4/24/15 MAE182A TA Session #4

Goals: Review chap 4: Higher Order Linear D.E.s

- Nth order linear O.D.E.
- Homogeneous Eqns. & Wronskian
  - Real & Unequal roots
  - Complex roots
  - Repeated roots
- Nonhomogeneous Eqs.
  - Undetermined coeff's
  - Variation of parameters

- Nth order ODE:

$$P_0(t)\frac{d^n y}{dt^n} + P_1(t)\frac{d^{(n-1)} y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t)$$

$$P_0(t) \neq 0 \Rightarrow L[y] = \frac{d^n y}{dt^n} + P_1(t)\frac{d^{(n-1)}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = g(t)$$

I.C.s. $y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots\ y^{(n-1)}(t_0) = y_0^{(n-1)}$

[Theorem 4.1.1] if $P_1, P_2, \cdots P_n$ & g are continuous on an interval I.

$\exists$ exactly one solution $y = \phi(t)$ $\ni$ satisfies the I.V.P. This solution exists throughout I.

- Homogeneous eqs

$$L[y] = \frac{d^n y}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = 0 \quad -(1)$$

if $y_1, y_2, \cdots, y_n$ are solns to (1)

$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ ~~are~~ is also soln.

I.C.s
$$\begin{cases} y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) + \cdots + c_n y_n(t_0) = y_0 \\ y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) + \cdots + c_n y_n'(t_0) = y_0' \\ \vdots \\ y^{(n-1)}(t_0) = c_1 y_1^{(n-1)}(t_0) + c_2 y_2^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

- Homogeneous Eqs & Wronskian

(1) has a unique solution iff

$$W(y_1, y_2, \cdots y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0$$

[Theorem 4.1.3]

$\{y_1, y_2, \cdots, y_n\}$ is a fundamental set of sols of

$L(y) = y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = 0$ on I

$\Leftrightarrow$ $\{y_1, \cdots, y_n\}$ are linearly indept. on I.

· Fundamental Solutions & Linear indept.

nth order ODE

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = 0$$

- $\{y_1 \cdots y_n\}$ are solns with $W(y_1, y_2, \cdots, y_n) \neq 0$ on I.
  is called a fundamental set of solns

- general solution:

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$$

[Abel's theorem]

$$y''' + p_1(t)\, y'' + p_2(t)\, y' + p_3(t)\, y = 0 \quad \text{— (2)}$$

y_1, y_2, y_3 are solns of (2)

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3 \qquad y(t_0) = y_0,\ y'(t_0) = y_0',\ y''(t_0) = y_0''$$

$$\begin{cases} y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) + c_3 y_3(t_0) = y_0 & \text{— (1)} \\ y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) + c_3 y_3'(t_0) = y_0' & \text{— (2)} \\ y''(t_0) = c_1 y_1''(t_0) + c_2 y_2''(t_0) + c_3 y_3''(t_0) = y_0'' & \text{— (3)} \end{cases}$$

P5

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) & y_3(t_0) \\ y_0' & y_2'(t_0) & y_3'(t_0) \\ y_0'' & y_2''(t_0) & y_3''(t_0) \end{vmatrix}}{W(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & y_3(t_0) \\ y_1'(t_0) & y_2'(t_0) & y_3'(t_0) \\ y_1''(t_0) & y_2''(t_0) & y_3''(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1 & y_0 & y_3 \\ y_1' & y_0' & y_3' \\ y_1'' & y_0'' & y_3'' \end{vmatrix}}{W(t_0)}, \quad c_3 = \frac{\begin{vmatrix} y_1 & y_2 & y_0 \\ y_1' & y_2' & y_0' \\ y_1'' & y_2'' & y_0'' \end{vmatrix}}{W(t_0)}$$

c_1, c_2 & c_3 exist $\Rightarrow W \neq 0$.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W' = \cancel{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} + \cancel{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

$$p_2 p_3 W' = \begin{vmatrix} p_3 y_1 & p_3 y_2 & p_3 y_3 \\ p_2 y_1' & p_2 y_2' & p_2 y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} p_3 y_1 & p_3 y_2 & p_3 y_3 \\ p_2 y_1' & p_2 y_2' & p_2 y_3' \\ y_1''' + p_2 y_1' + p_3 y_1 & y_2''' + p_2 y_2' + p_3 y_2 & y_3''' + p_2 y_3' + p_3 y_3 \end{vmatrix}.$$

$$\Rightarrow \cancel{p_2 p_3} W' = \begin{vmatrix} p_3 y_1 & p_3 y_2 & p_3 y_3 \\ p_2 y_1' & p_2 y_2' & p_2 y_3' \\ -p_1 y_1'' & -p_1 y_2'' & -p_1 y_3'' \end{vmatrix} = -p_1 p_2 p_3 \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -p_1 \cancel{p_2 p_3} W$$

$$\Rightarrow W' = -p_1 W \Rightarrow W = c \exp\left[-\int p_1(t)\,dt\right]$$

$$y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_n(t)\, y = 0$$ P6

By induction:

