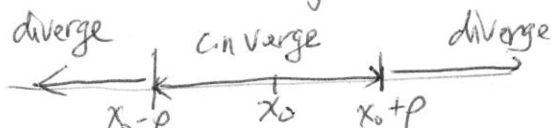
5/8/15 PD

TA Session Note #6

- Review chap 5 Series solutions
- Power Series: $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ expand about x_0

This series will converge at x_0 and in the interval of convergence $|x - x_0| < \rho$. $\rho > 0$: radius of convergence



- If a fun y has a Taylor series about $x = x_0$ w/ radius of convergence $\rho > 0$, it is <u>analytic</u>.
- It means y is <u>continuous</u> and <u>infinitely differentiable</u> on the interval of convergence
- Taylor Series expansion about $x = x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

- Goal is to solve

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0 \quad \text{around } x_0 \qquad \text{—(1)}$$

w/ y as a power series $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$(1) \Rightarrow \quad y'' + \frac{Q(x)}{P(x)} y' + \frac{R(x)}{P(x)} y = 0$$

$$y'' + p(x)\, y' + q(x)\, y = 0$$

{ ordinary point $P(x_0) \neq 0$.
{ singular point $P(x_0) = 0$ & $p(x)$, $q(x)$ become unbounded.

- Existence of solution – $p(x)$, $q(x)$ to be continuous on interval $|x-x_0| < \rho$

  for power series, we also need

  $p(x)$ & $q(x)$ to be analytic (p. 266 for proof)

- Radius of convergence is defined by the distance b/w the ordinary point x_0 & the nearest singular point.

- ordinary point ($x_0 = 0$) singular point

ordinary point ($x_0=0$)	singular point
$y = \sum_{n=0}^{\infty} a_n x^n$	$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{0}^{\infty} a_n x^{n+r}$
$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$	$y' = \sum_{n=1}^{\infty} a_n (n+r)\, x^{n+r-1}$
$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$	$y'' = \sum_{n=2}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-1}$

1° Find if x_0 is ordinary or singular point

2° choose appropriate solution ($y=\sum_0^\infty a_n x^n$ or $y=x^r\sum_0^\infty a_n x^n$)

3° plug into Diff. Eq

[Note] • Starting index of derivatives !
• Matching exponents

Ex ordinary point $\qquad P(x)=1, \quad x\in[-\infty,\infty]$

$$y''-(1+x)y=0 \Rightarrow y''-xy-y=0$$

$$y=\sum_{n=0}^\infty c_n x^n, \quad y'=\sum_{n=1}^\infty n c_n x^{n-1}, \quad y''=\sum_{n=2}^\infty n(n-1)c_n x^{n-2}$$

$$\Rightarrow \sum_{n=2}^\infty n(n-1)c_n x^{n-2} - x\sum_{n=0}^\infty c_n x^n - \sum_{n=0}^\infty c_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^\infty n(n-1)c_n x^{n-2} - \sum_{n=0}^\infty c_n x^{n+1} - \sum_{n=0}^\infty c_n x^n = 0$$

$k=n-2 \qquad k=n+1$

(match exponents) $\Rightarrow \sum_{k=0}^\infty (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^\infty c_{k-1}x^k - \sum_{k=0}^\infty c_k x^k = 0$

(starting points) $\Rightarrow 2c_2 + \sum_{k=1}^\infty (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^\infty c_{k-1}x^k - \left(c_0+\sum_{k=1}^\infty c_k x^k\right)=0$

$$\Rightarrow (2c_2-c_0)+\sum_{k=1}^\infty\left[(k+2)(k+1)c_{k+2}-c_{k-1}-c_k\right]x^k=0$$

$$\Rightarrow \begin{cases} 2c_2-c_0=0 & \Rightarrow c_2=\frac{1}{2}c_0 \\ (k+2)(k+1)c_{k+2}-c_{k-1}-c_k=0 & \Rightarrow c_{k+2}=\dfrac{c_k+c_{k-1}}{(k+2)(k+1)} \end{cases}$$

