5/1/15

MAE182A TA Session #5

P1

- Review chap 1 ~ 4.

1. Separable variables

ex Pr.3, P.16

$$\frac{dy}{dt} = -ay + b$$

$$\Rightarrow \frac{dy}{b-ay} = dt$$

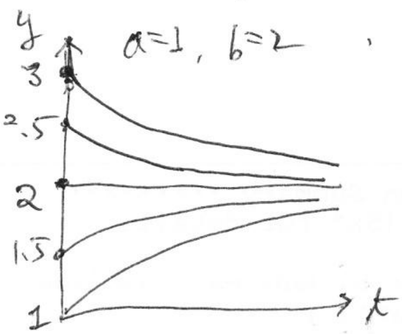
$$\Rightarrow -\frac{1}{a}\ln|b-ay| = t + c'$$

$$\Rightarrow |b-ay| = c e^{-at}$$

$$\Rightarrow b-ay = c e^{-at}$$

$$\Rightarrow y = \frac{b - c e^{-at}}{a}$$

@ eq. $\frac{dy}{dt} = 0 \Rightarrow y = \frac{b}{a}$



① $a\uparrow$ @ eq. $y = \frac{b}{a} \to 0$
convergence rate $a \uparrow \Rightarrow$ converge to equil. quicker

② $b\uparrow$ @ eq $y = \frac{b}{a}\uparrow \Rightarrow$ convergence rate same.

③ $a\uparrow, b\uparrow, \frac{b}{a} = \text{const}$ $y = \frac{b}{a} = \text{const}$.
Convergence rate $a\uparrow$

2. Integrating factor

ex. Pr. 15. P.40

$$ty' + 2y = t^2 - t + 1, \quad y(1) = \frac{1}{2}, \quad t > 0$$

① separable variables? ×

② Integrating factor

$$\ast\frac{1}{t} \Rightarrow \quad y' + 2t^{-1}y = t - 1 + t^{-1}$$

$$\ast\mu \Rightarrow \quad \mu y' + (2\mu t^{-1})y = \mu(t-1+t^{-1})$$

$$\Rightarrow \quad (\mu y)' = \mu y' + \mu' y = \mu(t-1+t^{-1})$$

$$\mu' = 2\mu t^{-1}$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \frac{2}{t}\,dt \Rightarrow \ln\mu = 2\ln|t| = \ln t^2 \Rightarrow \boxed{\mu = t^2}$$

$$(t^2 y)' = t^2 y' + 2ty = t^2(t-1+t^{-1}) = t^3 - t^2 + t$$

$$\Rightarrow \int d(t^2 y) = \int (t^3 - t^2 + t)\,dt$$

$$\Rightarrow \quad t^2 y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c$$

$$\Rightarrow \quad y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2}$$

I.C. $y(1) = \frac{1}{4} - \frac{1}{3} + \cancel{\frac{1}{2}} + \frac{c}{1} = \cancel{\frac{1}{2}}$

$$\Rightarrow \quad c = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

3. Exact eqs

$$M(x,y) + N(x,y)\, y' = 0 \quad —(1)$$

(1) is exact $\Longleftrightarrow$ $M_y(x,y) = N_x(x,y)$

$\exists\ \psi(x,y) \ni \psi_x(x,y) = M(x,y)$

$\psi_y(x,y) = N(x,y)$

solution for (1) $\psi(x,y) = c$

$$\frac{d\psi}{dx} = \underbrace{\psi_x}_{M(x,y)} \cancelto{1}{\frac{dx}{dx}} + \underbrace{\psi_y}_{N(x,y)} \frac{dy}{dx} = 0$$

$$\left.\begin{aligned} M_y &= \frac{\partial}{\partial y}\psi_x = \psi_{xy} \\ N_x &= \frac{\partial \psi_y}{\partial x} = \psi_{yx} \end{aligned}\right\} \quad \psi_{xy} = \psi_{yx} \Rightarrow M_y = N_x$$

ex Pr. 3. p. 133

$$\frac{dy}{dx} = \frac{2x+y}{3+3y^2-x}, \quad y(0) = 0$$

$$2x+y = (3+3y^2-x)\frac{dy}{dx} \Rightarrow \underbrace{(2x+y)}_{M} - \underbrace{(3+3y^2-x)}_{N}\frac{dy}{dx} = 0$$

$$\left.\begin{aligned} M(x,y) &= 2x+y \Rightarrow M_y = 1 \\ N(x,y) &= -3-3y^2+x \Rightarrow N_x = 1 \end{aligned}\right\} \quad M_y = N_x \Rightarrow \text{exact eq.}$$

$$\psi_x = M = 2x+y \Rightarrow \psi = \int (2x+y)\,dx + g(y)$$

$$\Rightarrow \psi(x,y) = x^2 + xy + g(y)$$

$$\psi_y = x + g'(y) = N = -3-3y^2+x \Rightarrow g'(y) = -3-3y^2$$

$$g(y) = \int (-3-3y^2)\,dy = -3y - y^3$$

$\therefore$ sol'n $\psi(x,y) = x^2 + xy - 3y - y^3 = c$, $y(0)=0 \Rightarrow c=0$

