TA Session 09

Ch 7 : Systems of 1st Order D.E.s.

1. Homogeneous
2. Non-homogeneous.

1. $\dot{\vec{x}} = A\vec{x}$ (Homogeneous Solution).

If λ_i is an e-val of A, and $\vec{v}_i$ is a corresponding e-vec to λ_i, then $\vec{x}_i = \vec{v}_i e^{\lambda_i t}$ is a solution of $\dot{\vec{x}} = A\vec{x}$.

Proof: $$(\vec{x}_i)' = (\vec{v}_i e^{\lambda_i t})' = \lambda_i \vec{v}_i e^{\lambda_i t} \overset{A\vec{v} = \lambda\vec{v}}{=\!=\!=} A\vec{v}_i e^{\lambda_i t} = A(\vec{v}_i e^{\lambda_i t}) = A\vec{x}_i$$

When you have a system with a dimension n, you need n solutions to form a general solution. (Fundamental set of solutions, Recall Chapter 4)

$$x = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$$

(Assuming $\lambda_1, \ldots \lambda_n$ are <u>distinct</u> real eigenvalues)

Question: what happened if
(1) $\lambda = \sigma + \omega i$ is a complex eigenvalue.
(2) $\lambda_i = \lambda_j$ are repeated eigenvalues

(1) Assume $\lambda_1 = \sigma + \omega i$, then $\vec{v}_1 \in \mathbb{C}$, $\vec{v}_1 = \vec{a} + \vec{b} i \; (\vec{a}, \vec{b} \in \mathbb{R})$

From previous theorem we have

$$\vec{x}_1 = \vec{v}_1 e^{\lambda_1 t} = (\vec{a} + \vec{b} i) e^{\sigma t} (\cos\omega t + i \sin\omega t)$$

$$= \left[(\vec{a}\cos\omega t - \vec{b}\sin\omega t) + i(\vec{a}\sin\omega t + \vec{b}\cos\omega t) \right] e^{\sigma t}$$

Both real and imaginary part are solutions, thus

$$c_1(\vec{a}\cos\omega t - \vec{b}\sin\omega t)e^{\sigma t} + c_2(\vec{a}\sin\omega t + \vec{b}\cos\omega t)e^{\sigma t}$$

is a solution for complex roots.

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(2) λ_i is a root that repeated m times

(Algebraic Multiplicity is m)

We need to check $(A - \lambda_i I)\vec{v}_i = \vec{0}$, where $A_{n\times n} \in \mathbb{R}$

If $(A - \lambda_i I)$ has q linearly dependent rows

$\Rightarrow$ q linearly independent eigenvectors can be found for λ_i

(Geometric Multiplicity is q)

which means

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \vec{v}_1 \cdot v_1 + \vec{v}_2 \cdot v_2 + \cdots + \vec{v}_q \cdot v_q$$

For geometric multiplicity of q, solution can be in the form of

$$c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t} + \cdots + c_q \vec{v}_q e^{\lambda t}$$

Q: What about the rest $(m-q)$ terms? $\boxed{\text{Assume } m-q=1}$

– instead of $\vec{x}_i = \vec{v}_i e^{\lambda_i t}$, assume $\vec{x}_i = \vec{\eta}_i e^{\lambda_i t} + \vec{v}_i t e^{\lambda_i t}$

($\vec{\eta}$ is NOT an e-vec of A)

Cont. plug into $\vec{x}' = A\vec{x}$, have

$$(\vec{X_i})' = \lambda_i \vec{\eta} e^{\lambda_i t} + \vec{V_i} e^{\lambda_i t} + \lambda_i \vec{V_i} t e^{\lambda_i t}$$

$$= (\lambda_i \vec{\eta} + \vec{V_i}) e^{\lambda_i t} + \lambda_i \vec{V_i} t e^{\lambda_i t} = A \overbrace{(\vec{\eta} e^{\lambda_i t} + \vec{V_i} t e^{\lambda_i t})}^{\vec{X_i}}$$

Collect terms, and <u>MURDER</u> the coefficients!

$te^{\lambda_i t}$: $\lambda_i \vec{V_i} - A\vec{V_i} = 0$ (Satisfied because $\vec{V_i}$ is an e-vec of λ_i)

$e^{\lambda_i t}$: $\boxed{(A - \lambda_i I)\vec{\eta} = \vec{V_i}}$ This is how you can find $\vec{\eta}$

Q: If I have q e-vecs. any one of them will work?

