INTRODUCTION TO DYNAMICS AND CONTROL

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CHAPTER 4

Response of First-Order and Second-Order Systems

4.1 INTRODUCTION

In Chapter 1 we discussed the response of linear systems in a general way by using a generic differential equation to describe the system behavior, without actually deriving the equation. In this chapter, we derive first the differential equations governing the behavior of simple mechanical and electrical systems and then produce their solution by the methods developed in Chapter 1. The discussion is confined to first-order and second-order systems. In deriving the differential equations describing mechanical and electrical systems, it becomes evident that the equations for these two classes of systems are entirely analogous. Hence, solutions obtained for mechanical systems are valid for electrical systems and vice-versa. Although for the most part the various concepts introduced are common to mechanical and electrical systems, applications presented in this chapter tend to favor mechanical systems.

The behavior of first-order systems is markedly different from the behavior of second-order systems. Specifically, the free response of first-order systems tends to have an aperiodic nature, in contrast to the response of second-order systems, which tends to be oscillatory. Exceptions to the latter are mechanical systems with relatively heavy damping and electrical systems with relatively large resistance.

Both first-order and second-order systems are mathematical idealizations of actual physical systems. As far as mechanical systems are concerned, first-order systems are less common. Nevertheless, on many occasions, first-order systems can provide useful information concerning system behavior. Moreover, they are of interest mathematically, as more complex systems can be formulated to resemble first-order systems. Hence, their study is fully warranted. Second-order systems are considerably more common, as they are used as mathematical models for a large variety of systems. Refined models of engineering systems are often of high order. In fact, distributed systems are of infinite order. However, in many instances it is possible to gain substantial insight into the behavior of systems from low-order models. Moreover, as shown later in this text, high-order systems can be decomposed into a set of low-order ones.

Although the behavior of first-order systems is different from the behavior of second-order systems, the mathematical techniques for obtaining their response are the same. For this reason, we choose to treat both first-order and second-order systems in a single chapter and let the nature of the excitation dictate the order of presentation of the material.

4.2 DIFFERENTIAL EQUATIONS OF MOTION FOR MECHANICAL SYSTEMS

In Chapter 3, we presented a very fundamental discussion of particle dynamics, beginning with Newton's laws and ending with the motion of planets and satellites. In the process, we introduced an entire spectrum of basic concepts, such as impulse, momentum, work, and energy. In this section, we expand the discussion by deriving the differential equations of motion for certain low-order mechanical systems of particular interest in vibrations and control. Then, in subsequent sections, we devote a great deal of attention to the solution of these equations of motion.

Before we can derive the system differential equations of motion, it will prove convenient to introduce certain definitions and notations. We wish to distinguish between variables and components, or elements. Variables refer to quantities describing excitation and response, and they are functions of time. Components, or elements, refer to parts of the system and they are identified with the system parameters. Although they can depend on time, only constant parameters will be considered here.

In the case of mechanical systems, the variables can be identified as the force and the displacement. At times, the velocity or the acceleration can play the role of variable. Mechanical components are of three types: two that store energy and one that dissipates energy. In particular, *masses* store kinetic energy, *springs* store potential energy, and *dampers* dissipate energy.

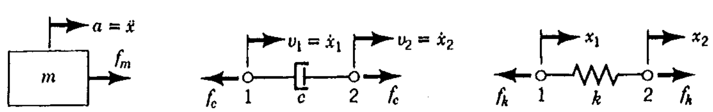
The relation between the excitation and response for the various mechanical components can be derived by means of the free-body diagrams shown in Fig. 4.1. Indeed, these relations are

$$f_m(t)=ma(t)=m\ddot{x}(t) \tag{4.1a}$$

$$f_c(t)=c[v_2(t)-v_1(t)]=c[\dot{x}_2(t)-\dot{x}_1(t)] \tag{4.1b}$$

$$f_k(t)=k[x_2(t)-x_1(t)] \tag{4.1c}$$

where the overdots represent derivatives with respect to time. Equation (4.1a) is merely an expression of Newton's second law of motion, and it states that a force



(a) (b) (c)

FIGURE 4.1

$f_m(t)$ causes the mass m to move with the acceleration $a(t)=\ddot{x}(t)$, where $x(t)$ is the displacement of the mass. Equation (4.1b) states that a force $f_c(t)$ applied at two terminal points 1 and 2 of a damper with the *coefficient of viscous damping c* will cause the two terminal points to separate with the relative velocity $v_2(t)-v_1(t)=\dot{x}_2(t)-\dot{x}_1(t)$. The equation is an expression of the fact that forces causing smooth shearing in viscous liquid are proportional to the relative velocity between the shearing layers. Viscous dampers are also known as *dashpots*. Finally, Eq. (4.1c) states that a force $f_k(t)$ applied at the two terminal points of a linear spring of *stiffness k* causes an elongation of the spring equal to $x_2(t)-x_1(t)$. The equation reflects the fact that in linear elasticity displacements are proportional to forces. The constant of proportionality k is also known as the *spring constant*. In SI units, the unit of force is the newton (N), the unit of mass is the kilogram (kg), and the unit of displacement is the meter (m). Of course, the unit of time is the second (s). It follows from Eqs. (4.1b) and (4.1c) that the unit of the viscous damping coefficient c is newton-seconds per meter (N·s/m) and that of the spring constant k is newtons per meter (N/m).

In the above discussion, it was assumed implicitly that the excitation–response relation is linear. This assumption is not always valid, and it is perhaps worth examining in detail. Letting $x_1=0$ and $x_2=x$ in Fig. 4.1*c*, a typical force–displacement relation for the spring is as shown in Fig. 4.2. For relatively small spring deflections, the deflections are proportional to the force, i.e., the spring is linear. Beyond a certain deflection $x=x_l$, however, small force increments produce relatively large deflection increments, so that beyond $x=x_l$ the spring becomes *nonlinear*. Such a spring is known as a *softening spring*. Note that a different type of nonlinear spring is the *hardening spring*, for which deflection increments require increasingly large force increments. Clearly, the spring can be regarded as linear provided the deflections satisfy the inequality $|x(t)|<x_l$. The range $-x_l<x(t)<x_l$ is known as the *linear range*.

In the above discussion, we have examined how various mechanical components act separately. We are now in a position to derive the differential equations governing the behavior of the assembled system. We confine ourselves to one of the simplest cases, namely, one in which only one variable is necessary to describe the system behavior. In the case of mechanical systems, this variable is ordinarily the displacement, referred to as a *coordinate*.

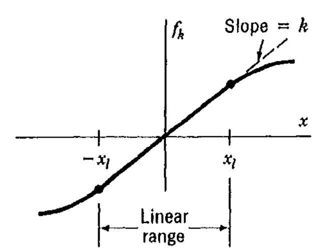
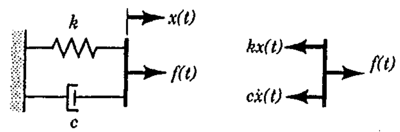


FIGURE 4.2



(a) (b)

FIGURE 4.3

Let us consider the system shown in Fig. 4.3*a*, which is known as a *damper–spring system*. Figure 4.3*b* shows a free-body diagram of the system. We assume that the bar transmitting the force $f(t)$ to the damper and spring is massless, so that Newton's second law, Eq. (3.4), reduces to

$$\sum F_x = f(t) - c\dot{x}(t) - kx(t) = 0 \tag{4.2}$$

which can be rearranged as follows:

$$c\dot{x}(t) + kx(t) = f(t) \tag{4.3}$$

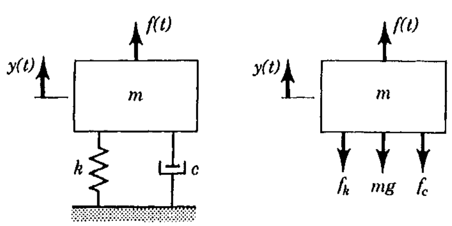
Equation (4.3) represents an ordinary differential equation of first order, so that the system is called a *first-order system*. The solution of Eq. (4.3) consists of two parts, the first corresponding to $f(t)=0$ and the second corresponding to $f(t)\neq 0$. They are known as the homogeneous solution and the particular solution, respectively. We shall discuss the solution of Eq. (4.3) later in this chapter.

Let us consider now the system of Fig. 4.4*a*. The system is commonly known as a *mass–damper–spring system* and is a simplified physical model representative of a large number of engineering systems, such as a piece of machinery on shock-absorbing mounts, or a buoy in viscous liquid. The corresponding free-body diagram is shown in Fig. 4.4*b*. Denoting the vertical displacement of the mass m from the *unstressed spring position* by $y(t)$, where the displacement is considered as positive in the upward direction, and using Newton's second law, we can write

$$\sum F_y = f(t) - f_c(t) - f_k(t) - mg = m\ddot{y}(t) \tag{4.4}$$

Letting $x_1 = 0$, $x_2 = y$ in Eqs. (4.1b) and (4.1c), introducing the results into Eq. (4.4), and rearranging, we obtain

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) + mg = f(t) \tag{4.5}$$



(a) (b)

FIGURE 4.4

which represents the system equation of motion. The equation can be simplified by eliminating the effect of the weight mg. Indeed, instead of measuring the displacement of m from the unstressed spring position, we can measure it from the *static equilibrium position*, the latter position being obtained from the former position by letting the mass undergo the static deflection (Fig. 4.5)

$$\delta_{st} = mg/k \tag{4.6}$$

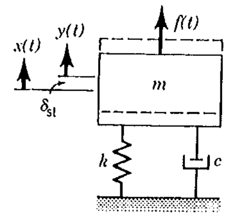


FIGURE 4.5

denoting the displacement from equilibrium by $x(t)$, introducing the coordinate transformation

$$y(t) = x(t) - \delta_{st} \tag{4.7}$$

into Eq. (4.5), and considering Eq. (4.6), we obtain

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \tag{4.8}$$

so that *by measuring the motion from the static equilibrium position, we can omit the weight mg*. The explanation for this fact is that in the process we omit not only the weight mg but also a prestress in the spring with a resultant force equal to $k\delta_{st}$, because these two forces cancel each other out according to Eq. (4.6).

The time derivative of highest order in Eq. (4.6) is the second derivative. Hence, the mass–damper–spring system is a *second-order system*. It is commonly referred to as a *single-degree-of-freedom system*.

Recalling the definition of linearity introduced in Section 1.3, we conclude that the systems described by Eqs. (4.3) and (4.8) are linear. In Section 1.4 we gave an example of a nonlinear differential equation and made the comment that it represented the equation of motion of a simple pendulum. At this point, it may prove of interest to verify the statement by deriving this equation. To this end, let us consider the simple pendulum shown in Fig. 4.6*a*. Using Newton's second law, in conjunction with the free-body diagram of Fig. 4.6*b*, we can write the equation of motion in the tangential direction

$$\sum F_t = -mg\sin\theta = a_t = mL\ddot{\theta} \tag{4.9}$$

where $a_t = L\ddot{\theta}$ is the acceleration of the mass m in the tangential direction, in which L is the length of the pendulum and $\ddot{\theta}$ is the angular acceleration. Writing the equation of motion in the tangential direction has the advantage that the string tension T, which is in the normal direction, does not appear in the equation.

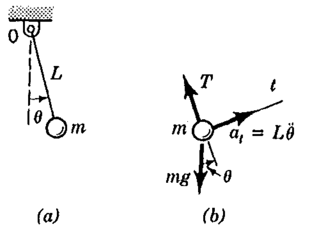


FIGURE 4.6

Note that we could have obtained essentially the same differential equation of motion by writing the moment equation about 0. Equation (4.9) can be rewritten in the form

$$mL\ddot{\theta} + mg \sin \theta = 0 \tag{4.10}$$

which is a nonlinear differential equation. Comparing Eqs. (4.8) and (4.10), we conclude that the simple pendulum represents an undamped second-order system with a nonlinear restoring force of magnitude $mg \sin \theta$. Moreover, comparing Eqs. (1.12) and (4.10), we conclude that Eq. (1.12) does indeed describe the motion of a simple pendulum, provided $c(t) = \theta(t)$, $a_0 = mL$, and $a_2 = mg$.

4.3 DIFFERENTIAL EQUATIONS FOR ELECTRICAL SYSTEMS

Electrical systems are encountered frequently in everyday life. Some of the most common ones are the light bulb, electric heaters, and toasters. They are also some of the simplest. A more complex one is the radio. Although not immediately evident, for the most part the behavior of electrical systems is analogous to the behavior of mechanical systems. In fact, it is possible to simulate mechanical systems by electrical analogs, and vice versa. Electrical systems are ordinarily known as *networks*, or *circuits*, and consist of arrays of electrical components, or elements. Quite often, electrical and mechanical elements are put together into so-called electromechanical devices. Typical examples of systems involving both electrical and mechanical elements are control systems (see Chapter 11). Before we proceed with the derivation of the differential equations describing the behavior of electrical systems, it is advisable to establish relations governing the behavior of the individual elements.

