MECHATRONICS

An Integrated Approach

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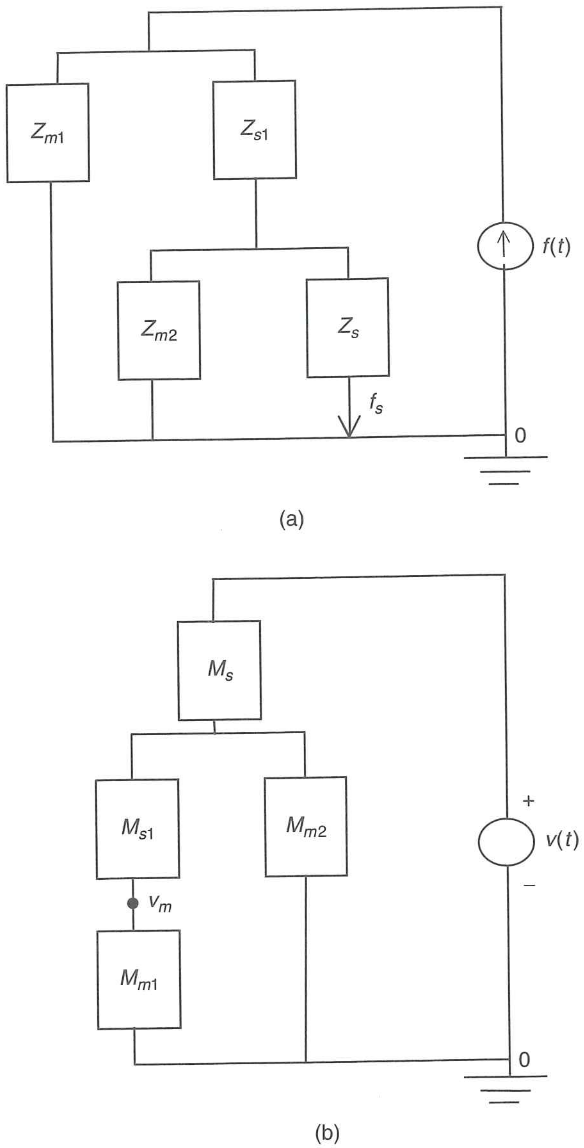

FIGURE 2.78
Impedance circuits of: (a) System in Figure 2.77(a); (b) System in Figure 2.77(b).

As a result, the motion transmissibility can be expressed as

$$T_m = \frac{V_m}{V} = \left[\frac{1}{M_s Z + 1}\right]\left[\frac{M_{m1}}{M_{s1} + M_{m1}}\right] \tag{iii}$$

It remains to show that $T_m = T_f$. To this end, let us examine the expression for T_m. Since $Z_s = 1/M_s$, T_m can be written as

$$T_m = \left[\frac{Z_s}{Z + Z_s}\right]\left[\frac{M_{m1}}{M_{s1} + M_{m1}}\right]$$

Note that

$$Z = \frac{1}{M_{s1} + M_{m1}} + Z_{m2}$$

Hence,

$$T_m = \left[\frac{Z_s}{\dfrac{1}{M_{s1} + M_{m1}} + Z_{m2} + Z_s}\right]\left[\frac{M_{m1}}{M_{s1} + M_{m1}}\right] = \left[\frac{M_{m1}}{\dfrac{1}{Z_{m2} + Z_s} + M_{s1} + M_{m1}}\right]\left[\frac{Z_s}{Z_{m2} + Z_s}\right]$$

$$= \left[\frac{1}{\dfrac{1}{M_{m1}}\left[\dfrac{1}{Z_{m2} + Z_s} + M_{s1}\right] + 1}\right]\left[\frac{Z_s}{Z_{m2} + Z_s}\right] = \left[\frac{1}{Z_{m1}\left[\dfrac{1}{Z_{m2} + Z_s} + \dfrac{1}{Z_{s1}}\right] + 1}\right]\left[\frac{Z_s}{Z_{m2} + Z_s}\right]$$

which is clearly identical to T_f as given in Equation ii, in view of Equation i.

The equivalence of T_f and T_m can be shown in a similar straightforward manner for higher degree-of-freedom systems as well.

2.13 Response Analysis and Simulation

An analytical model, which is a set of differential equations, has many uses. In particular, it can provide information regarding how the system responds when a specific excitation (input) is applied. Such a study may be carried out by

1. Solution of the differential equations (analytical)
2. Computer simulation (numerical)

In this section we will address these two approaches. A response analysis carried out using either approach, is valuable in many applications such as design, control, testing, validation, and qualification of mechatronic systems. For large-scale and complex systems, a purely analytical study may not be feasible, and we will have to increasingly rely on numerical approaches and computer simulation.

2.13.1 Analytical Solution

The response of a dynamic system may be obtained analytically by solving the associated differential equations, subject to the initial conditions. This may be done by

1. Direct solution (in the time domain)
2. Solution using Laplace transform

Consider a linear time-invariant model given by the input-output differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = u \tag{2.148}$$

At the outset, note that it is not necessary to specifically include derivative terms on the RHS; for example, $b_0u+b_1\frac{du}{dt}+\cdots+b_m\frac{d^mu}{dt^m}$ because, once we have the solution (say, y_s) for Equation 2.148 we can use the *principle of superposition* to obtain the solution for the general case, and is given by: $b_0y_s+b_1\frac{dy_s}{dt}+\cdots+b_m\frac{d^my_s}{dt^m}$. Hence, we will consider only the case of Equation 2.148.

2.13.1.1 Homogeneous Solution

The natural characteristics of a dynamic system do not depend on the input to the system. Hence, the natural behavior (or free response) of Equation 2.148 is determined by the homogeneous equation (i.e., the input = 0):

$$a_n\frac{d^ny}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_0y=0 \tag{2.149}$$

Its solution is denoted by y_h and it depends on the system initial conditions. For a linear system the natural response is known to take an exponential form given by

$$y_h=ce^{\lambda t} \tag{2.150}$$

where c is an arbitrary constant and, in general, λ can be complex. Substitute Equation 2.149 in Equation 2.150 with the knowledge that

$$\frac{d}{dt}e^{\lambda t}=\lambda e^{\lambda t} \tag{2.151}$$

and cancel the common term $ce^{\lambda t}$, since u cannot be zero at all times. Then we have

$$a_n\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_0=0 \tag{2.152}$$

This is called the *characteristic equation* of the system.

NOTE the LHS polynomial of Equation 2.152 is the *characteristic polynomial*. Equation 2.152 has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. These are called *poles* or *eigenvalues* of the system. Assuming that they are distinct (i.e., unequal), the overall solution to Equation 2.149 becomes

$$y_h=c_1e^{\lambda_1 t}+c_2e^{\lambda_2 t}+\cdots+c_ne^{\lambda_n} \tag{2.153}$$

The unknown constants $c_1, c_2, \ldots, c_n$ are determined using the necessary n initial conditions $y(0),\ \dot{y}(0),\ \ldots,\ \frac{d^{n-1}y(0)}{dt^{n-1}}$.

2.13.1.1.1 Repeated Poles

Suppose that at least two eigenvalues are equal. Without loss of generality suppose in Equation 2.153 that $\lambda_1 = \lambda_2$. Then the first two terms in Equation 2.153 can be combined into the single unknown $(c_1 + c_2)$. Consequently there are only $n-1$ unknowns in Equation 2.153 but there are n initial conditions. It follows that another unknown needs to be introduced for obtaining a complete solution. Since a repeated pole is equivalent to a double integration, the logical (and correct) solution for Equation 2.152 in the case $\lambda_1 = \lambda_2$ is

$$y_h=(c_1+c_2t)e^{\lambda_1 t}+c_3e^{\lambda_3 t}+\cdots+c_ne^{\lambda_t} \tag{2.154}$$

2.13.1.2 Particular Solution

The homogeneous solution corresponds to the "free" or "unforced" response of a system, and it does not take into account the input function. The effect of the input is incorporated into the particular solution, which is defined as one possible function for y that satisfies Equation 2.148. We denote this by y_p. Several important input functions and the corresponding form of y_p which satisfies Equation 2.148 are given in Table 2.12.

The parameters A, B, A_1, A_2, B_1, B_2, and D are determined by substituting the pair $u(t)$ and y_p into Equation 2.148 and then equating the like terms. This approach is called the *method of undetermined coefficients*.

The total response is given by

$$y=y_h+y_p \tag{2.155}$$

The unknown constants $c_1, c_2, \ldots, c_n$ in this result are determined by substituting the initial conditions of the system into Equation 2.155. Note that it is incorrect to first determine $c_1, c_2, \ldots, c_n$ by substituting the ICs into y_h and then adding y_p to the resulting y_h. Furthermore, when $u = 0$, the homogeneous solution is same as the free response, initial condition response, or zero-input response. When an input is present, however, the homogeneous solution is not identical to the other three types of response. These ideas are summarized in Table 2.13

TABLE 2.12

Particular Solutions for Useful Input Functions

Input $u(t)$	Particular Solution y_p
c	A
ct	$B_1t + B_2$
$\sin ct$	$A_1\sin ct+A_2\cos ct$
$\cos ct$	$B_1\sin ct+B_2\cos ct$
e^{ct}	De^{ct}

TABLE 2.13

Some Concepts of System Response

Total response (T)	= homogeneous solution (H) + particular integral (P)
	= free response (X) + forced response (F)
	= initial-condition response (X) + zero-initial-condition response (F)
	= zero-input response (X) + zero-state response (F)

Note: In general, $H \neq X$ and $P \neq F$
With no input (no forcing excitation), by definition, $H \equiv X$
At steady state, F becomes equal to P.