A: Yes.

Q: What's the form of the solution, something like "$c_1 \vec{V_1} e^{\lambda t} + c_2 \vec{\eta} t e^{\lambda t}$"?

P438 Pr 18. A: NO! It's $\boxed{c_1 \vec{V_1} e^{\lambda t} + c_2(\vec{\eta} e^{\lambda t} + \vec{V_1} t e^{\lambda t})}$

Q: What if $m - q = 2$? What about the next multiple root?

A: Assume $\vec{X_i} = \vec{\xi} e^{\lambda_i t} + \vec{\eta} e^{\lambda_i t} + \frac{1}{2} \vec{V_i} t^2 e^{\lambda_i t}$, plug in then you'll find $(A - \lambda_i I)\vec{\xi} = \vec{\eta}$, if you have even more roots, carry on and do $(A - \lambda_i I)\vec{\xi}_{j+1} = \vec{\xi}_j$ to find them all.

Q: Can I use this to find a solution regardless of q?

A: I don't think so. as $(A - \lambda_i I)$ has q linearly dependent rows, using $(A - \lambda_i I)\vec{\eta} = \vec{V_i}$ unnecessarily will lead to a non-exist solution case. Thus I recommend you to find all e-vecs from $(A - \lambda_i I)\vec{V_i} = \vec{0}$ before doing this.

※ Eigenvalue Decomposition & Jordan Form

Start with definition,

$$\lambda_1 \vec{V}_1 = A\vec{V}_1$$
$$\lambda_2 \vec{V}_2 = A\vec{V}_2$$
$$\vdots$$
$$\lambda_n \vec{V}_n = A\vec{V}_n$$

(Assume $m=q$, i.e. we can find n linearly independent e-vecs for all e-vals)

in matrix form,

$$\underbrace{\begin{bmatrix} \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{\Lambda} = A \underbrace{\begin{bmatrix} \vec{V}_1 & \vec{V}_2 & \cdots & \vec{V}_n \end{bmatrix}}_{T}$$

Left multiply by T^{-1}, we have

$$\Lambda = T^{-1}AT \quad \text{(Eigendecomposition)}$$

But what if $m > q$?

- We can't decomposite A into the form of Λ, but we can put A into Jordan form instead.

$$\underbrace{\begin{bmatrix} \vec{V}_1 & \vec{\eta} & \vec{\xi} & \cdots & \vec{V}_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & & & \vdots \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \cdots & & & \lambda_n \end{bmatrix}}_{J} = A \underbrace{\begin{bmatrix} \vec{V}_1 & \vec{\eta} & \vec{\xi} & \cdots & \vec{V}_n \end{bmatrix}}_{T}$$

assuming $\lambda_1 = \lambda_2 = \lambda_3$, $(A-\lambda_1 I)\vec{\eta} = \vec{V}_1$, $(A-\lambda_1 I)\vec{\xi} = \vec{\eta}$

This technique will be useful in non-homogeneous problem

Ex: P405 Pr. 18

$$\vec{x}' = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix} \vec{x}, \qquad x(0) = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

Solve the I.V.P and describe the behavior as $t \to \infty$

Eigenvalues: $\det(A - \lambda I)$

$$= \begin{vmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{vmatrix} = \underline{\lambda^2(4-\lambda)} - \underline{4+\lambda} = 0$$

$$\Rightarrow (\lambda^2 - 1)(\lambda - 4) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 4$$

Eigenvectors:

(1) $\lambda_1 = 1$, $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set $v_1 = 1$. then $v_3 = -1$, $v_2 = 2$.

$$\Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(2) $\lambda_2 = -1$: $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set $v_1 = 1$. then $v_3 = 1$. $v_2 = -2$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Cont. (3) $\lambda_3 = 4$. $(A - \lambda_3 I)\vec{V_3} = \vec{0}$

$$\begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Set $V_1 = 1$, then $V_3 = -4$, $V_2 = \frac{1}{2}$

$$\Rightarrow \vec{V_3} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ -4 \end{pmatrix}, \text{ or } \vec{V_3} = \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}$$

General solution given by

$$\vec{X} = C_1\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} + C_2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} + C_3\begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}$$

I.V.P.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ -1 & 1 & -8 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$$

Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 2 & -2 & 1 & 5 \\ -1 & 1 & -8 & 5 \end{array}\right] \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & -4 & -3 & -9 \\ 0 & 2 & -6 & 12 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 7 \\ 0 & 1 & \frac{3}{4} & \frac{9}{4} \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix}$$

$$\vec{X} = 6\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}e^{t} + 3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}e^{-t} - \begin{pmatrix} 2 \\ 1 \\ -8 \end{pmatrix}e^{4t}$$

as $t \to \infty$, $\vec{X} \to \infty$

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$$x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x \quad , \quad x(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Eigenvalues : $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{vmatrix} = (\lambda+7)(\lambda-1)+16 = \lambda^2+6\lambda+9 = 0.$$

$\Rightarrow \lambda_1 = \lambda_2 = -3.$ is a repeated root.