$k=1,2,3,\ldots$

(Recurrence relation)

$c_2 = \frac{1}{2} c_0$

$k=1, \quad c_3 = \frac{1}{3\cdot 2}(c_0 + c_1)$

$k=2, \quad c_4 = \frac{1}{4\cdot 3}(c_1 + c_2)$

$k=3, \quad c_5 = \frac{1}{5\cdot 4}(c_2 + c_3)$

} each coefficient is some multiple of c_0 & c_1

$y = c_0 y_1 + c_1 y_2$

We can find each soln y_1 & y_2 seperately since each is a sol. on its own.

let $c_0 \neq 0, c_1 = 0$

$c_2 = \frac{1}{2} c_0$

$c_3 = \frac{c_1 + c_0}{3\cdot 2} = \frac{c_0}{6}$

$c_4 = \frac{c_1 + c_2}{4\cdot 3} = \frac{c_0}{4\cdot 3\cdot 2} = \frac{c_0}{24}$

$c_5 = \frac{c_2 + c_3}{5\cdot 4} = \frac{\frac{1}{2}c_0 + \frac{1}{6}c_0}{5\cdot 4} = \frac{c_0}{30}$

$c_0 = 0, c_1 \neq 0$

$c_2 = \frac{1}{2} c_0 = 0$

$c_3 = \frac{c_0 + c_1}{3\cdot 2} = \frac{c_1}{6}$

$c_4 = \frac{c_1 + c_2}{4\cdot 3} = \frac{c_1}{12}$

$c_5 = \frac{c_2 + c_3}{5\cdot 4} = \frac{c_1}{120}$

$y = c_0 y_1 + c_1 y_2$

$\Rightarrow y = c_0 \left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots \right]$

$+ c_1 \left[x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots \right]$

Ex Singular point

$$2x^2y'' - xy' + (1+x)y = 0$$

$$y'' - \left(\frac{1}{2x}\right)y' + \left(\frac{1+x}{2x^2}\right)y = 0$$

(unbounded at $x=0$)

$$y = x^r \sum_{n=0}^{\infty} a_n x^n \Rightarrow 2x^2y'' - xy' + (1+x)y = 0$$

$$y' = \sum_{n=1}^{\infty} a_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$

$$\Rightarrow \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$$\Rightarrow a_0 2r(r-1)x^r + \sum_{n=1} 2a_n(n+r)(n+r-1)x^{n+r}$$
$$- a_0 r x^r - \sum_{n=1} a_n(n+r)x^{n+r}$$
$$+ a_0 x^r + \sum_{n=1} a_n x^{n+r}$$
$$+ \sum_{n=1} a_{n-1} x^{n+r} = 0$$

$$\Rightarrow a_0[2r(r-1) - r + 1]x^r$$
$$+ \sum \{[2(n+r)(n+r-1) - (n+r)]a_n + a_{n-1}\} x^{n+r} = 0$$

Indicial eq. $\Rightarrow 2r(r-1) - r + 1 = 0$

$$\Rightarrow 2r^2 - 3r + 1 = 0$$

$$\Rightarrow (2r-1)(r-1) = 0 \Rightarrow r_1 = 1,\ r_2 = \frac{1}{2}$$

- r_1 & r_2 called exponents at the singularity

· Recurrence relation

$$[2(n+r)(n+r-1)-(n+r)+1]a_n + a_{n-1} = 0$$

$$\Rightarrow \quad a_n = \frac{-a_{n-1}}{2(n+r)^2 - 3(n+r) + 1}$$

$$\Rightarrow \quad a_n = \frac{-a_{n-1}}{(n+r-1)(2(n+r)-1)}, \quad n = 1, 2, 3, \cdots$$

· Use each root to get a Recurrance relation.