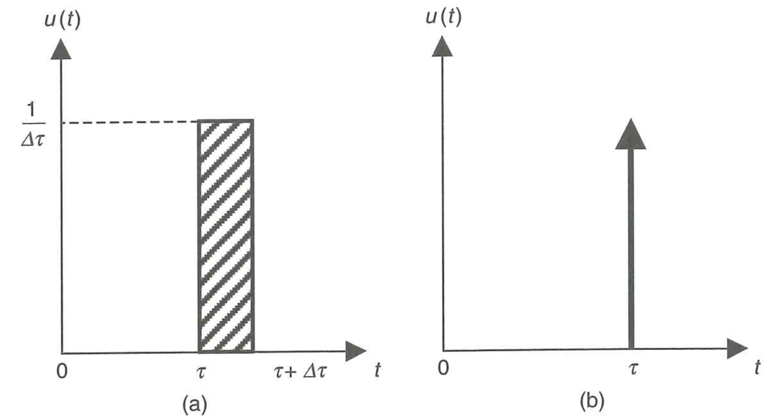

FIGURE 2.79
Illustration of: (a) Unit pulse; (b) Unit impulse.

2.13.1.3 Impulse Response Function

Consider a linear dynamic system. The principle of superposition holds. More specifically, if y_1 is the system response to excitation $u_1(t)$, and y_2 is the response to excitation $u_2(t)$, then $\alpha y_1 + \beta y_2$ is the system response to input $\alpha u_1(t) + \beta u_2(t)$ for any constants α and β and any excitation functions $u_1(t)$ and $u_2(t)$. This is true for both time-variant-parameter linear systems and constant-parameter linear systems.

A unit pulse of width $\Delta\tau$ starting at time $t = \tau$ is shown in Figure 2.79(a). Its area is unity. A unit impulse is the limiting case of a unit pulse for $\Delta\tau \to 0$. A unit impulse acting at time $t = \tau$ is denoted by $\delta(t-\tau)$ and is graphically represented as in Figure 2.79(b). In mathematical analysis, this is known as the *Dirac delta function*, and is defined by the two conditions:

$$\begin{aligned}\delta(t-\tau) &= 0 \quad \text{for} \quad t \neq \tau \\ &\to \infty \quad \text{at} \quad t = \tau\end{aligned} \tag{2.156}$$

and

$$\int_{-\infty}^{\infty} \delta(t-\tau)\,dt = 1 \tag{2.157}$$

The Dirac delta function has the following well-known and useful properties:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = f(\tau) \tag{2.158}$$

and

$$\int_{-\infty}^{\infty} \frac{d^n f(t)}{dt^n}\delta(t-\tau)\,dt = \left.\frac{d^n f(t)}{dt^n}\right|_{t=\tau} \tag{2.159}$$

for any well-behaved time function $f(t)$. The system response (output) to a unit-impulse excitation (input) acted at time $t = 0$, is known as the *impulse-response function* and is denoted by $h(t)$.

2.13.1.4 Convolution Integral

The system output in response to an arbitrary input may be expressed in terms of its impulse-response function. This is the essence of the impulse-response approach to determining the forced response of a dynamic system. Without loss of generality we shall assume that the system input $u(t)$ starts at $t = 0$; that is,

$$u(t) = 0 \quad \text{for} \quad t < 0 \tag{2.160}$$

For physically realizable systems, the response does not depend on the future values of the input. Consequently,

$$y(t) = 0 \quad \text{for} \quad t < 0 \tag{2.161}$$

and

$$h(t) = 0 \quad \text{for} \quad t < 0 \tag{2.162}$$

where $y(t)$ is the response of the system, to any general excitation $u(t)$.

Furthermore, if the system is a constant-parameter system, then the response does not depend on the time origin used for the input. Mathematically, this is stated as follows: if the response to input $u(t)$ satisfying Equation 2.160 is $y(t)$, which in turn satisfies Equation 2.161, then the response to input $u(t-\tau)$, which satisfies,

$$u(t-\tau) = 0 \quad \text{for} \quad t < \tau \tag{2.163}$$

is $y(t-\tau)$, and it satisfies

$$y(t-\tau) = 0 \quad \text{for} \quad t < \tau \tag{2.164}$$

This situation is illustrated in Figure 2.80. It follows that the delayed-impulse input $\delta(t-\tau)$, having time delay τ, produces the delayed response $h(t-\tau)$.

A given input $u(t)$ can be divided approximately into a series of pulses of width $\Delta\tau$ and magnitude $u(\tau)\cdot\Delta\tau$. In Figure 2.81, for $\Delta\tau \to 0$, the pulse shown by the shaded area becomes an impulse acting at $t = \tau$, having the magnitude $u(\tau)\cdot d\tau$. This impulse is given by $\delta(t-\tau)u(\tau)d\tau$. In a linear, constant-parameter system, it produces the response $h(t-\tau)u(\tau)d\tau$. By integrating over the entire time duration of the input $u(t)$, the overall response $y(t)$ is obtained as

$$\begin{aligned}y(t) &= \int_0^{\infty} h(t-\tau)u(\tau)\,d\tau \\ &= \int_0^{\infty} h(\tau)u(t-\tau)\,d\tau\end{aligned} \tag{2.165}$$

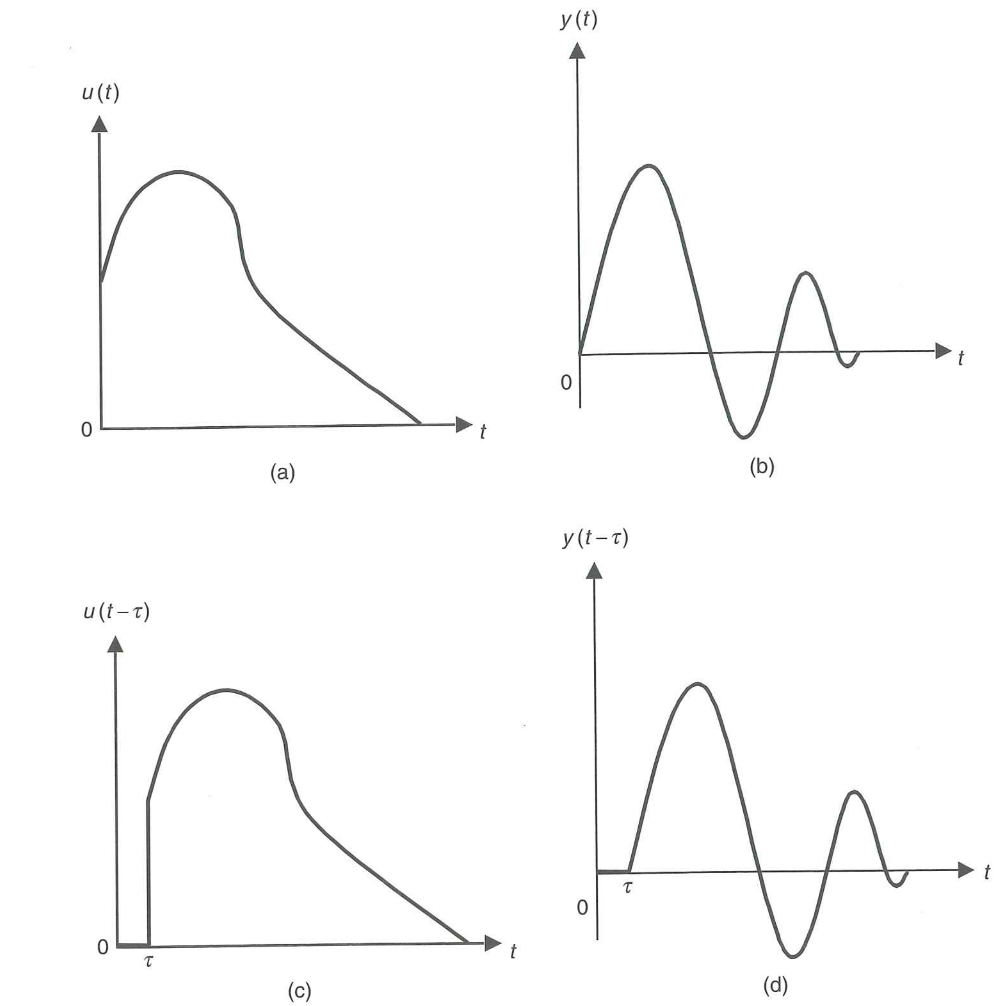

FIGURE 2.80
Response to a delayed input.

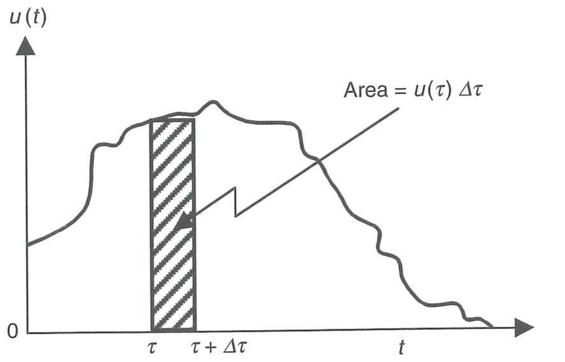

FIGURE 2.81
General input treated as a continuous series of impulses.