Electrical elements can be divided into three basic types: *inductors, resistors,* and *capacitors* (sometimes known as *condensers*). The inductors and capacitors store energy, and the resistors dissipate energy. Note that light bulbs, heaters, and toasters are mere resistors. Clearly, as in the case of mechanical elements, the electrical elements can be identified with the system parameters. As variables, we can identify the *voltage* $v(t)$ and the *current* $i(t)$.

The relations between the voltage and the current for the various electrical

elements are as follows:

$$v_L(t) = L\frac{di_L(t)}{dt} \tag{4.11a}$$

$$v_R(t) = Ri_R(t) \tag{4.11b}$$

$$v_C(t) = \frac{1}{C}\int i_C(t)\, dt \tag{4.11c}$$

The elements are shown in Figs. 4.7*a*, 4.7*b*, and 4.7*c* and can be identified as inductor, resistor, and capacitor, respectively. The parameters *L*, *R*, and *C* characterizing them are known as inductance, resistance, and capacitance, respectively. The units of voltage, current, inductance, resistance, and capacitance are volts, amperes (amp), henrys, ohms, and farads, respectively. Note that Eq. (4.11b) is the well-known *Ohm's law*.

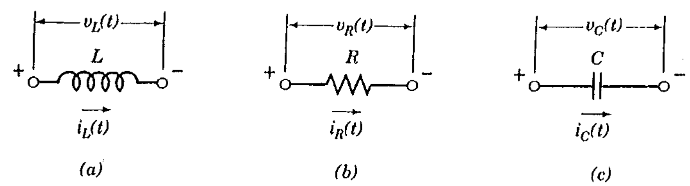


FIGURE 4.7

The analogy between the mechanical components and the electrical elements can be brought out by introducing the *charge* $q(t)$, which is related to the current by

$$i(t) = dq(t)/dt \tag{4.12}$$

where the unit of charge is the coulomb. Introducing Eq. (4.12) into Eqs. (4.11), the analogy becomes self-evident: The inductor is the electrical analog of the mass, the resistor is the analog of the damper, and the reciprocal of the capacitor is the analog of the spring. Moreover, a voltage source plays the role of a driving force. The analogy is made complete by observing that the resistor is the only electrical element dissipating energy.

The behavior of electrical networks is governed by *Kirchhoff's laws.* There are two such laws: the *voltage law* and the *current law.* In this chapter, we consider only the voltage law. The current law is introduced in Chapter 11.

The voltage law can be stated as follows: *The sum of voltage drops in the elements of a loop is equal to the sum of applied voltages.* A loop is an array of elements forming a closed circuit, such as the system shown in Fig. 4.8. This particular loop consists of a resistor *R*, a capacitor *C*, and a voltage source $v(t)$. Using Kirchhoff's law, we can write

$$v_R(t) + v_C(t) = v(t) \tag{4.13}$$

Introducing Eqs. (4.11b) and (4.11c) into Eq. (4.13) and recognizing that the current

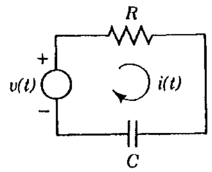


FIGURE 4.8

is the same within a given loop, we obtain

$$Ri(t)+\frac{1}{C}\int i(t)\,dt=v(t) \tag{4.14}$$

Moreover, considering Eq. (4.12), we can rewrite Eq. (4.14) in the form

$$R\dot{q}(t)+\frac{1}{C}q(t)=v(t) \tag{4.15}$$

which is entirely analogous to Eq. (4.3). Hence, the system of Fig. 4.8 is the electrical analog of the mechanical system of Fig. 4.3*a*. The system of Fig. 4.8 is known as an *RC* circuit.

Next, let us consider the electrical system shown in Fig. 4.9. It consists of an inductor *L*, a resistor *R*, a capacitor *C*, and a voltage source $v(t)$. We refer to this network as an *LRC* system. Using Kirchhoff's voltage law, we can write

$$v_L(t)+v_R(t)+v_C(t)=v(t) \tag{4.16}$$

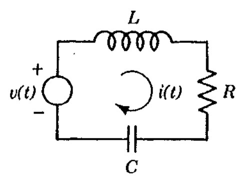


FIGURE 4.9

Introducing Eqs. (4.11) into Eq. (4.16), and recognizing that the current $i(t)$ is the same within a given loop, we obtain

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int i(t)\,dt=v(t) \tag{4.17}$$

Moreover, considering Eq. (4.12), we can rewrite Eq. (4.17) in the form

$$L\ddot{q}(t)+R\dot{q}(t)+\frac{1}{C}q(t)=v(t) \tag{4.18}$$

which is entirely analogous to Eq. (4.8) describing a mechanical system.

From the above discussion, we conclude that the behavior of the damper–spring system or of the mass–damper–spring system can be simulated by means of electrical analogs having the form of an *RC* network or of an *LRC* network, respectively. The analogous quantities become evident if one compares the coefficients of like derivatives in Eqs. (4.8) and (4.18).

The analogy between mechanical and electrical systems carries the implication that the response of the two types of systems to the same excitation is the same. Hence, in discussing the system response, it is not necessary to distinguish unduly between mechanical and electrical systems.

4.4 FREE RESPONSE OF FIRST-ORDER SYSTEMS

The *free response* is defined as the response of a system in the absence of external excitation. Hence, the free response represents simply the homogeneous part of the solution, which is due entirely to initial conditions. This definition is somewhat artificial, because quite often initial conditions are produced by some form of initial external excitation. Nevertheless, the definition is helpful, as in the case of linear systems it permits the derivation of the homogeneous solution independently of the particular solution.

In Section 4.3, we established an analogy between mechanical and electrical systems that allows us to ignore the distinction between the two types of systems and treat them as if they belonged to a larger single class. In view of this, we propose to classify systems according to the structure of the governing differential equations alone. This permits us to extend the analogy to a large variety of systems, as many mechanical systems are governed by differential equations that are similar in structure to Eq. (4.8), and the only difference lies in the system parameters. To carry out the joint analysis of similar systems, it will prove convenient to introduce certain groupings of parameters, some of them having the same units and some of them being nondimensional.

Let us consider a first-order homogeneous equation having the generic form

$$\dot{x}(t)+ax(t)=0 \tag{4.19}$$

where a is a parameter with the unit of reciprocal of seconds (s^{-1}). Equation (4.19) is subject to the initial condition

$$x(0)=x_0 \tag{4.20}$$

In the case of the mechanical system shown in Fig. 4.3*a* and described by Eq. (4.3), $x(t)$ is the displacement and

$$a=k/c \tag{4.21}$$

where k is the spring constant and c is the coefficient of viscous damping. In the case of the electrical system shown in Fig. 4.8 and described by Eq. (4.15), $x(t)$ is the charge and

$$a=1/RC \tag{4.22}$$

in which R is the resistance and C is the capacitance.

The classical approach to the solution of Eq. (4.19) is to assume a solution in the exponential form

$$x(t)=Ae^{\lambda t} \tag{4.23}$$

where A and λ are constant scalars. Introducing Eq. (4.23) into Eq. (4.19), we obtain

$$(\lambda+a)Ae^{\lambda t}=0 \tag{4.24}$$

Because $Ae^{\lambda t}$ cannot be zero for a nontrivial solution, Eq. (4.24) implies that

$$\lambda+a=0 \tag{4.25}$$

Equation (4.25) is known as the *characteristic equation* and has the solution

$$\lambda=-a \tag{4.26}$$

so that solution (4.23) becomes

$$x(t)=Ae^{-at} \tag{4.27}$$

where A is a constant of integration that can be determined by invoking the initial condition. Using Eq. (4.20) and introducing the notation

$$\tau=1/a \tag{4.28}$$

where τ is known as the system *time constant*, we can rewrite solution (4.27) in the form

$$x(t)=x_0e^{-t/\tau}u(t) \tag{4.29}$$

and we note that the solution was multiplied by the unit step function $u(t)$ in recognition of the fact that the response is zero for $t<0$. The solution of Eq. (4.29) is plotted in Fig. 4.10 for several values of τ. We note that the response has an aperiodic nature, as $x(t)$ approaches zero asymptotically for all time constants τ. For small time constants, it approaches zero faster.

The same solution can be obtained by the Laplace transform method (see Appendix). Recalling Eq. (A.3), the transform of Eq. (4.19) can be written as

$$sX(s)-x(0)+aX(s)=0 \tag{4.30}$$

where $X(s)$ is the Laplace transform of $x(t)$. Hence, using Eq. (4.20), we can rewrite Eq. (4.30) in the form

$$X(s)=\frac{1}{s+a}x_0 \tag{4.31}$$

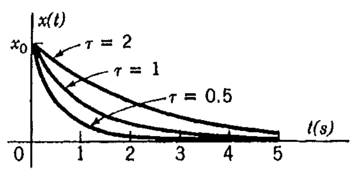


FIGURE 4.10

so that, using the table of Laplace transform pairs in Section A.7, we obtain the inverse transform of $X(s)$ in the form of Eq. (4.29). Note that the value $s=-a$ *is known as a simple pole of* $X(s)$ *and it coincides with the root of the characteristic equation.*

4.5 FREE RESPONSE OF UNDAMPED SECOND-ORDER SYSTEMS. THE HARMONIC OSCILLATOR

Consistent with the approach of Section 4.4, we wish to consider a generic differential equation, applicable to both mechanical and electrical systems. Nevertheless, when the situation demands, we shall favor terminology more common to mechanical engineering than to electrical engineering. Lack of damping implies that the elements associated with energy dissipation, namely, the damper and the resistor, are absent, so that there is no first-order derivative term in Eqs. (4.8) and (4.18). Hence, let us write the differential equation describing the behavior of a second-order undamped system in the form

$$\ddot{x}(t)+\omega_n^2x(t)=0 \tag{4.32}$$

where ω_n is known as the *natural frequency* of the system. The meaning of the term will become evident shortly. In the case of mechanical systems

$$\omega_n=\sqrt{k/m} \tag{4.33}$$

and in the case of electrical systems

$$\omega_n=\sqrt{1/LC} \tag{4.34}$$

The behavior of a large number of diverse physical and engineering systems can be described by Eq. (4.32). A classical example is the simple pendulum of Section 4.2, provided the motion is restricted to small angles, an assumption often referred to as the *small-motions assumption.* Invoking the small-motions assumption, which carries the implication that $\sin\theta\cong\theta$, we can rewrite Eq. (4.10) in the form

$$\ddot{\theta}(t)+\omega_n^2\theta(t)=0 \tag{4.35}$$

where the natural frequency of the pendulum is simply

$$\omega_n=\sqrt{g/L} \tag{4.36}$$

Equation (4.32) is one of the simplest second-order differential equations. Its general solution can be written in the form

$$x(t)=Ae^{\lambda t} \tag{4.37}$$

Introducing Eq. (4.37) into Eq. (4.32) and using the same reasoning as in Section 4.4, we obtain the characteristic equation

$$\lambda^2+\omega_n^2=0 \tag{4.38}$$

which has the roots

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \pm i\omega_n \tag{4.39}$$

Hence, the general solution (4.37) becomes

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} \tag{4.40}$$

where A_1 and A_2 are constants of integration. They are complex quantities. Because $x(t)$ must be real, however, A_2 must be the complex conjugate of A_1. It will prove convenient to introduce the notation

$$A_1 = \tfrac{1}{2}Ae^{-i\psi}, \qquad A_2 = \tfrac{1}{2}Ae^{i\psi} \tag{4.41}$$

where A and ψ are real. Inserting Eqs. (4.41) into Eq. (4.40), we obtain

$$x(t) = \tfrac{1}{2}A[e^{i(\omega_n t - \psi)} + e^{-i(\omega_n t - \psi)}] \tag{4.42}$$

so that, recalling formula (1.20) and its complex conjugate, we can reduce Eq. (4.42) to the real form

$$x(t) = A\cos(\omega_n t - \psi) \tag{4.43}$$

where the constants of integration are now A and ψ. It is easy to verify that the solution of Eq. (4.32) can also be expressed as

$$x(t) = B_1 \sin \omega_n t + B_2 \cos \omega_n t \tag{4.44}$$

where B_1 and B_2 are constants of integration. Then, comparing Eqs. (4.43) and (4.44) and recalling that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, we conclude that the two sets of constants of integration, B_1, B_2 and A, ψ, are related by

$$B_1 = A\sin\psi, \qquad B_2 = A\cos\psi \tag{4.45}$$

Equation (4.43) or Eq. (4.44) indicates that the free response of an undamped second-order system consists of simple sinusoidal oscillation. Sine and cosine functions are known as harmonic functions, and, consistent with this, Eq. (4.43) is said to describe *simple harmonic oscillation.* Moreover, systems governed by equations of the type (4.32) or (4.35) are called *harmonic oscillators.* The constants A and ψ are known as the *amplitude* and *phase angle* of the oscillation, respectively.