Eigenvectors : (1) $\lambda_1 = -3.$

$$(A - \lambda_1 I)\vec{v}_1 = \vec{0} \Rightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{v}_1$$

(2) $\lambda_2 = -3.$ $q = 1$ so we can't find more eigenvectors, turn to $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1$

$$\Rightarrow \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{3}{4} \end{pmatrix}, \text{ or } \vec{v}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

General Solution is :

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-3t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} \right]$$

Apply I.C.,
$$\begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,
$$\vec{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}$$
$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}$$

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2. Non-homogeneous.

$$\dot{\vec{x}} = A\vec{x} + B(t).$$

Q: Is this the same as the scalar case of D.E.?

A: Yes, and no.

Method: (1) Diagolization.

(2) Undetermined Coefficients

(3) Variation of Parameters

(4) Laplace Transform

$\}$ Contains rich information about linear dynamic system

(1) Diagolization

Idea: From Chapter 2, we can solve 1st Order D.E. in scalar form. so can we turn this into scalar form?

— Yes, eigendecomposition.

Define a coordinate transformation: $\vec{x} = T\vec{y}$, where $T = [v_1 \mid v_2 \mid \cdots \mid v_n]$.

$\dot{\vec{x}}' = A\vec{x} + B(t) \Rightarrow T\dot{\vec{y}}' = AT\vec{y} + B(t)$, left multiply by T^{-1},

$$\Rightarrow \dot{\vec{y}}' = T^{-1}AT\vec{y} + T^{-1}B(t) \Rightarrow \dot{\vec{y}}' = \Lambda\vec{y} + \bar{B}(t) \quad (\bar{B}(t) = T^{-1}B(t))$$

$$\Rightarrow \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} \bar{b}_1(t) \\ \vdots \\ \bar{b}_n(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1' = \lambda_1 y_1 + \bar{b}_1(t) \\ \vdots \\ y_n' = \lambda_n y_n + \bar{b}_n(t) \end{cases}$$

Now we can solve for $\vec{y}$ one by one, then transform back using $\vec{x} = T\vec{y}$ to obtain $\vec{x}$.

cont. For repeated roots, use Jordan Form, for example

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{bmatrix} \lambda_1 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_1 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ & & & & 1 \\ 0 & \cdots & & 0 & \lambda_1 \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} \bar{b}_1(t) \\ \vdots \\ \bar{b}_n(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} y_1' = \lambda_1 y_1 + y_2 + \bar{b}_1(t) \\ y_2' = \lambda_1 y_2 + y_3 + \bar{b}_2(t) \\ \vdots \\ y_n' = \lambda_1 y_n + \bar{b}_n(t) \end{cases}$$

Start with the LAST row up to the first row, plug in solved y_i when needed.

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(2) Undetermined Coefficients.

Idea: From Chapter 3, we know that $\vec{X} = \vec{X}_c + \vec{X}_p$, this theorem holds for system of D.E. as well.

We can solve $\dot{\vec{X}} = A\vec{X}$, but how to find $\vec{X}_p$? Should we use the same guessing as in Chapter 3?

— Yes, but take extra care when you have same form with homogeneous solution $e^{\lambda t}$, instead of guessing $\vec{a}te^{\lambda t}$, we need a full guess of $\underline{\vec{a}te^{\lambda t} + \vec{b}e^{\lambda t}}$

<u>Why</u>?

Because in scalar case, $y = c_1 e^{\lambda t} + Ae^{\lambda t} + Bte^{\lambda t} + \cdots$, coefficient A can be absorbed by c_1. However, in vector case, $\vec{X} = c_1\vec{V}_1 e^{\lambda t} + \vec{a}e^{\lambda t} + \vec{b}te^{\lambda t} + \cdots$, $\vec{a}$ may not be absorbed by $c_1\vec{V}_1$ so you need to keep it!