Equation 2.150 is known as the *convolution integral*. This is in fact the forced response, under zero initial conditions.

2.13.2 Stability

Many definitions are available for stability of a system. For example, a stable system may be defined as one whose natural response (i.e., free, initial-condition response) decays to zero. This is in fact the well-known *asymptotic stability.* If the initial-condition response oscillates within finite bounds we say the system is *marginally stable*. For a linear, time-invariant system of the type Equation 2.148, the free response is of the form Equation 2.153. Hence, if none of the eigenvalues λ_i have positive real parts, the system is considered stable, because in that case, the response Equation 2.153 does not grow unbounded. In particular, if the system has a single eigenvalue that is zero, or if the eigenvalues are purely imaginary, the system is marginally stable. If the system has two or more poles that are zero, we will have terms of the form $c_1 + ct$ as in Equation 2.154 and hence it will grow polynomially (not exponentially). Then the system will be *unstable*. Also note that, since physical systems have real parameters, their eigenvalues must occur as conjugate pairs, if complex. Since stability is governed by the sign of the real part of the eigenvalues, it can be represented on the eigenvalue plane (or the pole plane or root plane). This is illustrated in Figure 2.82.

2.13.3 First Order Systems

Consider the first order dynamic system with time constant τ, input u, and output y, as given by

$$\tau \dot{y} + y = u(t) \tag{2.166}$$

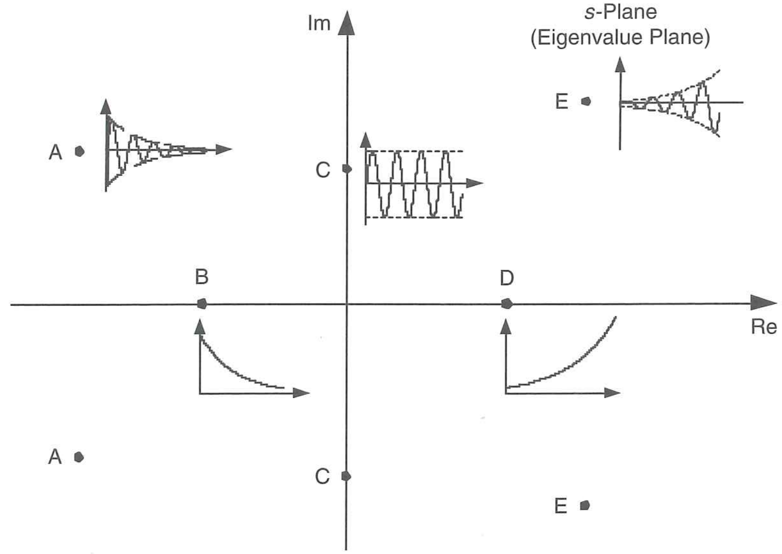

FIGURE 2.82
Dependence of stability on the pole location (**A** and **B** are stable; **C** is marginally stable; **D** and **E** are unstable).

Suppose that the system is starting from $y(0) = y_0$ and a step input of magnitude A is applied. The homogeneous solution is

$$y_h = ce^{-t/\tau}$$

The particular solution (see Table 2.12) is given by $y_p = A$. Hence, the total solution is

$$y = y_h + y_p = ce^{-t/\tau} + A$$

Substitute the IC: $y(0) = y_0$. We get $c + A = y_0$. Hence

$$y_{\text{step}} = \underbrace{(y_0 - A)e^{-t/\tau}}_{\substack{\text{Homogeneous}\\ y_h}} + \underbrace{A}_{\substack{\text{Particular}\\ y_p}} = \underbrace{y_0 e^{-t/\tau}}_{\substack{\text{Free Response}\\ y_x}} + \underbrace{A(1-e^{-t/\tau})}_{\substack{\text{Forced Response}\\ y_f}} \tag{2.167}$$

The steady-state value is given by $t \to \infty$. Hence

$$y_{ss} = A \tag{2.168}$$

It is seen from Equation 2.167 that the forced response to a unit step input (i.e., $A = 1$) is $(1-e^{-t/\tau})$. Due to linearity, the forced response to a unit impulse input is $\frac{d}{dt}(1-e^{-t/\tau}) = \frac{1}{\tau}e^{-t/\tau}$. Hence, the total response to an impulse input of magnitude P is

$$y_{\text{impulse}} = y_0 e^{-t/\tau} + \frac{P}{\tau}e^{-t/\tau} \tag{2.169}$$

This result follows from the fact that

$$\frac{d}{dt}(\text{Step Function}) = \text{Impulse Function}$$

and, due to linearity, when the input is differentiated, the output is correspondingly differentiated.

Note from Equation 2.167 and Equation 2.169 that if we know the response of a first order system to a step input, or to an impulse input, the system itself can be determined. This is known as *model identification*. We will illustrate this by an example.

2.13.4 Model Identification Example

Consider the first order system (model)

$$\tau\dot{y} + y = ku \tag{i}$$

Note the gain parameter k. The initial condition is $y(0) = y_0$.

Due to linearity, using Equation 2.167 we can derive the response of the system to a step input of magnitude A:

$$y_{\text{step}} = y_0 e^{-t/\tau} + Ak(1-e^{-t/\tau}) \tag{ii}$$

Now suppose that the unit step response of a first order system with zero ICs, was found to be (say, by curve fitting of experimental data)

$$y_{\text{step}} = 2.25(1-e^{-5.2t})$$

Then, it is clear from Equation ii that

$$k = 2.25 \text{ and } \tau = 1/5.2 = 0.192$$

2.13.5 Second Order Systems

A general high-order system can be represented by a suitable combination of first-order and second-order models, using the principles of modal analysis. Hence, it is useful to study the response behavior of second-order systems as well. Examples of second-order systems include mass-spring-damper systems and capacitor-inductor-resistor circuits, which we have studied in previous sections. These are called simple oscillators because they exhibit oscillations in the natural response (free response) when the level of damping is sufficiently low. We will study both free response and forced response.

2.13.5.1 Free Response of an Undamped Oscillator

We note that the equation of free (i.e., no excitation force) motion of an undamped simple oscillator is of the general form

$$\ddot{x} + \omega_n^2 x = 0 \tag{2.170}$$

For a mechanical system of mass m and stiffness k, we have

$$\omega_n = \sqrt{\frac{k}{m}} \tag{2.171}$$

For an electrical circuit with capacitance C and inductance L we have

$$\omega_n = \sqrt{\frac{1}{LC}} \tag{2.172}$$

To determine the time response x of this system, we use the trial solution:

$$x = A\sin(\omega_n t + \phi) \tag{2.173}$$

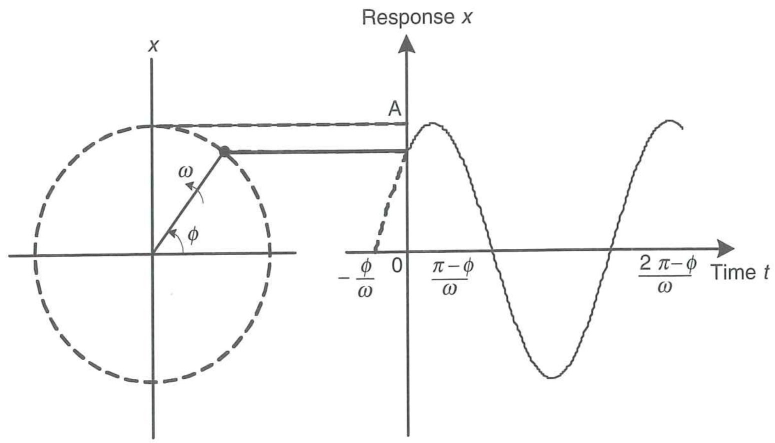

FIGURE 2.83
Free response of an undamped simple oscillator.

in which A and ϕ are unknown constants, to be determined by the initial conditions (for x and $\dot{x}$); say,

$$x(0) = x_o, \quad \dot{x}(0) = v_o \tag{2.174}$$

Substitute the trial solution into Equation (2.170). We get

$$\left(-A\omega_n^2 + A\omega_n^2\right)\sin(\omega_n t + \phi) = 0$$

This equation is identically satisfied for all t. Hence, the general solution of Equation 2.170 is indeed Equation 2.173, which is periodic and sinusoidal.

This response is sketched in Figure 2.83 (the subscript in ω_n is dropped for convenience). Note that this sinusoidal, oscillatory motion has a *frequency* of oscillation of ω (radians/s). Hence, a system that provides this type of natural motion is called a *simple oscillator*. In other words, the system response exactly repeats itself in time periods of T or at a *cyclic frequency* $f = \frac{1}{T}$(Hz). The frequency ω is in fact the *angular frequency* given by $\omega = 2\pi f$. Also, the response has an *amplitude* A, which is the peak value of the sinusoidal response. Now, suppose that we shift this response curve to the right through ϕ/ω. Consider the resulting curve to be the reference signal (with signal value = 0 at t = 0, and increasing). It should be clear that the response shown in Figure 2.83 leads the reference signal by a time period of ϕ/ω. This may be verified from the fact that the value of the reference signal at time t is the same as that of the signal in Figure 2.83 at time $t - \phi/\omega$. Hence ϕ is termed the *phase angle* of the response, and it is a *phase lead*.

