquad f(x) = \sum_{n=0}^{\infty} f_n(x-x_0)^n.$$

Theorem — Series Solution about an Ordinary Point

Suppose that x_0 is an ordinary point of the nth-order linear ordinary differential equation

$$y^{(n)} + p_{n-1}(x)\,y^{(n-1)} + p_{n-2}(x)\,y^{(n-2)} + \cdots + p_0(x)\,y = f(x),$$

i.e., the coefficients $p_0(x)$, $p_1(x)$, …, $p_{n-1}(x)$ and $f(x)$ are all analytic at $x=x_0$ and each can be expressed as a power series about x_0 convergent for $|x-x_0|<r$, $r>0$. Then every solution of this differential equation can be expanded in one and only one way as a power series in $(x-x_0)$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n, \qquad |x-x_0|<R,$$

where the radius of convergence $R \geqslant r$.

Example 9.5 — Legendre Equation

Find the power series solution in x of the Legendre equation

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0, \qquad p>0.$$

The differential equation can be written as

$$y'' + p_1(x)y' + p_0(x)y = 0, \qquad p_1(x) = -\frac{2x}{1-x^2}, \quad p_0(x) = \frac{p(p+1)}{1-x^2}.$$

Both $p_1(x)$ and $p_0(x)$ can be expanded in power series as

$$p_1(x) = -2x\cdot\frac{1}{1-x^2} = -2x\sum_{n=0}^{\infty}(x^2)^n = -2\sum_{n=0}^{\infty}x^{2n+1}, \qquad |x|<1,$$

$$p_0(x) = p(p+1)\cdot\frac{1}{1-x^2} = p(p+1)\sum_{n=0}^{\infty}(x^2)^n = p(p+1)\sum_{n=0}^{\infty}x^{2n}, \qquad |x|<1.$$

Hence, $x=0$ is an ordinary point and a unique power series solution exists

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad |x|<1,$$

where a_n, $n=0,1,\ldots$, are constants to be determined. Differentiating $y(x)$ with respect to x yields

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting y, y', and y'' into the differential equation yields

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0,$$

or, noting that

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2} \xrightarrow{n-2=m} \sum_{m=0}^{\infty} (m+2)(m+1)\,a_{m+2} x^m, \quad \text{Change the index of summation.}$$

one has

$$\sum_{n=0}^{\infty} (n+2)(n+1)\,a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)\,a_n x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} p(p+1)\,a_n x^n = 0.$$

For this equation to be true, the coefficient of x^n, $n=0,1,\ldots$, must be zero:

$$x^0:\quad 2\cdot 1\,a_2 + p(p+1)\,a_0 = 0 \quad\Longrightarrow\quad a_2 = -\frac{p(p+1)}{2!}\,a_0,$$

$$x^1:\quad 3\cdot 2\,a_3 - 2a_1 + p(p+1)\,a_1 = 0 \quad\Longrightarrow\quad a_3 = -\frac{(p-1)(p+2)}{3!}\,a_1.$$

For $n \geqslant 2$, the coefficient of x^n gives

$$(n+2)(n+1)\,a_{n+2} - n(n-1)\,a_n - 2n a_n + p(p+1)\,a_n = 0$$

$$\Longrightarrow\quad a_{n+2} = -\frac{(p-n)[p+(n+1)]}{(n+2)(n+1)}\,a_n.$$

Hence,

$$x^2:\quad a_4 = -\frac{(p-2)(p+3)}{4\cdot 3}\,a_2 = -\frac{(p-2)(p+3)}{4\cdot 3}\left[-\frac{p(p+1)}{2!}\,a_0\right]$$

$$= (-1)^2\,\frac{p(p+1)(p-2)(p+3)}{4!}\,a_0,$$

$$x^3:\quad a_5 = -\frac{(p-3)(p+4)}{5\cdot 4}\,a_3 = -\frac{(p-3)(p+4)}{5\cdot 4}\left[-\frac{(p-1)(p+2)}{3!}\,a_1\right]$$