4. Homogeneous & Nonhomogeneous eqs.

ex Pr 24. P.184

$$y'' + 2y' + 2y = \underbrace{3e^{-t}}_{g_1} + \underbrace{2e^{-t}\cos t}_{g_2} + \underbrace{4e^{-t}t^2\sin t}_{g_3}$$

1° Homogen. sol'n

$$y'' + 2y' + 2y = 0$$

charact. eq. $\Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2 \pm 2i}{2} = -1 \pm i$

$\therefore y_h = e^{-t}(c_1 \cos t + c_2 \sin t)$

2° Nonhomogeneous sol'ns

① $g_1(t) = e^{-t} \Rightarrow Y_1(t) = Ae^{-t}$

② $g_2(t) = e^{-t}\cos t$

first guess: $Y_2(t) = B_1 e^{-t}\cos t + B_2 e^{-t}\sin t$, but it's the same as y_h

$\overset{*t}{\Longrightarrow} Y_2(t) = te^{-t}(B_1 \cos t + B_2 \sin t)$

③ $g_3(t) = e^{-t}t^2 \sin t$

first guess: $Y_3(t) = e^{-t}(D_0 t^2 + D_1 t + D_2)\cos t + e^{-t}(E_0 t^2 + E_1 t + E_2)\sin t$

but $D_2 e^{-t}\cos t$ & $E_2 e^{-t}\sin t$ are the same as y_h

$\overset{*t}{\Longrightarrow} Y_3(t) = te^{-t}(D_0 t^2 + D_1 t + D_2)\cos t + te^{-t}(E_0 t^2 + E_1 t + E_2)\sin t$

$\therefore Y = Y_1 + Y_2 + Y_3 = Ae^{-t} + te^{-t}(B_1 \cos t + B_2 \sin t) + te^{-t}(D_0 t^2 + D_1 t + D_2)\cos t + te^{-t}(E_0 t^2 + E_1 t + E_2)\sin t$

* $B_1 te^{-t}\cos t$ & $D_2 te^{-t}\cos t$ can be combined
* $B_2 te^{-t}\sin t$ & $E_2 te^{-t}\sin t$ can be combined.

$\therefore Y = Ae^{-t} + te^{-t}(D_0 t^2 + D_1 t + D_2)\cos t + te^{-t}(E_0 t^2 + E_1 t + E_2)\sin t$.

calculate $Y', Y'' \Rightarrow Y'' + 2Y' + 2Y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2\sin t$

compare coeff's $\Rightarrow A = 3,\ D_0 = -\frac{2}{3},\ D_1 = 0,\ D_2 = 1,\ E_0 = 0,\ E_1 = 1,\ E_2 = 1$

3° general sol'n: $y(t) = y_h + Y = e^{-t}(c_1 \cos t + c_2 \sin t) + 3e^{-t} + te^{-t}(-\frac{2}{3}t^2 + 1)\cos t + te^{-t}(t+1)\sin t$

5. Higer order D.E.

- Undetermined Coeff^s.

ex. Pr 1. P.239

$$y''' - y'' - y' + y = 2e^{-t} + 3 \quad — (1)$$

1° Homogeneous sol'n : $y''' - y'' - y' + y = 0$

chact. eq. $\Rightarrow r^3 - r^2 - r + 1 = 0 \Rightarrow (r-1)^2(r+1) = 0 \Rightarrow r = -1, 1, 1.$

$$y_c(t) = c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t}$$

2° Nonhomogeneous sol'ns.

① $y''' - y'' - y' + y = 2e^{-t}$

first guess : $Y_1 = Ae^{-t}$ but it is the same as y_c

$\overset{*t}{\Rightarrow} Y_1 = Ate^{-t}$

② $y''' - y'' - y' + y = 3 \Rightarrow Y_2 = B$

$\therefore Y = Y_1 + Y_2 = Ate^{-t} + B$

$Y' = Ae^{-t} - Ate^{-t} = A(1-t)e^{-t}$

$Y'' = -Ae^{-t} - A(1-t)e^{-t} = A(-2+t)e^{-t}$

$Y''' = Ae^{-t} - A(-2+t)e^{-t} = A(3-t)e^{-t}$

$(1) \Rightarrow Y''' - Y'' - Y' + Y = A(3-t)e^{-t} - A(-2+t)e^{-t} - A(1-t)e^{-t} + Ate^{-t} + B = 2e^{-t} + 3$