$$W(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ \vdots & \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W'(y_1, y_2 \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ \vdots & & & & \\ y_1^{(n-2)} & y_2^{(n-2)} & y_3^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

$$p_2 p_3 \cdots p_n W' = -p_1 p_2 \cdots p_n W$$

$$\Rightarrow \quad W' = -p_1 W$$

$$\Rightarrow \quad W(y_1, y_2 \cdots y_n)(t) = c \exp\left[-\int p_1\, dt\right]$$

ex. Pr 24 p. 227

$$t^2 y^{(4)} + t y''' + y'' - 4y = 0$$

$$\Rightarrow \quad y^{(4)} + \underbrace{t^{-1}}_{= p_1} y''' + t^{-2} y'' - 4t^{-2} y = 0$$

$$p_1 = t^{-1} \Rightarrow W = c \exp\left[-\int \left(\tfrac{1}{t}\right) dt\right] = c\, e^{-\ln t} = \frac{c}{t}$$

ex Pr26. p234

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0$$

$$y = e^{rt} \Rightarrow r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$$

$$\begin{matrix} (r^2 & -6r & +6) \\ (r^2 & -1r & -6) \end{matrix}$$

$$\Rightarrow (r^2 - 6r + 6)(r^2 - r - 6) = 0$$

$$\Rightarrow (r^2 - 6r + 6)(r - 3)(r + 2) = 0$$

$$\Rightarrow r = -2, 3, \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

$$\therefore y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}$$

ex Pr28. p234

$$y^{(4)} + 6y''' + 17y'' + 22y' + 14y = 0$$

$$y = e^{rt} \Rightarrow r^4 + 6r^3 + 17r^2 + 22r + 14 = 0.$$

$$\Rightarrow \begin{matrix} r^2 & 4r & 7 \\ r^2 & 2r & 2 \end{matrix}$$

$$\Rightarrow (r^2 + 4r + 7)(r^2 + 2r + 2) = 0$$

$$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 28}}{2} = \frac{-4 \pm i\sqrt{3}}{2} = -2 \pm i\sqrt{3}, \quad \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm i2}{2} = -1 \pm i$$

$$\therefore y = e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-2t}(c_3 \cos\sqrt{3}t + c_4 \sin\sqrt{3}t)$$

- Nonhomogeneous eqs.

ex Pr. 5 p. 239

$$y^{(4)} - 4y'' = t^2 + e^t$$

1° homogeneous solution

$$y^{(4)} - 4y'' = 0$$

char. eq. $\Rightarrow r^4 - 4r^2 = 0 \Rightarrow r^2(r^2-4)=0 \Rightarrow r = 0, 0, -2, 2$

$$y_c(t) = c_1 e^{0t} + c_2 t e^{0t} + c_3 e^{-2t} + c_4 e^{2t} = c_1 + c_2 t + c_3 e^{-2t} + c_4 e^{2t}$$

2° nonhomogeneous solution

$$y^{(4)} - 4y'' = t^2$$

$$\Rightarrow y_{p_1} = t^2(At^2 + Bt + C)$$

$$y^{(4)} - 4y'' = e^t$$

$$\Rightarrow y_{p_2} = De^t$$

$$y = y_c + y_{p_1} + y_{p_2} = c_1 + c_2 t + c_3 e^{-2t} + c_4 e^{2t} + At^4 + Bt^3 + Ct^2 + De^t$$

$$\Rightarrow y = c_1 + c_2 t + c_3 e^{-2t} + c_4 e^{2t} - \frac{1}{48} t^4 - \frac{1}{16} t^2 - \frac{1}{3} e^t$$

ex Pr 18, p.239

$$y^{(4)}+2y'''+2y''=3e^{t}+2te^{-t}+e^{-t}\sin t$$

1° Homogeneous solutions

$$y^{(4)}+2y'''+2y''=0$$

char eq. $\Rightarrow$ $r^4+2r^3+2r^2=0 \Rightarrow r^2(r^2+2r+2)=0 \Rightarrow r=0,0,-1\pm i$

$$y_c=c_1e^{0t}+c_2te^{0t}+e^{-t}(c_3\cos t+c_4\sin t)=c_1+c_2t+c_3e^{-t}\cos t+c_4e^{-t}\sin t$$

2° Nonhomogeneous solutions

① $y^{(4)}+2y'''+2y''=3e^{t}$

$\Rightarrow y_{p_1}=Ae^{t}$

② $y^{(4)}+2y'''+2y''=2te^{-t}$

$\Rightarrow y_{p_2}=(Bt+C)e^{-t}$

③ $y^{(4)}+2y'''+2y''=e^{-t}\sin t$

$\Rightarrow y_{p_3}=Dte^{-t}\sin t+Ete^{-t}\cos t$

$\therefore y=y_c+y_{p_1}+y_{p_2}+y_{p_3}$

$\Rightarrow y=c_1+c_2t+c_3e^{-t}\cos t+c_4e^{-t}\sin t+Ae^{t}+(Bt+C)e^{-t}+t(De^{-t}\sin t+Ee^{-t}\cos t)$

ex Pr9. P.244

$$y''' + y' = \sec t; \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = -2$$