$$= (-1)^2\,\frac{(p-1)(p+2)(p-3)(p+4)}{5!}\,a_1,$$

$$x^4:\quad a_6 = -\frac{(p-4)(p+5)}{6\cdot 5}\,a_4$$

$$= -\frac{(p-4)(p+5)}{6\cdot 5}\left[(-1)^2\,\frac{p(p+1)(p-2)(p+3)}{4!}\,a_0\right]$$

$$= (-1)^3\,\frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!}\,a_0,$$

$$x^5:\quad a_7 = -\frac{(p-5)(p+6)}{7\cdot 6}\,a_5$$

$$= -\frac{(p-5)(p+6)}{7\cdot 6}\left[(-1)^2\,\frac{(p-1)(p+2)(p-3)(p+4)}{5!}\,a_1\right]$$

$$= (-1)^3\,\frac{(p-1)(p+2)(p-3)(p+4)(p-5)(p+6)}{7!}\,a_1,$$

... ...

In general,

$$a_{2k} = (-1)^k\,\frac{p(p+1)(p-2)(p+3)\cdots(p-2k+2)(p+2k-1)}{(2k)!}\,a_0$$

$$= \frac{(-1)^k}{(2k)!}\prod_{i=1}^{k}\big[(p-2i+2)(p+2i-1)\big]\,a_0,$$

$$a_{2k+1} = (-1)^k\,\frac{(p-1)(p+2)(p-3)(p+4)\cdots(p-2k+1)(p+2k)}{(2k+1)!}\,a_1$$

$$= \frac{(-1)^k}{(2k+1)!}\prod_{i=1}^{k}\big[(p-2i+1)(p+2i)\big]\,a_1.$$

Thus, the power series solution of Legendre equation is

$$y(x) = a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\prod_{i=1}^{k}\big[(p-2i+2)(p+2i-1)\big]\,x^{2k}$$

$$+\,a_1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\prod_{i=1}^{k}\big[(p-2i+1)(p+2i)\big]\,x^{2k+1}, \quad |x|<1,$$

where a_0 and a_1 are arbitrary constants.

Example 9.6

Find the power series solution in x of the equation $xy''+y\ln(1-x)=0$, $|x|<1$.

The differential equation can be written as

$$y'' + \frac{\ln(1-x)}{x}\,y = 0, \quad |x|<1,$$

which is of the form

$$y'' + p_1(x)y' + p_0(x)y = 0, \qquad p_1(x) = 0, \quad p_0(x) = \frac{\ln(1-x)}{x}.$$

Since

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x|<1,$$

integrating both sides of the equation with respect to x yields

$$\ln(1-x) = -\int\frac{1}{1-x}\,\mathrm{d}x = -\int\sum_{n=0}^{\infty}x^n\,\mathrm{d}x = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}, \quad |x|<1,$$

$$\therefore\quad p_0(x) = \frac{\ln(1-x)}{x} = -\sum_{n=0}^{\infty}\frac{x^n}{n+1}, \quad |x|<1.$$

Hence, both $p_0(x)$ and $p_1(x)$ can be expanded in power series, leading to $x=0$ being an ordinary point. The solution of the differential equation can be expressed in a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad |x| < 1,$$

where a_n, $n=0, 1, \ldots$, are constants to be determined. Differentiating $y(x)$ with respect to x gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \quad |x| < 1.$$

Substituting into the differential equation results in

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} \frac{x^n}{n+1} \cdot \sum_{n=0}^{\infty} a_n x^n = 0.$$

Noting that

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \xrightarrow{n-2=m} \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m,$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1} \cdot \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{x^m}{m+1} \cdot a_{n-m} x^{n-m} \right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right) x^n,$$

one obtains

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - \sum_{m=0}^{n} \frac{a_{n-m}}{m+1} \right] x^n = 0.$$

