MAE182A TA Session #10

• Review Chapter 5 ~ 7

[Note] p.276

① real & different roots

$y = c_1|x|^{r_1} + c_2|x|^{r_2}$

② real & equal roots

$y = (c_1 + c_2 \ln|x|)|x|^{r_1}$

③ complex roots

$y = |x|^{\lambda}[c_1\cos(\mu\ln|x|) + c_2\sin(\mu\ln|x|)]$

P.280 Pr.5

$$x^2y'' - xy' + y = 0$$

$$L[y] = x^2y'' - xy' + y = 0$$

$$\overset{y=x^r}{\Longrightarrow} L[x^r] = x^2(x^r)'' - x(x^r)' + x^r$$

$$= x^2 r(r-1)x^{r-2} - xrx^{r-1} + x^r$$

$$= x^r[\underbrace{r(r-1) - r + 1}] = 0$$

Quadratic eq: $F(r) = r(r-1) - r + 1 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1, 1$ (repeated root)

$$y = (c_1 + c_2\ln|x|)|x|$$

P.348 Pr.8

$$y'' + 4y = \sin t + 2\delta(t - \tfrac{\pi}{4}),\quad y(0) = 0,\quad y'(0) = 0$$

$$\overset{\mathcal{L}}{\Longrightarrow} \mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[\sin t] + 2\mathcal{L}[\delta(t - \tfrac{\pi}{4})]$$

$$\Rightarrow s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)} + 4Y(s) = \frac{1}{s^2+1} + 2e^{-\frac{\pi}{4}s}$$

$$\Rightarrow (s^2+4)Y(s) = \frac{1}{s^2+1} + 2e^{-\frac{\pi}{4}s}$$

$$\Rightarrow Y(s) = \frac{1}{(s^2+1)(s^2+4)} + \frac{2e^{-\frac{\pi}{4}s}}{(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{2e^{-\frac{\pi}{4}s}}{s^2+4}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow \quad 1 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$\Rightarrow \quad 1 = (A+C)s^3 + (B+D)s^2 + (4A+C)s + (4B+D)$$

$$\therefore \begin{cases} A+C=0 & -\ ① \\ B+D=0 & -\ ② \\ 4A+C=0 & -\ ③ \\ 4B+D=0 & -\ ④ \end{cases}$$

From ① ③ $\Rightarrow$ $A=0,\ C=0$

From ② ④ $\Rightarrow$ $B=\frac{1}{3},\ D=-\frac{1}{3}$

$$\therefore \quad Y(s) = \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4} + \frac{2e^{-\frac{\pi}{4}s}}{s^2+4}$$

$$= \frac{1}{3}\left(\frac{1}{s^2+1^2}\right) - \frac{1}{6}\left(\frac{2}{s^2+2^2}\right) + \underbrace{\left(\frac{2}{s^2+2^2}\right)}_{F(s)\ \Rightarrow\ f(t)=\sin 2t} e^{-\frac{\pi}{4}s}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u_{\frac{\pi}{4}}(t)\, f\left(t-\frac{\pi}{4}\right)$$

$$\Rightarrow \quad y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t + u\left(t-\frac{\pi}{4}\right)\sin 2\left(t-\frac{\pi}{4}\right)$$

[Note]

$$\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$$

$$\mathcal{L}[u_c(t) f(t-c)] = e^{-cs}F(s)$$

$$\mathcal{L}[\delta(t-c)] = e^{-cs}$$

$$\mathcal{L}[y''] = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}[y'] = sY(s) - y(0)$$

P.447. Pr.1

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x + \begin{pmatrix} e^t \\ t \end{pmatrix} = A\vec{x} + \vec{g}(t)$$

1° Homogeneous system $\vec{x}' = A\vec{x}$

$$\det(A - rI) = 0 \Rightarrow \begin{vmatrix} 2-r & -1 \\ 3 & -2-r \end{vmatrix} = 0$$

$$\Rightarrow (2-r)(-2-r) + 3 = 0 \Rightarrow -4 + r^2 + 3 = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$$

① $r_1 = 1$

$$(A - rI)\vec{\xi}_1 = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \xi_1 = \xi_2 \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\vec{X}_1 = \vec{\xi}_1 e^{r_1 t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$$

② $r_2 = -1$

$$(A - rI)\vec{\xi}_2 = 0 \Rightarrow \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 3\xi_1 - \xi_2 = 0 \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\vec{X}_2 = \vec{\xi}_2 e^{r_2 t} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$$\vec{X}_c = c_1\vec{X}_1 + c_2\vec{X}_2 = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

$\Rightarrow$ nonhomogeneous system

$$\Psi = [\vec{X}_1, \vec{X}_2] = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}$$

$$\Psi^{-1} = \frac{\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}}{\begin{vmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{vmatrix}} = \frac{1}{2}\begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}$$

$$\vec{u}(t) = \int_{t_0}^{t} \Psi^{-1}(s)\,\vec{g}(s)\,ds = \frac{1}{2}\int_{t_0}^{t} \begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^s & e^s \end{pmatrix}\begin{pmatrix} e^s \\ s \end{pmatrix} ds$$

$$\Rightarrow \vec{u}(t) = \frac{1}{2}\int_{t_0}^{t} \begin{pmatrix} 3 - se^{-s} \\ -e^{2s} + se^s \end{pmatrix} ds = \frac{1}{2}\begin{bmatrix} 3s + se^{-s} + e^{-s} \\ -\frac{1}{2}e^{2s} + se^s - e^s \end{bmatrix}_{t_0}^{t}$$

$$\Rightarrow \vec{u}(t) = \frac{1}{2}\begin{bmatrix} 3t + te^{-t} + e^{-t} \\ -\frac{1}{2}e^{2t} + te^t - e^t \end{bmatrix} + \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{=\vec{c}}$$

$$\vec{X} = \Psi\,\vec{u}(t)$$

$$\Rightarrow \vec{X} = \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 3t + te^{-t} + e^{-t} \\ -\frac{1}{2}e^{2t} + te^t - e^t \end{pmatrix} + \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \vec{X} = c_1\begin{pmatrix} e^t \\ e^t \end{pmatrix} + c_2\begin{pmatrix} e^{-t} \\ 3e^{-t} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3te^t + t + 1 - \frac{1}{2}e^t + t - 1 \\ 3te^t + t + 1 - \frac{3}{2}e^t + 3t - 3 \end{pmatrix}$$

$$\Rightarrow \vec{X} = \underbrace{c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^t + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{-t}}_{\text{general sol'n for homog. sys}} + \underbrace{\frac{1}{2}\begin{pmatrix} (3t - \frac{1}{2})e^t + 2t \\ (3t - \frac{3}{2})e^t + 4t - 2 \end{pmatrix}}_{\text{Particular sol'n for nonhomge. system}}$$