The left-hand-side portion of Figure 2.83 is the *phasor representation* of a sinusoidal response. In this representation, an arm of length A rotates in the counterclockwise direction at angular speed ω. This is the phasor. The arm starts at an angular position ϕ from the horizontal axis, at time t = 0. The projection of the arm onto the vertical (x) axis is the time response. In this manner, the phasor representation can conveniently indicate the amplitude, frequency, phase angle, and the actual time response (at any time t) of a sinusoidal motion.

2.13.5.2 Free Response of a Damped Oscillator

Energy dissipation may be added to a mechanical oscillator by using a damping element. For an electrical circuit, a resistor may be added to achieve this. In either case, the equation motion of the damped simple oscillator without an input, may be expressed as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{2.175}$$

Note that ζ is called the *damping ratio*.

Assume an exponential solution:

$$x = Ce^{\lambda t} \tag{2.176}$$

This is justified by the fact that linear systems have exponential or oscillatory (i.e., complex exponential) free responses. A more detailed justification will be provided later.

Substitute, Equation 2.176 into Equation 2.175. We get

$$\left[\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2\right] Ce^{\lambda t} = 0$$

Note that $Ce^{\lambda t}$ is not zero in general. It follows that, when λ satisfies the equation:

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{2.177}$$

then, Equation 2.176 will represent a solution of Equation 2.175. As noted before, Equation 2.177 is the *characteristic equation* of the system. This equation depends on the natural dynamics of the system, not the forcing excitation or the initial conditions. Solution of Equation 2.177 gives the two roots:

$$\begin{aligned} \lambda &= -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\ \omega_n \\ &= \lambda_1 \text{ and } \lambda_2 \end{aligned} \tag{2.178}$$

These are the *eigenvalues* or *poles* of the system. When $\lambda_1 \neq \lambda_2$, the general solution is

$$x = C_1 e^{\lambda_1} + C_2 e^{\lambda_2 t} \tag{2.179}$$

The two unknown constants C_1 and C_2 are related to the integration constants, and can be determined by two initial conditions which should be known.

If $\lambda_1 = \lambda_2 = \lambda$; we have the case of repeated roots. In this case, the general solution Equation 2.179 does not hold because C_1 and C_2 would no longer be independent constants, to be determined by two initial conditions. The repetition of the roots suggests that one term of the homogenous solution should have the multiplier t (a result of the double integration of zero). Then the general solution is,

$$x = C_1 e^{\lambda t} + C_2 t e^{\lambda t} \tag{2.180}$$

We can identify three ranges of damping, as discussed below, and the nature of the response will depend on the particular range of damping.

Case 1: Underdamped Motion ($\zeta < 1$)

In this case it follows from Equation 2.178 that the roots of the characteristic equation are

$$\lambda = -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n = -\zeta\omega_n \pm j\omega_d = \lambda_1 \text{ and } \lambda_2 \tag{2.181}$$

where the *damped natural frequency* is given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.182}$$

Note that λ_1 and λ_2 are complex conjugates, as required. The response (Equation 2.179), in this case, may be expressed as

$$x = e^{-\zeta\omega_n t}\left[C_1 e^{j\omega_d t} + C_2 e^{-j\omega_d t}\right] \tag{2.183}$$

The term within the square brackets of Equation 2.183 has to be real, because it represents the time response of a real physical system. It follows that C_1 and C_2 as well, have to be complex conjugates.

NOTE

$$e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t$$

$$e^{-j\omega_d t} = \cos\omega_d t - j\sin\omega_d t$$

So, an alternative form of the general solution would be

$$x = e^{-\zeta\omega_n t}[A_1\cos\omega_d t + A_2\sin\omega_d t] \tag{2.184}$$

Here A_1 and A_2 are the two unknown constants. By equating the coefficients it can be shown that

$$\begin{aligned} A_1 &= C_1 + C_2 \\ A_2 &= j(C_1 - C_2) \end{aligned} \tag{2.185}$$

Hence

$$\begin{aligned} C_1 &= \frac{1}{2}(A_1 - jA_2) \\ C_2 &= \frac{1}{2}(A_1 + jA_2) \end{aligned} \tag{2.186}$$

Initial Conditions:

Let
$x(0) = x_o$, $\dot{x}(0) = v_o$ as before. Then,

$$x_o = A_1 \quad \text{and} \quad v_o = -\zeta\omega_n A_1 + \omega_d A_2 \tag{2.187}$$

or,

$$A_2 = \frac{v_o}{\omega_d} + \frac{\zeta\omega_n x_o}{\omega_d} \tag{2.188}$$

Yet, another form of the solution would be:

$$x = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi) \tag{2.189}$$

Here A and ϕ are the unknown constants with

$$A = \sqrt{A_1^2 + A_2^2} \quad \text{and} \quad \sin\phi = \frac{A_1}{\sqrt{A_1^2 + A_2^2}}. \tag{2.190}$$

Also

$$\cos\phi = \frac{A_2}{\sqrt{A_1^2 + A_2^2}} \quad \text{and} \quad \tan\phi = \frac{A_1}{A_2} \tag{2.191}$$

Note that the response $x \to 0$ as $t \to \infty$. This means the system is *asymptotically stable.*

Case 2: Overdamped Motion ($\zeta > 1$)

In this case, roots λ_1 and λ_2 of the characteristic Equation 2.177 are real and negative. Specifically, we have

$$\lambda_1 = -\zeta\omega_n + \sqrt{\zeta^2 - 1} \quad \omega_n < 0 \tag{2.192}$$

$$\lambda_2 = -\zeta\omega_n - \sqrt{\zeta^2 - 1} \quad \omega_n < 0 \tag{2.193}$$

and the response Equation 2.179 is nonoscillatory. Also, since both λ_1 and λ_2 are negative, $x \to 0$ as $t \to \infty$. This means the system is asymptotically stable.

From the initial conditions $x(0) = x_o$, $\dot{x}(0) = v_o$ we get

$$x_o = C_1 + C_2 \tag{i}$$

and

$$v_o = \lambda_1 C_1 + \lambda_2 C_2 \tag{ii}$$

Multiply the first IC Equation i by λ_1: $\quad \lambda_1 x_o = \lambda_1 C_1 + \lambda_1 C_2 \quad$ (iii)

Subtract Equation iii from Equation ii: $\quad v_o - \lambda_1 x_o = C_2(\lambda_2 - \lambda_1)$

We get:

$$C_2 = \frac{v_o - \lambda_1 x_o}{\lambda_2 - \lambda_1} \tag{2.194}$$

Similarly, multiply the first IC Equation i by λ_2 and subtract from Equation ii. We get

$$v_o - \lambda_2 x_o = C_1(\lambda_1 - \lambda_2)$$

Hence

$$C_1 = \frac{v_o - \lambda_2 x_o}{\lambda_1 - \lambda_2} \tag{2.195}$$

Case 3: Critically Damped Motion ($\zeta = 1$)

Here, we have repeated roots, given by

$$\lambda_1 = \lambda_2 = -\omega_n \tag{2.196}$$

The response, for this case is given by (see Equation 2.180)

$$x = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} \tag{2.197}$$

Since the term $e^{-\omega_n t}$ goes to zero faster than t goes to infinity, we have

$$te^{-\omega_n t} \to 0 \text{ as } t \to \infty.$$

Hence the system is asymptotically stable.

Now use the initial conditions $x(0) = x_o$, $\dot{x}(0) = v_o$. We get,

$$x_o = C_1$$
$$v_o = -\omega_n C_1 + C_2$$

Hence

$$C_1 = x_o \tag{2.198}$$

$$C_2 = v_o + \omega_n x_o \tag{2.199}$$

NOTE When $\zeta = 1$ we have the critically damped response because below this value, the response is oscillatory (underdamped) and above this value, the response is nonoscillatory

TABLE 2.14

Free (natural) Response of a Damped Simple Oscillator

System Equation:

$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$

Undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}}$

Damping ratio $\zeta = \frac{b}{2\sqrt{km}}$

Characteristic Equation: $\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0$

Roots (eigenvalues or poles): λ_1 and $\lambda_2 = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$

Response: $x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ for unequal roots $(\lambda_1 \neq \lambda_2)$

$x = (C_1 + C_2 t)e^{\lambda t}$ for equal roots $(\lambda_1 = \lambda_2 = \lambda)$

Initial Conditions: $x(0) = x_0$ and $\dot{x}(0) = v_0$

Case 1: Underdamped ($\zeta < 1$)

Poles are complex conjugates: $-\zeta\omega_n \pm j\omega_d$

Damped natural frequency $\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$

$$x = e^{-\zeta\omega_n t}\left[C_1 e^{j\omega_d t} + C_2 e^{-j\omega_d t}\right] = e^{-\zeta\omega_n t}\left[A_1 \cos\omega_d t + A_2 \sin\omega_d t\right] = Ae^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$A_1 = C_1 + C_2$ and $A_2 = j(C_1 - C_2)$

$C_1 = \frac{1}{2}(A_1 - jA_2)$ and $C_2 = \frac{1}{2}(A_1 + jA_2)$

$A = \sqrt{A_1^2 + A_2^2}$ and $\tan\phi = \frac{A_1}{A_2}$

ICs give: $A_1 = x_o$ and $A_2 = \frac{v_o + \zeta\omega_n x_o}{\omega_d}$

Logarithmic Decrement per Radian: $\alpha = \frac{1}{2\pi n}\ln r = \frac{\zeta}{\sqrt{1-\zeta^2}}$

where $r = \frac{x(t)}{x(t+nT)}$ = decay ratio over n complete cycles. For small ζ: $\zeta \cong \alpha$

Case 2: Overdamped ($\zeta > 1$)

Poles are real and negative: $\lambda_1, \lambda_2 = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$

$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

$C_1 = \frac{v_0 - \lambda_2 x_0}{\lambda_1 - \lambda_2}$ and $C_2 = \frac{v_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}$

Case 3: Critically Damped ($\zeta = 1$)

Two identical poles: $\lambda_1 = \lambda_2 = \lambda = -\omega_n$

$x = (C_1 + C_2 t)e^{-\omega_n t}$ with $C_1 = x_0$ and $C_2 = v_0 + \omega_n x_0$

(overdamped). It follows that we may define the damping ratio as

$$\zeta = \text{damping ratio} = \frac{\text{damping constant}}{\text{damping constant for critically damped conditions}}$$

The main results for free (natural) response of a damped oscillator are given in Table 2.14. The response of a damped simple oscillator is shown in Figure 2.84.