find a solution y_1 for r_1

" " " y_2 for r_2

1° $r_1 = 1$, $a_n = \frac{-a_{n-1}}{(2n+1)n}$, $n = 1, 2, 3, \cdots$

$$a_1 = \frac{-a_0}{3}, \quad a_2 = \frac{-a_1}{5 \cdot 2} = \frac{a_0}{30}, \quad a_3 = \frac{-a_2}{7 \cdot 3} = -\frac{a_0}{630}, \cdots$$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = x(a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots)$$

$$= x\left(a_0 - \frac{a_0}{3}x + \frac{a_0}{30}x^2 - \frac{a_0}{630}x^3 + \cdots\right)$$

$$= a_0 \underbrace{\left[x - \frac{1}{3}x^2 + \frac{1}{30}x^3 - \frac{1}{630}x^4 + \cdots\right)}_{y_1}$$

2° $r_2 = \frac{1}{2}$, $a_n = \frac{-a_{n-1}}{(n-\frac{1}{2})(2n)} = \frac{-a_{n-1}}{n(2n-1)}$, $n = 1, 2, 3, \cdots$

$$a_1 = \frac{-a_0}{1}, \quad a_2 = \frac{-a_1}{2 \cdot 3} = \frac{a_0}{6}, \quad a_3 = -\frac{a_2}{3 \cdot 5} = -\frac{a_0}{90}$$

$$\therefore y = x^r \sum_{n=0} a_n x^n = x^{\frac{1}{2}}(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) = a_0 \underbrace{\left(x^{\frac{1}{2}} - x^{\frac{3}{2}} + \frac{1}{6}x^{\frac{5}{2}} - \frac{1}{90}x^{\frac{7}{2}} + \cdots\right)}_{y_2}$$

$$\therefore y = c_1 y_1 + c_2 y_2$$

Ex Singular point

$$xy'' + y = 0 \Rightarrow y'' + \underset{p}{0}\,y' + \underset{q}{\frac{1}{x}}\,y = 0, \quad P(x=0)=0$$

$p(x) = 0, \quad q(x) = \frac{1}{x}$

(a) $\lim_{x\to 0} x\,p(x) = 0, \quad \lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x = 0$ bounded at $x=0$

(b) $y = x^r \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=0}^{\infty} a_n (n+r)\, x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-2}$

$xy'' + y = 0$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$

(*x) $\Rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$

$\Rightarrow -$ $\quad k = n+1$ $\left(\sum_{k=1}^{\infty} a_{k-1} x^{k+r} \right)$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, x^{n+r} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$

$\Rightarrow a_0\, r(r-1)\, x^r + \sum_{n=1}^{\infty} \left[(n+r)(n+r-1) a_n + a_{n-1} \right] x^{n+r} = 0$

indicial eq: $r(r-1) = 0 \Rightarrow r_1 = 1,\ r_2 = 0 \quad (r_1 - r_2 = 1)$

recurrence relation: $(n+r)(n+r-1)\, a_n + a_{n-1} \Rightarrow a_n = \dfrac{-a_{n-1}}{(n+r)(n+r-1)}$

$r_1 - r_2 = 1 \Rightarrow$ integer $\Rightarrow$ sol'n is the larger value of r

(is → reponsed to)

1° $r_1 = 1$, $a_n = \dfrac{-a_{n-1}}{n(n+1)}$, $n = 1, 2, \cdots$

$a_1 = -\dfrac{a_2}{2}$

$a_2 = -\dfrac{a_1}{6} = \dfrac{a_0}{12}$

$a_3 = \dfrac{-a_2}{12} = \dfrac{-a_0}{144}$

$\therefore\ y = x^r \sum_{n=0}^{\infty} a_n x^n = x\left(a_0 - \frac{a_0}{2}x + \frac{a_0}{12}x^2 - \frac{a_0}{144}x^3 + \cdots\right)$

$\Rightarrow y = a_0\Big(\underbrace{x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots}_{y_1}\Big)$

$y_1 = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots$

$\therefore\ y = c_1 y_1$