Ex: Solve non-homogeneous system with undetermined coefficient method.

$$\vec{X}' = \begin{pmatrix} 3 & 0 \\ 5 & -2 \end{pmatrix} \vec{X} + \begin{pmatrix} -4\sin 2t \\ \cos 2t \end{pmatrix}$$

Homogeneous equation:

Eigenvalues: $\det(A - \lambda I) = (\lambda - 3)(\lambda + 2) = 0$

$$\Rightarrow \lambda_1 = -2, \quad \lambda_2 = 3.$$

Eigenvectors:

(1) $\lambda_1 = -2$.

$$\begin{pmatrix} 5 & 0 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(2) $\lambda_2 = 3$

$$\begin{pmatrix} 0 & 0 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Homogeneous solution:

$$\vec{X}_c = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t}$$

Particular Solution:

$$\vec{X}_p = \vec{a} \cos 2t + \vec{b} \sin 2t$$

$$\vec{X}_p' = -2\vec{a} \sin 2t + 2\vec{b} \cos 2t = A\vec{a} \cos 2t + A\vec{b} \sin 2t + \begin{pmatrix} -4 \\ 0 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t$$

$$\begin{cases} A\vec{b} + \begin{pmatrix} -4 \\ 0 \end{pmatrix} + 2\vec{a} = \vec{0} \\ A\vec{a} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2\vec{b} = \vec{0} \end{cases} \Rightarrow \begin{cases} \vec{a} = \begin{pmatrix} \frac{6}{13} \\ -\frac{7}{39} \end{pmatrix} \\ \vec{b} = \begin{pmatrix} \frac{12}{13} \\ \frac{173}{78} \end{pmatrix} \end{cases}$$

General Solution.

$$\vec{X} = \vec{X}_c + \vec{X}_p$$

$$= C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} \frac{8}{13} \\ -\frac{7}{39} \end{pmatrix} \cos 2t + \begin{pmatrix} \frac{12}{13} \\ \frac{173}{78} \end{pmatrix} \sin 2t$$

(3) Variation of Parameters.

Fundamental Matrix

$$\Psi(t) = \left(\vec{X}_1 \,\vdots\, \vec{X}_2 \,\vdots\, \cdots \,\vdots\, \vec{X}_n \right)$$

Fundamental Matrix satisfying the initial condition (Matrix Exponential)

$$\Phi(t) = e^{At} \quad \text{or} \quad \exp(At)$$

※ e^{At} is not taking exponential for all elements in A unless A is diagonal matrix !

How to find $\Phi(t)$ for A ?

(1) $\Phi(t) = e^{At} = e^{T\Lambda T^{-1}} = Te^{\Lambda}T^{-1}$, where $e^{\Lambda} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}$

(Diagonalization)

(2) $\Phi(t) = \mathcal{L}^{-1}[(sI - A)^{-1}]$

(Laplace Transform)

(3) $\Phi(t) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$

(Power Series)

(4) Relation between Ψ and Φ

$$\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$$

Time response of linear dynamic system

$$\vec{X} = \underbrace{\Psi(t)\Psi^{-1}(t_0)\vec{X}_0}_{\substack{\text{Solution of Homogeneous Eq.} \\ \text{(Zero Input response/ Free vibration)}}} + \underbrace{\Psi(t)\int_{t_0}^{t}\Psi^{-1}(\tau)B(\tau)\,d\tau}_{\substack{\text{Particular Solution} \\ \text{(Zero state response/convolution)}}}$$

or

$$\vec{X} = \overbrace{\Phi(t)\vec{X}_0} + \overbrace{\int_{t_0}^{t}\Phi(t-\tau)B(\tau)\,d\tau}$$

※ $\Phi(t)\Phi^{-1}(\tau) = \Phi(t-\tau)$

(4) Laplace Transform.

$$\dot{\vec{x}} = A\vec{x} + b(t)$$

Frequency response of linear dynamic system

$$\underline{X}(s) = \underbrace{(sI - A)^{-1}}_{\text{Transfer Function (describing stability and performance of the system)}} B(s)$$

Solution is given by

$$x(t) = \mathcal{L}^{-1}[\underline{X}(s)]$$

Note that the above equation is based on the assumption that $\vec{X}(0) = \vec{0}$, if you have non-zero initial condition please use $s\underline{X}(s) - x(0) = A\underline{X}(s) + G(s)$ instead.