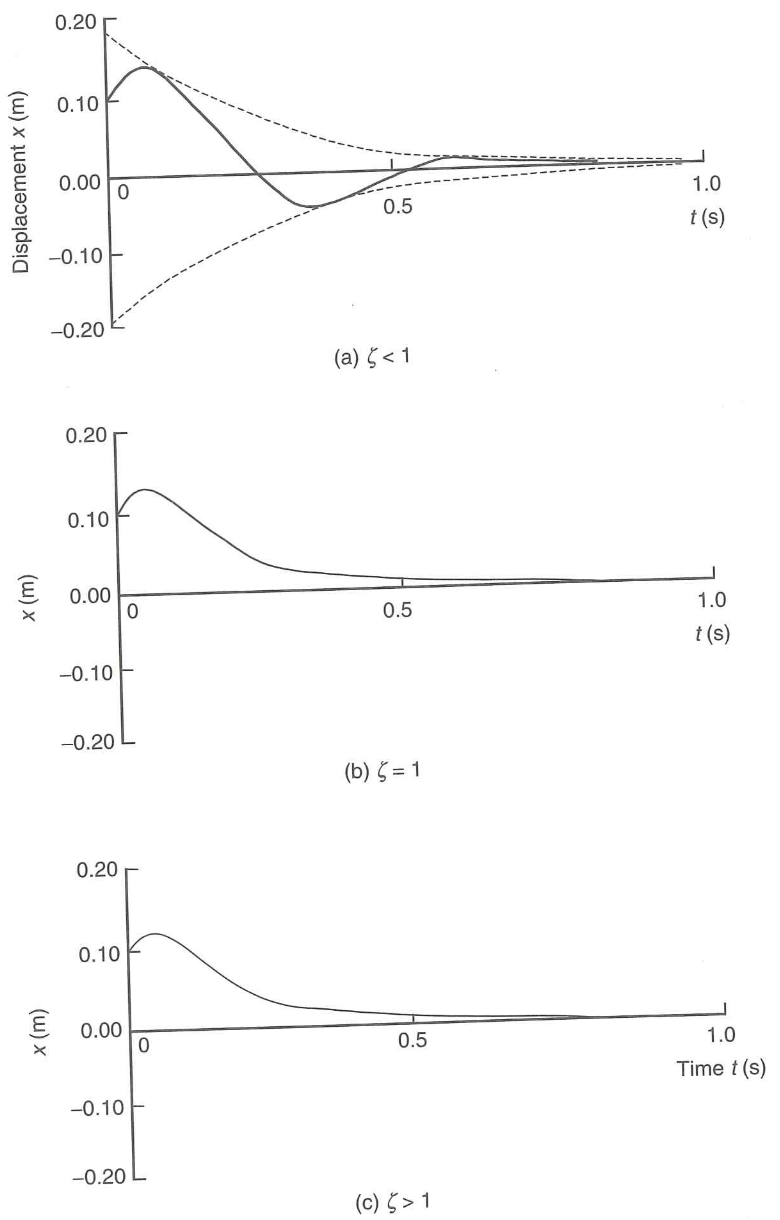

FIGURE 2.84
Free response of a damped oscillator: (a) Underdamped; (b) Critically Damped; (c) Overdamped.

2.13.5.3 Forced Response of a Damped Oscillator

The forced response depends on both the natural characteristics of the system (free response) and the nature of the input. Mathematically, as noted before, the total response is the sum of the homogeneous solution and the particular solution. Consider a damped simple oscillator, with input $u(t)$ scaled such that it has the same units as the response y; thus

$$\ddot{y}+2\zeta\omega_n\dot{y}+\omega_n^2 y=\omega_n^2 u(t) \quad (2.200)$$

We will consider the response of this system to three types of inputs:

1. Impulse input
2. Step input
3. Harmonic (sinusoidal) input

Impulse Response: Many important characteristics of a system can be studied by analyzing the system response to a baseline excitation such as an impulse, a step, or a sinusoidal (harmonic) input. Characteristics which may be studied in this manner may include system stability, speed of response, time constants, damping properties, and natural frequencies. As well, an insight into the system response to an arbitrary excitation can be gained. Responses to such test inputs can also serve as the basis for system comparison. For example, it is possible to determine the degree of nonlinearity in a system by exciting it with two input intensity levels, separately, and checking whether the proportionality is retained at the output; or when the excitation is harmonic, whether limit cycles are encountered by the response.

The response of the system (Equation 2.200) to a unit impulse $u(t)=\delta(t)$ may be conveniently determined by the Laplace transform approach (See Appendix A). However, in the present section we will use a time-domain approach, instead. First integrate Equation 2.200, over the almost zero interval from $t=0^-$ to $t=0^+$. We get

$$\dot{y}(0^+)=\dot{y}(0^-)-2\zeta\omega_n[y(0^+)-y(0^-)]-\omega_n^2\int_{0^-}^{0^+} y\,dt+\omega_n^2\int_{0^-}^{0^+} u(t)\,dt \quad (2.201)$$

Suppose that the system starts from rest. Hence, $y(0^-)=0$ and $\dot{y}(0^-)=0$. Also, when an impulse is applied over an infinitesimally short time period $[0^-,0^+]$ the system will not be able to move through a finite distance during that time. Hence, $y(0^+)=0$ as well, and furthermore, the integral of y on the RHS of Equation 2.201 also will be zero. Now by definition of a unit impulse, the integral of u on the RHS of Equation 2.201 will be unity. Hence, we have $\dot{y}(0^+)=\omega_n^2$. It follows that as soon as a unit impulse is applied to the system (Equation 2.200) the initial conditions will become

$$y(0^+)=0 \quad \text{and} \quad \dot{y}(0^+)=\omega_n^2 \quad (2.202)$$

Also, beyond $t=0^+$ the excitation $u(t)=0$, according to the definition of an impulse. Hence, the impulse response of the system (Equation 2.200) is obtained by its homogeneous solution (as carried out before, under free response), but with the initial conditions given by Equation 2.202. The three cases of damping ratio ($\zeta<1, \zeta>1$, and $\zeta=1$) should be considered separately. Then, we can conveniently obtain the following results:

$$y_{\text{impulse}}(t)=h(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}\exp(-\zeta\omega_n t)\sin\omega_d t \quad \text{for } \zeta<1 \quad (2.203a)$$

$$y_{\text{impulse}}(t)=h(t)=\frac{\omega_n}{2\sqrt{\zeta^2-1}}[\exp\lambda_1 t-\exp\lambda_2 t] \quad \text{for } \zeta>1 \quad (2.203b)$$

$$y_{\text{impulse}}(t)=h(t)=\omega_n^2 t\exp(-\omega_n t) \quad \text{for } \zeta=1 \quad (2.203c)$$

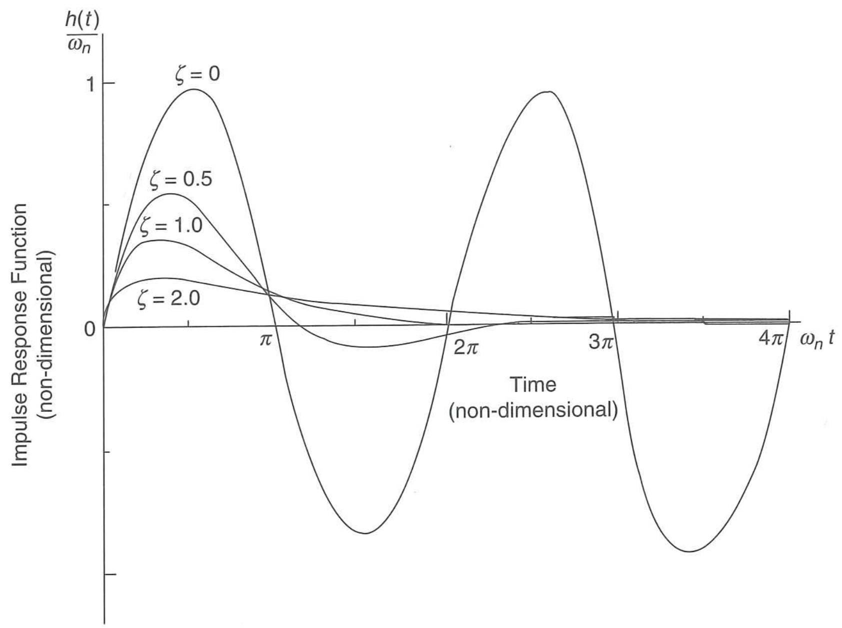

FIGURE 2.85
Impulse-response function of a damped oscillator.