For this equation to be true, the coefficient of x^n, $n=0, 1, \ldots$, must be zero:

$$a_{n+2} = \frac{1}{(n+2)(n+1)} \sum_{m=0}^{n} \frac{a_{n-m}}{m+1}.$$

Hence,

$$n=0:\ a_2 = \frac{1}{2 \cdot 1} a_0 = \frac{a_0}{2},$$

$$n=1:\ a_3 = \frac{1}{3 \cdot 2} \left(a_1 + \frac{a_0}{2} \right) = \frac{a_0}{12} + \frac{a_1}{6},$$

$$n=2:\ a_4 = \frac{1}{4 \cdot 3} \left(a_2 + \frac{a_1}{2} + \frac{a_0}{3} \right) = \frac{1}{12} \left(\frac{a_0}{2} + \frac{a_1}{2} + \frac{a_0}{3} \right) = \frac{5a_0}{72} + \frac{a_1}{24},$$

$$n=3:\ a_5 = \frac{1}{5 \cdot 4} \left(a_3 + \frac{a_2}{2} + \frac{a_1}{3} + \frac{a_0}{4} \right) = \frac{1}{20} \left[\left(\frac{a_0}{12} + \frac{a_1}{6} \right) + \frac{1}{2} \left(\frac{a_0}{2} \right) + \frac{a_1}{3} + \frac{a_0}{4} \right]$$

$$= \frac{7a_0}{240} + \frac{a_1}{40},$$

$$n=4:\ a_6 = \frac{1}{6 \cdot 5} \left(a_4 + \frac{a_3}{2} + \frac{a_2}{3} + \frac{a_1}{4} + \frac{a_0}{5} \right)$$

$$= \frac{1}{30} \left[\left(\frac{5a_0}{72} + \frac{a_1}{24} \right) + \frac{1}{2} \left(\frac{a_0}{12} + \frac{a_1}{6} \right) + \frac{1}{3} \left(\frac{a_0}{2} \right) + \frac{a_1}{4} + \frac{a_0}{5} \right] = \frac{43a_0}{2700} + \frac{a_1}{80}.$$

It is difficult to obtain the general expression for a_n. Stopping at x^6, the series solution is given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \frac{a_0}{2} x^2 + \left(\frac{a_0}{12} + \frac{a_1}{6} \right) x^3 + \left(\frac{5a_0}{72} + \frac{a_1}{24} \right) x^4$$

$$+ \left(\frac{7a_0}{240} + \frac{a_1}{40} \right) x^5 + \left(\frac{43a_0}{2700} + \frac{a_1}{80} \right) x^6 + \cdots$$

$$= a_0 \left(1 + \frac{x^2}{2} + \frac{x^3}{12} + \frac{5x^4}{72} + \frac{7x^5}{240} + \frac{43x^6}{2700} + \cdots \right)$$

$$+ a_1 \left(x + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{40} + \frac{x^6}{80} + \cdots \right),$$

where a_0 and a_1 are arbitrary constants.

Example 9.7

Find the power series solution in x of the equation $y''' - xy'' + (x-2)y' + y = 0$.

The differential equation is of the form

$$y''' + p_2(x) y'' + p_1(x) y' + p_0(x) y = 0, \quad p_2(x) = -x, \quad p_1(x) = x-2, \quad p_0(x) = 1.$$

Each of $p_0(x)$, $p_1(x)$ and $p_1(x)$ can be expressed in power series. Hence, $x=0$ is an ordinary point and there exists a unique power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad -\infty < x < \infty,$$

where a_n, $n=0, 1, \ldots$, are constants to be determined. Differentiating with respect to x yields, for $-\infty < x < \infty$,

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}, \quad y''' = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3}.$$

Substituting into the differential equation results in

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$+ \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the indices of summations

$$\sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} \xrightarrow{n-3=m} \sum_{m=0}^{\infty} (m+3)(m+2)(m+1) a_{m+3} x^m,$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} \xrightarrow{n-1=m} \sum_{m=1}^{\infty} (m+1) m a_{m+1} x^m,$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \xrightarrow{n-1=m} \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m,$$

one obtains

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1) a_{n+3} x^n - \sum_{n=1}^{\infty} (n+1) n a_{n+1} x^n$$
$$+ \sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\therefore \quad \sum_{n=0}^{\infty} \big[(n+3)(n+2)(n+1) a_{n+3} - 2(n+1) a_{n+1} + a_n\big] x^n$$
$$+ \sum_{n=1}^{\infty} \big[-(n+1) n a_{n+1} + n a_n\big] x^n = 0.$$