Solution (4.43) can be conveniently discussed by means of the geometric construction shown in Fig. 4.11. The amplitude A is represented in Fig. 4.11*a* by a vector **A** making an angle $\omega_n t - \psi$ with the vertical axis. Hence, at any time t the projection of the vector **A** on the vertical axis represents the solution $x(t)$, Eq. (4.43). The constants A_1 and A_2 can be interpreted as the Cartesian components of the vector **A**, so that **A** is the diagonal of the rectangle with sides A_1 and A_2, where the angle between A_1 and **A** is constant and can be recognized as the phase angle ψ. As time unfolds, the angle $\omega_n t - \psi$ increases linearly with it, causing the vector **A** to rotate in the plane with the angular velocity ω_n. In the process, the vertical projection of **A** varies harmonically with time. This projection is shown in Fig. 4.11*b*. At $t = 0$ the projection is A_1, and at $t = \psi/\omega_n$ the projection reaches its peak

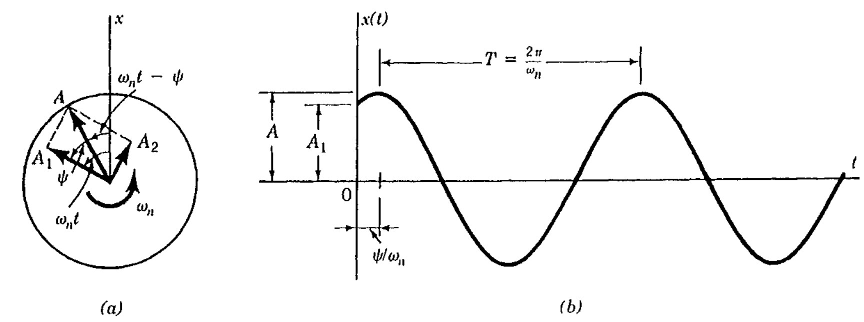


FIGURE 4.11

at a value equal to the amplitude A. Hence, we conclude from Fig. 4.11*b* that the phase angle ψ is a measure of the shift of the cosine function to the right. The solution $x(t)$ reaches a peak value of A once again after the time

$$T = 2\pi/\omega_n \tag{4.46}$$

has elapsed, and at every integer multiple of T thereafter, where T is known as the *period* of oscillation. Hence, the period represents the time between two consecutive peaks, or the time corresponding to one cycle of motion. It is commonly measured in seconds (s). From Eq. (4.46), we conclude that the frequency ω_n is measured in *radians per second* (rad/s). The natural frequency can also be defined as merely the reciprocal of the period, or

$$f_n = 1/T = \omega_n/2\pi \tag{4.47}$$

where f_n has units of *cycles per second* (cps). One cycle per second is commonly known as one *hertz* (Hz). Clearly, high frequencies imply short periods and vice versa.

It will prove of interest to examine the factors determining the period of the mass–spring system and of the simple pendulum. From Eqs. (4.33) and (4.46), we can write

$$T = 2\pi\sqrt{m/k} \tag{4.48}$$

so that the period T varies as the square root of m and is inversely proportional to the square root of k. Hence, the period T can be increased by increasing the mass or by decreasing the spring stiffness, or both. Similarly, using Eqs. (4.36) and (4.46), we obtain the pendulum period

$$T = 2\pi\sqrt{L/g} \tag{4.49}$$

But the quantity g represents the acceleration due to gravity. It is commonly assumed to be constant, although it varies with the altitude as measured from the sea level. For a given location, g can be regarded as constant, so that the period of the pendulum is proportional to the square root of its length. Hence, the interesting fact is that the period is affected not by the mass of the bob but only by the pendulum's length, a fact known to ancient Greeks.

Solution (4.43) indicates that no matter how the motion is initiated, free oscillation always occurs at the frequency ω_n. From Eqs. (4.33) and (4.36), we observe that ω_n depends only on the system parameters and not on external factors (gravity excluded), so that ω_n reflects a natural property of the system, which is the reason for it being called the natural frequency. Consistent with this, the simple harmonic oscillation at the natural frequency ω_n can be regarded as a *natural motion* of the harmonic oscillator.

The concept of harmonic oscillator represents a mathematical idealization more than a physical reality. Indeed, according to Eq. (4.43), once a motion is initiated, it will perpetuate itself ad infinitum. This is in contradiction to observed behavior, which indicates that the motion of a mass–spring system, or of a simple pendulum, will come to rest eventually if allowed to oscillate freely. This behavior can be attributed to the fact that every real system possesses some measure of damping. In the case of the pendulum, factors causing the motion to decay are air resistance and friction at the point of support. Nevertheless, the concept of a harmonic oscillator has its place. For some systems, damping is so small that the behavior is very close to that of a harmonic oscillator. In particular, if the interest lies in motion over a relatively short time interval compared to the period, then small damping may not have any noticeable effect over that interval.

Although the motion of a harmonic oscillator is always sinusiodal and the frequency of oscillation is always the natural frequency ω_n, the amplitude A and the phase angle ψ generally differ from case to case. Hence, the question remains as to what determines A and ψ. As mentioned earlier, A and ψ in Eq. (4.43) represent constants of integration. Mathematically, the determination of two constants of integration requires two conditions to be imposed on the solution. These conditions can be the value of the solution at two distinct times. More commonly, however, the two conditions are chosen as the value of the solution and of its first derivative at a given time, such as $t=0$. In this case, they represent physically the initial displacement and initial velocity. We denote them by

$$x(0)=x_0, \qquad \dot{x}(0)=v_0 \tag{4.50}$$

Introducing Eqs. (4.50) into Eq. (4.43), we obtain

$$x(0)=A\cos\psi=x_0, \qquad \dot{x}(0)=\omega_n A\sin\psi=v_0 \tag{4.51}$$

so that

$$A=\sqrt{x_0^2+(v_0/\omega_n)^2}, \qquad \psi=\tan^{-1}(v_0/\omega_n x_0) \tag{4.52}$$

As a matter of interest, we note that, by inserting Eqs. (4.51) into Eqs. (4.45) and by using Eq. (4.44), we can write the solution directly in terms of the initial conditions in the form

$$x(t)=x_0\cos\omega_n t+\frac{v_0}{\omega_n}\sin\omega_n t \tag{4.53}$$

Before considering the free vibration of damped systems, let us consider a solution of Eq. (4.32) by the Laplace transformation method. Transforming Eq. (4.32), we obtain

$$s^2X(s)-sx(0)-\dot{x}(0)+\omega_n^2X(s)=0 \tag{4.54}$$

which yields the transformed response

$$X(s)=\frac{s}{s^2+\omega_n^2}x(0)+\frac{1}{s^2+\omega_n^2}\dot{x}(0) \tag{4.55}$$

The inverse transformation of $X(s)$ can be obtained by the method of partial fractions described in Section A.3. If we were to expand $X(s)$ in terms of partial fractions, then we would factor out the denominator as follows:

$$s^2+\omega_n^2=(s-s_1)(s-s_2)=(s-i\omega_n)(s+i\omega_n) \tag{4.56}$$

where $s_1=i\omega_n$ and $s_2=-i\omega_n$ are simple poles of $X(s)$ (see Section A.3). Hence, from Eqs. (4.38) and (4.56), we conclude that *the simple poles of $X(s)$ are precisely the roots of the characteristic equation.* Because the functions $s/(s^2+\omega_n^2)$ and $1/s^2+\omega_n^2)$ are listed in the table of Laplace transform pairs given in Section A.7, expansion into partial fractions is actually not necessary. Indeed, using the table of Section A.7, we obtain directly

$$x(t)=x(0)\cos\omega_n t+\frac{\dot{x}(0)}{\omega_n}\sin\omega_n t \tag{4.57}$$

Equation (4.57) represents the response $x(t)$ expressed in terms of the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$, obtained earlier in the form of Eq. (4.53).

4.6 FREE RESPONSE OF DAMPED SECOND-ORDER SYSTEMS

Letting the external excitation be equal to zero, we can write the differential equation governing the free response of a damped second-order system in the form

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=0 \tag{4.58}$$

where ζ is a nondimensional parameter. Comparing Eqs. (4.8) and (4.58), we conclude that in the case of mechanical systems

$$\zeta=\frac{c}{2m\omega_n} \tag{4.59}$$

in which the natural frequency ω_n is given by Eq. (4.33). The nondimensional parameter ζ is known as the *viscous damping factor*. On the other hand, comparing Eqs. (4.18) and (4.58), it follows that for electrical systems

$$\zeta=\frac{R}{2L\omega_n} \tag{4.60}$$

where the natural frequency ω_n is given by Eq. (4.34). The solution of Eq. (4.58) is subject to the initial conditions (4.50) and can be obtained by the approach of Section 4.5.

Let us assume a solution of Eq. (4.58) in the exponential form

$$x(t) = Ce^{\lambda t} \tag{4.61}$$

where C and λ are constant scalars. Introducing Eq. (4.61) into Eq. (4.58) and using the same argument as for undamped systems, we conclude that the characteristic equation for damped systems is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{4.62}$$

Equation (4.62) has the roots

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n \tag{4.63}$$

so that the roots depend on the viscous damping factor ζ. We distinguish the following cases:

i. If $\zeta > 1$, the roots are real, negative, and distinct. They are in the form given by Eq. (4.63).

ii. If $\zeta = 1$, the roots are real, negative, and equal to one another, or

$$\lambda_1 = \lambda_2 = -\omega_n \tag{4.64}$$

iii. If $\zeta < 1$, the roots are complex conjugates with negative real part, or

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = (-\zeta \pm i\sqrt{1 - \zeta^2})\omega_n \tag{4.65}$$

The nature of the motion in each case depends on the roots λ_1 and λ_2, and hence on the viscous damping factor ζ. We now discuss the above three cases separately.

For $\zeta > 1$, the solution becomes

$$\begin{aligned} x(t) &= C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \\ &= C_1 \exp(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t + C_2 \exp(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t \\ &= (C_1 e^{\sqrt{\zeta^2 - 1}\omega_n t} + C_2 e^{-\sqrt{\zeta^2 - 1}\omega_n t})e^{-\zeta\omega_n t} \end{aligned} \tag{4.66}$$

where the constants of integration C_1 and C_2 depend on x_0 and v_0. Because $\zeta > \sqrt{\zeta^2 - 1}$, the response $x(t)$ decays exponentially with time. The motion is *aperiodic*, i.e., it approaches zero without oscillation. When $\zeta > 1$, the system is said to be *overdamped*.

For $\zeta = 1$ the two roots coincide, $\lambda_1 = \lambda_2 = -\omega_n$. In this case, the solution can be verified to have the form

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{4.67}$$

where C_1 and C_2 depend on x_0 and v_0. Once again, the motion can be shown to be aperiodic, approaching zero asymptotically. The case $\zeta = 1$ is known as *critical damping*. From Eq. (4.59), we conclude that in the critical damping case the coefficient of viscous damping has the value

$$c_{cr} = 2m\omega_n = 2\sqrt{km} \tag{4.68}$$

For $\zeta < 1$, the solution becomes

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = (C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t})e^{-\zeta\omega_n t} \tag{4.69}$$

where the notation

$$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n \tag{4.70}$$

has been introduced. Because $x(t)$ must be real, C_2 must be the complex conjugate of C_1, $C_2 = \bar{C}_1$, so that Eq. (4.69) reduces to

$$x(t) = 2 \operatorname{Re} C_1 e^{i\omega_d t} e^{-\zeta\omega_n t} \tag{4.71}$$

But $e^{i\omega_d t}$ represents a complex vector of unit magnitude rotating counterclockwise in the complex plane with the angular velocity ω_d, as demonstrated in Section 1.6. Hence, $2 \operatorname{Re} C_1 e^{i\omega_d t}$ represents the projection on the real axis of a rotating complex vector of magnitude $2|C_1|$. Recalling Fig. 4.11*a*, we conclude that $2 \operatorname{Re} C_1 e^{i\omega_d t}$ varies harmonically with time. On the other hand, $e^{-\zeta\omega_n t}$ represents a function decaying exponentially with time, approaching zero asymptotically. Letting

$$2C_1 = Ae^{-i\psi} \tag{4.72}$$

where A and ψ are real quantities, we can reduce Eq. (4.71) to

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \psi) \tag{4.73}$$

Equation (4.73) permits a simple interpretation of the motion. Indeed, $Ae^{-\zeta\omega_n t}$ can be regarded as a time-dependent amplitude, modulating the harmonic function $\cos(\omega_d t - \psi)$, where ω_d can be interpreted as the frequency *of the damped free vibration*. Moreover, ψ is merely a phase angle. Hence, Eq. (4.73) represents *damped harmonic motion*, with the oscillation being bounded by the envelope $\pm Ae^{-\zeta\omega_n t}$. Because the width of the envelope approaches zero asymptotically as $t \to \infty$, the system comes to rest eventually. The case $\zeta < 1$ is commonly known as the *underdamped* case. Example 4.2 presents a typical response of an underdamped system.

An interesting picture can be obtained by examining how the roots λ_1 and λ_2 change with ζ. Such a picture is shown in the complex λ-plane of Fig. 4.12. In the undamped case, $\zeta = 0$, the roots $\lambda_1 = i\omega_n$ and $\lambda_2 = -i\omega_n$ lie on the imaginary axis. As ζ increases, the roots move along a circle of radius ω_n, until they coalesce on the real axis, when ζ reaches unity. As ζ increases beyond $\zeta = 1$, the two roots split once again, moving along the negative real axis in opposite directions. Because the case $\zeta = 1$ represents merely a point in the λ-plane, critical damping should be regarded as being primarily of academic interest and representing the borderline case between overdamping and underdamping. Figure 4.12 represents a root-locus plot, a subject discussed extensively in Chapter 11.