An explanation concerning the dimensions of $h(t)$ is appropriate at this juncture. Note that $y(t)$ has the same dimensions as $u(t)$. Since $h(t)$ is the response to a unit impulse $\delta(t)$, it follows that these two have the same dimensions. The magnitude of $\delta(t)$ is represented by a unit area in the $u(t)$ versus t plane. Consequently, $\delta(t)$ has the dimensions of (1/time) or (frequency). It follows that $h(t)$ also has the dimensions of (1/time) or (frequency).

The impulse-response functions given by Equation 2.203 are plotted in Figure 2.85 for some representative values of damping ratio. It should be noted that, for $0 < \zeta < 1$, the angular frequency of damped vibrations is ω_d, which is smaller than the undamped natural frequency ω_n.

The Riddle of Zero Initial Conditions: For a second-order system, zero initial conditions correspond to $y(0)=0$ and $\dot{y}(0)=0$. It is clear from Equations 2.203 that $h(0)=0$, but $\dot{h}(0)\neq 0$, which appears to violate the zero-initial-conditions assumption. This situation is characteristic in a system response to an impulse and its higher derivatives. This may be explained as follows. When an impulse is applied to a system at rest (zero initial state), the highest derivative of the system differential equation momentarily becomes infinity. As a result, the next lower derivative becomes finite (nonzero) at $t=0^+$. The remaining lower derivatives maintain their zero values at that instant. When an impulse is applied to the mechanical system given by Equation 2.200 for example, the acceleration $\ddot{y}(t)$ becomes infinity, and the velocity $\dot{y}(t)$ takes a nonzero (finite) value shortly after its application ($t = 0^+$). The displacement $y(t)$, however, would not have sufficient time to change at $t=0^+$. The impulse input is therefore equivalent to a velocity initial condition in this case. This initial condition is determined by using the integrated version (Equation 2.201) of the system Equation 2.200, as has been done.

Step Response: A unit step excitation is defined by

$$\begin{aligned} \mathcal{U}(t) &= 1 \quad \text{for } t>0 \\ &= 0 \quad \text{for } t\leq 0 \end{aligned} \tag{2.204}$$

Unit impulse excitation $\delta(t)$ may be interpreted as the time derivative of $\mathcal{U}(t)$:

$$\delta(t) = \frac{d\mathcal{U}(t)}{dt} \tag{2.205}$$

Note that Equation 2.205 re-establishes the fact that for nondimensional $\mathcal{U}(t)$, the dimension of $\delta(t)$ is (time)$^{-1}$. Since a unit step is the integral of a unit impulse, the step response can be obtained directly as the integral of the impulse response; thus

$$y_{\text{step}}(t) = \int_0^t h(\tau)\,d\tau \tag{2.206}$$

This result also follows from the convolution integral (2.165) because, for a delayed unit step, we have

$$\begin{aligned} \mathcal{U}(t-\tau) &= 1 \quad \text{for } \tau<t \\ &= 0 \quad \text{for } \tau\geq t \end{aligned} \tag{2.207}$$

Thus, by integrating Equations 2.203 with zero initial conditions the following results are obtained for step response:

$$y_{\text{step}}(t) = 1-\frac{1}{\sqrt{1-\zeta^2}}\exp(-\zeta\omega_n t)\sin(\omega_d t+\phi) \quad \text{for } \zeta<1 \tag{2.208a}$$

$$y_{\text{step}} = 1-\frac{1}{2\sqrt{1-\zeta^2}\,\omega_n}[\lambda_1\exp\lambda_2 t-\lambda_2\exp\lambda_1 t] \quad \text{for } \zeta>1 \tag{2.208b}$$

$$y_{\text{step}} = 1-(\omega_n t+1)\exp(-\omega_n t) \quad \text{for } \zeta=1 \tag{2.208c}$$

with

$$\cos\phi = \zeta \tag{2.195}$$

The step responses given by Equations 2.208 are plotted in Figure 2.86, for several values of damping ratio.

Note that, since a step input does not cause the highest derivative of the system equation to approach infinity at $t=0^+$, the initial conditions which are required to solve the system equation remain unchanged at $t=0^+$, provided that there are no derivative terms on the input side of the system equation. If there are derivative terms in the input, then, a step will be converted into an impulse (due to differentiation), and the situation can change.

It should be emphasized that the response given by the convolution integral is based on the assumption that the initial state is zero. Hence, it is known as the *zero-state response*. In particular, the impulse response assumes a zero initial state. As we have stated, the

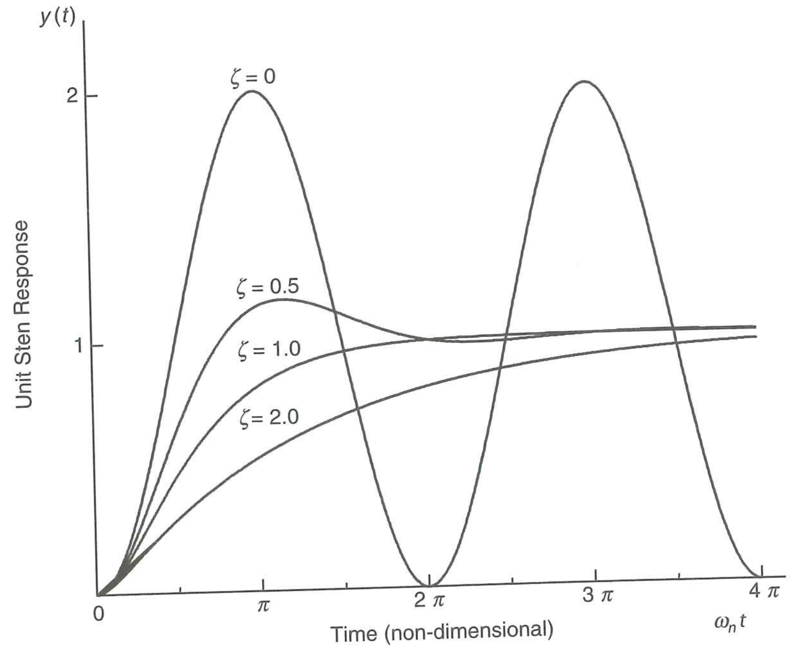

FIGURE 2.86
Unit step response of a damped oscillator.

zero-state response is not necessarily equal to the "particular solution" in mathematical analysis. Also, as t increases $(t \to \infty)$, this solution approaches the *steady-state response* denoted by y_{ss}, which is typically the particular solution. The impulse response of a system is the inverse Laplace transform of the transfer function. Hence, it can be determined using Laplace transform techniques (See Appendix A). Some useful concepts of forced response are summarized in Table 2.15.

2.13.5.4 *Response to Harmonic Excitation*

In many engineering problems the primary excitation typically has a repetitive periodic nature and in some cases this periodic input function may even be purely sinusoidal. Examples are excitations due to mass eccentricity and misalignments in rotational components, tooth meshing in gears, and electromagnetic devices excited by ac or periodic electrical signals. In basic terms, the frequency response of a dynamic system is the response to a pure sinusoidal excitation. As the amplitude and the frequency of the excitation are changed, the response also changes. In this manner the response of the system over a range of excitation frequencies can be determined, and this set of input-output data represents the frequency response. In this case frequency (ω) is the independent variable and hence we are dealing with the *frequency domain*.

Consider the damped oscillator with a harmonic input, as given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = a\cos\omega t = u(t) \tag{2.210}$$

TABLE 2.15
Useful Concepts of Forced Response

Convolution Integral: Response $y = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau$

where u = excitation (input) and h = impulse response function (response to a unit impulse input).

Damped Simple Oscillator: $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u(t)$

Poles (eigenvalues) $\lambda_1, \lambda_2 = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\,\omega_n$ for $\zeta \geq 1$

$= -\zeta\omega_n \pm j\omega_d$ for $\zeta < 1$

ω_n = undamped natural frequency, ω_d = damped natural frequency

ζ = damping ratio.

Note: $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

Impulse Response Function: (zero ICs)

$h(t) = \dfrac{\omega_n}{\sqrt{1-\zeta^2}}\exp(-\zeta\omega_n t)\sin\omega_d t$ for $\zeta < 1$

$= \dfrac{\omega_n}{2\sqrt{\zeta^2-1}}[\exp\lambda_1 t - \exp\lambda_2 t]$ for $\zeta > 1$

$= \omega_n^2 t\exp(-\omega_n t)$ for $\zeta = 1$

Unit Step Response: (zero ICs)

$y(t)_{step} = 1 - \dfrac{1}{\sqrt{1-\zeta^2}}\exp(-\zeta\omega_n t)\sin(\omega_d t + \phi)$ for $\zeta < 1$

$= 1 - \dfrac{1}{2\sqrt{\zeta^2 - 1}\,\omega_n}[\lambda_1\exp\lambda_2 t - \lambda_2\exp\lambda_1 t]$ for $\zeta > 1$

$= 1 - (\omega_n t + 1)\exp(-\omega_n t)$ for $\zeta = 1$

$\cos\phi = \zeta$

Note: Impulse Response $= \dfrac{d}{dt}$ (Step Response).

The particular solution x_p that satisfies Equation 2.210 is of the form (see Table 2.12)

$$x_p = a_1\cos\omega t + a_2\sin\omega t \text{ \{Except for the case: } \zeta = 0 \text{ and } \omega = \omega_n\} \tag{2.211}$$

where the constants a_1 and a_2 are determined by substituting Equation 2.211 into the system Equation 2.210 and equating the like coefficient; the method of undertermined coefficients. We will consider several important cases.

