MAE 182A TA Session # 7

- Review chapter 6 Laplace Transform
- Constant coefficients
  - characteristic eq.
  - $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
- Variable coefficients
  - series solution
  - $y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$
- Laplace transform is useful for equations w/ <u>discontinuous</u> or <u>impulsive</u> forcing terms.

$$\mathcal{L}[y(t)] = Y(s) \triangleq \int_0^{\infty} e^{-st} y(t)\,dt = \lim_{A\to\infty} \int_0^A e^{-st} y(t)\,dt$$

- y is <u>piecewise continuous</u> $\quad 0 < t \le A$
- $y(t) \le k e^{at}$

LT exists for $s > a$

ex

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{-st} e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad \underline{s > a}$$

$\mathcal{L}$ is a linear operator.

$\therefore \mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2]$

- Laplace of derivatives

$$\mathcal{L}[y'(t)] = \lim_{A\to\infty} \int_0^A e^{-st} y'(t)\,dt \overset{\text{Integration by parts}}{=} \lim_{A\to\infty} \Big[\underbrace{e^{-sA} y(A)}_{0 \text{ as } A\to\infty} - y(0) + s \underbrace{\int_0^A e^{-st} y(t)\,dt}_{\mathcal{L} \text{ as } A\to\infty} \Big]$$

$$= s\mathcal{L}[y] - y(0)$$

$$\mathcal{L}[y''(t)] = s\mathcal{L}[y'] - y'(0) = s[s\mathcal{L}[y] - y(0)] - y'(0)$$

$$= s^2\mathcal{L}[y] - sy(0) - y'(0).$$

$$\therefore \mathcal{L}[y^{(n)}(t)] = s^n \mathcal{L}[y] - s^{n-1} y(0) - \cdots - s y^{(n-2)}(0) - y^{(n-1)}(0)$$

- Solving I.V.P's
  - transfer DE into s-domain
  - solve equation algebraicly
  - invert back to original domain

ex. $y'' - y' - 2y = 0$, $y(0) = 1$, $y'(0) = 0$ (p3)

$$\overset{\mathcal{L}}{\Rightarrow} \mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0$$

$$\Rightarrow \left(s^2\mathcal{L}[y] - s y(0) - y'(0)\right) - \left(s\mathcal{L}[y] - y(0)\right) - 2\mathcal{L}[y] = 0$$

$$\Rightarrow (s^2 - s - 2)\,Y(s) + (1-s)\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0} = 0.$$

$$\Rightarrow Y(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}$$

Now convert back. Goal is to manipulate $Y(s)$ until it is in a recognizable term to use table.

partial fractions

$$\Rightarrow Y(s) = \frac{s-1}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$$

$\times (s-2)(s+1)$

$$\Rightarrow s-1 = a(s+1) + b(s-2)$$

$$\cancel{\Rightarrow s-1 = (a+b)s + (a-2b)}$$

$$s=2 \Rightarrow 1 = 3a \Rightarrow a = 1/3$$

$$s=-1 \Rightarrow -2 = -3b \Rightarrow b = 2/3$$

$$\therefore Y(s) = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} \mathcal{L}^{-1}[Y(s)] = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$$

table in p3-1

$$\Rightarrow y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$\mathcal{L}[e^{at}] = \frac{1}{s-a}$

ex $y'' + y = \sin 2t \quad y(0) = 2,\ y'(0) = 1$

$$\overset{\mathcal{L}}{\Rightarrow} \mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[\sin 2t]$$

$$\Rightarrow s^2 Y(s) - s y(0) - y'(0) + Y(s) = \frac{2}{s^2+2^2}$$

$$\Rightarrow (s^2+1) Y(s) = \frac{2}{s^2+4} + 2s + 1 = \frac{2s^3+s^2+8s+6}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$\times (s^2+1)(s^2+4)$

$$\Rightarrow 2s^3+s^2+8s+6 = (as+b)(s^2+4) + (cs+d)(s^2+1)$$

$$\Rightarrow 2s^3+s^2+8s+6 = (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$$

$$\Rightarrow \begin{cases} a+c = 2 \\ 4a+c = 8 \end{cases} \qquad \begin{cases} b+d = 1 \\ 4b+d = 6 \end{cases}$$

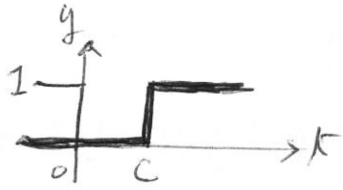
$$\Rightarrow \begin{cases} a = 2 \\ c = 0 \end{cases} \qquad \begin{cases} b = +5/3 \\ d = -2/3 \end{cases}$$

$$\therefore Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} + \frac{(-2/3)}{s^2+4}$$

$$\overset{\mathcal{L}^{-1}}{\Rightarrow} y(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

- Step functions

$$u_c(t) = \begin{cases} 0 & , t < c \\ 1 & , t \geq c \end{cases}$$

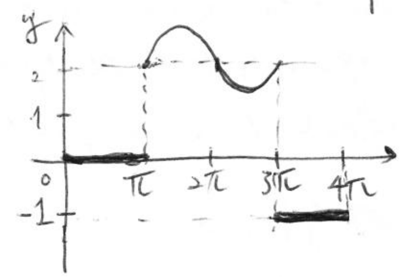


$u_c(t)$

$1 - u_c(t)$

- steps to 1
- steps at c
- allows us to deal w/ piecewise continuous functions

ex



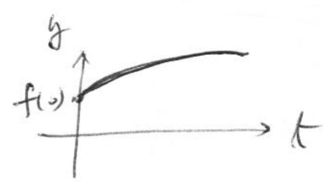
$$y(t) = \begin{cases} 0 & , 0 \leq t < \pi \\ 2\sin(t-\pi) & , \pi \leq t < 3\pi \\ -1 & 3\pi \leq t < 4\pi \end{cases}$$