$\Rightarrow (3A - \cancel{At} + 2A - \cancel{At} - A + \cancel{At} + \cancel{At})e^{-t} + B = 2e^{-t} + 3$

$\Rightarrow (4A)e^{-t} + B = 2e^{-t} + 3 \Rightarrow A = \frac{1}{2},\ B = 3$

$\therefore Y = \frac{1}{2}te^{-t} + 3$

3° general sol'n : $\underline{y(t) = y_c + Y = c_1 e^{-t} + c_2 e^{t} + c_3 t e^{t} + \frac{1}{2} t e^{-t} + 3}$

6. Method of variation of parameters

ex Pr. 5, p.244

$$y''' - y'' + y' - y = e^{-t}\sin t$$

1° homogeneous sol'n

$$y''' - y'' + y' - y = 0$$

charact. eq. $\Rightarrow \; r^3 - r^2 + r - 1 = 0 \Rightarrow (r-1)(r^2+1) = 0 \Rightarrow r = 1, \pm i$

$$\therefore \; y_c = c_1 e^t + c_2 \cos t + c_3 \sin t$$

2° $$W(e^t, \cos t, \sin t) = \begin{vmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{vmatrix}$$

$$\Rightarrow W = e^t \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = e^t \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\sin t & \cos t \\ 2 & 0 & 0 \end{vmatrix} = 2e^t$$

3° $$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1$$

$$W_2 = \begin{vmatrix} e^t & 0 & \sin t \\ e^t & 0 & \cos t \\ e^t & 1 & -\sin t \end{vmatrix} = e^t \begin{vmatrix} 1 & 0 & \sin t \\ 1 & 0 & \cos t \\ 1 & 1 & -\sin t \end{vmatrix} = -e^t(\cos t - \sin t) = e^t(\sin t - \cos t)$$

$$W_3 = \begin{vmatrix} e^t & \cos t & 0 \\ e^t & -\sin t & 0 \\ e^t & -\cos t & 1 \end{vmatrix} = e^t \begin{vmatrix} 1 & \cos t \\ 1 & -\sin t \end{vmatrix} = -e^t(\sin t + \cos t)$$

4°

$$u_1'(t) = \frac{g(t)W_1(t)}{W(t)} = \frac{e^{-t}\sin t}{2e^{t}} = \frac{e^{-2t}\sin t}{2}$$

$$u_2'(t) = \frac{g(t)W_2(t)}{W(t)} = \frac{(e^{-t}\sin t)(\cancel{e^{t}}(\sin t-\cos t))}{2\cancel{e^{t}}} = \frac{e^{-t}(\sin^2 t-\sin t\cos t)}{2}$$

$$u_3'(t) = \frac{g(t)W_3(t)}{W(t)} = \frac{-(e^{-t}\sin t)\cancel{e^{t}}(\sin t+\cos t)}{2\cancel{e^{t}}} = -\frac{e^{-t}(\sin^2 t+\sin t\cos t)}{2}$$

5°

$$u_1(t) = \frac{1}{2}\int e^{-2t}\sin t = -\frac{1}{10}e^{-2t}(\cos t+2\sin t)$$

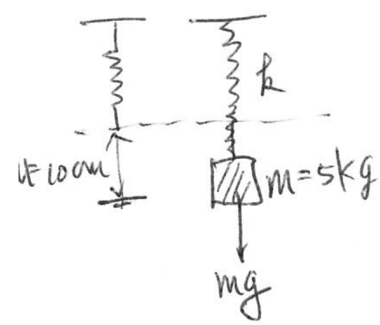
$$u_2(t) = -\frac{1}{4}e^{-t} + \frac{3}{20}e^{-t}\cos 2t - \frac{1}{20}\sin 2t$$

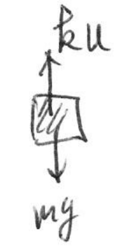
$$u_3(t) = \frac{1}{4}e^{-t} + \frac{1}{20}e^{-t}\cos 2t + \frac{3}{20}e^{-t}\sin 2t$$

$$\therefore Y = \sum_{i=1}^{3} y_i u_i = u_1 e^{t} + u_2\cos t + u_3\sin t$$

7. Forced Vibration

ex Pr 6.8. p.217

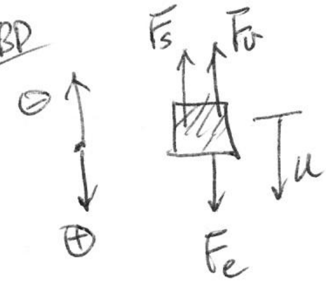


1° 

$mg = ku \Rightarrow k = \frac{mg}{u} = \frac{5\times 9.8}{0.1} = 490\ \frac{N}{m}$

2° $F_v = r u' \Rightarrow r = \frac{F_v}{u'} = \frac{2N}{0.04\ m/s} = 50\ \frac{N\cdot s}{m}$

FBD



3° $\sum_i F_i = m u'' = F_e - F_s - F_v$

$\Rightarrow 5u'' = 10\sin(\frac{t}{2}) - 490u - 50u'$

$\Rightarrow 5u'' + 50u' + 490u = 10\sin(\frac{t}{2})$.