1. *Undamped Oscillator with Excitation Frequency ≠ Natural Frequency:*

We have

$$\ddot{x} + \omega_n^2 x = a\cos\omega t \quad \text{with} \quad \omega \neq \omega_n \tag{2.212}$$

Homogeneous solution:

$$x_h = A_1\cos\omega_n t + A_2\sin\omega_n t \tag{2.213}$$

Particular solution:

$$x_p = \frac{a}{(\omega_n^2-\omega^2)}\cos\omega t \tag{2.214}$$

NOTE It can be easily verified that x_p given by Equation 2.214 satisfies the forced system Equation 2.210, with $\zeta = 0$. Hence it is a particular solution.

Complete solution:

$$x = \underbrace{A_1\cos\omega_n t + A_2\sin\omega_n t}_{H} + \underbrace{\frac{a}{(\omega_n^2-\omega^2)}\cos\omega t}_{P} \tag{2.215}$$

Satisfies the homogeneous equation. Satisfies the equation with input.

Now A_1 and A_2 are determined using the initial conditions (ICs):

$$x(0) = x_o \quad \text{and} \quad \dot{x}(0) = v_o \tag{2.216}$$

Specifically, we obtain

$$x_o = A_1 + \frac{a}{\omega_n^2-\omega^2} \tag{2.217a}$$

$$v_o = A_2\omega_n \tag{2.217b}$$

Hence, the complete response is

$$x = \underbrace{\left[x_o - \frac{a}{(\omega_n^2-\omega^2)}\right]\cos\omega_n t + \frac{v_o}{\omega_n}\sin\omega_n t}_{H} \quad \underbrace{+\frac{a}{\omega_n^2-\omega^2}\cos\omega t}_{P} \tag{2.218a}$$

Homogeneous solution. Particular solution.

$$= \underbrace{x_o\cos\omega_n t + \frac{v_o}{\omega_n}\sin\omega_n t}_{X} + \underbrace{\frac{a}{(\omega_n^2-\omega^2)}\underbrace{[\cos\omega t - \cos\omega_n t]}_{2\sin\frac{(\omega_n+\omega)}{2}t\sin\frac{(\omega_n-\omega)}{2}t}}_{F} \tag{2.218b}$$

Free response (Depends only on ICs) Comes from x_h; Sinusodal at ω_n

*Forced response (depends on input) Comes from both x_h and x_p.
*Will exhibit a beat phenomenon for small $\omega_n - \omega$; i.e., $\frac{(\omega_n+\omega)}{2}$ wave "modulated" by $\frac{(\omega_n-\omega)}{2}$ wave.

This is a "stable" response in the sense of bounded-input bounded-output (BIBO) stability, as it is bounded and does not increase steadily.

NOTE If there is no forcing excitation, the homogeneous solution H and the free response X will be identical. With a forcing input, the natural response (the homogeneous solution) will be influenced by it in general, as clear from Equation 2.218a.

2. ***Undamped Oscillator with $\omega = \omega_n$ (Resonant Condition):***
This is the degenerate case. In this case the x_p that was used before is no longer valid because, otherwise the particular solution could not be distinguished from the homogeneous solution and the former would be completely absorbed into the latter. Instead, in view of the "double-integration" nature of the forced system equation when $\omega = \omega_n$, we use the particular solution (P):

$$x_p = \frac{at}{2\omega}\sin\omega t \tag{2.219}$$

This choice of particular solution is strictly justified by the fact that it satisfies the forced system equation.
Complete solution:

$$x = A_1\cos\omega t + A_2\sin\omega t + \frac{at}{2\omega}\sin\omega t \tag{2.220}$$

ICs:

$$x(0) = x_o \quad \text{and} \quad \dot{x}(0) = v_o.$$

By substitution we get

$$x_o = A_1 \tag{2.221}$$

$$v_o = \omega A_2 \tag{2.222}$$

The total response:

$$\underbrace{x = x_o\cos\omega t + \frac{v_o}{\omega}\sin\omega t}_{X} \quad \underbrace{+\frac{at}{2\omega}\sin\omega t}_{F} \tag{2.223}$$

Free response (Depends on ICs) *Sinusoidal with frequency ω. Forced response (Depends on Input) *Amplitude increases linearly.

Since the forced response increases steadily, this is an unstable response in the bounded-input-bounded-output (BIBO) sense. Furthermore, the homogeneous solution H and the free response X are identical, and the particular solution P is identical to the forced response F in this case.

Note that the same system (undamped oscillator) gives a bounded response for some excitations while producing an unstable (steady linear increase) response when the excitation frequency is equal to its natural frequency. Hence, the system is not quite

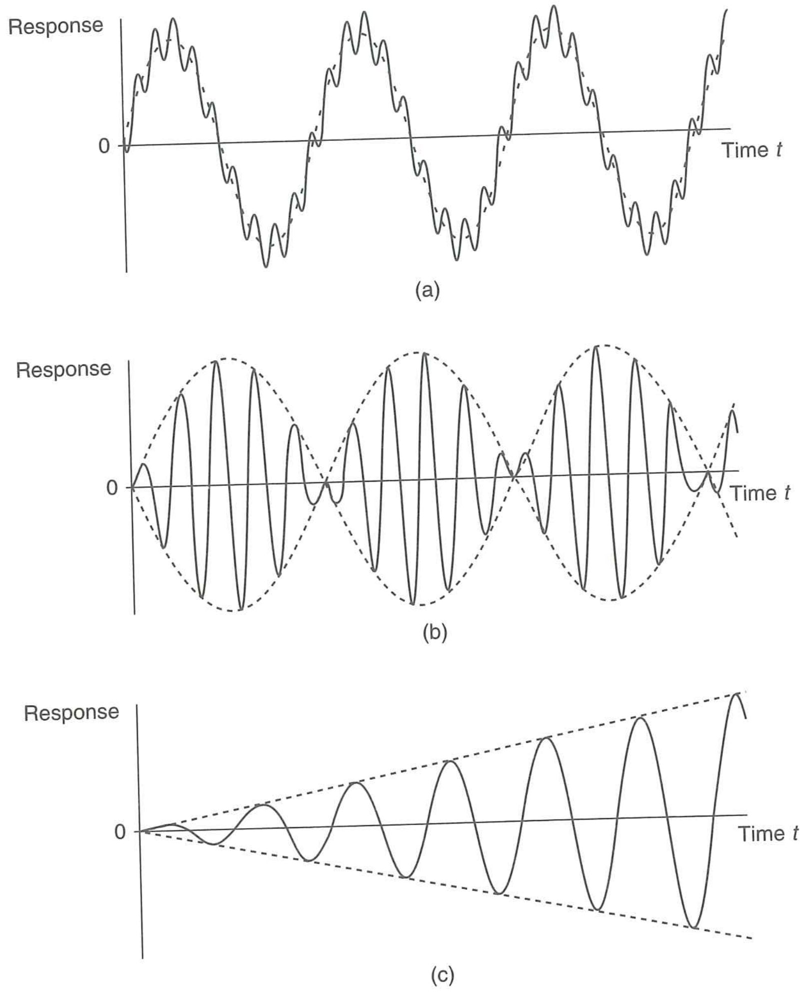

FIGURE 2.87
Forced response of a harmonic-excited undamped simple oscillator: (a) For a large frequency difference; (b) For a small frequency difference (beat phenomenon); (c) Response at resonance.

unstable, but is not quite stable either. In fact, the undamped oscillator is said to be marginally stable. When the excitation frequency is equal to the natural frequency it is reasonable for the system to respond in a complementary and steadily increasing manner because this corresponds to the most "receptive" excitation. Specifically, in this case, the excitation complements and reinforces the natural response of the system. In other words, the system is "in resonance" with the excitation, and the condition is called a *resonance*. Later on we will address this aspect for the more general case of a damped oscillator.

Figure 2.87 shows typical forced responses of an undamped oscillator for a large difference in excitation and natural frequencies (Case 1); for a small difference in excitation and natural frequencies (also Case 1), where a beat-phenomenon is clearly manifested; and for the resonant case (Case 2).