For this equation to be true, the coefficient of x^n, $n=0, 1, \ldots$, must be zero. When $n=0$, one has

$$3\cdot 2\cdot 1 a_3 - 2\cdot 1 a_1 + a_0 = 0 \implies a_3 = -\frac{a_0}{6} + \frac{a_1}{3}.$$

For $n \geqslant 1$, one obtains

$$\big[(n+3)(n+2)(n+1) a_{n+3} - 2(n+1) a_{n+1} + a_n\big] + \big[-(n+1) n a_{n+1} + n a_n\big] = 0,$$

$$\therefore \quad a_{n+3} = -\frac{a_n}{(n+3)(n+2)} + \frac{a_{n+1}}{n+3}.$$

Hence,

$$n=1: \quad a_4 = -\frac{a_1}{4\cdot 3} + \frac{a_2}{4} = -\frac{a_1}{12} + \frac{a_2}{4},$$

$$n=2: \quad a_5 = -\frac{a_2}{5\cdot 4} + \frac{a_3}{5} = -\frac{a_2}{20} + \frac{1}{5}\Big(-\frac{a_0}{6} + \frac{a_1}{3}\Big) = -\frac{a_0}{30} + \frac{a_1}{15} - \frac{a_2}{20},$$

$$n=3: \quad a_6 = -\frac{a_3}{6\cdot 5} + \frac{a_4}{6} = -\frac{1}{30}\Big(-\frac{a_0}{6} + \frac{a_1}{3}\Big) + \frac{1}{6}\Big(-\frac{a_1}{12} + \frac{a_2}{4}\Big)$$
$$= \frac{a_0}{180} - \frac{a_1}{40} + \frac{a_2}{24}.$$

It is difficult to obtain the general expression for a_n. Stopping at x^6, the series solution is given by

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \Big(-\frac{a_0}{6} + \frac{a_1}{3}\Big) x^3 + \Big(-\frac{a_1}{12} + \frac{a_2}{4}\Big) x^4$$
$$+ \Big(-\frac{a_0}{30} + \frac{a_0}{15} - \frac{a_2}{20}\Big) x^5 + \Big(\frac{a_0}{180} - \frac{a_1}{40} + \frac{a_2}{24}\Big) x^6 + \cdots$$

$$= a_0\Big(1 - \frac{x^3}{6} - \frac{x^5}{30} + \frac{x^6}{180} + \cdots\Big) + a_1\Big(x + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{15} - \frac{x^6}{40} + \cdots\Big)$$
$$+ a_2\Big(x^2 + \frac{x^4}{4} - \frac{x^5}{20} + \frac{x^6}{24} + \cdots\Big),$$

where a_0, a_1, and a_2 are arbitrary constants.

9.3 Series Solution about a Regular Singular Point

Definition — Singular Point

Consider the nth-order linear homogeneous ordinary differential equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + p_{n-2}(x)\, y^{(n-2)} + \cdots + p_0(x)\, y = 0.$$

- A point x_0 is called a *singular point* of the given differential equation if it is not an ordinary point, i.e., not all of the coefficients $p_0(x)$, $p_1(x)$, …, $p_{n-1}(x)$ are analytic at $x=x_0$.
- A point x_0 is a *regular singular point* of the given differential equation if it is not an ordinary point, i.e., not all of the coefficients $p_k(x)$ are analytic, but all of $(x-x_0)^{n-k} p_k(x)$ are analytic for $k=0, 1, \ldots, n-1$.
- A point x_0 is an *irregular singular point* of the given differential equation if it is neither an ordinary point nor a regular singular point.