In all three cases discussed above, the motion is fully determined as soon as the constants of integration are evaluated in terms of the initial conditions. We do

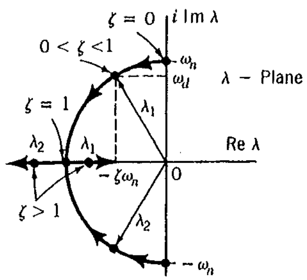


FIGURE 4.12

not pursue this approach here. Instead, we obtain the solution by the Laplace transformation method.

Transforming Eq. (4.58), we can write the transformed response $X(s)$ in the form

$$X(s)=\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}x_0+\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}v_0 \tag{4.74}$$

where $x_0=x(0)$ and $v_0=\dot{x}(0)$. The simple poles of $X(s)$ are the roots of the characteristic equation, Eq. (4.62), or

$$s_1=(-\zeta+\sqrt{\zeta^2-1})\omega_n, \qquad s_2=(-\zeta-\sqrt{\zeta^2-1})\omega_n \tag{4.75}$$

and we note once again that the poles s_1 and s_2 coincide with the roots λ_1 and λ_2 of the characteristic equation. But from Section A.3 we can write

$$\begin{aligned}\mathscr{L}^{-1}\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}&=\frac{s_1+2\zeta\omega_n}{s_1-s_2}e^{s_1t}+\frac{s_2+2\zeta\omega_n}{s_2-s_1}e^{s_2t}\\&=\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{(-\zeta+\sqrt{\zeta^2-1})\omega_nt}-\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{(-\zeta-\sqrt{\zeta^2-1})\omega_nt}\\&=\frac{1}{\sqrt{\zeta^2-1}}(\zeta\sinh\sqrt{\zeta^2-1}\,\omega_nt\\&\quad+\sqrt{\zeta^2-1}\cosh\sqrt{\zeta^2-1}\,\omega_nt)e^{-\zeta\omega_nt}\end{aligned} \tag{4.76}$$

and

$$\begin{aligned}\mathscr{L}^{-1}\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}&=\frac{1}{s_1-s_2}e^{s_1t}+\frac{1}{s_2-s_1}e^{s_2t}\\&=\frac{1}{\sqrt{\zeta^2-1}\,\omega_n}e^{-\zeta\omega_nt}\sinh\sqrt{\zeta^2-1}\,\omega_nt\end{aligned} \tag{4.77}$$

so that the response to the initial displacement x_0 and initial velocity v_0 is

$$\begin{aligned}x(t)=&\frac{x_0}{\sqrt{\zeta^2-1}}(\zeta\sinh\sqrt{\zeta^2-1}\,\omega_nt+\sqrt{\zeta^2-1}\cosh\sqrt{\zeta^2-1}\,\omega_nt)e^{-\zeta\omega_nt}\\&+\frac{v_0}{\sqrt{\zeta^2-1}\,\omega_n}e^{-\zeta\omega_nt}\sinh\sqrt{\zeta^2-1}\,\omega_nt\end{aligned} \tag{4.78}$$

The response in all three cases, $\zeta>1$, $\zeta=1$, and $\zeta<1$, can be derived from Eq. (4.78).

Example 4.1

The damped system described by Eq. (4.58) has the following parameters:

$$\zeta=1.2, \qquad \omega_n=5 \quad \text{rad/s} \tag{a}$$

Plot the response to the initial excitation

$$x_0=0.1 \quad \text{m}, \qquad v_0=0 \tag{b}$$

Introducing Eqs. (a) and (b) into Eq. (4.78), we obtain

$$x(t)=(0.1809\sinh 3.3166t+0.1\cosh 3.3166t)e^{-6t} \tag{c}$$

The plot $x(t)$ versus t is shown in Fig. 4.13; it confirms the aperiodic nature of the motion for this overdamped case.

Example 4.2

The damped system described by Eq. (4.58) has the following parameters:

$$\zeta=0.1, \quad \omega_n=5 \quad \text{rad/s} \tag{a}$$

Plot the response to the initial excitation

$$x_0=0, \qquad v_0=0.2 \quad \text{m/s} \tag{b}$$

Because $\zeta<1$, we wish to introduce the notation

$$\sqrt{\zeta^2-1}\,\omega_n=i\sqrt{1-\zeta^2}\,\omega_n=i\omega_d \tag{c}$$

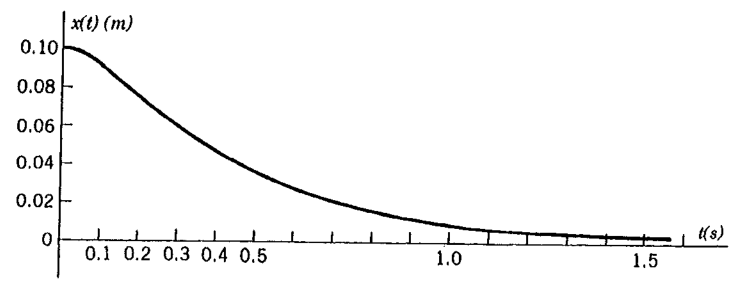


FIGURE 4.13

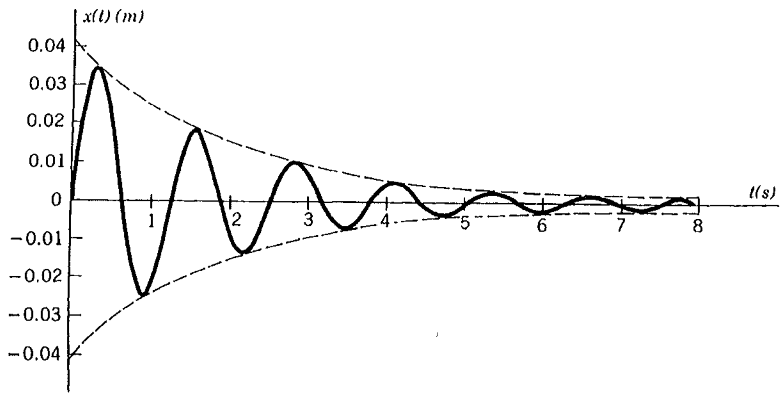


FIGURE 4.14

where ω_d is the frequency of the damped oscillation. Introducing Eq. (c) into Eq. (4.78), we can write

$$x(t)=\frac{v_0}{i\omega_d}e^{-\zeta\omega_n t}\sinh i\omega_d t=\frac{v_0}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{d}$$

Hence, inserting Eqs. (a) and (b) into Eq. (d), we obtain

$$x(t)=0.0402e^{-0.5t}\sin 4.9749t \tag{e}$$

The plot $x(t)$ versus t is shown in Fig. 4.14, and it represents the damped oscillatory motion characterizing an underdamped system.

4.7 THE LOGARITHMIC DECREMENT

Quite often the amount of damping in a system is not known and must be determined experimentally. This can be done by disturbing the system initially in some fashion and measuring the response, so that the question reduces to how to determine the amount of damping from the observed response. We are interested in a viscously damped system, and in particular in an underdamped system, so that the response has the form of an exponentially decaying oscillation, such as the one shown in Fig. 4.15. Clearly, the rate of decay depends on the amount of damping. Hence, we propose to determine the damping by relating it to an established measure of the decay.

Let us denote the time at which the response reaches a peak by t_1 (Fig. 4.15). Because the motion is periodic, albeit damped, the subsequent peak is reached at the time $t_2=t_1+T$, where T is the period given by

$$T=2\pi/\omega_d \tag{4.79}$$

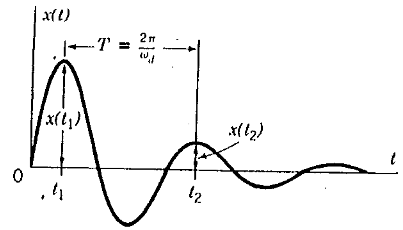


FIGURE 4.15

From Eq. (4.73), the corresponding peak responses have the expressions

$$x(t_1)=Ae^{-\zeta\omega_n t_1}\cos(\omega_d t_1-\psi) \tag{4.80a}$$

$$x(t_2)=Ae^{-\zeta\omega_n t_2}\cos(\omega_d t_2-\psi)=Ae^{-\zeta\omega_n(t_1+T)}\cos[\omega_d(t_1+T)-\psi] \tag{4.80b}$$

where T is given by Eq. (4.79). But, recalling Eq. (4.70), we obtain

$$\begin{aligned}e^{-\zeta\omega_n(t_1+T)}&=e^{-\zeta\omega_n t_1}e^{-\zeta\omega_n T}=e^{-\zeta\omega_n t_1}e^{-2\pi\zeta\omega_n/\omega_d}\\&=e^{-\zeta\omega_n t_1}e^{-2\pi\zeta/(1-\zeta^2)^{1/2}}\end{aligned} \tag{4.81}$$

Moreover,

$$\begin{aligned}\cos[\omega_d(t_1+T)-\psi]&=\cos(\omega_d t_1-\psi)\cos\omega_d T-\sin(\omega_d t_1-\psi)\sin\omega_d T\\&=\cos(\omega_d t_1-\psi)\cos 2\pi-\sin(\omega_d t_1-\psi_1)\sin 2\pi\\&=\cos(\omega_d t_1-\psi)\end{aligned} \tag{4.82}$$

Using Eqs. (4.80)–(4.82), we can write the ratio between two peak values

$$\frac{x(t_1)}{x(t_2)}=\frac{Ae^{-\zeta\omega_n t_1}\cos(\omega_d t_1-\psi)}{Ae^{-\zeta\omega_n t_2}\cos(\omega_d t_2-\psi)}=e^{2\pi\zeta/(1-\zeta^2)^{1/2}} \tag{4.83}$$

In view of the exponential form of the above ratio, it is convenient to introduce the notation

$$\delta=\ln\frac{x(t_1)}{x(t_2)}=\frac{2\pi\zeta}{(1-\zeta^2)^{1/2}} \tag{4.84}$$

where δ is known as the *logarithmic decrement*. Clearly, the logarithmic decrement can be obtained from the response curve by taking the natural logarithm of the ratio of two consecutive peak values, not necessarily the first two. Then, the viscous damping factor can be calculated by solving Eq. (4.84) for ζ. The result can be shown to be

$$\zeta=\frac{\delta}{[(2\pi)^2+\delta^2]^{1/2}} \tag{4.85}$$

For damping sufficiently small that $\delta<<2\pi$, the viscous damping factor can be approximated by

$$\zeta \cong \frac{\delta}{2\pi} \tag{4.86}$$

The viscous damping factor ζ can be determined by measuring peaks separated by any number of periods. Indeed, it is easy to verify that

$$\frac{x(t_1)}{x(t_2)}=\frac{x(t_2)}{x(t_3)}=\frac{x(t_3)}{x(t_4)}=\cdots \tag{4.87}$$

so that considering peaks at t_1 and at $t_{k+1}=t_1+kT$, where k is a given integer, we conclude that

$$\frac{x(t_1)}{x(t_{k+1})}=\frac{x(t_1)}{x(t_2)}\frac{x(t_2)}{x(t_3)}\frac{x(t_3)}{x(t_4)}\cdots\frac{x(t_k)}{x(t_{k+1})}=\left[\frac{x(t_1)}{x(t_2)}\right]^k \tag{4.88}$$

Hence, inserting Eq. (4.88) into Eq. (4.84), we obtain the logarithmic decrement in the form

$$\delta=\ln\frac{x(t_1)}{x(t_2)}=\ln\left[\frac{x(t_1)}{x(t_{k+1})}\right]^{1/k}=\frac{1}{k}\ln\frac{x(t_1)}{x(t_{k+1})} \tag{4.89}$$

Then, the viscous damping factor ζ can be determined from Eq. (4.85) or from Eq. (4.86).

It should be pointed out that Eq. (4.83) remains valid even when t_1 and t_2 are any two instants separated by a period T and not necessarily corresponding to peak values for $x(t)$. Measuring peak values, however, is more convenient than measuring arbitrary amplitudes.

Example 4.3

After two complete periods the peak amplitude of a viscously damped second-order system has fallen by 60%. Calculate the viscous damping factor by using both Eqs. (4.85) and (4.86), compare results, and draw conclusions.

Using Eq. (4.89) with $k=2$, we obtain the logarithmic decrement

$$\delta=\frac{1}{2}\ln\frac{x(t_1)}{x(t_3)}=\frac{1}{2}\ln\frac{1}{0.4}=0.45815 \tag{a}$$

so that, from Eq. (4.85), we can write

$$\zeta=\frac{\delta}{[(2\pi)^2+\delta^2]^{1/2}}=\frac{0.45815}{[(2\pi)^2+0.45815^2]^{1/2}}=0.0727 \tag{b}$$

and, from Eq. (4.86), we have

$$\zeta=\frac{\delta}{2\pi}=\frac{0.45815}{2\pi}=0.0729 \tag{c}$$

Comparing Eqs. (b) and (c), we conclude that in the case under consideration Eq. (4.86) yields a value for the damping factor differing by about 0.27% from the value given by the more accurate Eq. (4.85), so that Eq. (4.86) is entirely adequate.