$\Rightarrow u'' + 10u + 98u = 2\sin(\frac{t}{2})$

4° Homoge. sol'n

$u'' + 10u + 98u = 0$

charact. eq. $\Rightarrow r^2 + 10r + 98 = 0$

$\Rightarrow r = \frac{-10 \pm i2\sqrt{73}}{2} = -5 \pm i\sqrt{73}$

$\therefore u_c(t) = A e^{-5t}\cos\sqrt{73}t + B e^{-5t}\sin\sqrt{73}t$

5° Nonhomog. sol'n

$u'' + 10u + 98u = 2\sin(\frac{t}{2})$

Particular sol'n : $U(t) = c_1\cos(\frac{t}{2}) + c_2\sin(\frac{t}{2})$

$U',\ U''$

$U'' + 10U + 98U = 2\sin(\frac{t}{2})$

$\underbrace{(\quad)}_{0}\cos(\frac{t}{2}) + \underbrace{(\quad)}_{2}\sin(\frac{t}{2}) = 2\sin(\frac{t}{2}) \Rightarrow c_1, c_2 = ?$

6° General sol'n

$$u(t) = u_c(t) + U(t)$$

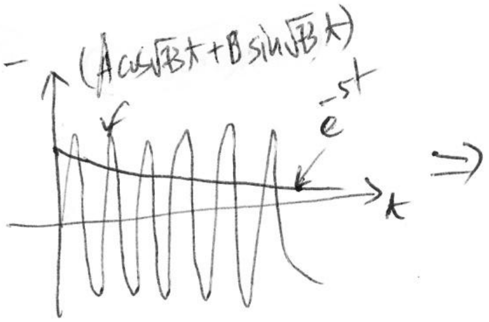
$$\Rightarrow u(t) = \underbrace{A}\, e^{-5t} \cos\sqrt{73}\,t + \underbrace{B}\, e^{-5t} \sin\sqrt{73}\,t + c_1 \cos\left(\tfrac{t}{2}\right) + c_2 \sin\left(\tfrac{t}{2}\right)$$

I.C.s: $u(0) = 0$, $u'(0) = 3\ \mathrm{cm/s} = 0.03\ \mathrm{m/s}$

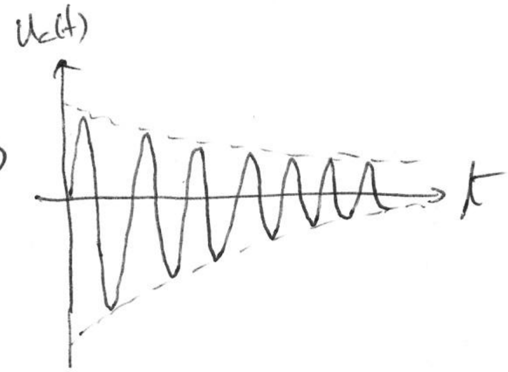
$$\Rightarrow A, B = ?$$

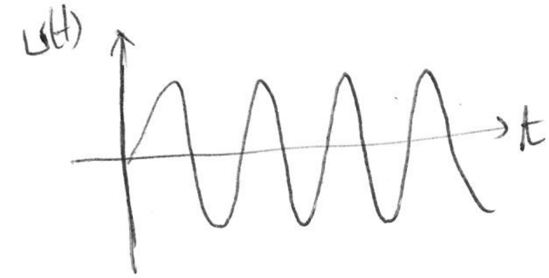
7° $u_c(t) = A \underbrace{e^{-5t}}_{\text{decay}} \cos\sqrt{73}\,t + B \underbrace{e^{-5t}}_{\text{decay}} \sin\sqrt{73}\,t$: transient state

$U(t) = c_1 \cos\left(\tfrac{t}{2}\right) + c_2 \sin\left(\tfrac{t}{2}\right)$: steady state



$\Rightarrow$





8° $mu'' + \gamma u' + ku = F_0 \cos\omega t$

(p210) Amplitude $R = \dfrac{F_0}{\Delta}$, $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$, $\omega_0^2 = \dfrac{k}{m}$

R max $\Rightarrow \Delta$ min $\Rightarrow \dfrac{d\Delta}{d\omega} = 0 \Rightarrow \omega = ?$

9° damping ratio $\xi = ?$ (Lec #6 slide)