1° Homogeneous solutions

$$y''' + y' = 0$$

char eq. $\Rightarrow$ $r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \Rightarrow r = 0, +i, -i$

$$\Rightarrow y_c = c_1 + c_2 \cos t + c_3 \sin t$$

$$W(1, \cos t, \sin t) = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1$$

$$W_1(t) = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1, \quad W_2 = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t$$

$$W_3(t) = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t$$

$$u_1' = \frac{\sec t\, W_1}{W} = \sec t, \quad u_2' = \frac{\sec t\, W_2}{W} = -1, \quad u_3' = \frac{\sec t\, W_3}{W} = -\frac{\sin t}{\cos t}$$

$$\Rightarrow u_1 = \ln(\sec t + \tan t), \quad u_2 = -t, \quad u_3 = \ln(\cos t)$$

P.242 $\Rightarrow Y(t) = \ln|\sec t + \tan t) - t\cos t + \sin t \ln(\cos t)$

ex Pr 16, p.245

$$y''' - 3y'' + 3y' - y = g(t), \quad g(t) = t^{-2}e^{t} \quad Y(t) = ?$$

char. eq. $r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0 \Rightarrow r = 1, 1, 1$

$$y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$$

$$W(e^t, te^t, t^2e^t) = \begin{vmatrix} e^t & te^t & t^2e^t \\ e^t & (t+1)e^t & (t^2+2t)e^t \\ e^t & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix}$$

$$= e^{3t}\begin{vmatrix} 1 & t & t^2 \\ 1 & t+1 & t^2+2t \\ 1 & t+2 & t^2+4t+2 \end{vmatrix} = e^{3t}\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 2 & 4t+2 \end{vmatrix}.$$

$$= e^{3t}\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2e^{3t}$$

$$W_1 = \begin{vmatrix} 0 & te^t & t^2e^t \\ 0 & (t+1)e^t & (t^2+2t)e^t \\ 1 & (t+2)e^t & (t^2+4t+2)e^t \end{vmatrix} = t^2e^{2t}$$

$$W_2 = \begin{vmatrix} e^t & 0 & t^2e^t \\ e^t & 0 & (t^2+2t)e^t \\ e^t & 1 & (t^2+4t+2)e^t \end{vmatrix} = -2te^{2t}, \quad W_3 = \begin{vmatrix} e^t & te^t & 0 \\ e^t & (t+1)e^t & 0 \\ e^t & (t+2)e^t & 1 \end{vmatrix} = e^{2t}.$$

$$u_1' = \frac{g(t)W_1}{W} = \frac{g(t)t^2 e^{2t}}{2e^{3t}} = \frac{g(t)t^2 e^{-t}}{2}$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{-g(t)\,2t\,e^{2t}}{2e^{3t}} = -g(t)\,t\,e^{-t}$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{g(t)\,e^{2t}}{2e^{3t}} = \frac{g(t)\,e^{-t}}{2}$$

$$Y(t) = \sum_{m=1}^{3} y_m(t) \int_{t_0}^{t} \frac{g(s)\,W_m(s)}{W(s)}\,ds.$$

$$\Rightarrow Y(t) = e^t \int_{t_0}^{t} \frac{g(s)s^2e^{-s}}{2}\,ds + te^t \int_{t_0}^{t} -g(s)se^{-s}\,ds + t^2e^t \int_{t_0}^{t} \frac{g(s)e^{-s}}{2}\,ds$$

$$\Rightarrow Y(s) = \int_{t_0}^{t} \frac{g(s)\,e^{t-s}(s^2-2ts+t^2)}{2}\,ds = \int_{t_0}^{t} \frac{g(s)\,e^{t-s}(s-t)^2}{2}\,ds$$

$$g(t) = t^{-2}e^t \Rightarrow Y(s) = \int_{t_0}^{t} \frac{s^{-2}e^{s}\cdot e^{t-s}(s-t)^2}{2}\,ds = e^t \int_{t_0}^{t} \frac{s^{-2}(s-t)^2}{2}\,ds$$

$$\Rightarrow Y(s) = e^t \int_{t_0}^{t} \frac{1}{s^2}\frac{(s^2-2st+t^2)}{2}\,ds = e^t \int \left(\frac{1}{2} - \frac{t}{s} + \frac{t^2}{2s^2}\right)ds$$

$$\Rightarrow Y(s) = e^t\left[\left(\cancel{\tfrac{1}{2}t} - \underbrace{t\ln t} - \cancel{\tfrac{t^2}{2}\tfrac{1}{t}}\right) - \left(\underbrace{\tfrac{1}{2}t_0e^t - \ln|t_0|\,te^t - \tfrac{1}{2t_0}t^2e^t}_{t_0 \text{ is const. this part can combine w/ } y_c}\right)\right]$$

$$\Rightarrow Y(s) = -te^t\ln t$$