Consider the second-order linear homogeneous ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

If $x=0$ is a regular singular point, then $xP(x)$ and $x^2Q(x)$ can be expanded as power series

$$xP(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \quad |x| < r,$$

which leads to

$$P(x) = \sum_{n=0}^{\infty} P_n x^{n-1}, \quad Q(x) = \sum_{n=0}^{\infty} Q_n x^{n-2}, \quad |x| < r, \ x \neq 0.$$

Seek the power series solution of the differential equation of the form

$$y(x) = x^{\alpha} \cdot \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \quad 0 < x < r,$$

which is called a *Frobenius* series solution. Differentiating with respect to x yields

$$y'(x) = \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}, \quad y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2}.$$

Substituting into the differential equation results in

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} + \sum_{n=0}^{\infty} P_n x^{n-1} \cdot \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1}$$
$$+ \sum_{n=0}^{\infty} Q_n x^{n-2} \cdot \sum_{n=0}^{\infty} a_n x^{n+\alpha} = 0.$$

Noting that

$$\sum_{n=0}^{\infty} P_n x^{n-1} \cdot \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_{n-m} x^{n-m-1} \cdot (m+\alpha) a_m x^{m+\alpha-1}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} (m+\alpha) P_{n-m} a_m \right] x^{n+\alpha-2},$$

$$\sum_{n=0}^{\infty} Q_n x^{n-2} \cdot \sum_{n=0}^{\infty} a_n x^{n+\alpha} = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Q_{n-m} a_m \right) x^{n+\alpha-2},$$

one obtains

$$\sum_{n=0}^{\infty} \left\{ (n+\alpha)(n+\alpha-1) a_n + \sum_{m=0}^{n} \left[(m+\alpha) P_{n-m} + Q_{n-m} \right] a_m \right\} x^{n+\alpha-2} = 0.$$

For this equation to be true, the coefficient of $x^{n+\alpha-2}$, $n=0, 1, \ldots$, must be zero. For $n=0$, one has

$$\left[\alpha(\alpha-1) + \alpha P_0 + Q_0 \right] a_0 = 0,$$

which implies either $a_0=0$ or $\alpha(\alpha-1)+\alpha P_0+Q_0=0$. For $n \geqslant 1$, one obtains

$$(n+\alpha)(n+\alpha-1) a_n + \sum_{m=0}^{n} \left[(m+\alpha) P_{n-m} + Q_{n-m} \right] a_m = 0.$$

$$\therefore \quad a_n = -\frac{1}{(n+\alpha)(n+\alpha-1)+(n+\alpha)P_0+Q_0} \sum_{m=0}^{n-1} \left[(m+\alpha) P_{n-m} + Q_{n-m} \right] a_m.$$

Case 1. If $a_0=0$, then $a_1=a_2=\cdots=0$, resulting in the zero solution $y(x)=0$.

Case 2. If $a_0 \neq 0$, then

$$\alpha(\alpha-1) + \alpha P_0 + Q_0 = 0,$$

which is called the *indicial equation*. Solving this quadratic equation for α, one obtains two roots α_1 and α_2.

Hence, in order to have a nonzero solution, it is required that $a_0 \neq 0$ and α is a root of the indicial equation.

Remarks: If a series solution about a point $x=x_0 \neq 0$ is to be determined, one can change the independent variable to $t=x-x_0$ and then solve the resulting differential equation about $t=0$. If a solution valid for $x<0$ is to be determined, let $t=-x$ and then solve the resulting differential equation.

Fuchs' Theorem — Series Solution about a Regular Singular Point

For the second-order linear homogeneous ordinary differential equation

$$y''(x) + P(x) y'(x) + Q(x) y(x) = 0,$$

if $x=0$ is a regular singular point, then

$$x P(x) = \sum_{n=0}^{\infty} P_n x^n, \quad x^2 Q(x) = \sum_{n=0}^{\infty} Q_n x^n, \quad |x|<r.$$

Suppose that the indicial equation

$$\alpha(\alpha-1) + \alpha P_0 + Q_0 = 0$$

has two real roots α_1 and α_2, $\alpha_1 \geqslant \alpha_2$. Then the differential equation has at least one *Frobenius* series solution given by

$$y_1(x) = x^{\alpha_1} \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \quad 0<x<r,$$

where the coefficients a_n can be determined by substituting $y_1(x)$ into the differential equation. A second linearly independent solution is obtained as follows:

1. If $\alpha_1-\alpha_2$ is not equal to an integer, then a second Frobenius series solution is given by

$$y_2(x) = x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r,$$

in which the coefficients b_n can be determined by substituting $y_2(x)$ into the differential equation.