In fact, it is easy to verify that the logarithmic decrement can reach the value $\delta=0.9$ and the error arising from using Eq. (4.86) is still only about 1%.

4.8 RESPONSE OF FIRST-ORDER SYSTEMS TO HARMONIC EXCITATION. FREQUENCY RESPONSE

Let us consider now the case in which the first-order system described by Eq. (4.3) is subjected to the harmonic excitation given by

$$f(t)=f_0 e^{i\omega t}=Ake^{i\omega t} \tag{4.90}$$

where A is a real constant having units of displacement and k is the spring constant. The notation $f_0=Ak$ has the advantage that it permits expressing the response in terms of a nondimensional ratio, as we shall see shortly. Moreover, because $e^{i\omega t}=\cos\omega t+i\sin\omega t$, the notation of Eq. (4.90) enables us to derive the response to $f_0\cos\omega t$ and $f_0\sin\omega t$ simultaneously. Inserting Eq. (4.90) into Eq. (4.3) and dividing through by c, we can write the equation of motion in the form

$$\dot{x}+ax=Aae^{i\omega t} \tag{4.91}$$

where, according to Eq. (4.21), $a=k/c$. The response of a general linear system whose dynamic characteristics are described by a differential operator $D(t)$ to the excitation given by Eq. (4.90) was virtually evaluated in Section 1.7. Indeed, the response was given by Eq. (1.27), so that letting $c(t)=x(t)$ and $r_0=Ak$ in Eq. (1.27) we have

$$x(t)=X(i\omega)e^{i\omega t} \tag{4.92}$$

where

$$X(i\omega)=\frac{Aa}{Z(i\omega)} \tag{4.93}$$

in which $Z(i\omega)$ is the system impedance. Note that the particular solution given by Eq. (4.92) represents a steady-state solution.

For the first-order system at hand, the impedance is

$$Z(i\omega)=a+i\omega \tag{4.94}$$

Dividing the top and bottom of the right side of Eq. (4.93) by a and recalling the definition (4.28) of the time constant, namely, $\tau=1/a=c/k$, we obtain

$$X(i\omega)=\frac{A}{1+i\omega\tau} \tag{4.95}$$

It will prove convenient to introduce the nondimensional ratio

$$G(i\omega)=\frac{X(i\omega)}{A}=\frac{1}{1+i\omega\tau}=\frac{1-i\omega\tau}{1+\omega^2\tau^2} \tag{4.96}$$

where $G(i\omega)$ is recognized as the frequency response (Section 1.6). Inserting Eqs.

(4.95) and (4.96) into Eq. (4.92), we obtain the harmonic response

$$x(t)=AG(i\omega)e^{i\omega t} \tag{4.97}$$

But, the frequency response, as any complex expression, can be written in the form

$$G(i\omega)=|G(i\omega)|e^{i\phi} \tag{4.98}$$

where $|G(i\omega)|$ is the magnitude of $G(i\omega)$ and ϕ is a phase angle.* Introducing Eq. (4.98) into Eq. (4.97), the response becomes

$$x(t)=A|G(i\omega)|e^{i(\omega t+\phi)} \tag{4.99}$$

Note that the nature of the present phase angle ϕ is different from that encountered in the free response.

Comparing Eqs. (4.90) and (4.99), we conclude that the phase angle ϕ represents a measure of the time interval by which the response leads the excitation. As shown later in this section, in the case of the first-order system considered here the phase angle is negative, so that the response lags behind the excitation.

Equation (4.99) contains in essence the response to $Ak\cos\omega t$ and $Ak\sin\omega t$ in a single expression, as anticipated. The two responses can be separated from one another by considering the real and imaginary parts of Eq. (4.99), so that the response to the harmonic excitation $Ak\cos\omega t$ is simply

$$\text{Re}\, x(t)=A|G(i\omega)|\cos(\omega t+\phi) \tag{4.100a}$$

and the response to the harmonic excitation $Ak\sin\omega t$ is

$$\text{Im}\, x(t)=A|G(i\omega)|\sin(\omega t+\phi) \tag{4.100b}$$

Later in this chapter we shall present a geometric interpretation of solutions (4.99)–(4.100).

Next, let us examine how the response of the system behaves as the driving frequency ω varies. To this end, we wish to plot the magnitude $|G(i\omega)|$ and the phase angle ϕ of the frequency response $G(i\omega)$ as functions of ω. From complex algebra, we refer to Eq. (4.96) and write

$$|G(i\omega)|=[\text{Re}^2\, G(i\omega)+\text{Im}^2\, G(i\omega)]^{1/2}=\left[\left(\frac{1}{1+\omega^2\tau^2}\right)^2+\left(\frac{-\omega\tau}{1+\omega^2\tau^2}\right)^2\right]^{1/2}$$

$$=\frac{1}{(1+\omega^2\tau^2)^{1/2}} \tag{4.101}$$

The plot of $|G(i\omega)|$ versus $\omega\tau$ is displayed in Fig. 4.16. Moreover, recognizing that $e^{i\phi}=\cos\phi+i\sin\phi$ and recalling Eqs. (4.96) and (4.98), we obtain the phase angle

$$\phi=\tan^{-1}\frac{\text{Im}\, G(i\omega)}{\text{Re}\, G(i\omega)}=\tan^{-1}\frac{-\omega\tau/(1+\omega^2\tau^2)}{1/(1+\omega^2\tau^2)}=\tan^{-1}(-\omega\tau) \tag{4.102}$$

The plot of ϕ versus $\omega\tau$ is shown in Fig. 4.17. It should be pointed out that the plots of Figs. 4.16 and 4.17 are known as *frequency response plots*. They are used extensively in vibrations and in control.

*In texts on vibrations, the phase angle is defined as the negative of the one here. The definition given here is consistent with the one given in Chapter 11, and is the definition ordinarily used in control.

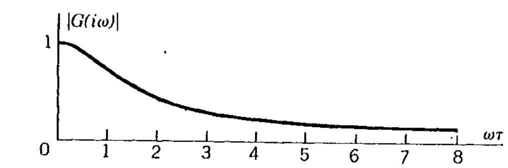


FIGURE 4.16

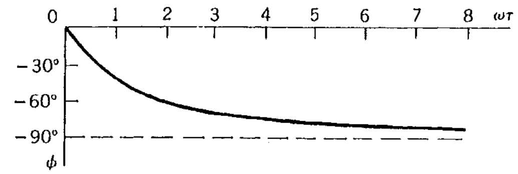


FIGURE 4.17

The magnitude $|G(i\omega)|$ of the frequency response can be given a physical interpretation. Indeed, using Eq. (4.99), we observe that the magnitude of the force in the spring is

$$|f_s(t)|=k|x(t)|=Ak|G(i\omega)| \tag{4.103}$$

Moreover, the magnitude of the external force is simply $|f(t)|=Ak$. Hence, we can write

$$|G(i\omega)|=\frac{|f_s(t)|}{|f(t)|} \tag{4.104}$$

or, the magnitude of the frequency response represents the nondimensional ratio of the magnitude of the spring force $f_s(t)$ to the magnitude of the external force $f(t)$.

We observe from the plot of $|G(i\omega)|$ versus $\omega\tau$ in Fig. 4.16 that for relatively large values of $\omega\tau$ the response is attenuated greatly. Hence, for a given τ, the system acts like a *filter*, leaving low-frequency inputs largely unaffected and attenuating high-frequency inputs. In many applications, electrical signals are contaminated by undesirable external factors called *noise*. In general, signals have low frequencies and noise has high frequencies. Then, the RC circuit discussed in Section 4.3 can be used as a filter reducing the amplitude of the undesirable noise relative to the amplitude of the signal. In view of this, such an RC circuit is called a *low-pass filter*. Note that in this case the time constant is $\tau=RC$.

4.9 RESPONSE OF SECOND-ORDER SYSTEMS TO HARMONIC EXCITATION

The response of second-order systems to harmonic excitation can be obtained in a way analogous to that used in Section 4.8 for the response of first-order systems. Indeed, the mathematical analogy is complete, and the difference lies in the manner in which the two types of systems respond.

Let us consider the second-order system

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=A\omega_n^2e^{i\omega t} \tag{4.105}$$

where A is a constant having the same units as $x(t)$. Then, following the procedure outlined in Section 4.8, the response can be shown to be

$$x(t)=X(i\omega)e^{i\omega t} \tag{4.106}$$

where

$$X(i\omega)=\frac{A}{1-(\omega/\omega_n)^2+i2\zeta\omega/\omega_n} \tag{4.107}$$

Defining the frequency response for this second-order system in the form of the nondimensional ratio

$$G(i\omega)=\frac{X(i\omega)}{A}=\frac{1}{1-(\omega/\omega_n)^2+i2\zeta\omega/\omega_n} \tag{4.108}$$

we can write the response once again in the form

$$x(t)=A|G(i\omega)|e^{i(\omega t+\phi)} \tag{4.109}$$

where $|G(i\omega)|$ is the magnitude and ϕ is the phase angle of the frequency response $G(i\omega)$.

To study the nature of the response, let us examine the dependence of $|G(i\omega)|$ and ϕ on the driving frequency ω. To this end, let us write

$$|G(i\omega)|=[\mathrm{Re}^2\,G(i\omega)+\mathrm{Im}^2\,G(i\omega)]^{1/2}$$
$$=\frac{1}{\{[1-(\omega/\omega_n)^2]^2+(2\zeta\omega/\omega_n)^2\}^{1/2}} \tag{4.110}$$

Plots of $|G(i\omega)|$ versus ω/ω_n for various values of ζ are shown in Fig. 4.18. For small ζ, the amplitude increases appreciably in the neighborhood of $\omega/\omega_n=1$ and then it falls off as ω/ω_n continues to increase. Note that $|G(i\omega)|$ is called the *magnification factor*, in spite of the fact that for certain values of ω the amplitude of the response is actually reduced instead of being magnified. The curves $|G(i\omega)|$ versus ω/ω_n reach peak values for certain ω/ω_n. To obtain the peak value for any of the curves, we use the standard technique of calculus for the determination of maxima and write

$$\frac{dG(i\omega)}{d(\omega/\omega_n)}=-\frac{1}{2}\frac{2[1-(\omega/\omega_n)^2](-2\omega/\omega_n)+8\zeta^2\omega/\omega_n}{\{[1-(\omega/\omega_n)^2]^2+(2\zeta\omega/\omega_n)^2\}^{3/2}}=0 \tag{4.111}$$

which yields

$$\frac{\omega}{\omega_n}=(1-2\zeta^2)^{1/2} \tag{4.112}$$

so that the peaks occur for $\omega/\omega_n<1$. The proximity of the peaks to $\omega/\omega_n=1$ depends on how small ζ is. Moreover, peaks occur only if $1-2\zeta^2$ is positive. Inserting

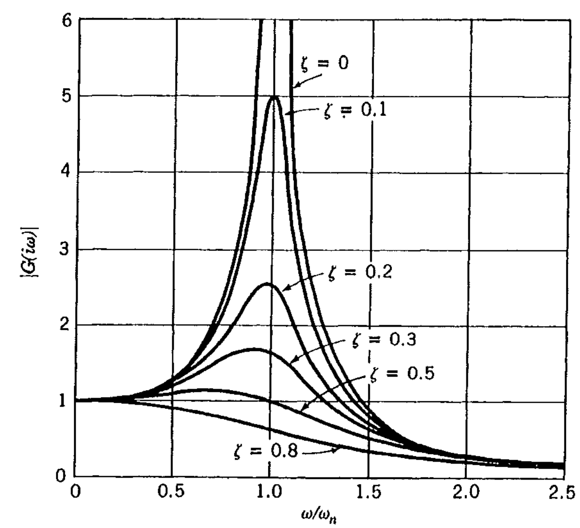


FIGURE 4.18

Eq. (4.112) back into Eq. (4.110), we obtain the peak amplitude

$$Q=|G(i\omega)|_{\max}=\frac{1}{2\zeta(1-\zeta^2)^{1/2}},\qquad \zeta<\frac{1}{\sqrt{2}} \tag{4.113}$$

which, for small ζ, can be approximated by

$$Q\cong\frac{1}{2\zeta} \tag{4.114}$$

The peak amplitude Q is known as the *quality factor*.

For very small values of ζ, Q becomes very large. In fact, as $\zeta\to 0$, $Q\to\infty$. For $\zeta=0$, we have no longer a peak but a discontinuity. It is easy to verify from Eq. (4.110) that the discontinuity occurs at $\omega=\omega_n$, at which driving frequency the amplitude becomes infinite. Of course, this is impossible for real physical systems, for which the displacement must remain finite. In fact, for our analysis to remain valid, the displacement must remain sufficiently small to stay within the linear range. Nevertheless, this serves as an indication that undamped systems experience violent vibrations at $\omega=\omega_n$, a phenomenon known as *resonance*. It should be pointed out that solution (4.109) is not valid at resonance, so that a separate solution for the case $\zeta=0$, $\omega=\omega_n$ must be produced. This is done later in this section. In many engineering systems, the driving frequency is not constant but increases from zero to a given steady operating value, such as when starting a motor driving the system. If the operating value of ω is larger than the natural frequency ω_n, then some high-amplitude vibration can be expected when ω is close to ω_n. This points to the desirability of a certain amount of damping in the system to prevent resonance.