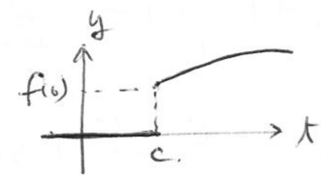
$$y(t) = 2u_\pi(t)\sin(t-\pi) - 2u_{3\pi}(t)\sin(t-\pi) - u_{3\pi}(t)$$

$$\mathcal{L}[u_c(t)] = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt = \frac{e^{-cs}}{s} \quad s > 0$$

From previous ex. We know

$$\text{if } y = \begin{cases} 0 & , t < c \\ f(t-c) & , t \geq c \end{cases} \Rightarrow y = u_c(t) f(t-c)$$





[Theorem]

$$\mathcal{L}[u_c(t)\,y(t-c)] = e^{-cs}\mathcal{L}[y(t)] = e^{-cs}Y(s)$$

or

$$\Rightarrow \quad u_c(t)\,y(t-c) = \mathcal{L}^{-1}[e^{-cs}Y(s)]$$

* To shift a fcn, we multiply Laplace by e^{-cs}.

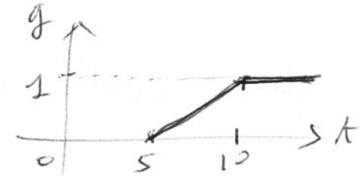
$$\mathcal{L}[e^{ct}g(t)] = Y(s-c)$$

* substituting $(s-c)$ in Laplace is equivalent to multiplying original fcn by e^{ct}

$$\Rightarrow \quad e^{ct}g(t) = \mathcal{L}^{-1}[Y(s-c)]$$

ex $\quad y'' + 4y = g(t), \quad y(0)=0, \quad y'(0)=0$

$$g(t) = \begin{cases} 0 & , 0 \le t < 5 \\ \frac{t-5}{5} & , 5 \le t < 10 \\ 1 & , 10 \le t \end{cases}$$



rewrite:

$$g(t) = \frac{u_5(t-5) - u_{10}(t)(t-10)}{5}$$

$$\overset{\mathcal{L}}{\Rightarrow} \quad s^2Y(s) - s\,\cancel{y(0)} - \cancel{y'(0)} + 4Y(s) = \mathcal{L}[g(t)]$$

$$\Rightarrow \quad (s^2+4)Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

$$\Rightarrow \quad Y(s) = \frac{e^{-5s} - 10^{-10s}}{5}H(s). \quad H(s) = \frac{1}{s^2(s^2+4)}$$

$$\Rightarrow \quad y(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

$$H(s) = \frac{1}{s^2(s^2+4)} = \frac{(\frac{1}{4})}{s^2} - \frac{(\frac{1}{4})}{s^2+4} \Rightarrow h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

$$\therefore \quad y(t) = \frac{1}{5}\left[u_5(t)\left(\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)\right) - u_{10}(t)\left(\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)\right)\right]$$

Impulse functions

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\,dt$$

- $g(t)$ occurs in very short interval
- $g(t) = 0$ outside τ

$$= \int_{-\infty}^{\infty} g(t)\,dt$$

Unit Impulse

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

dirac delta fun

if impluse occurs at t_0, $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

$$\mathcal{L}[\delta(t-t_0)] = e^{-st_0}$$

ex $2y'' + y' + 2y = \delta(t-5)$, $y(0)=0$, $y'(0)=0$ p8

$$\Rightarrow (2s^2+s+2)Y(s) = e^{-5s}$$

$$\Rightarrow Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2(s^2+\frac{1}{2}s+1)} = \frac{e^{-5s}}{2}\underbrace{\left[\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}\right]}_{H(s)}$$

$$Y(s) = \frac{1}{2}e^{-5s}H(s)$$

$$\xrightarrow{\mathcal{L}^{-1}} y(t) = \frac{1}{2}\cdot u_5(t)\,h(t-5)$$

$$H(s) = \frac{1}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2} = \frac{\frac{4}{\sqrt{15}}\cdot(\frac{\sqrt{15}}{4})}{[s-(-\frac{1}{4})]^2+(\frac{\sqrt{15}}{4})^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} h(t) = \frac{4}{\sqrt{15}}e^{-\frac{1}{4}t}\sin\frac{\sqrt{15}}{4}t$$

$$\therefore y(t) = \frac{1}{2}u_5(t)\,h(t-5) = \left(\frac{1}{2}\right)\left(\frac{4}{\sqrt{15}}\right)u_5(t)\cdot e^{-\frac{1}{4}(t-5)}\sin\frac{\sqrt{15}}{4}(t-5)$$

$$\Rightarrow y(t) = \frac{2}{\sqrt{15}}u_5(t)\,e^{-\frac{(t-5)}{4}}\sin\frac{\sqrt{15}}{4}(t-5) = \begin{cases} 0, & t<5 \\ \frac{2}{\sqrt{15}}e^{-\frac{(t-5)}{4}}\sin\frac{\sqrt{15}}{4}(t-5), & t\geq 5\end{cases}$$