2. If $\alpha_1=\alpha_2=\alpha$, then

$$y_2(x) = y_1(x) \ln x + x^{\alpha} \sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r,$$

in which the coefficients b_n can be determined by substituting $y_2(x)$ into the differential equation, once $y_1(x)$ is known. In this case, the second solution $y_2(x)$ is not a Frobenius series solution.

3. If $\alpha_1-\alpha_2$ is a positive integer, then

$$y_2(x) = a\, y_1(x) \ln x + x^{\alpha_2} \sum_{n=0}^{\infty} b_n x^n, \quad 0<x<r,$$

where the coefficients b_n and a can be determined by substituting y_2 into the differential equation, once y_1 is known. The parameter a may be zero, in which case the second solution $y_2(x)$ is also a Frobenius series solution.

The general solution of the differential equation is then given by

$$y(x) = C_1 y_1(x) + C_2 y_2(x).$$

Example 9.8

Obtain series solution about $x=0$ of the equation

$$2x^2y''+x(2x+1)y'-y=0.$$

The differential equation is of the form

$$y''+P(x)y'+Q(x)y=0,\quad P(x)=\frac{2x+1}{2x},\quad Q(x)=-\frac{1}{2x^2}.$$

Obviously, $x=0$ is a singular point. Note that

$$xP(x)=\frac{2x+1}{2}=\tfrac{1}{2}+x+0\cdot x^2+0\cdot x^3+\cdots \implies P_0=\tfrac{1}{2},$$

$$x^2Q(x)=-\tfrac{1}{2}=-\tfrac{1}{2}+0\cdot x+0\cdot x^2+0\cdot x^3+\cdots \implies Q_0=-\tfrac{1}{2}.$$

Both $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$ and can be expanded as power series that are convergent for $|x|<\infty$. Hence, $x=0$ is a regular singular point.

The indicial equation is $\alpha(\alpha-1)+\alpha P_0+Q_0=0$:

$$\alpha(\alpha-1)+\alpha\cdot\tfrac{1}{2}-\tfrac{1}{2}=0 \implies (\alpha+\tfrac{1}{2})(\alpha-1)=0 \implies \alpha_1=1,\ \alpha_2=-\tfrac{1}{2}.$$

Thus the equation has a Frobenius series solution of the form

$$y_1(x)=x^{\alpha_1}\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+1},\quad a_0\neq 0,\quad 0<x<\infty,$$

where a_n, $n=0,1,\ldots$, are constants to be determined. Differentiating with respect to x yields

$$y_1'(x)=\sum_{n=0}^{\infty}(n+1)a_nx^n,\quad y_1''(x)=\sum_{n=1}^{\infty}(n+1)na_nx^{n-1}.$$

Substituting y_1, y_1', and y_1'' into the differential equation results in

$$\sum_{n=1}^{\infty}2(n+1)na_nx^{n+1}+\sum_{n=0}^{\infty}2(n+1)a_nx^{n+2}+\sum_{n=0}^{\infty}(n+1)a_nx^{n+1}-\sum_{n=0}^{\infty}a_nx^{n+1}=0.$$

Changing the indices of the summations

$$\sum_{n=1}^{\infty}2(n+1)na_nx^{n+1}\xrightarrow{n+1=m}\sum_{m=2}^{\infty}2m(m-1)a_{m-1}x^m,$$

$$\sum_{n=0}^{\infty}2(n+1)a_nx^{n+2}\xrightarrow{n+2=m}\sum_{m=2}^{\infty}2(m-1)a_{m-2}x^m,$$

$$\sum_{n=0}^{\infty}na_nx^{n+1}\xrightarrow{n+1=m}\sum_{m=1}^{\infty}(m-1)a_{m-1}x^m,$$

one obtains

$$\sum_{n=2}^{\infty}\big[2n(n-1)a_{n-1}+2(n-1)a_{n-2}\big]x^n+\sum_{n=1}^{\infty}(n-1)a_{n-1}x^n=0.$$