Next, let us examine the dependence of the phase angle ϕ on ω. Following the procedure of Section 4.5, we obtain the phase angle

$$\phi=\tan^{-1}\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}=\tan^{-1}\left[-\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right] \tag{4.115}$$

Figure 4.19 shows plots of ϕ versus ω/ω_n for various values of ζ. We observe that all curves pass through the point $\phi=-\pi/2$, $\omega/\omega_n=1$. Moreover, $\phi>-\pi/2$ for $\omega/\omega_n<1$ and $\phi<-\pi/2$ for $\omega/\omega_n>1$. As $\omega/\omega_n\to0$, $\phi\to0$, and as $\omega/\omega_n\to\infty$, $\phi\to-\pi$. Hence, because the phase angle is negative, except for $\zeta=0$, $\omega<\omega_n$, we conclude from Eqs. (4.105) and (4.109) that the response of damped systems lags behind the excitation. For $\zeta=0$, the plot exhibits a discontinuity at $\omega/\omega_n=1$. In the undamped case, $\zeta=0$, the response reduces to

$$x(t)=\frac{A}{|1-(\omega/\omega_n)^2|}e^{i(\omega t+\phi)} \tag{4.116}$$

where $\phi=0$ for $\omega/\omega_n<1$ and $\phi=-\pi$ for $\omega/\omega_n>1$. Hence, Eq. (4.116) can be written in the form

$$x(t)=\frac{A}{1-(\omega/\omega_n)^2}e^{i\omega t} \tag{4.117}$$

Equation (4.117) states that the displacement is *in phase* with the excitation for $\omega<\omega_n$ and that it is *180° out of phase* with the excitation for $\omega>\omega_n$.

Finally, let us examine the resonance case, which occurs when a harmonic oscillator is driven at the natural frequency. Letting $c=0$, $\omega=\omega_n$ in Eq. (4.105), considering only the real part of the excitation, and dividing through by m, we obtain

$$\ddot{x}(t)+\omega_n^2x(t)=\omega_n^2A\cos\omega_n t \tag{4.118}$$

We shall produce a particular solution of Eq. (4.118) by the Laplace transformation

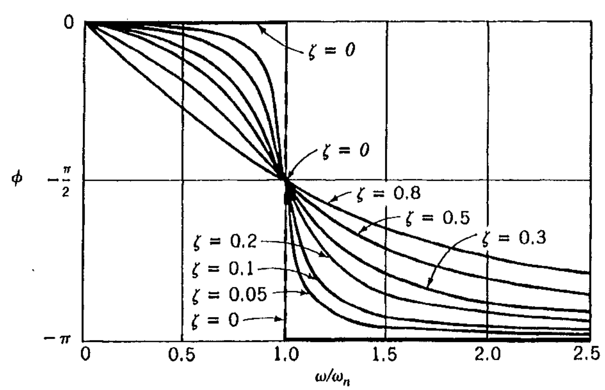


FIGURE 4.19

method. Transforming both sides of Eq. (4.118), letting $x(0)=\dot{x}(0)=0$, and using the Laplace transforms table in the Appendix, we can write

$$(s^2+\omega_n^2)X(s)=\omega_n^2A\frac{s}{s^2+\omega_n^2} \tag{4.119}$$

or

$$X(s)=\frac{\omega_n^2As}{(s^2+\omega_n^2)^2} \tag{4.120}$$

Once again using the Laplace transforms table, we can write the inverse transformation

$$x(t)=\mathscr{L}^{-1}X(s)=\frac{A}{2}\omega_n t\sin\omega_n t \tag{4.121}$$

The response lends itself to relatively easy interpretation. The term $(A/2)\omega_n t$ can be regarded as a time-dependent amplitude, modulating the harmonic function $\sin\omega_n t$. Hence, the response will be bounded by the envelope defined by the two straight lines $\pm(A/2)\omega_n t$. As the width of the envelope increases with time, the response is characterized by increasingly large amplitudes (Fig. 4.20). At a certain point, however, the linear range of the spring will be exceeded, at which point either the system breaks down, as in the case of a softening spring, or the motion is contained, as in the case of a stiffening spring. Of course, when the system exceeds the linear range, one must abandon the linear analysis as invalid and consider nonlinear analysis. Because the excitation is a cosine function and the response is a sine function and the two functions are related by the identity $\sin\omega t\equiv\cos(\omega t-\pi/2)$, it follows that the phase angle ϕ has the value $-\pi/2$. Hence the plot ϕ versus ω/ω_n for $\zeta=0$ consists of the straight line $\phi=0$ for $\omega<\omega_n$, the point $\phi=-\pi/2$ for $\omega=\omega_n$, and the straight line $\phi=-\pi$ for $\omega>\omega_n$.

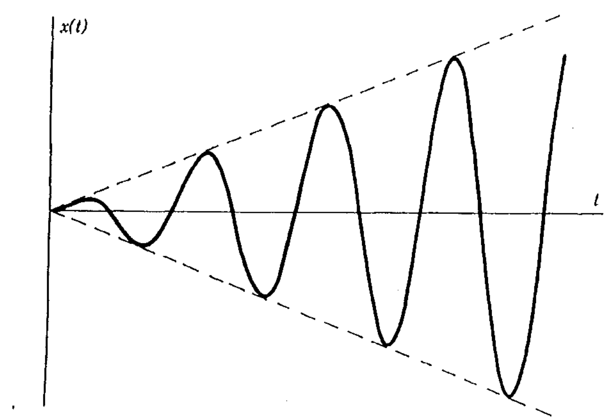


FIGURE 4.20

4.10 GEOMETRIC INTERPRETATION OF THE RESPONSE TO HARMONIC EXCITATION

Equation (4.91), governing the response of a first-order system to harmonic excitation, can be given an interesting geometric interpretation by representing it in the complex plane. Referring to Eq. (4.99), we can write

$$\dot{x}(t)=i\omega A|G(i\omega)|e^{i(\omega t+\phi)} \tag{4.122}$$

so that, considering the identity

$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=e^{i\pi/2} \tag{4.123}$$

we can rewrite Eq. (4.122) in the form

$$\dot{x}(t)=\omega A|G(i\omega)|e^{i(\omega t+\phi+\pi/2)} \tag{4.124}$$

Hence, $\dot{x}(t)$ is a vector whose magnitude is equal to the magnitude of $x(t)$ multiplied by ω and whose direction makes an angle $\pi/2$ with the direction of $x(t)$. In view of this, Eq. (4.91) can be satisfied vectorially, as shown in the diagram of Fig. 4.21. Note that, as time unfolds, the entire diagram rotates counterclockwise in the complex plane with the angular velocity ω. The response to the excitation $Aa \cos \omega t$ can be obtained by taking the projection of $x(t)$ on the real axis and the response to $Ak \sin \omega t$ can be obtained by taking the projection of $x(t)$ on the imaginary axis, so that the complex representation of motion yields the two solutions simultaneously.

The geometric interpretation of the response of a second-order system to harmonic excitation can be obtained analogously. From Eq. (4.99), we can write

$$\ddot{x}(t)=\omega^2 A|G(i\omega)|e^{i(\omega t+\phi+\pi)} \tag{4.125}$$

and because

$$-1=\cos\pi+i\sin\pi=e^{i\pi} \tag{4.126}$$

we have

$$\ddot{x}(t)=\omega^2 A|G(i\omega)|e^{i(\omega t+\phi+\pi)} \tag{4.127}$$

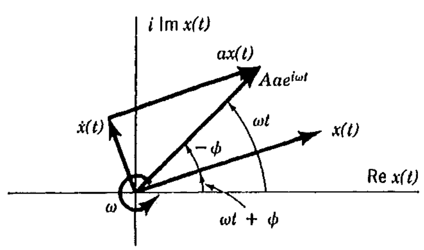


FIGURE 4.21

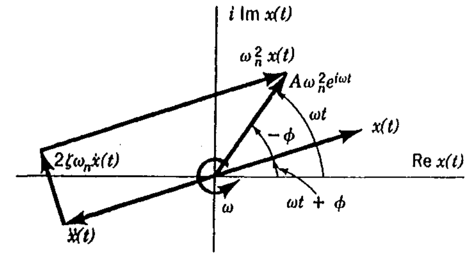


FIGURE 4.22

Hence, $\ddot{x}(t)$ is a vector whose magnitude is equal to the magnitude of $x(t)$ multiplied by ω^2 and whose direction is opposite to that of $x(t)$. The vector diagram describing Eq. (4.105) is shown in Fig. 4.22.

4.11 ROTATING UNBALANCED MASSES

Many engineering systems contain rotating unbalanced masses, sometimes by design but quite often inadvertently. Such masses produce harmonic excitation that can lead to excessive vibration and possible damage.

Let us consider a system consisting of a principal mass $M-m$ supported by two equal springs of combined stiffness k and a damper with coefficient of viscous damping c. Two equal eccentric masses $m/2$ rotate in opposite sense with constant angular velocity ω about symmetrically placed points at distances R from the masses, so that at any time the angle between the horizontal and the rigid links carrying the masses is ωt (Fig. 4.23*a*). Figures 4.23*b* and 4.23*c* show free-body diagrams for the principal mass and for the right eccentric mass, respectively. From Fig. 4.23*b*, if we measure the displacement $x(t)$ from the static equilibrium position, we can write Newton's second law for the principal mass in the form

$$-2F_x-c\dot{x}(t)-kx(t)=(M-m)\ddot{x}(t) \tag{4.128}$$

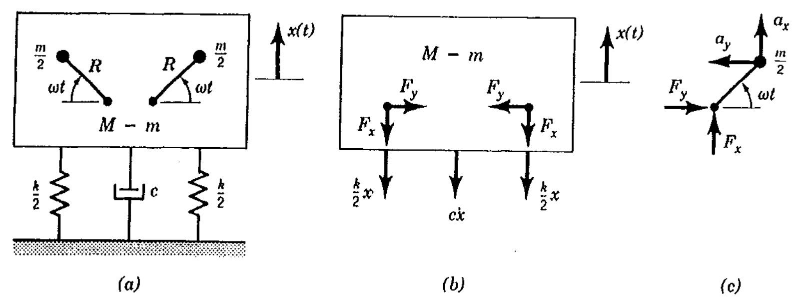


FIGURE 4.23

where F_x represents the vertical force exerted by one eccentric mass on the principal mass. Of course, for the two masses, the vertical forces add up to $2F_x$, whereas the horizontal forces, F_y and $-F_y$, cancel out. To obtain the expression for F_x, we consider the free-body diagram for the right rotating mass shown in Fig. 4.23c and write Newton's second law. Observing that the displacement of the mass is $x(t)+R\sin\omega t$, we can write

$$F_x=\frac{m}{2}a_x=\frac{m}{2}\frac{d^2}{dt^2}[x(t)+R\sin\omega t]=\frac{m}{2}[\ddot{x}(t)-R\omega^2\sin\omega t] \quad (4.129)$$

It must be pointed out that, by measuring $x(t)$ from equilibrium, we were able to cancel out the effect of the weights $(M-m)g$ and $mg/2$ in Eqs. (4.128) and (4.129), respectively. Introducing Eq. (4.129) into Eq. (4.128) and rearranging, we obtain the system equation of motion

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=\frac{mR\omega^2}{M}\sin\omega t \quad (4.130)$$

where

$$2\zeta\omega_n=\frac{c}{M}, \qquad \omega_n^2=\frac{k}{M} \quad (4.131)$$

Hence, the rotating unbalanced masses produce a harmonic excitation of the system, where the excitation has the frequency ω. Note that, although the system involves three masses, the motion of the reciprocating masses relative to the principal mass is prescribed, so that this is a single-degree-of-freedom system.

The solution of an equation similar to Eq. (4.130) was derived earlier in the form of Eq. (4.109), and to use this solution it is only necessary to recognize that in this case

$$A=\frac{m}{M}\left(\frac{\omega}{\omega_n}\right)^2R \quad (4.132)$$

Hence, retaining the imaginary part of the solution (4.109), with A as indicated by Eq. (4.132), we obtain

$$x(t)=\frac{mR}{M}\left(\frac{\omega}{\omega_n}\right)^2|G(i\omega)|\sin(\omega t+\phi) \quad (4.133)$$

where $|G(i\omega)|$ and ϕ are given by Eqs. (4.110) and (4.115), respectively.