3. *Damped Oscillator:*

The equation of forced motion is

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = a\cos\omega t \tag{2.224}$$

Particular Solution: Since derivatives of both odd order and even order are present in this equation, the particular solution should have terms corresponding to odd and even derivatives of the forcing function (i.e., sin ωt and cos ωt). Hence, the appropriate particular solution will be of the form:

$$x_p = a_1\cos\omega t + a_2\sin\omega t \tag{2.225}$$

Substitute Equation 2.225 into Equation 2.224. We get

$$-\omega^2 a_1\cos\omega t - \omega^2 a_2\sin\omega t + 2\zeta\omega_n[-\omega a_1\sin\omega t + \omega a_2\cos\omega t]$$
$$+\omega_n^2[a_1\cos\omega t + a_2\sin\omega t] = a\cos\omega t$$

Equate like coefficients:

$$-\omega^2 a_1 + 2\zeta\omega_n\omega a_2 + \omega_n^2 a_1 = a$$
$$-\omega^2 a_2 - 2\zeta\omega_n\omega a_1 + \omega_n^2 a_2 = 0$$

Hence, we have

$$\left(\omega_n^2-\omega^2\right) a_1 + 2\zeta\omega_n\omega a_2 = a \tag{2.226a}$$

$$-2\zeta\omega_n\omega a_1 + \left(\omega_n^2-\omega^2\right) a_2 = 0 \tag{2.226b}$$

This can be written in the vector-matrix form:

$$\begin{bmatrix}(\omega_n^2-\omega^2) & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & (\omega_n^2-\omega^2)\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}a\\0\end{bmatrix} \tag{2.226c}$$

Solution is

$$\begin{bmatrix}a_1\\a_2\end{bmatrix}=\frac{1}{D}\begin{bmatrix}(\omega_n^2-\omega^2) & -2\zeta\omega_n\omega \\ 2\zeta\omega_n\omega & (\omega_n^2-\omega^2)\end{bmatrix}\begin{bmatrix}a\\o\end{bmatrix} \tag{2.227}$$

with the determinant

$$D=\left(\omega_n^2-\omega^2\right)^2+(2\zeta\omega_n\omega)^2 \tag{2.228}$$

TABLE 2.16
Harmonic Response of a Simple Oscillator

Undamped Oscillator: $\ddot{x}+\omega_n^2 x = a\cos\omega t$; $x(0) = x_o$, $\dot{x}(0) = v_o$

For $\omega \neq \omega_n$:

$$x = \underbrace{x_0\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t}_{X} + \underbrace{\frac{a}{\omega_n^2-\omega^2}[\cos\omega t - \cos\omega_n t]}_{F}$$

For $\omega = \omega_n$ **(resonance):** $x = \text{Same } X + \frac{at}{2\omega}\sin\omega t$

Damped Oscillator: $\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = a\cos\omega t$

$$x = H + \underbrace{\frac{a}{|\omega_n^2-\omega^2+2j\zeta\omega_n\omega|}\cos(\omega t-\phi)}_{P}$$

where, $\tan\phi = \frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}$; $\phi =$ phase lag.

Particular solution P is also the steady-state response.

Homogeneous solution $H = A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}$

where, λ_1 and λ_2 are roots of $\lambda^2+2\zeta\omega_n\lambda+\omega_n^2=0$ (characteristic equation)

A_1 and A_2 are determined from ICs: $x(0)=x_0$, $\dot{x}(0)=v_0$

Resonant Frequency: $\omega_r = \sqrt{1-2\zeta^2}\,\omega_n$

The magnitude of P will peak at resonance.

Damping Ratio: $\zeta = \frac{\Delta\omega}{2\omega_n} = \frac{\omega_2-\omega_1}{\omega_2+\omega_1}$ for low damping

where, $\Delta\omega =$ half-power bandwidth $= \omega_2-\omega_1$

Note: Q-factor $= \frac{\omega_n}{\Delta\omega} = \frac{1}{2\zeta}$ for low damping

On simplification, we get

$$a_1 = \frac{(\omega_n^2-\omega^2)}{D}a \quad (2.229a)$$

$$a_2 = \frac{2\zeta\omega_n\omega}{D}a \quad (2.229b)$$

This is the method of "undetermined coefficients."

Some useful results on the frequency response of a simple oscillator are summarized in Table 2.16.

2.13.6 Response Using Laplace Transform

Transfer function concepts were discussed in previous sections, and transform techniques are outlined in Appendix A. Once a transfer function model of a system is available, its

response can be determined using the Laplace transform approach. The steps are:

1. Using Laplace transform table (Appendix A) determine the Laplace transform ($U(s)$) of the input.
2. Multiply by the transfer function ($G(s)$) to obtain the Laplace transform of the output: $Y(s)=G(s)U(s)$
3. Convert the expression in Step 2 into a convenient form (e.g., by partial fractions).
4. Using Laplace transform table, obtain the inverse Laplace transform of $Y(s)$, which gives the response $y(t)$.

Let us illustrate this approach by determining again the step response of a simple oscillator.

2.13.6.1 Step Response Using Laplace Transforms

Consider the oscillator system given by Equation 2.200. Since $\mathcal{L}\mathcal{U}(t) = 1/s$, the unit step response of the dynamic system (Equation 2.200) can be obtained by taking the inverse Laplace transform of

$$Y_{\text{step}}(s) = \frac{1}{s}\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)} \quad (2.230a)$$

To facilitate using the Laplace transform table, partial fractions of Equation 2.230 are determined in the form

$$\frac{a_1}{s}+\frac{a_2+a_3 s}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

in which, the constants a_1, a_2, and a_3 are determined by comparing the numerator polynomial; thus,

$$\omega_n^2 = a_1(s^2+2\zeta\omega_n s+\omega_n^2)+s(a_2+a_3 s)$$

Then, $a_1=1$, $a_2=-2\zeta\omega_n$, and $a_3=-1$.
Hence,

$$Y_{\text{Step}}(s) = \frac{1}{s}+\frac{-s-\zeta\omega_n}{(s^2+2\zeta\omega_n s+\omega_n^2)} \quad (2.230b)$$

Next, using Laplace transform tables, the inverse transform of Equation 2.230b is obtained, and verified to be identical to Equation 2.208.

2.13.7 Computer Simulation

Simulation of the response of a dynamic system by using a digital computer is perhaps the most convenient and popular approach to response analysis. An important advantage is that any complex, nonlinear, and time variant system may be analyzed in this manner.

The main disadvantage is that the solution is not analytic and valid only for a specific excitation. Of course, symbolic approaches of obtaining analytical solutions using a digital computer are available as well. We will consider here numerical simulation only.

The digital simulation typically involves integration of a differential equation of the form

$$\dot{y} = f(y, u, t) \tag{2.231}$$

The most straightforward approach to digital integration of this equation is by using *trapezoidal rule*, which is Euler's method, as given by

$$y_{n+1} = y_n + f(y_n, u_n, t_n)\Delta t \quad n = 0, 1, \ldots \tag{2.232}$$

Here t_n is the nth time instant, $u_n = u(t_n)$, $y_n = y(t_n)$; and Δt is the integration time step ($\Delta t = t_{n+1} - t_n$). This approach is generally robust. But depending on the nature of the function f, the integration can be ill behaved. Also, Δt has to be chosen sufficiently small.

For complex nonlinearities, a better approach of digital integration is the Runge-Kutta method. In this approach, in each time step, first the following four quantities are computed:

$$g_1 = f(y_n, u_n, t_n) \tag{2.233a}$$

$$g_2 = f\left[\left(y_n + g_1\frac{\Delta t}{2}\right), u_{n+\frac{1}{2}}, \left(t_n + \frac{\Delta t}{2}\right)\right] \tag{2.233b}$$

$$g_3 = f\left[\left(y_n + g_2\frac{\Delta t}{2}\right), u_{n+\frac{1}{2}}, \left(t_n + \frac{\Delta t}{2}\right)\right] \tag{2.233c}$$

$$g_4 = f[(y_n + g_3\Delta t), u_{n+1}, t_{n+1}] \tag{2.233d}$$

Then, the integration step is carried out according to

$$y_{n+1} = y_n + (g_1 + 2g_2 + 2g_3 + g_4)\frac{\Delta t}{6} \tag{2.234}$$

Note that $u_{n+\frac{1}{2}} = u\left(t_n + \frac{\Delta t}{2}\right)$.

Other sophisticated approaches of digital simulation are available as well. Perhaps the most convenient computer-based approach to simulation of a dynamic model is by using a graphic environment that uses block diagrams. Several such environments are commercially available. One that is widely used is SIMULINK, which is an extension to MATLAB (See Appendix B).

2.14 Problems

2.1 What is a "dynamic" system, a special case of any general system?
A typical input variable is identified for each of the following examples of dynamic systems. Give at least one output variable for each system.

a. Human body: neuroelectric pulses
b. Company: information
c. Power plant: fuel rate
d. Automobile: steering wheel movement
e. Robot: voltage to joint motor
f. Highway bridge: vehicle force

2.2 Real systems are nonlinear. Under what conditions a linear model is sufficient in studying a real systems?
Consider the following system equations:

a. $\ddot{y} + (2\sin\omega t + 3)\dot{y} + 5y = u(t)$
b. $3\dddot{y} - 2y = u(t)$
c. $3\ddot{y} + 2\dot{y}^3 + y = u(t)$
d. $5\ddot{y} + 2\dot{y} + 3y = 5u(t)$
    i. Which ones of these are linear?
    ii. Which ones are nonlinear?
    iii. Which ones are time-variant?

2.3 Give four categories of uses of dynamic modeling.
List advantages and disadvantages of experimental modeling over analytical modeling.

2.4 What are the basic lumped elements of
i. a mechanical system
ii. an electrical system?

Indicate whether a distributed-parameter method is needed or a lumped-parameter model is adequate in the study of following dynamic systems:

a. vehicle suspension system (motion)
b. elevated vehicle guideway (transverse motion)
c. oscillator circuit (electrical signals)
d. environment (weather) system (temperature)
e. aircraft (motion and stresses)
f. large transmission cable (capacitance and inductance).

NOTE: Variables/parameters of interest are given in parentheses.

2.5 Write down the order of each of the systems shown in Figure P2.5.

2.6

a. Give logical steps of the analytical modeling process for a general physical system.
b. Once a dynamic model is derived, what other information would be needed for analyzing its time response (or for computer simulation)?
c. A system is divided into two subsystems, and models are developed for these subsystems. What other information would be needed to obtain a model for the overall system?

2.7 Various possibilities of model development for a physical system are shown in Figure P2.7. Give advantages and disadvantages of the SM approach of developing an approximate model in comparison to a combined DM+MR approach.