For this equation to be true, the coefficient of x^n, $n=1,2,\ldots$, must be zero. For $n=1$, one has

$$0\cdot a_0=0 \implies a_0\neq 0 \text{ is arbitrary; take } a_0=1.$$

For $n\geqslant 2$, one has

$$2n(n-1)a_{n-1}+2(n-1)a_{n-2}+(n-1)a_{n-1}=0 \implies a_{n-1}=-\frac{2a_{n-2}}{2n+1}.$$

Hence,

$$n=2:\quad a_1=-\frac{2a_0}{2\cdot 2+1}=-\frac{2}{5},$$

$$n=3:\quad a_2=-\frac{2a_1}{2\cdot 3+1}=(-1)^2\frac{2^2}{7\cdot 5},$$

$$\vdots$$

$$n+1:\quad a_n=-\frac{2a_{n-1}}{2(n+1)+1}=(-1)^n\frac{2^n}{(2n+3)(2n+1)\cdots 5}=(-1)^n\frac{3\cdot 2^n}{(2n+3)!!},$$

where $(2n+3)!!=(2n+3)(2n+1)\cdots 5\cdot 3\cdot 1$ is the double factorial. The first Frobenius series solution is

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+1}=\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 2^n}{(2n+3)!!}x^{n+1},\quad 0<x<\infty.$$

Since $\alpha_1-\alpha_2=\frac{3}{2}$, according to Fuchs' Theorem, a second linearly independent solution is also a Frobenius series given by

$$y_2(x)=x^{\alpha_2}\sum_{n=0}^{\infty}b_nx^n=\sum_{n=0}^{\infty}b_nx^{n-\frac{1}{2}},\quad b_0\neq 0,\quad 0<x<\infty,$$

$$y_2'(x)=\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)b_nx^{n-\frac{3}{2}},\quad y_2''(x)=\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)b_nx^{n-\frac{5}{2}}.$$

Substituting y_2, y_2', and y_2'' into the differential equation leads to

$$2x^2\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)b_nx^{n-\frac{5}{2}}+(2x^2+x)\sum_{n=0}^{\infty}\left(n-\tfrac{1}{2}\right)b_nx^{n-\frac{3}{2}}-\sum_{n=0}^{\infty}b_nx^{n-\frac{1}{2}}=0,$$

$$\sum_{n=0}^{\infty}\Big\{\Big[2\left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)+\left(n-\tfrac{1}{2}\right)-1\Big]b_nx^{n-\frac{1}{2}}+2\left(n-\tfrac{1}{2}\right)b_nx^{n+\frac{1}{2}}\Big\}=0.$$

Multiplying this equation by $x^{\frac{1}{2}}$ yields

$$\sum_{n=0}^{\infty}\Big[n(2n-3)\,b_n x^n+(2n-1)\,b_n x^{n+1}\Big]=0.$$

Changing the index of the summation

$$\sum_{n=0}^{\infty}(2n-1)\,b_n x^{n+1}\xRightarrow{n+1=m}\sum_{m=1}^{\infty}(2m-3)\,b_{m-1}x^m,$$

one obtains

$$\sum_{n=0}^{\infty}n(2n-3)\,b_n x^n+\sum_{n=1}^{\infty}(2n-3)\,b_{n-1}x^n=0.$$

For this equation to be true, the coefficient of x^n, $n=0,1,\ldots$, must be zero. For $n=0$, one has

$$0\cdot(-3)\,b_0=0 \implies b_0\neq 0 \text{ is arbitrary; take } b_0=1.$$

For $n\geqslant 1$, one has

$$n(2n-3)\,b_n+(2n-3)\,b_{n-1}=0 \implies b_n=-\frac{b_{n-1}}{n}.$$

Hence,

$$b_1=-\frac{b_0}{1}=-\frac{1}{1},\quad b_2=-\frac{b_1}{2}=(-1)^2\frac{1}{2!},\quad b_3=-\frac{b_2}{3}=(-1)^3\frac{1}{3!},\quad\cdots$$

$$\therefore\quad b_n=-\frac{b_{n-1}}{n}=(-1)^n\frac{1}{n!}.$$

Thus, a second linearly independent solution is

$$y_2(x)=x^{-\frac{1}{2}}\sum_{n=0}^{\infty}b_n x^n=x^{-\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}=x^{-\frac{1}{2}}\mathrm{e}^{-x}.$$