Next, let us examine the manner in which the amplitude and phase angle of the response vary with the driving frequency ω. Examining Eq. (4.133), we conclude that the magnification factor in this case requires some modification. Indeed, now the indicated nondimensional ratio is

$$\frac{|x(t)|M}{Rm}=\left(\frac{\omega}{\omega_n}\right)^2|G(i\omega)| \quad (4.134)$$

Plots of $(\omega/\omega_n)^2|G(i\omega)|$ versus ω/ω_n are shown in Fig. 4.24 for various values of the damping factor ζ. Clearly, the plots ϕ versus ω/ω_n for various values of ζ remain as in Fig. 4.19.

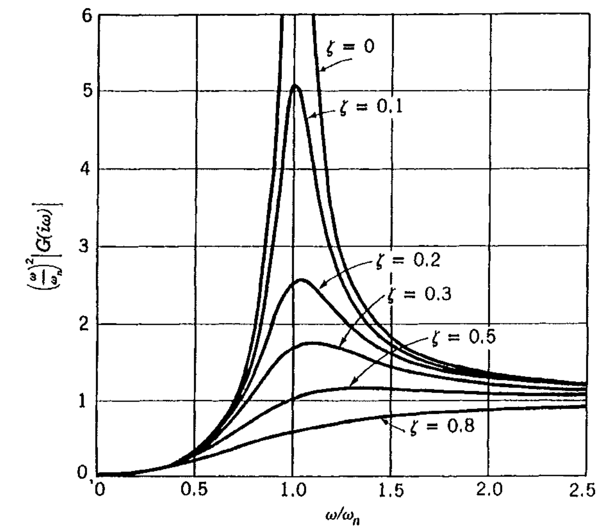


FIGURE 4.24

We note from Fig. 4.24 that the effect of multiplying $|G(i\omega)|$ by $(\omega/\omega_n)^2$ is to shift the peaks from values of ω smaller than ω_n to values larger than ω_n. We also note that $(\omega/\omega_n)^2|G(i\omega)|\to 0$ as $\omega/\omega_n\to 0$ and that $(\omega/\omega_n)^2|G(\omega)|\to 1$ as $\omega/\omega_n\to\infty$. This latter statement leads to an interesting result. From Fig. 4.19, we conclude that $\phi\to-\pi$ as $\omega/\omega_n\to\infty$, which implies that the excitation and response are 180° out of phase in this case. Hence, as $\omega/\omega_n\to\infty$ and $(\omega/\omega_n)^2|G(i\omega)|\to 1$, the displacement of $M-m$ becomes

$$x(t)=\frac{mR}{M}\sin(\omega t-\pi)=-\frac{mR}{M}\sin\omega t \quad (4.135)$$

On the other hand, under the same circumstances, the vertical displacement of the masses $m/2$ becomes

$$x(t)+R\sin\omega t=\frac{M-m}{M}R\sin\omega t \quad (4.136)$$

But in general the position of the mass center of a system of masses is defined as (see Section 5.3)

$$x_C=\frac{\sum_{i=1}^n m_ix_i}{\sum_{i=1}^n m_i} \quad (4.137)$$

which in this particular case yields

$$x_C=\frac{1}{M}\left[(M-m)\left(-\frac{mR}{M}\sin\omega t\right)+2\frac{m}{2}\frac{M-m}{M}R\sin\omega t\right]=0 \quad (4.138)$$

so that for very large driving frequencies ω the principal mass and the two eccentric masses move in such a way that the center of mass remains at rest in the static equilibrium position.

Examples of systems containing rotating unbalanced masses are very common, and in most cases the effect of the imbalance is undesirable. Washing machines and clothes dryers with rotating drums can serve as such examples if the clothes are not spread uniformly around the drum. It is assumed that the clothes do not move relative to the drum. Automobiles with unbalanced tires are other examples. Because normal operation involves increasing the driving frequency ω from zero to well beyond $\omega = \omega_n$, corrective measures are necessary if vibration is to be eliminated, such as spreading the clothes evenly and balancing the tires. In the discussion just preceding, it was tacitly assumed that washing machines, clothes dryers, and automobiles can be modeled as single-degree-of-freedom systems, which must be regarded only as a crude assumption. Nevertheless, the phenomenon described above is commonly encountered in these systems, so that the assumption has some measure of validity, at least for the purpose of explaining this phenomenon.

4.12 MOTION OF VEHICLES OVER WAVY TERRAIN

Let us consider a vehicle traveling with uniform velocity v over a wavy terrain. We shall model the vehicle as a damped single-degree-of-freedom system and the terrain as the function

$$y(x) = A \sin \frac{2\pi x}{L} \tag{4.139}$$

where L is the wavelength (Fig. 4.25*a*). The forward motion of the vehicle on the wavy terrain $y(x)$ results in a vertical motion $y(t)$ of the wheel. Because uniform forward motion implies the relation $x = vt$, the vertical motion of the wheel is simply

$$y(t) = A \sin \frac{2\pi v t}{L} \tag{4.140}$$

Considering the free-body diagram of Fig. 4.25*b*, we can use Newton's second law to write

$$-c[\dot{z}(t) - \dot{y}(t)] - k[z(t) - y(t)] = m\ddot{z}(t) \tag{4.141}$$

yielding

$$m\ddot{z}(t) + c\dot{z}(t) + kz(t) = c\dot{y}(t) + ky(t) \tag{4.142}$$

Dividing Eq. (4.142) through by m and using Eq. (4.140), we obtain

$$\ddot{z}(t) + 2\zeta\omega_n \dot{z}(t) + \omega_n^2 z(t) = \omega_n^2 A[\sin \omega t + 2\zeta(\omega/\omega_n) \cos \omega t] \tag{4.143}$$

where ζ and ω_n have the customary meaning and

$$\omega = 2\pi v/L \tag{4.144}$$

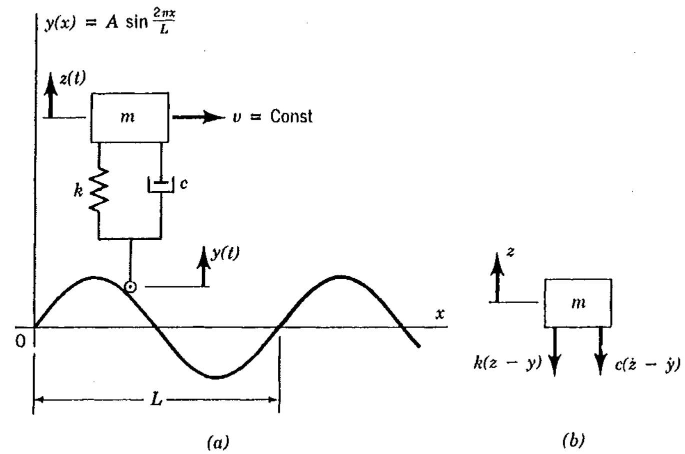


FIGURE 4.25

is the driving frequency. Introducing the notation

$$2\zeta\omega/\omega_n = \tan \alpha \tag{4.145}$$

we can reduce Eq. (4.143) to

$$\begin{aligned}\ddot{z}(t) + 2\zeta\omega_n \dot{z}(t) + \omega_n^2 z(t) &= \omega_n^2 A \frac{1}{\cos \alpha} (\sin \omega t \cos \alpha + \cos \omega t \sin \alpha) \\ &= \omega_n^2 A[1 + (2\zeta\omega/\omega_n)^2]^{1/2} \sin(\omega t + \alpha)\end{aligned} \tag{4.146}$$

so that α can be identified as an excitation phase angle.

The solution to an equation similar to Eq. (4.146), namely, Eq. (4.130), was obtained in Section 4.11 in the form of Eq. (4.133). Hence, we shall produce the solution to Eq. (4.146) by adapting solution (4.133) to the case at hand. Indeed, comparing Eqs. (4.130) and (4.146), we observe that in this case the amplitude of the excitation is multiplied by $[1 + (2\zeta\omega/\omega_n)^2]^{1/2}$ and that the sine function contains the phase angle α. With this in mind, we can modify solution (4.133) and write the solution to Eq. (4.146) directly in the form

$$z(t) = A[1 + (2\zeta\omega/\omega_n)^2]^{1/2} |G(i\omega)| \sin(\omega t + \phi_1) \tag{4.147}$$

where

$$\phi_1 = \phi + \alpha \tag{4.148}$$

is the response phase angle. Note that $|G(i\omega)|$ and ϕ are given by Eqs. (4.110) and (4.115), respectively.

A measure of the magnitude of the response can be obtained from the nondimensional ratio

$$\frac{|z(t)|}{A} = [1 + (2\zeta\omega/\omega_n)^2]^{1/2} |G(i\omega)| \tag{4.149}$$

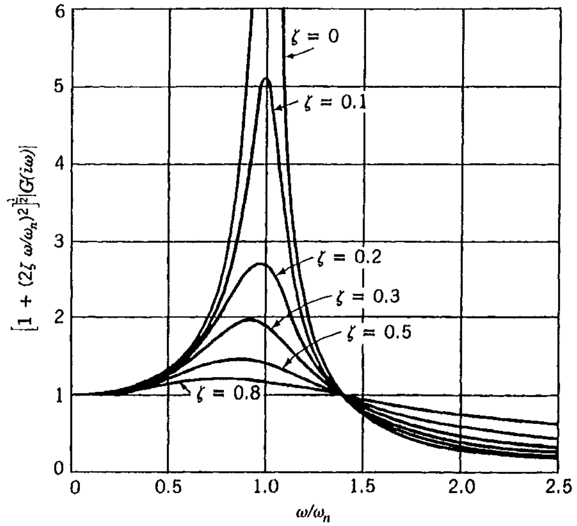


FIGURE 4.26

which is known as *transmissibility*. Figure 4.26 shows plots of $[1+(2\zeta\omega/\omega_n)^2]^{1/2}|G(i\omega)|$ versus ω/ω_n for various values of the damping factor ζ. We note that for small ζ the curves experience peaks in the neighborhood of $\omega=\omega_n$, but for $\omega<\omega_n$. Moreover, the response approaches zero as $\omega/\omega_n\to\infty$, and it does so faster for smaller damping. The response will be the largest for a given ratio of the velocity v to the wavelength L. Indeed, for small damping, the critical ratio is approximately equal to the natural frequency $f_n=\omega_n/2\pi$, which can be verified by substituting $\omega=\omega_n$ in Eq. (4.144). Shock absorbers in automobiles are really dampers designed to reduce vibration. They generally possess heavy damping. When they are worn out, however, vibration magnification like that discussed above can occur.

To calculate the response phase angle ϕ_1, let us write first

$$\tan\phi_1=\tan(\phi+\alpha)=\frac{\tan\phi+\tan\alpha}{1-\tan\phi\,\tan\alpha}$$

$$=\frac{-(2\zeta\omega/\omega_n)/[1-(\omega/\omega_n)^2]+2\zeta\omega/\omega_n}{1+\{(2\zeta\omega/\omega_n)/[1-(\omega/\omega_n)^2]\}2\zeta\omega/\omega_n}=-\frac{2\zeta(\omega/\omega_n)^3}{1-(\omega/\omega_n)^2+(2\zeta\omega/\omega_n)^2} \tag{4.150}$$

so that the phase angle ϕ_1 is

$$\phi_1=\tan^{-1}\left[-\frac{2\zeta(\omega/\omega_n)^3}{1-(\omega/\omega_n)^2+(2\zeta\omega/\omega_n)^2}\right] \tag{4.151}$$

Plots of ϕ_1 versus ω/ω_n for various values of ζ are shown in Fig. 4.27.

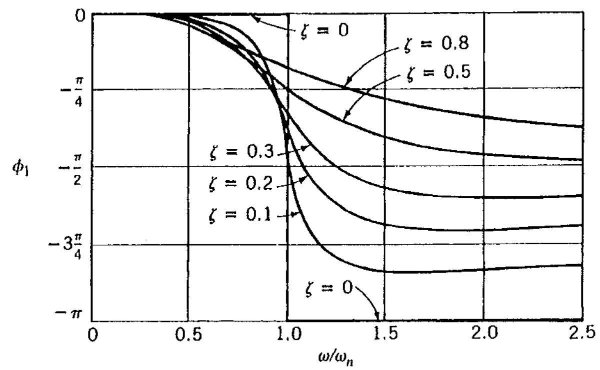


FIGURE 4.27

4.13 IMPULSE RESPONSE

The impulse response, denoted $g(t)$, was defined in Section 1.9 as the response to a unit impulse applied at $t=0$, with the initial conditions being equal to zero. Also in Section 1.9 it was shown that the impulse response is equal to the inverse Laplace transform of the transfer function $G(s)$, or

$$g(t)=\mathscr{L}^{-1}G(s) \tag{4.152}$$

In this section, we propose to derive the impulse response both for a first-order and a second-order system.

The equation of motion for a damper–spring system was shown in Section 4.2 to have the form

$$c\dot{x}(t)+kx(t)=f(t) \tag{4.153}$$

Taking the Laplace transform of both sides of Eq. (4.153), while letting $x(0)$ be equal to zero, we obtain

$$(cs+k)X(s)=F(s) \tag{4.154}$$

so that, using the analogy with Eq. (1.32), we conclude that the transfer function for the first-order system in question has the expression

$$G(s)=\frac{X(s)}{F(s)}=\frac{1}{cs+k}=\frac{1}{c(s+a)} \tag{4.155}$$

where $a=k/c$. Considering Eq. (4.152) and using the table of Laplace transform pairs (Section A.7), we obtain the impulse response

$$g(t)=\mathscr{L}^{-1}G(s)=\mathscr{L}^{-1}\frac{1}{c}\frac{1}{s+a}=\frac{1}{c}e^{-t/\tau}u(t) \tag{4.156}$$

where $\tau=1/a=c/k$ is recognized as the time constant and $u(t)$ is the unit step function. Note that we multiplied the response by $u(t)$ in recognition of the fact that it must be zero for $t<0$. An equation similar to Eq. (4.156) was obtained in Section 4.4 in connection with the free response of a first-order system. Indeed, comparing Eqs. (4.29) and (4.156) we conclude that the *impulse response of a first-order system is equivalent to the response to an initial excitation.* In the case of the mechanical system at hand the initial excitation is an initial displacement, $x(0)=1/c$. In the case of an electrical system, such as the RL circuit, the initial excitation is an initial charge, $q(0)=1/R$, where R is the resistance.

Next, let us consider a second-order system in the form of the mass–damper–spring system of Fig. 4.5. The differential equation for the system is given by Eq. (4.8), so that it can be verified easily that the transfer function has the form

$$G(s)=\frac{1}{ms^2+cs+k}=\frac{1}{m(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{4.157}$$

where ζ is the damping factor and ω_n is the frequency of undamped oscillation. Hence, the impulse response for the mass–damper–spring system is simply

$$g(t)=\mathscr{L}^{-1}G(s)=\mathscr{L}^{-1}\frac{1}{m(s^2+2\zeta\omega_n s+\omega_n^2)} \tag{4.158}$$

To obtain the inverse Laplace transform, it will prove convenient to expand $G(s)$ into partial fractions. It is not difficult to show that

$$G(s)=\frac{1}{m(s_1-s_2)}\left(\frac{1}{s-s_1}-\frac{1}{s-s_2}\right) \tag{4.159}$$

where, assuming that $\zeta<1$,

$$\left.\begin{matrix}s_1\\ s_2\end{matrix}\right\}=-\zeta\omega_n\pm i\omega_d \tag{4.160}$$

are the simple poles of $G(s)$, in which $\omega_d=(1-\zeta^2)^{1/2}\omega_n$ is the frequency of damped oscillation. Using the table of Laplace transform pairs (Section A.7), we can write

$$g(t)=\frac{1}{m(s_1-s_2)}(e^{s_1 t}+e^{s_2 t}) \tag{4.161}$$

Inserting s_1 and s_2 from Eqs. (4.160) into Eq. (4.161), we obtain the impulse response for the mass–damper–system in the form

$$g(t)=\frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t\,\mathscr{u}(t) \tag{4.162}$$

Note that the impulse response, Eq. (4.162), could have been obtained directly from Eq. (4.158) and the table of Laplace transform pairs in Section A.7.

An equation similar to Eq. (4.162) was obtained in Example 4.2. Comparing Eq. (d) of Example 4.2 with Eq. (4.162), we conclude that *the impulse response of the mass–damper–spring system is equal to the response to the initial velocity* $\dot{x}(0)=v_0=1/m$. Similarly, it is not difficult to see that for the LRC circuit of Fig. 4.9 the impulse response is equal to the response to the initial current $i(0)=1/L$.

4.14 STEP RESPONSE

The step response, denoted by $\mathscr{s}(t)$, is defined as the response to a unit step function applied at $t=0$, with the initial conditions being equal to zero. It was shown in Section 1.10 that the step response of a general linear system has the expression

$$\mathscr{s}(t)=\mathscr{L}^{-1}\frac{G(s)}{s} \tag{4.163}$$

where $G(s)$ is the transfer function.

Recalling Eq. (4.155), we can write the step response of the damper–spring system described by Eq. (4.153) in the form

$$\mathscr{s}(t)=\mathscr{L}^{-1}\frac{1}{cs(s+a)} \tag{4.164}$$

It is not difficult to show that the partial fractions expansion of the function on the right side of Eq. (4.164) is

$$\frac{1}{cs(s+a)}=\frac{1}{k}\left(\frac{1}{s}-\frac{1}{s+a}\right) \tag{4.165}$$

so that, using the table of Laplace transform pairs (Section A.7), we can write

$$\mathscr{s}(t)=\frac{1}{k}(1-e^{-t/\tau})\mathscr{u}(t) \tag{4.166}$$

where $\tau=1/a=c/k$ is the time constant and $\mathscr{u}(t)$ is the unit step function. The step response is plotted in Fig. 4.28.

Next, let us determine the step response of an undamped second-order system. The transfer function of such a system is obtained by simply letting $c=0$ in Eq. (4.157), so that the step response is

$$\mathscr{s}(t)=\mathscr{L}^{-1}\frac{1}{ms(s^2+\omega_n^2)} \tag{4.167}$$

But

$$\frac{1}{ms(s^2+\omega_n^2)}=\frac{1}{k}\left[1-\frac{1}{2(s-i\omega_n)}-\frac{1}{2(s+i\omega_n)}\right] \tag{4.168}$$

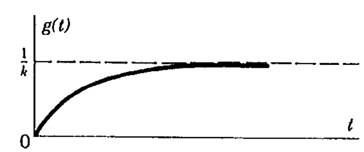


FIGURE 4.28

FIGURE 4.29

so that, using the table of Laplace transform pairs once again, we obtain the step response

$$\mathscr{o}(t)=\frac{1}{k}\left(1-\frac{1}{2}e^{i\omega_n t}-\frac{1}{2}e^{-i\omega_n t}\right)u(t)=\frac{1}{k}(1-\cos\omega_n t)\,u(t) \tag{4.169}$$

It should be pointed out that the resolution into partial fractions was not really necessary here, because the function on the right side of Eq. (4.167) can be found in the table of Laplace transform pairs in Section A.7. The response is plotted in Fig. 4.29.

4.15 RESPONSE TO ARBITRARY EXCITATION

The response to any arbitrary excitation can be obtained by means of the convolution integral derived in Section 1.11. Because we have used the symbol τ for the time constant, we rewrite Eq. (1.59) in the form

$$x(t)=\int_0^t g(t-\sigma)f(\sigma)\,d\sigma=\int_0^t g(\sigma)f(t-\sigma)\,d\sigma \tag{4.170}$$

where σ is a dummy variable of integration.

As an application of the convolution integral, let us consider the response of the damper–spring system to a force in the form of the ramp function shown in Fig. 4.30. The force can be expressed in the form

$$f(t)=\frac{f_0}{T}\,t\,u(t) \tag{4.171}$$

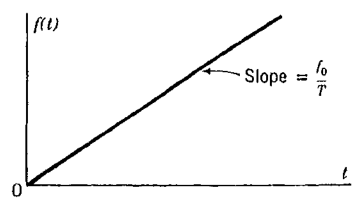


FIGURE 4.30

In Section 4.13, we showed that the impulse response of the damper–spring system is

$$g(t)=\frac{1}{c}e^{-t/\tau}u(t) \tag{4.172}$$

Inserting Eqs. (4.171) and (4.172) into the first form of the convolution integral, we can write

$$\begin{aligned} x(t)&=\frac{f_0}{Tc}\int_0^t \sigma e^{-(t-\sigma)/\tau}\,d\sigma=\frac{f_0}{Tc}e^{-t/\tau}\int_0^t \sigma e^{\sigma/\tau}\,d\sigma \\ &=\frac{f_0}{Tc}e^{-t/\tau}\left[\frac{e^{\sigma/\tau}}{(1/\tau)^2}\left(\frac{\sigma}{\tau}-1\right)\right]\Bigg|_0^t u(t)=\frac{f_0}{Tk}\left[t-\tau(1-e^{-t/\tau})\right]u(t) \end{aligned} \tag{4.173}$$

The response is plotted in Fig. 4.31. Note that, compared with an equivalent displacement input equal to $f(t)/k$, the output $x(t)$ exhibits a steady-state error equal to f_0c/Tk^2.

As a second illustration, let us consider the response of a mass–spring system to the rectangular pulse shown in Fig. 4.32. The force can be expressed in the form

$$f(t)=\begin{cases} f_0, & 0<t<T \\ 0, & \text{everywhere else} \end{cases} \tag{4.174}$$

Moreover, letting $\zeta=0$ in Eq. (4.162), we obtain the impulse response

$$g(t)=\frac{1}{m\omega_n}\sin\omega_n t\;u(t) \tag{4.175}$$

Inserting Eqs. (4.174) and (4.175) into the first form of the convolution integral,

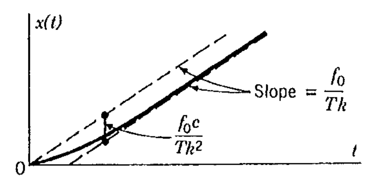


FIGURE 4.31

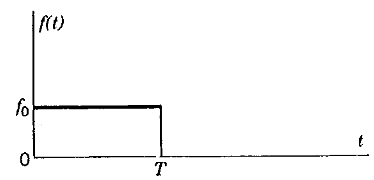


FIGURE 4.32

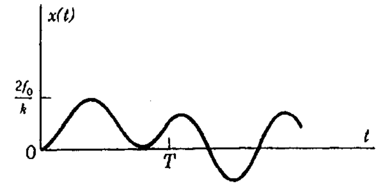


FIGURE 4.33

Eq. (4.170), we obtain

$$x(t)=\frac{f_0}{m\omega_n}\int_0^t \sin\omega_n(t-\sigma)\,d\sigma=\frac{f_0}{m\omega_n}\,\mathrm{Im}\int_0^t e^{i\omega_n(t-\sigma)}\,d\sigma,\qquad 0<t<T \tag{4.176a}$$

$$x(t)=\frac{f_0}{m\omega_n}\int_0^T \sin\omega_n(t-\sigma)\,d\sigma=\frac{f_0}{m\omega_n}\,\mathrm{Im}\int_0^T e^{i\omega_n(t-\sigma)}\,d\sigma,\qquad t>T \tag{4.176b}$$

Evaluation of the intergrals yields

$$x(t)=\begin{cases}\dfrac{f_0}{k}(1-\cos\omega_n t), & 0<t<T\\[2mm] \dfrac{f_0}{k}[\cos\omega_n(t-T)-\cos\omega_n t], & t>T\end{cases} \tag{4.177}$$

A typical plot is shown in Fig. 4.33.

The above result can be obtained, perhaps in a more direct fashion, by regarding the rectangular pulse as a superposition of two step functions. Indeed, recalling developments from Section 1.8, we can write the input in the form

$$f(t)=f_0[\mathscr{u}(t)-\mathscr{u}(t-T)] \tag{4.178}$$

so that the response can be expressed as the superposition of two step responses as follows:

$$x(t)=f_0[\mathscr{s}(t)-\mathscr{s}(t-T)] \tag{4.179}$$

Hence, using Eq. (4.169), we obtain

$$x(t)=\frac{f_0}{k}\{(1-\cos\omega_n t)\mathscr{u}(t)-[1-\cos\omega_n(t-T)]\mathscr{u}(t-T)\} \tag{4.180}$$

which is identical to Eqs. (4.177).

PROBLEMS

4.1 The system shown in Fig. 4.34 consists of two linear springs arranged in series. Determine the equivalent spring constant, defined as $k_{eq}=f/\delta$.

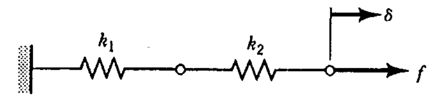


FIGURE 4.34

4.2 Figure 4.35 depicts a mass–damper system. Show that the system can be described by a first-order differential equation, and give an expression for the time constant. Then, indicate the electrical analog, the corresponding differential equation, and the time constant.

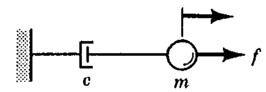


FIGURE 4.35

4.3 A massless rigid bar hinged at point 0 is supported by two linear springs, as shown in Fig. 4.36. Derive the differential equation for the angular motion θ under the assumption that the angle θ is sufficiently small that $\sin\theta\cong\theta$ and $\cos\theta\cong 1$. Determine the static equilibrium position θ_{st}, show how the effect of the weight Mg can be eliminated from the equation of motion, and calculate the natural frequency of oscillation about the equilibrium position.

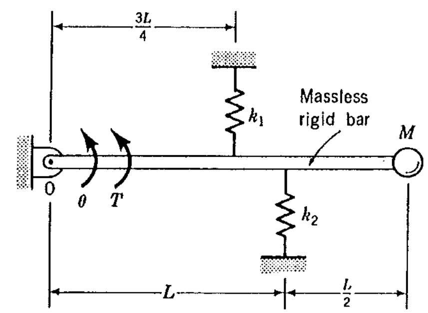


FIGURE 4.36

4.4 A mass m is suspended through a pulley-and-spring mechanism, as shown in Fig. 4.37. Let the spring be linear, and derive the differential equation of

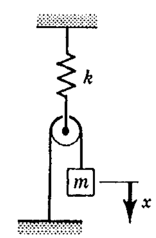


FIGURE 4.37