The general solution of the differential equation is

$$y(x)=C_1y_1(x)+C_2y_2(x)=C_1\sum_{n=0}^{\infty}(-1)^n\frac{3\cdot 2^n}{(2n+3)!!}x^{n+1}+C_2x^{-\frac{1}{2}}\mathrm{e}^{-x}.$$

9.3.1 *Bessel's Equation and Its Applications*

9.3.1.1 Solutions of Bessel's Equation

Bessel's equation of the form

$$x^2y''+xy'+(x^2-\nu^2)\,y=0,\quad x>0,$$

where $\nu\geqslant 0$ is a constant, is of great importance in applied mathematics and has numerous applications in engineering and science. Furthermore, in solving Bessel's equation using series, it exhibits all possibilities in Fuchs' Theorem. As a

result, it is an excellent example to illustrate the procedure and nuances for solving a second-order differential equation using series about a regular singular point.

Bessel's equation is of the form

$$y''+P(x)y'+Q(x)y=0,\quad P(x)=\frac{1}{x},\quad Q(x)=\frac{x^2-\nu^2}{x^2}.$$

It is obvious that $x=0$ is a singular point. Since

$$xP(x)=1=1+0\cdot x+0\cdot x^2+\cdots \implies P_0=1,$$

$$x^2Q(x)=x^2-\nu^2=-\nu^2+0\cdot x+x^2+0\cdot x^3+0\cdot x^4+\cdots \implies Q_0=-\nu^2,$$

both $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$ and can be expanded as power series convergent for $|x|<\infty$. Hence, $x=0$ is a regular singular point.

The indicial equation is $\alpha(\alpha-1)+\alpha P_0+Q_0=0$:

$$\alpha(\alpha-1)+\alpha\cdot 1-\nu^2=0 \implies \alpha-\nu^2=0 \implies \alpha_1=\nu,\quad \alpha_2=-\nu.$$

Bessel's equation has a Frobenius series solution of the form

$$y_1(x)=x^{\nu}\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+\nu},\quad a_0\neq 0,\quad 0<x<\infty.$$

Differentiating with respect to x yields

$$y_1'(x)=\sum_{n=0}^{\infty}(n+\nu)\,a_nx^{n+\nu-1},\quad y_1''(x)=\sum_{n=0}^{\infty}(n+\nu)(n+\nu-1)\,a_nx^{n+\nu-2}.$$

Substituting y_1, y_1', and y_1'' into Bessel's equation results in

$$x^2\sum_{n=0}^{\infty}(n+\nu)(n+\nu-1)\,a_nx^{n+\nu-2}+x\sum_{n=0}^{\infty}(n+\nu)\,a_nx^{n+\nu-1}+(x^2-\nu^2)\sum_{n=0}^{\infty}a_nx^{n+\nu}=0.$$

Changing the index of the summation

$$\sum_{n=0}^{\infty}a_nx^{n+\nu+2}\xRightarrow{n+2=m}\sum_{m=2}^{\infty}a_{m-2}x^{m+\nu}=\sum_{n=2}^{\infty}a_{n-2}x^{n+\nu},$$

one obtains

$$x^{\nu}\left\{\sum_{n=0}^{\infty}\big[(n+\nu)(n+\nu-1)+(n+\nu)-\nu^2\big]a_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n\right\}=0,$$

$$x^{\nu}\neq 0 \implies \sum_{n=0}^{\infty}n(n+2\nu)\,a_nx^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0.$$

For this equation to be true, the coefficient of x^n, $n=0,1,\ldots$, must be zero:

$$x^0:\quad 0\cdot(0+2\nu)\,a_0=0 \implies a_0\neq 0 \text{ is arbitrary},$$